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THE FUNDAMENTAL PRINCIPLE OF EHRENPREIS - PALAMODOV

New proofs, and essential uniqueness of the construction

by
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În amintirea lui Bogdan și a Anei

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New proofs, and essential uniqueness of the construction

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The fundamental principle for a system p(D) of linear constant coefficient partial differential operators states, roughly speaking, that the solutions in convex sets of the homogeneous equation $p(D)u = 0$, are superpositions of polynomial exponential solutions, for that equation. Unlike the case of ordinary differential operators, this is a deep result, and was stated (in L.Ehrenpreis [1]) and proved rather recently by L.Ehrenpreis and V.P.Palamodov.

It turns out, that the fundamental principle is closely related to the solvability in holomorphic germs of systems of form

$$\sum_{j=1}^{m} p_{ij}(z)\, h_j(z) = g_i(z), \quad i=1,\dots,s \qquad (1)$$

where the p_{ij} are polynomials, and the g_i are holomorphic germs.

The solvability of (1) can be studied with the aid of algebraic Noetherian operators. An algebraic Noetherian operator, is a couple (V, ∂), where V is an algebraic variety, and ∂ is a s-vector of differential operators in $\partial/\partial z_r$, $r=1,\dots,n$ with polynomial coefficients. It is possible to prove, that for every matrix p_{ij}, there is a collection (V^k, ∂^k), $k=1,\dots,\mu$ of Noetherian operators, such that the following two assertions are equivalent:

(i) if g_i are germs of holomorphic functions near z^o, then the restriction to the germ of V^k at z^o of $\sum \partial_i^k g_i$ vanishes identically.

(ii) there are h_j, holomorphic near z^o such that $ph = g$.

If p(D) is as above, and $p\ (=p_{ij}(z))$ is the polynomial matrix, associated with p(D), by changing $\partial/\partial x_r$ with $-i\ z_r$, $i^2=-1$, we can therefore find algebraic Noetherian operators for the matrix ${}^t p$ (the matrix transposed to p) such that for them (i) and (ii) are equivalent. The relation of these operators to the fundamental principle, is the following: if u is a C^∞ solution of $p(D)u = 0$ which is defined in Ω , $\Omega \subset R^n$ a convex set, then for every convex compact $B \subset \Omega$ and for every $b \in R$, which is great enough, there are Radon measures μ_k, concentrated on V^k, such that $\int d|\mu_k| < \infty$, and such that $u(x) = \int \sum_k \partial^k (\exp i\langle x,z\rangle)\, d\mu_k(z) / \exp(H_B(\mathrm{Im}\, z) + b\ln(1+|z|))$. H_B is here the support function of B.

In this paper, we present a new proof for the fundamental principle. The main new features which appear, are related to the fact, that we study the essential uniqueness of Noetherian operators, associated with a matrix, and that we study the structure of distributions concentrated on algebraic varieties.

Some results from this paper, are perhaps of independente interest.

Notations.

Thoughout this paper, we use without further explanation the following notations (which are standard, except the notation $d_V f = 0$):

- N, R, R_+, C, are the natural, real, real positive, complex numbers.
- $X^n = \prod_{i=1}^{n} X$. (Sometimes, $X^n = \prod_{i=0}^{n-1} X$).
- Variables in C^n are denoted $z = (z_1, \ldots, z_n)$. For special choices of coordinates, $z = (z', t)$, $z = (x, y)$, where z', t, x, y, denote groups of variables. z' is usually $z' = (z_1, \ldots, z_{n-1})$. x stands also for the variables in R^n.
- O_x, F_x are, for $x \in C^k$, the germs of holomorphic functions at z respectively their completion in the Krull topology, the formal power series in complex indeterminates. k should be clear from context. For $x = 0$ we also write $O_o = O$.
- $O(U)$ are the holomorphic functions in the domain $U \subset C^n$.
- $V \subset U$ is called an analytic variety in U, if there are $f_1, \ldots, f_k \in O(U)$ such that $V = \{z \in U;\ f_i(z) = 0\}$.
- $I_z(V) = \{f \in O_z;\ f$ vanishes on the germ of V at $z\}$.

- $I_z^F(V)$ is the ideal generated by $I_z(V)$ in F_z.
- P denotes the polynomials in n variables, with complex coefficients, and in the indeterminate z.
- I(V). For an algebraic variety $V \subset C^n$, we denote $I(V) = \{f \in P;$ f vanishes on $V\}$. Let us note here the following: if V is an algebraic variety, and z is in V, then $f \in F_z$ is in $I_z^F(V)$, if and only if there are $f_1, \ldots, f_k \in I(V)$ and $g_i \in F_z$ such that $f = \sum f_i g_i$.
- $d_V f = 0$. If V is a germ of an analytic variety at z, and $f \in F_z$, we say that $d_V f = 0$ if $f \in I_z^F(V)$. When $f \in O_z$, this means that the restriction of f to V vanishes.
- $C^\infty(C^n)$ are the infinitely differentiable functions on $C^n = R^{2n}$.
- $C_0^\infty(\Omega)$ is the subspace in $C^\infty(C^n)$ of elements with compact support in Ω.
- $S(C^n), S'(C^n)$ is the subspace in $C^\infty(C^n)$ of elements which decrease rapidly (in the sense of L.Schwartz) at infinity, respectively the dual of $S(C^n)$, the tempered distributions.
- $E'(C^n)$ are the distributions with compact support in C^n.
- $H^z(C^n)$ is the Sobolev space of order z, with norm $\| \|_z$.
- $\hat{u}$ is the Fourier transform of u.
- If $f: X \to Y$, then ${}^t f$ is the transpose of f (when this makes sense) This notation is also used for matrices.
- D stands generically for $\partial/\partial x_j$ when we work in R^n, and for $(\partial/\partial z_j, \partial/\partial \bar{z}_j)$, when we work in C^n. The elements from F_x depend only on z. Therefore in part I, practically, $D \rightsquigarrow \partial/\partial z_j$.
- If $p(z,D) = \sum_{|\alpha| \le m} a^\alpha (\partial/\partial z)^\alpha$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum \alpha_i$, $\alpha_i \in N$, and $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in N$, then we denote

  $$p(z,D)^{(\beta)} = \sum_{\alpha_i \le \beta_i} a_\alpha (\alpha!/(\alpha-\beta)!) (\partial/\partial z)^{\alpha-\beta} \text{ and}$$

  $$p(z,D)_{(\beta)} = \sum_{|\alpha| \le m} ((\partial/\partial z)^\beta a_\alpha(z))(\partial/\partial z)^\alpha.$$

PART I. The construction of Noetherian operators.

§1. Preliminaries.

1. Consider $z\in C^n$ and $A\subset F_z$ a ring which is closed under differentiation by $D_k=(\partial/\partial z_k)$, $k=1,\dots,n$. $A(D)$ denotes the ring of finite order linear partial differential operators of form $a(D)=\sum_{|\alpha|\leq\sigma} a_\alpha D^\alpha$, $a_\alpha\in A$, $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$. For $f\in F_z$ we define $a(D)f=\sum_\alpha a_\alpha D^\alpha f$.

Theorem 1.1. If A is Noetherian, then so is $A(D)$.

This theorem is due to I.B.Lopatinski. For a proof cf., e.g. B.Malgrange [5] or V.P.Palamodov [3] .

2. Consider $L_{ij}\in F_z(D)$, $i=1,\dots,m$, $j=1,\dots,s$. We are interested in solvability conditions for the system

$$\sum_j L_{ij}\ g_j=f_i,\quad i=1,\dots,m \tag{1}$$

where $f_i\in F_z$.

To obtain these solvability conditions, denote

$$R_z=\{(r_1,\dots,r_m)\in[C(D)]^m\ ;\ d_z\sum_i r_iL_{ij}=0, j=1,\dots,s)\}.$$

Here $C(D)$ is $A(D)$ for $A=C$.

Proposition 1.2. For every $f\in F_z^m$ there are equivalent:

(i) there is $g\in F_z^s$ such $\sum L_{ij}\ g_j=f_i$, $i=1,\dots,m$.

(ii) for every $r\in R_z$, $d_z\sum r_if_i=0$.

This proposition is an easy consequence of the following result, which we recall from L.Hörmander [3] , lemma 6.3.7.

Lemma 1.3. Consider $A_1,A_2,\dots$ a sequence of linear forms with complex coefficients, in the complex variables $\xi_1,\xi_2,\dots$, and suppose that each A_i depends only on a finite number of variables ξ_j. Then the system $A_j(\xi)=b_j$, $j=1,2,\dots$ is solvable, if and only if, every finite number of equations in the system has a solution.

Proof of proposition 1.2. (i)$\Longrightarrow$(ii) is trivial. To prove (ii)$\Longrightarrow$(i), we first ease the notations, by setting $z=0$. We then write $g_j = \gamma_j^\alpha z^\alpha$, $f_i = \varphi_i^\alpha z^\alpha$, $r_i = \sum_{|\beta|\leq\nu} \rho_i^\beta D^\beta$, $L_{ij} = \sum_{\alpha,|\beta|\leq\nu} \lambda_{ij}^{\alpha\beta} z^\alpha D^\beta$, with γ_j^α, φ_i^α, ρ_i^β, $\lambda_{ij}^{\alpha\beta}$ constants.

The equations $\sum L_{ij}\, g_j = f_i$ now reduce to

$$\sum_j \sum \lambda_{ij}^{\alpha\beta}\, (\varkappa!/(\varkappa-\beta)!)\, \gamma_j^\varkappa = \varphi_i^\sigma \qquad \forall\sigma, \forall i$$

where the second sum is extended over all $(\alpha,\beta,\varkappa) \in B_\sigma = \{(\alpha,\beta,\varkappa);\ |\beta|\leq\nu, \beta\leq\varkappa, \alpha+\varkappa-\beta=\sigma\}$.

In view of Lemma 1.3 this infinite system can be solved if and only if

$$\sum_{i,\sigma} \rho_i^\sigma \sum_{\substack{(\alpha,\beta,\varkappa)\in B_\sigma\\ \alpha,\beta}} \lambda_{ij}^{\alpha\beta}/(\sigma-\alpha)! = 0 \quad \text{for all } j \text{ and } \varkappa$$

implies $\sum_{i,\sigma} \rho_i^\sigma\ \varphi_i^\sigma = 0$. Here ρ_i^σ is different from zero only for a finite set of indices.

To see if this condition is satisfied, it remains to rewrite (ii) First note that $r_i\, L_{ij} = \sum_\sigma \rho_i^\sigma \sum_{\alpha,\beta} \lambda_{ij}^{\alpha\beta} \sum_{\mu\leq\sigma,\mu\leq\alpha} z^{\alpha-\mu}$ $(1/\mu!)(\alpha!/(\alpha-\mu)!)(\sigma!/(\sigma-\mu)!)D^{\beta+\sigma-\mu}$ and therefore $d_0 \sum_i r_i L_{ij} = 0$ comes to $\sum_i \sum_{\substack{\beta+\sigma-\alpha=\varkappa\\ \sigma\geq\alpha}} \rho_i^\sigma\ \lambda_{ij}^{\alpha\beta}\ \sigma!/(\sigma-\alpha)! = 0$ for all $\varkappa$ and j. Similarily, $\sum r_i f_i = 0$, is equivalent with $\sum_{i,\sigma} \rho_i^\sigma\ \varphi_i^\sigma\ \sigma! = 0$. The proposition now follows.

<u>3. Proposition 1.4.</u> Consider $L_{ij} \in O(|z|<\eta)(D)$ and let V be an analytic variety defined in $|z|<\eta$. Denote $R_V = \{(r_1,\ldots,r_m) \in [O(|z|<\eta)(D)]^m;\ d_V(\sum_i r_i L_{ij})=0, \text{for all } j\}$.

Then there is $W\subset V$ with the following properties:

- W is a countable union of analytic varieties, all defined in $|z|<\eta$, such that $V\setminus W$ is dense in V.
- for $z\in V\setminus W$, the natural map $d_z R_V \longrightarrow R_z$ is surjective.

Proof. Consider $R_V^k = \{(r_1,\ldots,r_m) \in R_V;$ all r_i are of order less then $k\}$, and similarily, $R_z^k = \{(r_1,\ldots,r_m) \in R_z;$ all r_i are of order less then $k\}$. It suffices to find an analytic variety W_k, such that $V\setminus W_k$ is dense in V and such that

$d_z: R^k_V \longrightarrow R^k_z$ is surjective for $z \in V \setminus W_k$.

Now we write down $d_y \sum r_i L_{ij} = 0$ explicitely, and arrive at a system $d_y \sum_{i, |\alpha| \leq k} \rho_i^\alpha \; \lambda_{ij}^{\alpha\beta} = 0$ for all j and β, for some analytic functions $\lambda_{ij}^{\alpha\beta}$, only a finite number of them being different from zero. It remains therefore to compare solutions of $\sum \rho_i^\alpha \; \lambda_{ij}^{\alpha\beta}(y)=0$, ρ_i^α constants, with solutions of $d_V \sum \rho_i^\alpha(z) \; \lambda_{ij}^{\alpha\beta}(z) = 0$. It follows that $d_z: R_V \longrightarrow R_z$ is surjective at the points where the rank of matrices $(\lambda_{ij}^{\alpha\beta}(y))_{\alpha, i}$ is maximal in V for all β and j. This clearly happens outside a set W_k with the desired properties.

4. Regarding the solvability of the system (1), we finally need the following result:

<u>Proposition 1.5.</u> Consider $1 \leq k \leq n$ and suppose that $L_{ij} \in O(D)$ $i=1,\ldots,m$, $j=1,\ldots,s$ are operators which contain only derivations in $D_1,\ldots,D_k$. Denote $R = \{ (r_1,\ldots,r_m) \in [O(D)]^m \; ; \; \sum_i r_i L_{ij} = 0, j=1,\ldots,s \}$. Then R is generated, as an O(D) module, by the subset of elements which contain only derivations in $D_1,\ldots,D_k$.

Proof. Denote $D_x = (D_1,\ldots,D_k), D_y = (D_{k+1},\ldots,D_n)$, suppose that $r \in R$ and write $r_i = \sum_{|\beta| \leq \sigma} r_i^\beta(z, D_x) D_y^\beta$. The proof is by increasing induction in σ. In fact, if for all j, $\sum_i r_i L_{ij} = 0$, then clearly $\sum_i r_i^\beta L_{ij} = 0$ for all j and all β such that $|\beta| = \sigma$. Then $(r_i - \sum_{|\beta| = \sigma} D_y^\beta r_i^\beta)_i$ is in R, and is an operator of order less then σ in D_y.

5. In the remaining part of this paragraph, we recall some results from analytic geometry and local algebra.

<u>Definition 1.6.</u> Consider A a ring and E an A module. For $f = (f_1,\ldots,f_m) \in A^m$ denote R(f,E) the submodule in E^m of those vectors $e = (e_1,\ldots,e_m)$ such that $\sum f_i e_i = 0$. E is called flat over A if $R(f,E) = R(f,A)E$ for all f.

Theorem 1.7. F_z is flat over O_z and P. If $|z|<\eta$, then O_z is flat over $O(|z|<\eta)$.

cf. N. Bourbaki [1] .

Corollary 1.8. Consider $V\subset C^n$ an algebraic variety and $p_{ij}\in P$, $i=1,\ldots,m$, $j=1,\ldots,s$. For $z\in V$ denote $G_1 = \{ f\in F_z^s;\ d_V\sum p_{ij} f_j = 0,\ i=1,\ldots,m\}$ and $G_2 = \{ f\in P^s;\ d_V\sum p_{ij} f_j = 0,\ i=1,\ldots,m\}$. Then G_1 is generated by G_2 as a F_z module.

Proposition 1.9. Consider $V\subset C^n$ an algebraic variety, $z\in V$ and denote $G_z = \{\lambda\in O_z(D);\ f\in F_z,\ d_V f = 0 \Rightarrow d_V\lambda f = 0\}$, $G = \{\lambda\in P(D)\ ;\ \forall\ z\in C^n, f\in F_z, d_V f = 0 \Rightarrow d_V\lambda f = 0\}$.

Then for every $\lambda\in G_z$ there are $f_i\in O_z$ and $\lambda_i\in G$ such that $\lambda = \sum f_i\lambda_i$.

Proof. $\lambda\in G_z$ is equivalent with $d_V\lambda^{(\beta)} p = 0$ for all β and all $p\in I(V)$. (To see this, we observe that $\lambda^{(\beta)} z_j p = z_j\lambda^{(\beta)} p + \lambda^{(\beta')} p$, $\beta_i' = \beta_i, i\neq j$, $\beta_j = \beta_j'+1$, and apply induction in $|\beta|$). Now write $\lambda = \sum_{|\alpha|\leq\nu}\lambda_\alpha D^\alpha$, $d_V\lambda^{(\beta)} p = 0$ then gives $d_V\sum_{\alpha\geq\beta}\lambda_\alpha(\alpha!/(\alpha-\beta)!)D^{\alpha-\beta}\ p = 0$. Denote $p_i, i=1,\ldots,q$, generators for $I(V)$. Then $\lambda\in G_z$ is equivalent with $d_V\sum_{\alpha\geq\beta}\lambda_\alpha\ (\alpha!/(\alpha-\beta)!)D^{\alpha-\beta}p_i = 0$, $i=1,\ldots,q$, $\forall\ \beta$. This means that for ν fixed, the functions λ_α satisfy certain relations, with polynomial coefficients. We can now apply corollary 1.8 and conclude that the polynomial solutions of the same relations generate the solutions in germs.

Lemma 1.10. Consider $I\subsetneqq O$ a primary ideal. Then we can choose coordinates (z',t), a Weierstrass pseudopolynomial, $p = t^q + \sum_{j<q} c_j(z')t^j$, $c_j\in O, c_j(0) = 0$, and $\alpha\in N$, such that p has no multiple factors, $p^\alpha\in I$, but such that $\partial p/\partial t$ is not in the radical of I. In particular, the discriminant of p with respect to t is not in this radical. A similar assertion holds, when I is a polynomial ideal. In fact in this case p can be chosen to be a polynomial.

Proof. ($I \subset O$). We choose convenient coordinates and apply the Weierstrass preparation theorem, to find g in the radical of I of form $g = t^q + \sum_{j<q} c_j(z')t^j$, $c_j(0) = 0$. Among all g with these two properties, we choose one of minimum degree. This implies that g has no multiple factors, and also that $\partial g/\partial t$ is not in the radical of I. Finally we apply the Nullstellensatz.

§2. Formal properties.

1. In this paragraph A denotes a ring such that:

- $A \subset O(|z|<\eta)$ for some η.
- A is stable under differentiation.
- F_z is flat over A for any z with $|z|<\eta$.

Theorem 2.1. Suppose there are given

- $L_{ij}(D), L_j(D)$, $i=1,\dots,m$, $j=1,\dots,s$, all in $A(D)$
- $U \subset \{|z|<\eta\}$ a countable union of analytic varieties of codimension greater then one, defined in $|z|<\eta$.

Suppose that the following property is satisfied: for every $z \in \complement U$, $|z|<\eta$, and every $u_1,\dots,u_s \in F_z$, for which $\sum_j L_{ij} u_j = 0$ for $i=1,\dots,m$, we also have $\sum L_j u_j = 0$.

Then there are $f \in A, f\neq 0$ and $f_i \in A(D)$ such that

$f L_j = \sum_i f_i L_{ij}$ for $j=1,\dots,s$.

Moreover, if the L_j, L_{ij} are operators in $D_1,\dots,D_k$ only, then the f_i can be chosen to depend also only on $D_1,\dots,D_k$.

2. We prepare the proof with the following lemma:

Lemma 2.2. Suppose that for some z, $|z|<\eta$, there are given $f \in F_z, f\neq 0, f_i \in F_z(D)$ such that $f L_j = \sum_i f_i L_{ij}$, $j=1,\dots,s$, $L_j, L_{ij} \in A(D)$. Then there are $\tilde{f} \in A$, $\tilde{f}\neq 0$ and $\tilde{f}_i \in A(D)$ such that $\tilde{f} L_j = \sum_i \tilde{f}_i L_{ij}$, $j=1,\dots,s$. Moreover, if the f_i are operators only in $D_1,\dots,D_k$, then the $\tilde{f}_i$ can be chosen to depend only on $D_1,\dots,D_k$.

Proof. We write $f_i = \sum_{|\beta|\leq\sigma} f_{i\beta} D^\beta$, $f_{i\beta} \in F_z$ and denote $L_{ij}^\beta = D^\beta L_{ij}$. Then $f_i L_{ij} = \sum_\beta f_{i\beta} L_{ij}^\beta$. If $\varkappa$ is the maximum order of differentiation in L_j, L_{ij}^β and if $\varkappa$ is the number of multiindices of length less then $\varkappa$, then L_j, L_{ij}^β can be regarded in an obvious way as elements in $A^\varkappa$. $fL_j = \sum_{i,\beta} f_{i,\beta} L_{ij}^\beta$, $j=1,\dots,s$ may therefore be regarded as a relation in $\prod_1^s A^\varkappa$, which gives the lemma, in view of the hypothesis on A.

It is now clear that theorem 2.1 follows by induction from the propositions 2.3 and 2.5 from below.

<u>Proposition 2.3.</u> Let $k \in N$, $1 \leq k \leq n$ be given, and suppose that theorem 2.1 is proved, whenever the operators L_j, L_{ij} contain only derivations in $D_1,\dots,D_{k-1}$. Let further $F_1,\dots,F_m, F \in O(D)$ be differential operators in $D_1,\dots,D_k$ (with coefficients in n variables) and suppose that there are $\eta > 0$ and U with the following properties:

- $F_i, F \in O(|z| < \eta)(D)$,
- U is a countable union of analytic varieties, of codimension ≥ 1, all defined in $|z| < \eta$,
- $z \in \complement U$, $u \in F_z$, $F_i u = 0, i=1,\dots,m \Rightarrow Fu = 0$.

Then there is $y \in C^n$, $|y| < \eta$, $f \in F_y$ $f \neq 0$ and $f_i \in F_y(D)$ which are operators in $D_1,\dots,D_k$ only, such that $fF = \sum_i f_i F_i$.

Note that for k=1 the assertion follows from linear algebra.

Proof. (first part). Choose variables such that

$$F_1(z,D) = a(z)D_k^q + \sum_{r<q} a_r(z,D')D_k^r, \quad a \neq 0, \quad D' = (D_1,\dots,D_{k-1}).$$

We may shrink η and change the origin, such that we can suppose that $a(z) \neq 0$ for $|z| < \eta$. Dividing F_1 with $a(z)$, we assume therefore that $F_1(z,D) = D_k^q + \sum_{r<q} c_r(z,D')D_k^r$.

We now prove a lemma.

Lemma 2.4. For every $P \in O(D_1,\dots,D_k)$ there is $\mu \in N$ and η such that for every $u \in F_y$, $|y|<\eta$ which satisfies $F_1 u = 0$, the following two assertions are equivalent:

(i) $Pu = 0$

(ii) $d_{\{z_k = y_k\}} D_k^t Pu = 0$, for $t=0,\dots,\mu$.

Proof. Denote $M = O(D_1,\dots,D_k)/O(D_1,\dots,D_k)F_1$. M is a left $O(D')$ module, which is clearly generated by the classes of $1, D_k,\dots,D^{q-1}$ in M. Therefore, the image of $O(D_1,\dots,D_k)P$ in M is a finitely generated $O(D')$ module, and we may obviously choose generators of form $D_k^t P$, $t \leqslant \mu$ for some $\mu \in N$. It remains therefore to observe, that $Pu = 0$ is equivalent with $d_{\{z_k = y_k\}} D_k^t Pu = 0$ for all t.

Proof of proposition 2.3. (continuation). For $i \geqslant 2$ (if such i exist) we choose $w_{tr}^i \in O(D')$, $w_r \in O(D')$ such that

$$D_k^t F_i = \sum_{0\leqslant r<q} w_{tr}^i D_k^r + q_t^i F_1 \qquad i \geqslant 2, t \leqslant \mu$$

$$F = \sum w_r D_k^r + \tilde{q} F_1.$$

It is now easy to see that for $y \in \complement U$, $g_r \in F_y$

$$\sum_r w_{tr}^i(z,D')\, g_r(z) = 0, \qquad i \geqslant 2, t \leqslant \mu \qquad (2)$$

implies (when $m = 1$, there are no conditions on the g_r)

$$\sum_r w_r(z,D')\, g_r(z) \qquad (3).$$

In fact, for y_k fixed, we can apply the Cauchy-Kowalewskaia theorem and find $u \in F_y$ such that $F_1 u = 0$, $d_{\{z_k = y_k\}} D_k^r u = d_{\{z_k = y_k\}} g_r$. It follows from the properties of the g_j, and from lemma 2.4 that $F_i u=0$, $i \geqslant 2$, and therefore that $Fu = 0$, which gives (3).

For the implication (2) $\Longrightarrow$ (3), we can now apply the induction hypothesis. This means that $hw_r = \sum_{i\geqslant 2, t\leqslant\mu} h_i(z,D') w_{tr}^i(z,D')$, and therefore we also obtain $h \sum_r w_r(z,D')\, D_k^r = \sum h_{it} w_{tr}^i(z,D') D_k^r$. The proposition now follows.

3. Proposition 2.5. Suppose theorem 2.1 is proved for $m\times\sigma$ systems, when $\sigma < s$ (m arbitrary). Then it is true also for $m\times s$ systems.

Proof. Suppose U, η, L_{ij}, L_j, $i=1,\dots,m$, $j=1,\dots,s$ satisfy the conditions from the hypothesis of theorem 2.1. The same is then true for L_{ij}, L_j, $i=1,\dots,m$, $j=1,\dots,s-1$, U and η. For this situation, we can apply the induction hypothesis from the proposition and may therefore suppose that

$$f L_j = \sum_i f_i L_{ij} \qquad j=1,\dots,s-1$$

for some $f \in O(|z|<\eta)$ and $f_i \in (|z|<\eta)(D)$, $f \neq 0$.

Let us now denote R the O(D) module of vectors $r = (r_1,\dots,r_m)$, $r_i \in O(|z|<\eta)(D)$ such that $\sum r_i L_{ij} = 0$, for all $j = 1,\dots,s-1$. This module is finitely generated, say by the elements $r^1,\dots,r^\lambda$ from R. Denote Q the matrix $Q = (r^1,\dots,r^\lambda)$. For some η', $r_i^\nu \in O(|z|<\eta')(D)$ and for $|z|<\eta'$, $z \in \complement U$, and $z \in \complement W$, where U is from the hypothesis and W is a countable union of analytic varieties which appears in proposition 1.4, the following is then valid:

$$u_s \in F_z, \qquad Q\circ\begin{pmatrix} L_{1s} \\ \vdots \\ L_{ms}\end{pmatrix} u_s = 0 \implies (f L_s - \sum f_i L_{is}) u_s = 0.$$

In fact, when u_s satisfies $Q\circ\begin{pmatrix} L_{is} \\ L_{ms}\end{pmatrix} u_s = 0$ and z avoids W, then there are $u_1,\dots,u_{s-1}$ such that $\sum_{j<s} L_{ij} u_j = -L_{is} u_s$. Therefore also $\sum_{j\leq s} L_j u_j = 0$, and $(fL_s - \sum f_i L_{is})u_s = 0$ then follows. For the implication which we have just proved, we can now apply the induction hypothesis and find $g \in O$, $g\neq 0$ and $g_\nu \in O(D)$ such that $g(fL_s - \sum_i f_i L_{is}) = (g_1,\dots,g_\lambda)\cdot Q\circ\begin{pmatrix} L_{1s} \\ L_{ms}\end{pmatrix}$. The proposition now follows when we rewrite the last equality.

4. In the sequel we need a variant of theorem 2.1.

Proposition 2.6. a) Suppose that there are given

- $L_{ij}, L_j \in O(|z|<\eta)(D)$
- irreducible analytic varieties V^i, V, defined in $|z|<\eta$,

and suppose the following is satisfied:

when $u \in F_z^s$, and $d_{V^i}(\sum_j L_{ij}u_j)=0 \; \forall i$, then also $d_V(\sum_j L_ju_j) = 0$.

Then there are $f \in O(|z|<\eta), f_i \in O(|z|<\eta)(D)$, with the following properties:

- $d_V f \neq 0$ (4)
- $d_{V^i} g = 0 \Longrightarrow d_V f_i g = 0$ for all $g \in O(|z|<\eta)$ (5)
- $d_V fL_j = d_V \sum f_i L_{ij}$ $j=1,\ldots,s$ (6).

b) If in the above, $L_{ij}, L_j \in P(D)$, if the V^i, V are algebraic varieties, and if the hypothesis from a) is valid for any $z \in C^n$, then in the conclusion, f and f_i can be chosen in P, respectively $P(D)$

Proof. Arguing as in the proof of lemma 2.2, we conclude again that it suffices to find $z \in V$ and $f \in F_z, f_i \in F_z(D)$, with the properties (4),(5),(6). In fact, if we denote $q_1,\ldots,q_\mu, q_1^i,\ldots,q_\mu^i \in O(|z|<\eta)$ generators for $I(V)$, respectively $I(V^i)$, then (6) can be written in the following form:

there are $h_{\nu j} \in O(|z|<\eta)(D)$ such that $fL_j = \sum f_iL_{ij} + \sum q_\nu h_{\nu j}$

and (5) can be written as

$f_i^{(\beta)} q_\nu^i = \sum_\lambda q_\lambda^i h_{i\nu\lambda}^\beta$, $h_{i\nu\lambda}^\beta \in O(|z|<\eta)$ for every β and all i and ν.

Again we would like to find a solution in $O(|z|<\eta)$ of a system with coefficients in $O(|z|<\eta)$, when we already know that a formal solution for this system exists. By flatness, this formal solution is a combination of solutions in $O(|z|<\eta)$, and therefore, in particular, there must also be a solution in $O(|z|<\eta)$ for which $d_V f \neq 0$. The details are left for the reader. The same proof also shows that part b) of the proposition follows from part a).

5. Our next concern in the proof of proposition 2.6 is to reduce ourselves to the case when $V^i = \{|z|<\eta\}$ for all i. In fact, when $q_1^i,\ldots,q_\mu^i$ generate $I(V^i)$, then $d_{V^i}(\sum_j L_{ij}u_j) = 0$ means exactly

that there are $v_{i\nu} \in F_z$ such that $\sum_j L_{ij} u_j + \sum q_\nu^i v_{i\nu} = 0$.

The hypothesis is therefore, that $\sum_j L_{ij} u_j + \sum q_\nu^i v_{i\nu} = 0$ for $i=1,\ldots,m$, implies $d_V(\sum_j L_j u_j) = 0$.

We may now shrink η, change the origin, and change coordinates analytically, such that we may suppose that for $|z|<\eta$, V is given by $z_{d+1} = \cdots = z_n = 0$ for some d. For simplicity, we introduce the notations $x=(z_1,\ldots,z_d)$, $y=(z_{d+1},\ldots,z_n)$, $z=(x,y)$, and write $L_j = \sum_{|\beta|\le\sigma} L_{j\beta}(z,D_x)D_y^\beta$. Further, we use Taylor expansion in y of order σ for u_j at points in V: $u_j(z) = \sum_{|\beta|\le\sigma} u_{j\beta}(x)y^\beta + \sum_{|\beta|=\sigma+1} y^\beta u_\beta(z)$, and therefore $d_V \sum L_j u_j = 0$ means that $\sum \beta!(d_V L_{j\beta} u_{j\beta}(x)) = 0$.

We conclude that the hypothesis of the proposition can be written in the following form:

$$\begin{cases} \sum_j \sum_{|\beta|\le\sigma} L_{ij} y^\beta u_{j\beta} + \sum_{j,\,|\beta|=\sigma+1} L_{ij} y^\beta u_{j\beta} + \sum_\nu q_\nu^i v_{i\nu} = 0, \ \forall i \\ \operatorname{grad}_y u_{j\beta} = 0, \quad \forall j, \ \forall |\beta| \le \sigma \end{cases} \tag{7}$$

$$\Longrightarrow \sum_{j,|\beta|\le\sigma} \beta!\, d_V L_{j\beta} u_{j\beta} = 0.$$

The written implication is valid for any point in V.

6. In the last equation $u_{j\beta}$, $|\beta| = \sigma+1$ and $v_{i\nu}$ do not appear, and therefore it is only natural to eliminate them completely. To do so, denote R_V the set of vectors $r = (r_1,\ldots,r_m)$ such that

$$- d_V \sum_i r_i L_{ij} y^\beta g = 0, \quad \forall j, \ \forall \beta \text{ with } |\beta| = \sigma+1, \text{and all } g \tag{8}$$

$$- d_V r_i q_\nu^i g = 0, \ \forall i, \forall \nu, \ \forall g \tag{9}$$

R_V is obviously an $O(x)(D_x)$ module, and it is easy to see that the following implication holds for all x outside some countable union of analytic subvarieties in the x-space:

$$\begin{cases} d_V \sum_i r_i \sum_j \sum_{|\beta|\le\sigma} L_{ij} y^\beta u_{j\beta} = 0, \ \forall r \in R_V \\ \operatorname{grad}_y u_{j\beta} = 0, \ \forall j, \ \forall \beta \end{cases} \tag{10}$$

implies $\sum_{j,\ |\beta|\le\sigma} \beta!\, L_{j\beta} u_{j\beta} = 0$.

In fact, when (10) holds, and when x avoids a conveniently chosen countable union of analytic varieties, then it follows from proposition 1.4 that there are $u_{j\beta}$, $|\beta|=\sigma+1$ and $v_{i\nu}$ such that (7) holds.

The operators $d_V \Sigma r_i L_{ij} y^{\beta}$ reduce to operators in $O(|x|<\eta')(D_x)$, when applied on elements with $\mathrm{grad}_y u = 0$, and when $r \in R_V$, the set of these operators is an $O(x)(D_x)$ module. Therefore there are finitely many vectors $r^1,\dots,r^{\lambda}$ such that for x outside some countable union of analytic subvarieties, and for $u_{j\beta} \in F_x$,

$$d_V \sum_i r_i^{\varkappa} \sum_j \sum_{|\beta|\leq\sigma} L_{ij}\, y^{\beta} u_{j\beta} = 0, \qquad \varkappa=1,\dots,\lambda$$

$$\Rightarrow \sum_{j,\beta} \beta!\, L_{j\beta} u_{j\beta} = 0.$$

We can now apply theorem 2.1 and it follows that there are $h \in O(|x|<\eta')$, $h \neq 0$ and $h_{\varkappa} \in O(|x|<\eta')(D_x)$ such that $h\, d_V L_{j\beta} y^{\beta} g = \sum_{i,\varkappa} h_{\varkappa} d_V\, r_i\, L_{ij} y^{\beta} g$ for all j and $|\beta|\leq\sigma$ and all g. It follows easily from (8) that $h\, d_V L_j u = \sum_{i,\varkappa} h_{\varkappa}\, d_V r_i L_{ij} u$ for all u and all j. It is also clear that (9) implies (5).

<u>6. Proposition 2.7.</u> Consider irreducible algebraic varieties $V \subset W \subset C^n$, and let $\lambda \in P(D)$ have the property that $d_V \lambda f = 0$ if $d_W f = 0$. Then there is $\mu \in N$ and λ_i, λ_i', $i=1,\dots,\mu$, all in P(D) such that $\lambda = \Sigma\, \lambda_i\, \lambda_i'$ and such that $d_V f = 0$ implies $d_V \lambda_i f = 0$ and $d_W g = 0$ implies $d_W \lambda_i' g = 0$.

Proof. If we choose z in the regular part of V, use convenient coordinates and apply the arguments from the proof of proposition 2.6, then we obtain $\tilde{\lambda}_i$, $\tilde{\lambda}_i' \in O_z(D)$ such that $\lambda = \Sigma \tilde{\lambda}_i\, \tilde{\lambda}_i'$ and such that $d_V f = 0 \Rightarrow d_V \tilde{\lambda}_i f = 0$, $d_W g = 0 \Rightarrow d_W \tilde{\lambda}_i' g = 0$. We can now apply proposition 1.9 and may therefore suppose that $\tilde{\lambda}_i' \in P(D)$. Then we apply proposition 2.6 for $(V, \tilde{\lambda}_i')$ and (V, λ).

Proposition 2.7 will not be used essentially in this paper. In fact, we will only use it in §5, but by an inversation of an argument, we could have avoided it.

§3. Remarks concerning the Weierstrass preparation theorem.

1. Consider $p(z',t) = t^q + \sum_{j<q} c_j(z')t^j$, $c_j \in O(|z'| < \eta')$ a polynomial in t without multiple factors, and let $\Delta(z')$ be the discriminant of p with respect to t.

Definition 3.1. Consider $U \subset C^n$ an open set. We say that U has property (W) (with respect to p) if there is η, $\eta' \geqslant \eta > 0$ such that:

- the projection of U on C^{n-1} is $|z'| < \eta$,
- for each z', $|z'| < \eta$, the number of roots $\tilde{t}$ of the equation $p(z',t) = 0$ for which $(z',\tilde{t}) \in U$ is the same. We denote this number $q(U)$. (k-tuple roots are counted k times). We also denote

$$p_U(z',t) = \prod_{\substack{p(z',t_i)=0 \\ (z',t_i)\in U}} (t - t_i).$$

2. Suppose that U has property (W), and consider $\alpha \in N$ and $f \in O(U)$. The global Weierstrass preparation theorem (cf. also proposition 3.3 from below) shows that there are $g_j \in O(|z'| < \eta)$ and $h \in O(U)$ such that

$$f = \sum_{0 \leqslant j \leqslant \alpha q(U)-1} g_j(z')\, t^j + p^\alpha h \qquad (1).$$

Clearly (1) determinates g_j uniquely and we denote

$\varepsilon^U : O(U) \to \prod_{i=0}^{\alpha q(U)-1} O(|z'| < \eta)$ the map with components

$\varepsilon^U_j(f) = g_j$, $j = 0, \dots, \alpha q(U)-1$.

3. The Weierstrass preparation theorem holds also in the set up of formal power series, but then we must work locally.

Thus choose y with $p(y) = 0$, $|y'| < \eta'$, and denote p_y the Weierstrass pseudopolynomial associated with p at y. This means that for some $q(y)$, $p_y = (t-y_n)^{q(y)} + \sum_{j<q(y)} c'_j(z')(t-y_n)^j$, $c'_j(y') = 0$, that p/p_y is holomorphic near y and that $(p/p_y)(y) \neq 0$.

For every $f \in F_y$ we can now apply the Weierstrass preparation theorem and find $\tilde{g}_j \in F_{y'}$ and $h \in F_y$ such that

$$f = \sum_{j < \alpha q(y)} \tilde{g}_j(z')(t-y_n)^j + p_y^{\alpha} h \qquad (2).$$

We now consider $g_j \in F_{y'}$ such that $\sum_{j<\alpha q(y)} \tilde{g}_j(z')(t-y_n)^j =$

$= \sum_{j < \alpha q(y)} g_j(z')\, t^j$ and again introduce a map

$\varepsilon^y : F_y \longrightarrow \prod_0^{\alpha q(y)-1} F_{y'}$, by $\varepsilon^y_j(f) = g_j$. ε^y is continuous if we endow $F_y, F_{y'}$ with the Krull topologies.

4. The Weierstrass preparation theorem is a classical tool to reduce "assertions in n variables" to "assertions in n-1 variables". We will use it with this purpose in §4.In that paragraph we need several properties of the Weierstrass decompositions (1),(2).Most of these properties are standard.We mention them explicitely for later convenience.

<u>Proposition 3.2.</u> Consider U with (W).

a) Let r be in $O(|z'| < \eta)$. Then $\varepsilon^U_j(rf) = r\, \varepsilon^U_j(f)$.

b) Consider $r \in O(U)$ and denote $r_{ij} = \varepsilon^U_i(t^j r)$, $0 \leq i,j \leq \alpha q(U)-1$. Then for every $f \in O(U)$, $\varepsilon^U_i(rf) = \sum_j r_{ij}\, \varepsilon^U_j(f)$.

c) There are $T_j \in O(U)$, polynomials in t, such that $\varepsilon^U_j(f) = \varepsilon^U_{\alpha q(U)-1}(T_j f)$.

Similar assertions remain valid for ε^y (in the last part, the T_j are then Weierstrass pseudopolynomials at y).

Proof. Only c) requires comments. Denote $p_j \in O(|z'| < \eta)$ such that $t^{\alpha q(U)} = \sum_{j < \alpha q(U)} p_j(z') t^j + p_U^{\alpha}$. This shows that

$$tf - \sum_{j < \alpha q(U)-1} \varepsilon^U_j(f)\, t^{j+1} - \varepsilon^U_{\alpha q(U)-1}(f) \sum_{j < \alpha q(U)} p_j t^j = p_U^{\alpha} h,$$

whence $\varepsilon^U_{\alpha q(U)-i}(tf) = \varepsilon^U_{\alpha q(U)-i-1}(f) + \varepsilon^U_{\alpha q(U)-1}(p_i f)$. This gives inductively $T_{\alpha q(U)-i-1} = t\, T_{\alpha q(U)-i} + p_i$.

5. To state the next proposition, we introduce a notation: let U be with (W), and consider $y \in U$, $p(y) = 0$. Then we introduce $i^U_y : \prod_0^{\alpha q(y)-1} F_{y'} \longrightarrow \prod_0^{\alpha q(U)-1} F_{y'}$ in the following way:

$i^U_y(g)_k$ is the coefficient of t^k in $(p_U/p_y)^{\alpha} \sum_{j<\alpha q(y)} g_j t^j$

Proposition 3.3. Suppose U has property (W) and consider $f \in O(U)$ and $y \in U$.

a) $\varepsilon^U(f) = \sum_{y_n, p_U(y', y_n)=0} i^U_y\, \varepsilon^y(f\,(p_y/p_U)^\alpha)$.

b) $\varepsilon^U_{\alpha q(U)-1}(f) = \sum_{y_n, p_U(y', y_n)=0} \varepsilon^y_{\alpha q(y)-1}(f\,(p_y/p_U)^\alpha)$.

c) If $\Delta(y') \neq 0$ and if $t_1(z'), \ldots, t_{q(U)}(z')$ are distinct germs in $O_{y'}$ such that $p(z', t_i(z')) = 0$, $(z', t_i(z')) \in U$,

$$\varepsilon^U_{\alpha q(U)-1}(f) = 1/(\alpha-1)! \sum_k (\partial/\partial t)^{\alpha-1} (f/\prod_{i\neq k}(t-t_i(z'))^\alpha \qquad (3).$$

Proof. a) By definition, $f(p_y/p_U)^\alpha = \sum \varepsilon^y_j(f(p_y/p_U)^\alpha)t^j + p^\alpha_y h'_y$, which gives $f = (p_U/p_y)^\alpha (\sum \varepsilon^y_j(p_y/p_U)^\alpha)t^j + p^\alpha_y h'_y$.
It follows that $f - \sum_{y_n} (p_U/p_y)^\alpha (\sum \varepsilon^y_j(f(p_y/p_U)^\alpha)t^j$ is divisible by p_U.

b) follows from a).

c) With $t_i(z')$ as in c), it is clear that we can determinate $\varepsilon^y_j(f)$ for $y = (y', t_i(y'))$ inductively by

$$\rho!\, \varepsilon^y_\rho(f) = (\partial/\partial t)^\rho f(z', t_i(z')) - \sum_{j, \rho < j < \alpha q(y)-1} j!/(j-\rho)!\, t_i^{j-\rho}(z')\, \varepsilon^y_j(f).$$

In fact the ε^y_j then satisfy

$$\rho < \alpha \Rightarrow (\partial/\partial t)^\rho (f - \sum \varepsilon^y_j(f)(z')t^j)(z', t_i(z')) = 0.$$

c) follows therefore from b).

6. When we want to relate ε^U to ε^y, the following technical lemma is useful

Lemma 3.4. Consider y, $|y'| < \eta$, $p(y) = 0$. There is $\eta'' > 0$ such that for every $k \in N$ there is a unique function $g^y \in O(|z'-y'| < \eta'' \times C)$ with the following properties:

- g^y is a polynomial of degree $kq(U)-1$ in t
- $p_y(z', t) = 0 \Rightarrow g^y(z', t) = 1$
- $p_U(z', t)/p_y(z', t) = 0 \Rightarrow g^y(z', t) = 0$

- $p_U(z',t) = 0 \Longrightarrow (\partial/\partial t)^{\rho} \quad g^y(z',t) = 0$ for $1 \leq \rho \leq k$
- for y with $\Delta(y') \neq 0$, there is S, a polynomial in t and z, with coefficients in $O_{y'}$, such that $g^y(z',t) = S(z',t,y_n)$.

Proof. Write $(p_y/p_U)^k = \sum_{0 \leq j < kq(y)} g_j t^j + p_y^k h$. It is easy to see that $g^y = (p_U/p_y)^k(\sum g_j t^j)$ satisfies the properties from the lemma.

<u>Proposition 3.5.</u> Let g^y be the functions from lemma 3.4 for $k \geq \alpha$.

a) $\varepsilon^y(g^y f) = \varepsilon^y(f)$ for all $f \in F_y$.

b) $\varepsilon^U(g^y f) = i_y^U \varepsilon^y(g^y f(p_y/p_U)^{\alpha})$ for all $f \in O(U)$.

Proof. a) is easy to verify for $f \in O_y$. It follows by density therefore for general f. b) is a consequence of a).

<u>Proposition3.6.</u> Consider y' with $\Delta(y') \neq 0$, U with (W), $s \in N$ and let P be a polynomial in t and z with coefficients in $O_{y'}$. Denote $t_1, \dots, t_{q(U)} \in O_{y'}$, distinct germs such that $p(z',t_i(z'))=0$, and such that $(z',t_i(z')) \in U$. Further denote g^y the functions from lemma 3.4 for some $k \geq s$.

Then there is $\sigma \in N$ and R, a polynomial in t, with coefficients in $O_{y'}$, such that

$$\sum_k g^{(y',tk(y'))} P(z',t,t_k(z'))(p_U/p_{(y',t_k(y'))})^{s}(z',t) = R/\Delta^{\sigma}.$$

Moreover, when p, P are defined for $|y'| < \eta$ (when p, P are polynomials), then we may choose R to be defined in $|y'| < \eta$ (to be a polynomial).

Proof. $p_U/p_{(y',t_k(z'))} = \prod_{i \neq k} (t - t_{i(z')})$ and therefore $g^y(p_y/p_U)^s$ is a polynomial in t. The coefficients of $t^{\varkappa}$ in the above sum are for every $\varkappa$ symmetric polynomial combinations of the $t_1, \dots, t_{q(U)}$. This is clear from symmetry considerations using the last part of lemma 3.4. The proposition now follows.

7. One of the main objectives of this paragraph is to compute $\partial/\partial z_s \; \varepsilon_j^U(f)$, when $\Delta(y') \neq 0$, $1 \leq s \leq n-1$. We may reduce ourselves to

the computation of $\partial/\partial z_s\, \varepsilon^U_{\alpha q(U)-1}(f)$, for which we have formula (3). Let us also remark that if $t(z') \in O_y$, satisfies $p(z',t(z')) \equiv 0$, then it follows, differentiating, that

$$\partial t/\partial z_s = -(\partial p/\partial z_s)(z',t(z'))/(\partial p/\partial t)(z',t(z')). \qquad (4).$$

Proposition 3.7. For $s< n$, $\Delta(y') \neq 0$, there are $Q_s, R_s \in O_{y'}[t]$ polynomials in t, with coefficients in $O_{y'}$ and $\sigma \in N$ such that

$$(\partial/\partial z_s)\varepsilon^U_{\alpha q(U)-1}(f) = \varepsilon^U_{\alpha q(U)-1}((\partial/\partial z_s + (Q_s/\Delta^\sigma)\partial/\partial t + R_s/\Delta^\sigma)f).$$

When $p \in P$, Q_s and R_s can be chosen in P.

Proof. We consider functions $t_1,\dots,t_{q(U)}$ as in proposition 3.3 c) and use formula (3). In (3) we have to derivate a sum of composite functions and obtain $(\alpha-1)!\,(\partial/\partial z_s)\varepsilon^U_{\alpha q(U)-1}(f) =$

$$\sum_k (\partial/\partial t)^{\alpha-1}((\partial/\partial z_s + (\partial t_k(z')/\partial z_s)\partial/\partial t)$$

$$(f/\prod_{i\neq k}(t-t_i(z'))^\alpha)(z',t_k(z')) = \sum_k (\partial/\partial t)^{\alpha-1}((\partial f/\partial z_s +$$

$$+\partial t_k/\partial z_s\, \partial f/\partial t)/\prod_{i\neq k}(t-t_i(z'))^\alpha -$$

$$-\alpha(f/\prod_{i\neq k}(t-t_i(z'))^\alpha(\sum_{i\neq k}\frac{\partial t_k/\partial z_s - \partial t_i/\partial z_s}{t-t_i(z')})(z',t_k(z')).$$

We now observe that $\sum_k (\partial/\partial t)^{\alpha-1}(\partial t_k/\partial z_s\, \partial f/\partial t\,/$

$$\prod_{i\neq k}(t-t_i(z'))^\alpha)(z',t_k(z')) = \sum_k (\partial/\partial t)^{\alpha-1}(\sum_{k'} g^{(y',t_{k'})}$$

$\partial t_{k'}/\partial z_s\, \partial f/\partial t)/\prod_{i\neq k}(t-t_i(z'))^\alpha$ if g^y are the functions from lemma 3.4 for some $k \geq 2\alpha+1$ and a similar assertion is true for

$$\sum_k(\partial/\partial t)^{\alpha-1}(f\sum_{i\neq k}\frac{\partial t_k/\partial z_s - \partial t_i/\partial z_s}{t-t_i(z')})/\prod_{i\neq k}(t-t_i(z'))^\alpha(z',t_k(z'))$$

The proposition now follows when we apply (4) and proposition 3.6.

Corollary 3.8. Consider $\delta_0,\dots,\delta_{\alpha q(U)-1} \in O(|y'|<\eta)(D')$, $D' = (D_1,\dots,D_{n-1})$. Then there exists $\sigma \in N$ and a differential operator $\partial \in O(U)(D)$ such that

a) $\Delta^\sigma \sum_k \delta_k\, \varepsilon^U_k(f) = \varepsilon^U_{\alpha q(U)-1}(\partial f)$ for all $f \in O(U)$.

b) When g^y denote the functions from lemma 3.4 for some great k, then for all $f \in F_y$

$$\Delta^\sigma(\delta i^U_y\, \varepsilon^U((g^y f)(p_U/p_y)^\alpha) = \varepsilon^y_{\alpha q(y)-1}((p_U/p_y)^\alpha \partial g^y f).$$

Proof. a) follows from proposition 3.7 and proposition 3.2. To prove b) suppose first that $f \in O(U)$. Then we have

$\Delta^{\sigma}\delta i_y^U \ \varepsilon^y(g^y f(p_U/p_y)^{\alpha}) =$ (in view of proposition 3.5) $\Delta^{\sigma}\delta\varepsilon^U(g^y f) =$
$= \varepsilon^U_{\alpha q(U)-1}(\partial(g^y f)) = \varepsilon^y_{\alpha q(y)-1}((p_U/p_y)^{\alpha}(\partial g^y f))$.

8. Consider $U, p, p_1, \ldots, p_m \in O(U)$, and $\alpha \in N$ such that:-

-p has no multiple factors and is of form $p = t^q + \sum_{j<q} c_j(z')t^j$,

- U has property (W) with respect to p,

- p^{α} is in the ideal I generated by $p_1, \ldots, p_m$ in $O(U)$.

$\varepsilon^U(I) \subset \prod_0^{\alpha q(U)-1} O(z', |z'|<\eta)$ is an $O(z', |z'|<\eta)$ module, and it is clear that $f \in O(U)$ is in I if and only if $\varepsilon^U(f) \in \varepsilon^U(I)$. With the aid of proposition 3.2 it is also easy to compute an $(\alpha q(U)-1) \times b$ matrix p', $b = (\alpha q(U)-1) \times m$, such that

$\varepsilon^U(I) = p' \, [O(z', |z'|<\eta)]^b$.

Further we introduce the following notations :

$M_y = \{g \in \prod_0^{\alpha q(y)-1} O_{y'} \ ; \ g$ is in the $O_{y'}$ module generated by $\varepsilon^y(\sum_{i \geq 1} p_i O_y)\}$,

$\widetilde{M}_y = \{g \in \prod_0^{\alpha q(y)-1} F_{y'} \ ; \ g$ is in the $F_{y'}$ module generated by $M_y\}$,

$M_{U,y'} = \{g \in \prod_0^{\alpha q(U)-1} O_{y'} \ ; \ g$ is in the $O_{y'}$ module generated by $\varepsilon^U(I)\}$,

$\widetilde{M}_{U,y'} = \{g \in \prod_0^{\alpha q(U)-1} F_{y'} \ ; \ g$ is in the $F_{y'}$ module generated by $M_{U,y'}\}$.

<u>Proposition 3.9.</u> a) $\bigcap_{y_n, p(y', y_n) = 0} i^U_{(y', y_n)}(M_y) = M_{U,y'}$

b) $\bigcap_{y_n, p(y', y_n)=0} i^U_{(y', y_n)}(\widetilde{M}_y) = \widetilde{M}_{U,y'}$.

In particular, $\varepsilon^y(f) \in M_y \Longleftrightarrow i_y^U \ \varepsilon^y(f) \in \widetilde{M}_{U,y'}$.

Proof. b) follows from a) by flatness, so we must only prove a).

To prove the nontrivial inclusion, consider $g \in i^U_{(y',y_n)}(M_y)$. This means that $\sum_j g_j t^j = \sum_i (p_U/p_y)^\alpha p_i h_i^y + p_y^\alpha h_y$ for some $h_i^y, h_y \in O_{y'}$, which are polynomials in t. Denote g^y the functions from lemma 3.4 for $k=\alpha$. It follows that $\sum g_j t^j - \sum_i (\sum_{y_n} g^y h_i^y) p_i$ is divisible by p_y^α for every y. This implies $g \in M_{U,y'}$.

9. We conclude the paragraph with a result which relates $\varepsilon^y_{\alpha q(y)-1}(f)$ to the remainder of the Weierstrass division algoritm when $\alpha = 1$. Thus denote $\tilde{\varepsilon}_j(f)$, $j<q(y)$, functions such that $p_y(z)=0$ implies $(f - \sum \tilde{\varepsilon}_j(f)\, t^j)(z) = 0$. When $\Delta(z')\neq 0$,

$$\tilde{\varepsilon}_{q(y)-1}(f) = \sum_k \Big(f/\prod_{i\neq k}(t-t_i(z'))\Big)(z',t_k(z')).$$

Now observe that

$$\varepsilon^y_{\alpha q(y)-1}(f) = \sum_k \sum_{j<\alpha} \big(1/((\alpha-j-1)!\,j!)\big)(\partial/\partial t)^j f\,(\partial/\partial t)^{\alpha-j-1}\Big(1/\prod_{i\neq k}(t-t_i)^\alpha\Big)(z',t_k(z')) =$$

$$= \big(1/((\alpha-j-1)!\,j!)\big)\frac{(\partial/\partial t)^j f(z',t_k(z'))}{\prod_{i\neq k}(t_k(z')-t_i(z'))}\,.$$

$$\cdot(\partial/\partial t)^{\alpha-j-1}\Big(1/\prod_{i\neq k}(t-t_i(z'))^\alpha\Big)(z',t_k(z'))\prod_{i\neq k}(t_k(z')-t_i(z')).$$

We can now use lemma 3.4 as in the proof of proposition 3.7 and conclude:

<u>Lemma 3.10.</u> There are $R_j(z',t)$, polynomials in t, with coefficients in $O_{y'}$, and $\sigma \in N$ such that

$$\varepsilon^y_{\alpha q(y)-1}(f) = \sum_{j<\alpha} \tilde{\varepsilon}_{q(y)-1}((\partial/\partial t)^j f)\,R_j(z',t)/\Delta^\sigma).$$

§4 .The construction of Noetherian operators.

1.Definition 4.1. Consider $z \in C^n$, V a germ of an analytic variety and $\partial = (\partial_1, \ldots, \partial_s) \in O_z^s(D)$.

a) (V,∂) is called a Noetherian operator. For $f \in F_z^s$ we say that $(V,\partial) f = 0$ when $\sum \partial_i f_i \in I_z^F(V)$. If $((V,\partial^1),(V,\partial^2)$ are Noetherian operators, we write $(V,\partial^1) = (V,\partial^2)$ if for all f, $(V, \partial^1 - \partial^2)f = 0$.

b) We say that (V,∂) is an algebraic Noetherian operator, if V is algebraic and if $\partial_i \in P(D)$.

Theorem 4.2.(L.Ehrenpreis-V.P.Palamodov). Consider a $s \times m$ matrix $p = (p_{ij})$, $p_{ij} \in O_z$. Then there exists $\eta > 0$ and a collection of Noetherian operators (V^k,∂^k), $k=1,\ldots,\mu$ with the following properties:

- all coefficients of the ∂^k, and all the functions p_{ij} are analytic functions for $|y-z| < \eta$. The V^k are analytic varieties defined in $|y-z| < \eta$.
- for every y with $|y-z| < \eta$, and every $f \in F_y^s$, the following two properties are equivalent:

  (i) there is $g \in F_y^m$ such that $pg = f$

  (ii) $(V^k,\partial^k)f = 0$ for $k=1,\ldots,\mu$.

Moreover, when $p_{ij} \in P$, then we can find algebraic Noetherian operators (V^k,∂^k), $k=1,\ldots,\mu$, for which (i) and (ii) are equivalent for every $y \in C^n$.

Remark 4.3. a) (ii) is a nontrivial condition only when $y \in \cup V^k$. Further, denote V the germ at z of the set $V = \{ y;$ the rank of the matrix $p_{ij}(y)$ is $< s \}$. V is a germ of an analytic variety, which will be called the variety of p (in the polynomial case, V will be an algebraic variety). When $y \in \complement V$, then (i) is satisfied for all f. Therefore at such points y,

$\partial^k f \in I_y^F(V^k)$ for all f. This implies easily (cf. the proof of porposition 4.6 below) that the coefficients of ∂_i^k are all in $I_y(V^k)$. In particular, we may change the operators (V^k, ∂^k) with $(V^k \cap V, \partial^k)$.

b) When $V^k = V' \cup V''$ for some analytic varieties, then the condition $(V^k, \partial^k)\, f = 0$ is equivalent with the condition : $(V', \partial^k) f = 0$ and $(V'', \partial^k) f = 0$.

<u>Definition 4.4.</u> We call (V^k, ∂^k) ,$k=1,\ldots,\mu$ a collection of Noetherian operators associated with p, if for it (i) and (ii) in theorem 4.2 are equivalent.

Proposition 2.6 shows that Noetherian operators associated with a given p are essentially unique:

<u>Proposition 4.5.</u> Consider $p = (p_{ij})$ a s×m matrix and (V^k, ∂^k), $k=1,\ldots,\mu$, (W^r, δ^r) ,$r=1,\ldots,\nu$,two collections of Noetherian operators, associated both with p.

Then there exist : $h_k \in O_z$, $d_{Wk}\, h_k \neq 0$ and $h_{kr} \in O_z(D)$ such that $(V^k, h_k\, \partial^k) = (V^k, \Sigma h_{kr}\, \delta^r)$ and such that $d_{Wr}\, f = 0$ implies $d_{Vk}\, h_{kr}\, f = 0$. When (V^k, ∂^k), (W^r, δ^r) are algebraic, then we may choose $h_k \in P$, $h_{kr} \in P(D)$.

2. Before embarking on the proof of theorem 4.2, we briefly study modules defined with the aid of Noetherian operators.

<u>Proposition 4.6.</u> Consider (V^k, ∂^k) $k=1,\ldots,\mu$ a collection of Noetherian operators and denote:

$M = \{f \in F_z^s;\ (V^k, \partial^k) f = 0, k=1,\ldots,\mu\}$.

a) Suppose M is an O_z module. Then $f \in M$ implies $(V^k, \partial^{k(\beta)}) f = 0$ for all β and all k. Conversely, if for all $f \in M$, $(V^k, \partial^{k(\beta)}) f = 0$, $\forall \beta$, $k=1,\ldots,\mu$, then M is an F_z-module.

b) Suppose that all the V^k are irreducible, and suppose that M is an O_z module. Suppose that $r \notin \bigcup_k I(V^k)$, for some $r \in O_z$ and that $rf \in M$. Then $f \in M$.

c) Suppose $V^k = V$, $k=1,\dots,\mu$ that V is irreducible, and suppose that M is an O_z module. Then M is primary.

Proof. The first assertion in a) results by increasing induction in $|\beta|$, using the fact, that $f \in M$ implies $z^\alpha f \in M$. The second assertion in a) is obvious.

b) We know that $(V^k, \partial^{k(\beta)})rf = 0$ for all k and β, and want to conclude that $(V^k, \partial^{k(\beta)})f = 0$ for all k and β. This follows by decreasing induction in $|\beta|$. In fact, for $|\beta|$ great, $(V^k, \partial^{k(\beta)})f = 0$ is trivial, and then we use $\partial^{k(\beta)}rf = r\partial^{k(\beta)}f + \sum_{\gamma\neq 0} (\partial/\partial z)^\gamma r\, \partial^{k(\beta+\gamma)}f/\gamma!$. We can now conclude by induction, that $r\,\partial^{k(\beta)}f \in I_z^F(V^k)$, and since the latter is prime (by a theorem of Nagata-Zariski, cf. B. Malgrange[3]), we obtain $\partial^{k(\beta)}f \in I_z^F(V^k)$. c) follows from b).

3. The proof of theorem 4.2 is essentially by induction in n and s. We separately formulate the two main steps in the proof.

<u>Proposition 4.7.</u> Suppose:

- theorem 4.2 is proved in n-1 variables,
- there are given $p_1,\dots,p_m \in O_z$.

Then there exists a collection of Noetherian operators (V^k,∂^k), $k=1,\dots,\mu$ which is associated with the $1\times m$ matrix $(p_1,\dots,p_m)$. For $p_i \in P$ the (V^k,∂^k) can be chosen algebraic.

<u>Proposition 4.8.</u> Let $s \in N$ be given. Suppose that theorem 4.2 is proved for any $\sigma\times m$ matrix, with $\sigma < s$ (for arbitrary m). Then theorem 4.2 is true for any $s\times m$ matrix.

Proof of proposition 4.7.

The first thing to do, is to reduce the proof to the case, when the ideal I generated by $p_1,\dots,p_m$ in O_z is primary. In fact, we may write I as a finite intersection of primary ideals, and construct for every such ideal, an associated collection of

Noetherian operators (proposition 4.7 is in an obvious sense related to ideals!).The collection of operators which appear,is then associated with I (this follows by flatness).

We now apply lemma 1.10 and obtain $p=t^q+\sum_{j<q} c_j(z')t^j$, α, $p^{\alpha}\in I$, but such that the discriminant Δ of p in t is not in the radical of I. For any fixed $\sigma\in N$, it follows that $f=\sum p_j g_j$, $g_j\in F_y$, if and only if $\Delta^{\sigma} f=\sum p_j g_j'$, for some $g_j'\in F_y$. σ will be chosen below.

Further, we choose a neighborhood U of z, which has property (W) with respect to p, and denote ε^U, ε^z, the operators constructed in §3 for p^{α}. We conclude from §3.8, and the induction hypothesis, that there are Noetherian operators (W^k,δ^k), $k=1,\dots,\mu$, in n-1 variables, W^k irreducible, such that $f'\in[F_{y'}]^{\alpha q}$ is in the module generated by $\varepsilon^U(\sum p_i O(U))$ in $[F_{y'}]^{\alpha q}$ if and only if $(W^k,\delta^k)f'=0$, $k=1,\dots,\mu$. We can labbel (W^k,δ^k) in such a way, that for some μ', $1\leq\mu'\leq\mu$, $\Delta\notin I(W^k)$ for $1\leq k\leq\mu'$, but $\Delta\in I(W^k)$ for $\mu'<k\leq\mu$ (if there are such k). We now apply corollary 3.8 and obtain σ' and $\partial^k\in O(U)(D)$ such that $\Delta^{\sigma'}\sum\delta_j^k\varepsilon_j^U(f)=\varepsilon^U(\partial^k f)$ for $f\in O(U)$.

The proof of proposition 4.7 comes to an end, when we prove:

Proposition 4.9. With the notations from above,

$((W^k\times C)\cap V,\ (\partial/\partial t)^r\partial^{k(\beta)})$, $k=1,\dots,\mu'$, $r<\alpha$, $\forall\beta$,

is a collect_ion of Noetherian operators associated with $p_1,\dots,p_m$.

Proof.(We freely use notations from §3.).

When $f_i\in O(U)$, it is clear that $(V,\partial)p_if_i=0$ for any operator (V,∂) from the above. By density, we also have $(V,\partial)p_if_i=0$ for $f_i\in F_y$. To prove the converse, we first note the following:

Lemma 4.10. The coefficients of $(\partial/\partial t)^r\partial^{k(\beta)}$ (written in the standard form $\sum a_{\gamma}(\partial/\partial z)^{\gamma}$) vanish on $W^k\times C\cap\{p=0\}\cap V$.

In fact, for any $f=\sum p_if_i$, $f_i\in O(U)$, and any β we have $d_{W^k}\,\varepsilon^U_{\alpha q-1}(\partial^{k(\beta)}f)=0$, which shows that $(\partial/\partial t)^r\partial^{k(\beta)}f$

vanishes for all f on $W^k \times C \cap \{p = 0\} \cap \complement V$.

To return to the proof of proposition 4.9, suppose that for some $f \in F_y$, $((W^k \times C) \cap V, (\partial/\partial t)^r \partial^{k(\beta)})f = 0$, for $k=1,\dots,\mu'$, $r < \alpha$, and all β. This also implies that $((W^k \times C) \cap V, (\partial/\partial t)^r \partial^{k(\beta)}) \Delta^\sigma f = 0$ for any σ, which, in view of the preceding lemma, shows that $d_{W^k} \varepsilon^y_{\alpha q(y)-1}(\partial^{k(\beta)} \Delta^\sigma f) = 0$ for all β and $k=1,\dots,\mu'$. We now use decreasing induction in $|\beta|$, as in the proof of prpposition 4.6 and obtain $d_{W^k} \varepsilon^y_{\alpha q(y)-1}(\partial^k(g^y \Delta^\sigma f)(p_U/p_y)^\alpha) = 0$ (here g^y is from lemma 3.4). This gives in particular,

$(W^k, \Delta^{\sigma'} \delta^k)\, i^U_y \varepsilon^y (g^y \Delta^\sigma f) = 0$ for $k = 1,\dots,\mu'$ which implies $(W^k, \delta^k)\, i^U_y \varepsilon^y (g^y \Delta^\sigma f) = 0$. When σ is great enough, we also have $(W^k, \delta^k)\, i^U_y \varepsilon^y (g^y \Delta^\sigma f) = 0$ for $\mu' < k \leq \mu$. In view of the choice of the (W^k, δ^k) this shows that $g^y \Delta^\sigma f = \sum p_i h_i$ for $h_i \in F_y$. Finally we remark that we may divide out g^y.

In the case, when $p_i \in P$, the proof of the corresponding part in the theorem is practically the same. We then choose $U = C^n$, and apply lemma 1.10 in the polynomial case.

Note that the proof is much easier, when dim V = 0.

Remark 4.11. a) In the proof of proposition 4.7 we could have used induction in dim V(I). This is natural to do in part II.

b). Suppose that the ideal I generated by $p_1,\dots,p_m$ in O_z is primary and that dim V = k. Then the projection on the subspace $z_n = z_n^o$ is for any z_n^o a variety W, which also has dimension k (note that the projection is proper in view of the fact that p vanishes on V and has the form $p = t^q + \sum_{j<q} c_j(z')t^j$, $t=z_n$).

V and W are not necesarily irreducible at points close to z^o, but in any case they are unions of irreducible varieties, all of dimension k, at such points (this is a theorem of C. Chevalley. cf. A. Grothendieck [1]). It is then easy to show that, in the notations from the proof of proposition 4.9, already the already the Noetherian operators with $\dim W^k = \dim W$ are a collection of Noetherian operators associated with $\varepsilon^{z^o}(I)$. In fact, this follows from proposition 4.6. To see this, denote $M' = \{f; (W^k, \delta^k)f = 0, \dim W^k = \dim W\}$, $M'' = \{f; (W^k, \delta^k)f = 0, \dim W^k < \dim W\}$. Then $\varepsilon^z(I) = M' \cap M''$. Since $\varepsilon^z(I)$ has a primary decomposition formed by modules associated with varieties of dimension k, we must have $M'' = M$.

Proof of proposition 4.8.

We first introduce a number of notations.

Thus denote $p^1 = (p_{11}, \ldots, p_{1m})$, $p' = (p_{ij})_{i \geqslant 2}$, and consider matrices q^1, q' with entries in O_z such that the sequences $O_z^{k_1} \xrightarrow{q^1} O_z^m \xrightarrow{p^1} O_z$, $O_z^{k'} \xrightarrow{q'} O_z \xrightarrow{p'} O_z^{s-1}$, are exact.

By flatness it follows, for all y close to z, that the sequences $F_y^{k_1} \xrightarrow{q^1} F_y^m \xrightarrow{p^1} F_y$, $F_y^{k'} \xrightarrow{q'} F_y^m \xrightarrow{p'} F_y^{s-1}$ are also exact.

It is clear, that $pg = f$ has a solution, if and only if there is a solution $\tilde{g}$ for $p^1 \tilde{g} = f_1$, and if for any solution $\tilde{g}$ for $p^1 \tilde{g} = f$, there is a solution h for $p'q^1 h = f' - p'\tilde{g}$, $f' = (f_2, \ldots, f_s)$. The solvability of $p^1 \tilde{g} = f_1$ can be characterized with the aid of conditions $(V^k, \partial^k)f_1 = 0, k = 1, \ldots, \mu$, for some Noetherian operators (V^k, ∂^k). To study the solvability of $p'q^1 h = f' - p'\tilde{g}$, denote (V^k, ∂^k), $k = \mu+1, \ldots, \nu$, Noetherian operators associated with $p'q^1$, and (V^k, ∂^k), $k = \nu+1, \ldots, \varkappa$, Noetherian operators associated with $p^1 q'$. Such operators exist, in view of the induction hypothesis.

We now have the following easy remark:

<u>Remark 4.11.</u> $p^1 u \in p^1 q' F_y^{k'} \Longleftrightarrow p'u \in p'q^1 F_y^{k_1}$.

In fact, both inclusions are equivalent with $u \in (q^1, q')F^{k_1 + k'}$.

In view of this remark, $(V^r, \partial^r p^1)u = 0$, $r = \nu+1, \ldots, \varkappa$, is equivalent with $(V^k, \partial^k p')u = 0$, $k = \mu+1, \ldots, \nu$. It follows that there are $\lambda_k \in O_z$, $d_{V^k} \lambda_k \neq 0$ and $\lambda_{kr} \in O_z(D)$ such that $(V^k, \lambda_k \partial^k p') = (V^k, \sum_r \lambda_{kr} \partial^r p^1)$ for $k = \mu+1, \ldots, \nu$.

When $p^1\tilde{g} = f_1$, we can therefore characterize the solvability of $p'q^1 h = f' - p'\tilde{g}$ with the aid of Noetherian operators applied only on f. In fact, $(V^k, \partial^k)(f' - p'g) = 0$ for $k=\mu+1,\ldots,\nu$ is equivalent with $0 = (V^k, \lambda_k \partial^k)(f' - p'g) = (V^k, \lambda_k \partial^k)f' - (V^k, \sum_r \lambda_{kr} \partial^r) f_1 = 0$, for the same k.

The solvability condition for pg = f is therefore $(V^k, \partial^k)f_1=0$ $k=1,\ldots,\mu$, $d_{V^k}(\lambda_k \partial^k f' - \sum_r \lambda_{kr}\partial^r f_1) = 0$, $k=\mu+1,\ldots,$.

§ 5. The local extension theorem.

At this moment it is possible to give a proof of the fundamental principle, along the lines of the proofs given by L. Ehrenpreis [2], V.P. Palamodov [2], and more recently, J.E. Björck [1]. In these proofs, the following theorem plays (in different variants) a central role (cf. e.g. J.E. Björck [1]).

Theorem 5.1. ("the local extemsion theorem"). Let p be a $s \times m$ matrix of polynomials, such that the module pP^m is primary, and consider $\rho > 0$ and $\delta \geq 0$. Then there exist:

- $\rho' > 0$, $\delta' \geq 0$, $C > 0$, $K > 0$,
- a polynomial R which does not vanish identically on the variety V of p,
- a set (V^k, ∂^k), $k=1,\ldots,\mu$ of algebraic Noetherian operators, associated with p,

such that the following is true:

if $z^0 \in V$ and if $f \in [O(|z - z^0| < \rho(1 + |z^0|)^{-\delta})]^s$ satisfies $|\Sigma\partial_i^k f_i(z)| \leq 1$ for all k and $z \in V$, $|z - z^0| < \rho(1 + |z^0|)^{-\delta}$, then there is $\tilde{f} \in [O(|z - z^0| < \rho'(1 + |z^0|)^{-\delta'})]^s$ such that $|\tilde{f}(z)| \leq C(1 + |z^0|)^K$ for $|z - z^0| < \rho'(1 + |z^0|)^{-\delta'}$ and such that $\tilde{f} - Rf \in p[O(|z - z^0| < \rho'(1 + |z^0|)^{-\delta'}]^m$.

Using ch. VII from L. Hörmander [3], it is possible to show that the fundamental principle follows from the local extension theorem (cf. § 2 from J.E. Björk [1]).

For completeness, we scetch a proof of theorem 5.1. In part II of this paper, we will turn, with more details, to another proof of the fundamental principle, in which the main emphasis is set on the structure of distributions concentrated on algebraic varieties.

The proof of theorem 5.1 relies on ch. VII from L.Hörmander [3]. We first mention four lemmas.

Lemma 5.2. Suppose the local extension theorem is valid for $I(V)$ for some irreducible algebraic variety V, and consider $\rho > 0$, $\delta \geqslant 0, Q \in P, Q \notin I(V)$. There are constants C, K, ρ', δ' such that if $h \in O(|z - z^o| < \rho(1 + |z^o|)^{-\delta})$ satisfies $|Qh(z)| \leqslant 1$, for $z \in V$, $|z - z^o| \cap \rho(1 + |z^o|)^{-\delta}$, then $|h(z)| \leqslant C(1 + |z^o|)^K$ on $V \cap \{|z - z^o| < \rho'(1 + |z^o|)^{-\delta'}\}$.

Proof. Choose $R \notin I(V)$ and $\tilde{h} \in O(|z - z^o| < \tilde{\rho}(1 + |z^o|)^{-\tilde{\delta}})$ such that $|\tilde{h}(z)| \leqslant \tilde{C}(1 + |z^o|)^{\tilde{K}}$ on $|z - z^o| < \tilde{\rho}(1 + |z^o|)^{-\tilde{\delta}}$ and such that $\tilde{h} = RQh$ on $V \cap |z - z^o| < \tilde{\rho}(1 + |z^o|)^{-\tilde{\delta}}$. Further, we use theorem 7.5.11 from L.Hörmander [3] and write $\tilde{h} = RQh' + \sum p_i h_i$, with p_i generators for $I(V)$ and h', h_i, satisfying $|h'(z)| + \sum |h_i(z)| \leqslant C(1 + |z^o|)^K$ on $V \cap |z - z^o| < \rho'(1 + |z^o|)^{-\delta'}$. Obviously, $h = h'$ on the latter set.

Lemma 5.3. Consider $V \subset W$ algebraic varieties, and suppose theorem 5.1 is valid for the irreducible components of V and W. Let $\partial \in P(D)$ be such that ∂f vanishes on V if f vanishes on W. Then for every ρ, δ, there are ρ', δ', C, K such that the following is true: if $h \in O(|z - z^o| < \rho(1 + |z^o|)^{-\delta}$ satisfies $|h(z)| \leqslant 1$ on $W \cap |z - z^o| < \rho(1 + |z^o|)^{-\delta}$, then $|\partial h(z)| \leqslant C(1 + |z^o|)^K$ on $V \cap |z - z^o| < \rho'(1 + |z^o|)^{-\delta'}$.

Proof. In view of proposition 2.7, we may assume that $V = W$, and we may also assume that V is irreducible. Now choose $R \notin I(V)$ and $\tilde{h}$ such that $|\tilde{h}(z)| \leqslant C(1 + |z^o|)^{-\delta}$ on $|z - z^o| < \rho(1 + |z^o|)^{-\delta}$ and such that $Rh - \tilde{h}$ vanishes on V. Therefore, $\partial Rh = \partial \tilde{h}$, when $z \in V$.

Further we observe, that if ∂ satisfies the hypothesis of the proposition, then so does $\partial^{(B)}$ for any B (induction in $|B|!$). We now prove inductively that $|R\partial^{(B)} h(z)| \leq C'(1+|z^o|)^{K'}$, and finally apply the preceding lemma.

Lemma 5.4. Consider p of form $p = t^q + \sum_{j<q} c_j(z')t^j$. Then for every ρ, δ there are ρ', δ' with the following property:

- if z^o satisfies $p(z^o) = 0$, then there is $\rho/q < c < \rho$ such that the set $\{z;\ |z' - z'_o| \leq \rho'(1+|z'_o|)^{-\delta'},\ |t - t_o| \leq c(1+|z'_o|)^{-\delta}$ has property (W) with respect to p.

This follows, e.g. from §2, ch. IV. in B. Malgrange [4] .

Lemma 5.5. Suppose that $U \subset C^n$ has property (W) with respect to p and is of form $U = \{z;\ |z' - z'_o| \leq \rho'(1+|z'_o|)^{-\delta'},$ $|t - t_o| \leq c(1+|z'_o|)^{-\delta}$. Denote Δ the discriminant of p with respect to t and suppose $W \subset C^{n-1}$ is an irreducible algebraic variety such that $\Delta \notin I(W)$. Then there are $\rho'', \delta'', C, K, \sigma$, such that if $h \in O(U)$ satisfies $|(\partial/\partial t)^r f(z)| \leq 1$ for $z \in U \cap \{W \times C\} \cap \{p=0\}$, $r < \alpha$, then $|\Delta^{\sigma}\varepsilon^U(f)(z')| \leq C(1+|z'|)^K$ for $|z' - z'_o| \leq \rho''(1+|z'_o|)^{-\delta''} \cap W$.

This is not hard to prove, if we use (3) from § 3.

Theorem 5.1 can now be proved with arguments which are parallel to those from §4. To do this, it is however convenient to get rid of the R from the statement (to make R = 1). This can be done with the arguments from §1.2 in J.E. Björck [1]). We can now extend the theorem also to the case of modules which are no more primary (this extension can be avoided), and then we start induction in n and s. What should be done in the induction in s is obvious. For the induction in n, we work on sets like those from lemma 5.4 , p associated with the ideal under study as in §4. We may then project with the operator ε^U to n-1 variables and apply the local extension lemma there.

PART II. The proof of the fundamental principle.

§ 1. Distributions concentrated on algebraic varieties. Their projections and liftings.

1. Definition 1.1. For $A \subset C^n$ a compact set, $V \subset C^n$ an algebraic variety, $p \in P$ without multiple factors of form $p = t^q + \sum_{j<q} c_j(z')t^j$, and $\alpha \in N$, we denote

- $S'(V,A) = \{u \in S'(C^n)\ ;\ \mathrm{supp}\, u \subset A, ru = 0 \text{ for all } r \in I(V)\}$,
- $S'(p,\alpha,A) = \{u \in S'(C^n)\ ;\ \mathrm{supp}\, u \subset A,\ p^\alpha u = 0\}$,
- $\tilde{S}'(p,\alpha,A) = \{u \in S'(C^n)\ ;\ \mathrm{supp}\, u \subset A, u(g) = 0 \text{ for all } g \in C_0^\infty(C^n)$ for which $p(z)=0, \rho<\alpha$, implies $(\partial/\partial t)^\rho g(z) = 0\}$.

We obtain a first information for the above spaces using:

Theorem 1.2. Let Q be a matrix of polynomials, and consider R such that $P^k \xrightarrow{{}^tR} P^r \xrightarrow{{}^tQ} P^t$ is exact. For every τ, $a > 0$, $\nu > 0$ there are constants τ', C, K such that if $u \in [S'(C^n; \{|z_i - z_i^o| \leq a\})]^r$ for some $z^o \in C^n$ satisfies $Ru = 0$ and $\|u\|_\tau \leq 1$, then there is $v \in [S'(C^n; \{|z_i - z_i^o| \leq a+\nu\})]^t$ such that $u = Qv$ and $\|v\|_{\tau'} \leq C(1 + |z^o|)^K$.

This is a consequence of wellknown results of L. Hörmander [1], S. Lojasiewicz [1], and B. Malgrange [1] about the division of distributions by polynomials. In the present situation, theorem 1.2 follows quite easily, if we know that it is true for the case, when Q is a single polynomial. For the convenience of the reader, we indicate this reduction in §5 from below.

2. Proposition 1.3. Denote $A = \{z; |z_i - z_i^o| \leq a\}$, $A' = \{z; |z_i - z_i^o| \leq a+\nu\}$. V, V^j, W, W' are algebraic varieties.

a) Suppose $V = \bigcup V^j$. Then for every $u \in S'(V,A)$, there are $u_j \in S'(V^j,A')$ such that $u = \sum u_j$.

b) Suppose V is irreducible and consider $\lambda \in P \setminus I(V)$. Then for every $u \in S'(V,A)$, there is $v \in S'(V,A')$ such that $\lambda v = u$. Further if $g \in P(D)$, then there are $v_\beta \in S'(V,A')$ such that $gu = \lambda \sum_\beta g^{(\beta)} v_\beta$.

c) Suppose $g \in P(D)$ is such that $d_W f = 0$ implies $d_V g f = 0$ for all $f \in P$. If $u \in S'(V,A)$, then ${}^t g\, u \in S'(W,A)$

d) Suppose $\lambda \in P$ vanishes on $V \setminus W'$. Then for every $u \in S'(V,A)$ there is $w \in S'(W' \cap V, A')$ such that $\lambda u = \lambda w$.

Moreover, for every τ, there are constants τ', C, K, which do not depend on z^0, such that if $\|u\|_\tau \leqslant 1$, then we can choose u_j, v, v_β, w such that

$$\sum_j \|u_j\|_{\tau'} + \|v\|_{\tau'} + \sum_\beta \|v_\beta\|_{\tau'} + \|w\|_{\tau'} + \|{}^t g u\|_{\tau'} \leqslant C(1 + |z^0|)^K.$$

Proof. a) For $u \in S'(V,A)$, we want to find $u_j \in S'(V^j, A')$ such that $\sum u_j = u$ and such that $q \in I(V^j)$ implies $q w_j = 0$. By theorem 1.2 this is possible if and only if $\lambda u = 0$ for all $\lambda \in \bigcap_j I(V^j)$. Therefore a) follows from $I(V) = \bigcap_j I(V^j)$.

b) We search for $v \in S'(V,A)$ which satisfies $\lambda v = u$ and $qv = 0$ for all $q \in I(V)$. The compatibility conditions of the corresponding system are that $r \in I(V)$ implies $ru = 0$, i.e. the hypothesis on u. The second assertion in b) follows from the first, if g is of order zero. We may therefore apply induction in the order of g. To do so, first choose v such that $\lambda v = u$. Then $g u = g \lambda v = \lambda g v - \sum_{\gamma \neq 0} 1/\gamma!\; g^{(\gamma)} \lambda_{(\gamma)}\, v$ and we can apply the induction.

c) is trivial

d) Denote $V^1 = V \cap \{\lambda = 0\}$. Then $V = V^1 \cup (V \cap W')$, and we apply a).

The last assertion in the proposition follows checking constants in the preceding proofs.

3. The main tool which we use in the study of the spaces $S'(V,A)$ is projection on C^{n-1}. We will consider here projections of elements from $S'(p,\alpha,A)$.

Here and everywhere in the rest of this paragraph, $p \in P$ is without multiple factors of form $p = t^q + \sum_{j<q} c_j(z')t^j$.

<u>Definition 1.4.</u> We define $\pi_k : S'(p,\alpha,A) \longrightarrow S'(C^{n-1},B)$, B the projection of A on C^{n-1} by

$$(\pi_k u)(g) = u(t^k g) \qquad g \in S(C^{n-1}).$$

The map $S'(p,\alpha,A) \longrightarrow \prod_{0}^{\alpha q-1} S'(C^{n-1},B)$ defined by the components $\pi_o,\ldots,\pi_{\alpha q-1}$ will be denoted π.

The next lemma is dual to lemma 3.2.part I.

Lemma 1.5.a) Consider r a polynomial in n-1 variables. Then $\pi_o(r\, t^j u) = r\, \pi_j u$.

b) Consider $r \in P$ and denote r_{jk} the coefficients of t^k in $t^j r = \sum r_{jk} t^k + p^\alpha h$. Then $\pi_j(ru) = \sum_k r_{jk} \pi_k u$.

When $W \subset C^{n-1}$ is an algebraic variety and $r \in I(\{W\times C\} \cap \{p = 0\})$, then, for $\alpha = 1$, $r_{jk} \in I(W)$.

The proof is obvious.

Lemma 1.6. Consider $A = \{z;\ |z_i - z_i^o| \leqslant a\}$. For every τ there are τ', γ, C such that for $u \in S'(p,\alpha,A)$ which satisfies $\|u\|_\tau \leqslant 1$, the following two assertions are equivalent:

(i) there is $v \in S'(p,\alpha,A)$ such that $u = \partial/\partial\bar t\ v$ and such that $\|v\|_{\tau'} \leqslant C$

(ii) $\pi_k u = 0$, $k = 0,\ldots,\alpha q-1$.

Proof. (i) $\Longrightarrow$ (ii) is trivial. To prove (ii) $\Longrightarrow$ (i) it suffices to show that for all $g \in C^\infty(C^n)$ which satisfy $\partial/\partial\bar t\ g = 0$, we have $u(g) = 0$. In fact, we then apply theorem 3.22 from L.Hörmander [2] and obtain $u = \partial/\partial\bar t\ v$ which has suppoart in A and which can be estimated suitably. In view of $\partial/\partial\bar t\ p^\alpha v = (= p^\alpha\, \partial/\partial\bar t\ v =) = 0$ we have $p^\alpha v = 0$. Thus suppose g satisfies $\partial/\partial\bar t\ g = 0$. We can then apply the Weierstrass preparation theorem and find $g_o,\ldots,g_{\alpha q-1} \in C^\infty(C^{n-1})$ such that $g = \sum_j g_j t^j + p^\alpha h$ for some $h \in C^\infty(C^n)$. It follows that $u(g) = \sum u(g_j\, t^j) = 0$.

4. In order to define an inverse to the map π, we first must extend some constructions from § 3, part I.

At first, we define for $f \in C^\infty(C^n)$ functions $g_j = \varepsilon_j(f)$ in n-1 variables, which are defined when $\Delta(z') \neq 0$ and are C^∞ there, such that

$z \in V,\ \Delta(z') \neq 0, \rho < \alpha \Longrightarrow (\partial/\partial t)^{\rho}(f - \sum \varepsilon_j(f) t^j)(z) = 0.$

Here $V = \{z;\ p(z) = 0\}$ and Δ is the discriminant of p with respect to t. In the notations from § 3, part I, ε corresponds to ε^U, $U = C^n$. Obviously, we could have defined ε for functions defined on more general sets with property (W), but we do not need such extensions here essentially.

Many results from § 3, part I, can be obtained for the present situation, using the same arguments.

Thus there are polynomials T_j such that $\varepsilon_j(f) = \varepsilon_{\alpha q-1}(T_j f)$, and when $\Delta(y') \neq 0$ and $t_1,\dots,t_q \in O_{y'}$ are distinct germs which satisfy $p(z', t_i(z')) = 0$, then close to y',

$$\varepsilon_{\alpha q-1}(f)(z') = (1/(\alpha-1)!)\sum_k (\partial/\partial t)^{\alpha-1}(f/\prod_{i\neq k}(t-t_i(z'))^{\alpha})(z', t_k(z')) \quad (1)$$

This shows that, for σ great enough, the functions $\Delta^{\sigma}\varepsilon_j(f)$ can be defined also on $\Delta(y') = 0$, and that the extension is of any given degree of regularity, if we choose σ conveniently. More precisely, for every k, there are $\sigma, \varkappa, C, K$ such that $\Delta^{\sigma}\varepsilon_j(f)$ is k-times differentiable and such that

$$\sup_{|\gamma|\leqslant k} |D^{\gamma}_{z',\bar z'}\Delta^{\sigma}\varepsilon_j(f)(z')| \leqslant C(1+|z'|)^K \sup_{\substack{|\varkappa|\leqslant \varkappa \\ i=1,\dots,q}} |D^{\varkappa}_{z,\bar z} f(z)(z', t_i(z'))| \quad (2)$$

Here we have used that $|\Delta^{\sigma} D^{\gamma} t_i(z')| \leqslant \tilde C(1 + |z'|)^K$. For $\gamma = 0$ this is wellknown and for $\gamma \neq 0$ it follows by differentiation of $p(z', t_i(z')) = 0$, using induction.

<u>Proposition 1.7.</u> For every a, k, there are $\sigma, \varkappa, c, C, K$ such that if $u_0,\dots,u_{\alpha q-1} \in S'(C^{n-1}, \{|z_i - z_i^o| \leqslant a, i=1,\dots,n-1\})$ and if $|u_j(g)| \leqslant \sup_{|\gamma|\leqslant k} \sup_{z' \in C^{n-1}} |D^{\gamma} g(z')|$ for all $g \in C_o^{\infty}(C^{n-1})$, then

$\eta(\Delta^{\sigma}u_0,\dots,\Delta^{\sigma}u_{\alpha q-1}) : C_o^{\infty}(C^n) \longrightarrow C$ defined by

$\eta(\Delta^{\sigma}u_0,\dots,\Delta^{\sigma}u_{\alpha q-1})(f) = \sum_j u_j(\Delta^{\sigma}\varepsilon_j(f))$ is a distribution in $\tilde S'(p, \alpha, \{|z_i - z_i^o| \leqslant a, i=1,\dots,n-1\} \times \{|z_n| \leqslant c(1 + |z_o'|)^K\})$ such that $\eta(\Delta^{\sigma}u_0,\dots,\Delta^{\sigma}u_{\alpha q-1})(f) \leqslant C \sup_{\substack{|\gamma|\leqslant\varkappa \\ z \in C^n}} (1+|z^o|)^K |D^{\gamma} f(z)|$

Further, η is an inverse for Π, in the sense that

$\eta(\Pi_0 \Delta^\sigma u, \ldots, \Pi_{\alpha q-1} \Delta^\sigma u) = \Delta^\sigma u$, for all $u \in S'(p,\alpha,A)$.

The proposition follows easily from (2).

Proposition 1.8. a) For every τ, there is τ', σ, c, C, K such that, if in the proposition 1.7, $\| u_j \|_\tau \leqslant 1$, then

$\| \eta(\Delta^\sigma u_0, \ldots, \Delta^\sigma u_{\alpha q-1}) \|_{\tau'} \leqslant C(1 + |z'_0|)^K$.

b) There are polynomials R_s, $0 \leqslant s < \alpha$, and $\sigma' \in N$, such that, if we denote $\tilde{\eta}$ the operator of type η associated with p for $\alpha = 1$, then

$\Delta^{\sigma'} \eta(0,\ldots,0,\Delta^\sigma u) = \sum_s (\partial/\partial t)^s R_s \tilde{\eta}(0,\ldots,0,\Delta^\sigma u)$.

c) When $\alpha = 1$, $W \subset C^{n-1}$, $u_j \in S'(W, |z_i - z_i^0| \leqslant a, i=1,\ldots,n-1)$, then

$\tilde{\eta}(\Delta^\sigma u_0 u, \ldots, \Delta^\sigma u_{\alpha q-1}) \in S'(\{W \times C\} \cap V, \{|z_i - z_i^0| \leqslant a, i \neq n\} \times \{|t| \leqslant c(1+|z'_0|)^K\})$.

d) $\operatorname{supp}(\Delta^\sigma \Pi_0 u, \ldots, \Delta^\sigma \Pi_{\alpha q-1} u) \subset \operatorname{supp} u$.

Proof. a) follows from proposition 1.7, b) from lemma 3.10, part I, and c) is immediate. It remains to check d).

Since we may multiply with $\Delta^{\sigma'}$ for some conveni*n*et σ', it suffices to show that, for all sufficiently small sets U with property (W) we have $\operatorname{supp} \eta(\Delta^\sigma \Pi_0 u, \ldots, \Delta^\sigma \Pi_{\alpha q-1} u) \cap \{\Delta(z') \neq 0\} \subset \subset U \cap \{\Delta(z') \neq 0\}$, if $\operatorname{supp} u \subset U$. $\operatorname{supp} u \subset U$ implies $p_U^\alpha u = 0$, as is easily seen.

Further, it is immediate that $\eta(\Delta^\sigma \Pi_0 u, \ldots, \Delta^\sigma \Pi_{\alpha q-1} u) \subset B \times C$, where B is the projection of supp u on C^{n-1}. It suffices therefore to prove that $\eta(\Delta^\sigma \Pi_0 u, \ldots, \Delta^\sigma \Pi_{\alpha q-1} u)(f) = 0$ for any $f \in C_0^\infty(U' \times C \setminus \Delta(z') = 0)$, U' the projection of U on C^{n-1} and which satisfies $\operatorname{supp} f \cap U = \emptyset$.

Now choose $t_1, \ldots, t_q \in O(C^{n-1} \setminus \Delta(z')=0)$, distinct functions such that $p(z', t_i(z')) \equiv 0$. If the support of f is sufficiently small, then we can labbel these functions in such a way, that $(z', t_i(z')) \in U, i=1,\ldots,k, (z', t_i(z')) \notin U, i=k+1, \ldots, q$, for any z' in the projection on C^{n-1} of supp f. Now we write $f = \prod_{i \leqslant k} (t - t_i(z')) f'$ for some $f' \in C_0^\infty(C^n)$ and choose

$g_j(z') \in C_o^{\infty}(C^{n-1}), j \leq \alpha(q-k)-1$ such that $k < i \leq q, \rho < \alpha$ implies $(\partial/\partial t)^{\rho}(f' - \sum g_j t^j)(z', t_i(z')) = 0$. In view of the unicity of the $\varepsilon_j(f)$, it follows that $\sum \varepsilon_j(f) t^j =$

$= \prod_{i \leq k} (t - t_i(z'))^{\alpha} (\sum g_j(z') t^j)$. Therefore,

$\eta(\Delta^{\sigma}\pi_o u, \ldots, \Delta^{\sigma}\pi_{\alpha q-1} u)(f) = \sum \Delta^{\sigma}\pi_j u(\varepsilon_j(f)) = u(\Delta^{\sigma} \sum \varepsilon_j(f) t^j) =$

$= u(\Delta^{\sigma} \prod_{i \leq k} (t - t_i(z'))^{\alpha} (\sum g_j t^j) = 0$, in view of the fact, that $p_U = \prod_{i \leq k} (t - t_i(z'))^{\alpha}$ on supp $\sum g_j t^j$.

5. We need a commutation relation for $\partial/\partial z_s \eta$ which is dual to corollary 3.7 from part I. We first observe that, with $Q_s, R_s \in P$ those from lemma 3.6, part I, and for $\Delta(y') \neq 0$

$$(\partial/\partial z_s)\, \varepsilon_{\alpha q-1}(f)(y') = \varepsilon_{\alpha q-1}((\partial/\partial z_s + Q_s/\Delta^{\sigma}\, \partial/\partial t + R_s/\Delta^{\sigma})f) \qquad (3).$$

In fact, when we derivate (1), then we arrive, since $\partial \bar{t}_i/\partial z_s = 0$, at exactly the expressions which occur in the proof of lemma 3.6 from part I.

We conclude from (3) that there is a first order differential operator p_s, with polynomial coefficients, such that (still for $\Delta(y') \neq 0$)

$$\Delta^{\sigma}\, \partial/\partial z_s\, \varepsilon_{\alpha q-1}(f)(y') = \varepsilon_{\alpha q-1}(p_s f)(y') \qquad (4).$$

We now obtain the following

Proposition 1.9. Consider $\delta_o, \ldots, \delta_{\alpha q-1}$ differential operators in n-1 variables, with polynomial coefficients. Then there is σ' and $\partial \in P(D)$ such that

$$\Delta^{\sigma'} \sum_k \delta_k\, \varepsilon_k(f)(y') = \varepsilon_{\alpha q-1}(\partial f)(y') \quad \text{when } \Delta(y') \neq 0.$$

Moreover, for every $\tau \in R$ there is $\tilde{\sigma}$ such that if $w \in S'(C^{n-1}, A) \cap H^{\tau}$, then

$$\eta({}^t\delta_o \Delta^{\sigma'+\tilde{\sigma}} w, \ldots, {}^t\delta_{\alpha q-1}\Delta^{\sigma'+\tilde{\sigma}} w) = {}^t\partial\, \eta(0, \ldots, 0, \Delta^{\tilde{\sigma}} w).$$

(this means in particular, that both sides of the equality are welldefined.)

Proof. When σ' is great enough, we may write δ_k in the form $\Delta^{\sigma'}\delta_k = \sum_{\beta} a_\beta (\Delta^{\sigma}\partial/\partial z_1)^{\beta_1}\cdots(\Delta^{\sigma}\partial/\partial z_{n-1})^{\beta_{n-1}}$ for some a_β, which are polynomials in n-1 variables. The first assertion therefore follows from (4) and $\varepsilon_j(f) = \varepsilon_{\alpha q-1}(T_j f)$. The second assertion follows by dualization.

6. Proposition 1.10. For every a, z, there are τ', σ, k, C, K with the following property: denote $A = \{z;\ |z_i - z_i^0| \leq a\}$ for some $z^0 \in C^n$. Then

a) For every $u \in S'(p,\alpha,A)$ with $\|u\|_\tau \leq 1$, there are $u_j \in \tilde{S}'(p,\alpha,A)$ such that $\Delta^\sigma u = \sum_{0\leq j\leq k} (\partial/\partial\bar{t})^j u_j$, $\|u_j\|_{\tau'} \leq C(1+|z^0|)^K$.

b) Suppose that for some $u_j \in \tilde{S}'(p,\alpha,A)$, $u_j \in H_{\tau'}$, $\sum_{j\leq k} (\partial/\partial\bar{t})^j u_j = 0$. Then $\Delta^\sigma u_j = 0$.

Proof. When σ is great enough, we can apply lemma 1.6 and proposition 1.7 and find $v_1 \in S'(p,\alpha,A)$, $\|v_1\|_{\tilde{\tau}} \leq \tilde{C}(1+|z^0|)^{\tilde{K}}$ such that $\Delta^{\tilde{\sigma}} u = \eta(\Delta^{\tilde{\sigma}}\pi_0 u, \ldots, \Delta^{\tilde{\sigma}}\pi_{\alpha q-1} u) + (\partial/\partial\bar{t})v_1$.

We may continue this procedure, and arrive at a representation $\Delta^\omega u = \sum_{j\leq k} (\partial/\partial\bar{t})^j u_j + (\partial/\partial\bar{t})^{k+1} v_{k+1}$ for some $u_j \in \tilde{S}'(p,\alpha,A)$, $v_{k+1} \in S'(p,\alpha,A)$, all estimable. It suffices therefore to show that, when k is great enough, then $\operatorname{supp} v_{k+1} \subset \{z;\ \Delta(z') = 0\}$.

To see this, we first observe that, for some $\varkappa \in N$, $D^\beta f(z) = 0$ for all $|\beta| \leq \varkappa$ and all z with $p(z) = 0$, implies $u(f) = 0$, and that $(\partial/\partial\bar{t})^j(\partial/\partial t)^\rho f(z) = 0$ for $\rho < \alpha$, $p(z) = 0$ implies $u_j(f) = 0$.

For every $f \in C_0^\infty(C^n \setminus \{\Delta(z')=0\})$ we now consider $f_{j\lambda} \in C_0^\infty(C^{n-1})$ such that

$p(z) = 0, \varkappa \leq k, \rho < \alpha \Rightarrow (\partial/\partial\bar{t})^\varkappa (\partial/\partial t)^\rho (f - \sum_{s\leq k, \lambda \leq \alpha q-1} \bar{t}^s t^\lambda f_{s\lambda}) = 0.$

Therefore, if k is great enough, $\Delta^\omega u(f) = \Delta^\omega u(\sum \bar{t}^s t^\lambda f_{s\lambda})(z') =$
$= \sum_j ((\partial/\partial\bar{t})^j u_j(\sum \bar{t}^s t^\lambda f_{s\lambda}) = \sum_j ((\partial/\partial\bar{t})^j u_j)(f)$. This proves a).

b) follows from the proof of a).

7. Definition 1.11. Consider $\rho > 0$, $\delta \geqslant 0$ and sets $A, B \subset C^n$. We write $A \subset_{(\rho,\delta)} B$ and say that B is a (ρ,δ) neighborhood of A if the set $\{z; \text{distance}(z,A) \leqslant \rho(1+|z|)^{-\delta}\} \subset B$.

Proposition 1.12. Consider $V \subset C^n$ an algebraic variety, $a > 0$, $\nu > 0$, $\tau \in R$. Then there are $\rho > 0$, $\delta \geqslant 0$, τ', C, K and coordinates, with the following property:

if $u \in S'(C^n; \{|z_i - z_i^o| \leqslant a\})$: $\|u\|_\tau \leqslant 1$, then there are $v, w \in S'(C^n, \{|z_i - z_i^o| \leqslant a+\nu\})$, $\|v\|_{\tau'} + \|w\|_{\tau'} \leqslant C(1 + |z^o|)^K$ such that $u = v + (\partial/\partial\bar{z}_n)w$, $\operatorname{supp} v \subset_{(\rho,\delta)} \complement V$.

We postpone the proof to the end of §3 from below.

§2. The fundamental principle in $S'(C^n, A)$.

1. Theorem 2.1. Consider $a > 0$, $\nu > 0$ and $p = (p_{ij})$ an $s \times m$ matrix of polynomials and (V^k, ∂^k), $k=1,\dots,\mu$ a collection of alg. Noetherian operators associated with p.

Then for every τ there is τ', C, K such that if $u \in [S'(C^n, \{|z_i - z_i^o| \leqslant a\})]^s$, $\|u\|_\tau \leqslant 1$ satisfies

$$^t p u = 0$$

then there are $v_k \in S'(V^k, \{|z_i - z_i^o| \leqslant a+\nu\})$, $k=1,\dots,\mu$ such that $\|v_k\|_{\tau'} \leqslant C(1 + |z^o|)^K$ and such that

$$u = \sum_k {}^t\partial^k v_k .$$

Note that the converse is also true: if (V,∂) is an algebraic Noetherian operator such that $(V,\partial)ph = 0$ for all $h \in P^m$, and if $v \in S'(V,A)$, then ${}^t p\, {}^t\partial v = 0$.

Before starting the proof of theorem 2.1, which is dual to §4, part I, we mention the following corollary of proposition 1.3.

Proposition 2.2. Consider $L_j, L_{kj} \in P(D)$, $k=1,\dots,m$, $j=1,\dots,s$ and V, V^k, $k=1,\dots,\mu$ algebraic varieties. Suppose that $(V, h L_j) = (V, \sum h_k L_{kj})$ for $h \in P$, $d_V h \neq 0$, $h_k \in P(D)$. Then for every τ, there are τ', C, K such that the following is true:

$d_{V^k} f = 0 \Rightarrow d_V h_k f = 0.$

for every $w \in S'(V,\{|z_i - z_i^o| \leq a\}), \|w\|_{\tau} \leq 1$, there are $w_k \in S'(V^k, \{|z_i - z_i^o| \leq a+\nu\}), \|w_k\|_{\tau'} \leq C(1 + |z^o|)^K$ such that ${}^tL_j\, w = \sum_k {}^tL_{kj}\, w_k$.

2. Proof of theorem 2.1.

In view of the preceding result, it suffices to find algebraic Noetherian operators $(V^k, \partial^k), k=1,\ldots,\mu$ and estimable $v_k \in S'(V^k, |z_i - z_i^o| \leq a+\nu')$ such that $(V^k, \partial^k p) = 0$ and such that $u = \sum {}^t\partial^k\, v_k$. This will be done with arguments parallel to those from §4, part I. As in that paragraph, there is induction in n and s. Since the induction in s is completely parallel to the arguments from §4, part I, we will only perform the induction in n.

a) Thus suppose, theorem 2.1 is proved in n-1 variables, and suppose $u \in S'(C^n, |z_i - z_i^o| \leq a)$ satisfies $p_1 u = \cdots = p_m u = 0$ for some polynomials $p_i \in P$. We can reduce ourselves to the situation when the ideal I generated by $p_1,\ldots,p_m$ in P is primary. We now choose $p \in P$ and $\alpha \in N$ such that p has no multiple factors, $p = t^q + \sum_{j<q} c_j(z')t^j$, $p^\alpha \in I$ and such that the discriminant Δ of p with respect to t does not vanish identically on $V = \{z;\ p_i(z) = 0,\ i=1,\ldots,m\}$.

Further, we observe that it suffices to prove the assertion from the statement for $\Delta^\sigma u$, where $\sigma \in N$ is some fixed number, which will be chosen later. In fact, if we know that $\Delta^\sigma u = \sum {}^t\partial^k\, v_k$, $v_k \in S'(V^k, |z_i - z_i^o| \leq a+\nu/3)$, for some Noetherian operators (V^k, ∂^k) associated with $p_1,\ldots,p_m$, then we can apply proposition 1.3 for those V^k with $\Delta \notin I(V^k)$ and write ${}^t\partial^k\, v_k = \Delta^\sigma \sum_B {}^t\partial^{k(B)}\, v_{kB}$ for $v_{kB} \in S'(V^k, |z_i - z_i^o| \leq a+2\nu/3)$.

Thus $v = u - \sum'_{k,B} {}^t\partial^{k(B)}\, v_{kB}$, - the ' in Σ' means that the summation is only for those k for which $\Delta \notin I(V^k)$-, satisfies $\Delta^\sigma v = 0$, for some great σ and also $p_i v = 0$. We may therefore apply induction in the dimension of the associated varieties.

b) We now apply proposition 1.10 and write $\Delta^\sigma u = \sum_{j\leq k} (\partial/\partial \bar{t})^j u_j$,

for some $u_j \in \tilde{S}'(p,\alpha, |z_i - z_i^o| \leq a+\bar{\nu})$ which satisfies

$\|u_j\|_{\tilde{\tau}} \leq \tilde{C}(1 + |z^o|)^{\tilde{K}}$. It also follows that $p_i \Delta^\sigma u_j = 0$, for

$i=1,\ldots,m$ and $j \leq k$.

It clearly suffices to prove the theorem for all u_j.

To sum up, we see that we have reduced the proof of theorem 2.1 to the following

Lemma 2.3. For every τ, a, ν there are τ', σ, C, K such that if $u \in \tilde{S}'(p,\alpha, |z_i - z_i^o| \leq a)$ satisfies $p_i u = 0, i=1,\ldots,m$, and $\|u\|_\tau \leq 1$, then there are $v_k \in S'(V^k, |z_i - z_i^o| \leq a+\nu)$ such that $\Delta^\sigma u = \sum {}^t\partial^k v_k$ and $\|v_k\|_{\tau'} \leq C(1 + |z^o|)^K$.

Proof. We apply the induction hypothesis in n, and may therefore write $\pi u = \sum {}^t\delta^k w_k$ for $w_k \in S'(W^k, \{|z_i - z_i^o| \leq a+\nu/2, i=1,\ldots,n-1\})$, $\|w_k\|_{\tilde{\tau}} \leq \tilde{C}(1 + |z_o'|)^{\tilde{K}}$. Here $W^k \subset C^{n-1}$ and (W^k, δ^k) appear as in §4, part I. When σ is chosen great enough, then $\Delta^{\sigma} {}^t\delta^k w_k = 0$ for all k for which Δ vanishes identically on W^k. It follows that $\Delta^\sigma \pi u = \sum \Delta^\sigma {}^t\delta^k w_k = \sum_{k,\beta}' {}^t\delta^{k(\beta)} w_{k\beta}$, $w_{k\beta}$ suitable. Here Σ' means again that the sum is extended only over those k for which $d_{Wk}\Delta \neq 0$.

Further we multiply with $\Delta^{\sigma'}$ and arrive at a representation $\Delta^{\sigma+\sigma'} \pi u = \pi(\Delta^{\sigma+\sigma'} u) = \Sigma' {}^t\delta^{k(\beta)} \Delta^{\varkappa'+\varkappa} w'_{k\beta}$, where $\varkappa', \varkappa$ are as great as we want, if we choose σ' great enough, and where $w'_{k\beta} \in S'(W^k, |z_i - z_i^o| \leq a + \nu/2)$ satisfy an estimate $\|w'_{k\beta}\|_{\bar{\tau}} \leq C'(1 + |z^o|)^{K'}$, $\bar{\tau}$ independente of $\varkappa'$ and $\varkappa$.

For σ' great enough, this gives $\Delta^{\sigma+\sigma'} u =$ $= \eta(\Sigma' {}^t\delta^{k(\beta)} \Delta^{\varkappa+\varkappa'} w'_{k\beta}) = \Sigma' {}^t\partial^{k\beta} \eta(0,\ldots,0,\Delta^\varkappa w'_{k\beta})$ for some $\partial^{k\beta} \in P(D)$. Further we may apply proposition 1.8 to write,

if $\varkappa$ is great enough, $\eta(0,\ldots,0,\Delta^\varkappa w'_{k\beta}) =$ $= \sum_{j<\alpha} (\partial/\partial t)^j R_j \tilde{\eta}(0,\ldots,0,\Delta^{\tilde{\varkappa}} w'_{k\beta})$ for some $R_j \in P$ and some $\tilde{\varkappa}$.

Now we apply lemma 4.10 from part I (or rather its proof) and proposition 1.3 d) and conclude that there are $w''_{k\beta} \in$

$S'(\{W^k \times C\} \cap \{p=0\} \cap V, |z_i - z_i^o| \leqslant a+\nu)$, which can be estimated, such that $^t\partial^{k\beta}(\partial/\partial t)^j R_j \tilde{\eta}(0,\ldots,0,\Delta^{\varkappa} w'_{k\beta}) = {}^t\partial^{k\beta}(\partial/\partial t)^j w''_{k\beta}$.

This proves the lemma.

3. Theorem 2.4. Let V be an irreducible algebraic variety, $W \subsetneqq V$ a subvariety, and $\tau \in R, a > 0, > 0$. There are $\rho, \delta, \tau', C, K$ with the following property:

if $u \in S'(V, |z_i - z_i^o| \leqslant a)$, $\|u\|_\tau \leqslant 1$, then there are

$v, v_j \in S'(V, |z_i - z_i^o| \leqslant a+\nu)$, $j=1,\ldots,n$ such that

- $u = v + \sum (\partial/\partial \bar{z}_j) v_j$
- $\operatorname{supp} v \subset_{(\rho,\delta)} \{W$
- $\|v\|_{\tau'} + \sum \|v_j\|_{\tau'} \leqslant C(1+|z^o|)^K$.

We postpone the proof to the end of § 3.

§ 3. Distribution spaces with rapid decay at infinity.

1. In this paragraph we introduce distribution spaces, which are characterized by the fact, that their elements decay rapidly at infinity. Several possibilities are here at hand. We prefer a combination of L_2-type estimates with sup-norm estimates, which leads most rapidly to the desired results.

Definition 3.1. Consider $\varphi : C^n \longrightarrow R_+$ a function such that $|\varphi(z^1) - \varphi(z^2)| \leqslant |z^1 - z^2|$ and let be : $\beta \in R_+$, $B \subset R^n$ a compact convex set and $b, \tau \in R$. We denote $H^\tau_{\beta,B,b}(C^n, -\varphi)$ the space of distributions $u \in S'(C^n)$ for which there is some $e \in C_o^\infty(C^n)$, $e \not\equiv 0$ and C such that

$$\|e(z-y)u(z)\|_\tau \leqslant C \exp-(\beta\varphi(y) + H_B(\operatorname{Im} y) + b \ln(1+|y|)) \qquad (1).$$

Here H_B is the support function of B.

If (1) is true for some e, C, then it is true, for every e, for some C, which depends on e. This follows from proposition 3.3 from below. In particular, $H_{\beta,B,b}$ has a natural Banach space topology.

We first introduce a technical definition.

<u>Definition 3.2.</u> Let $\rho>0$, $\delta \geqslant 0$, $c > 0$, $d \geqslant 0$ be constants. $e \in C_0^{\infty}(C^n)$ is called a (ρ,δ) test function of order (c,d), if $y \in \operatorname{supp} e \Rightarrow \operatorname{supp} e \subset \{z;\ |z - y| \leqslant \rho(1+|y|)^{-\delta}\}$ and $|D^{\gamma} e(z)| \leqslant c^{|\gamma|+1}\rho^{-|\gamma|}(1 + |z|)^{\delta|\gamma|}$ for $|\gamma| \leqslant d$.

When e is a (ρ,δ) test function of order $(c, |\tau| + 2n + 1)$, then for $y \in \operatorname{supp} e$

$$|\zeta|^{|\tau|+2n+1}\,|\hat{e}(\zeta)| \leqslant C(c,\rho,\delta)(1 + |y|)^{\delta(|\tau|+2n+1)}$$

and it follows that

$$\int |\hat{e}(\zeta)|(1+|\zeta|)^{|\tau|}\,d\zeta d\bar{\zeta} \leqslant \tilde{C}(c,\rho,\delta)(1+|y|)^{\delta(|\tau|+2n+1)}.$$

Therefore, when $v \in H^{\tau}(C^n)$, and e is a (ρ,δ) test function of order $(c, |\tau| + 2n + 1)$, then

$$\|ev\|_{\tau} \leqslant \tilde{C}(1+|y|)^{\delta(|\tau| + 2n + 1)}\,\|v\|_{\tau}. \qquad (2).$$

In fact, this follows from (cf. L. Hörmander [2] theorem 2.2.5),

$$\|ev\|_{\tau} \leqslant \left(\int |\hat{e}(\zeta)|(1+|\zeta|)^{|\tau|}d\zeta d\bar{\zeta}\right)\|v\|_{\tau}.$$

The main tool for testing if $u \in H_{\beta,B,b}(C^n,-\varphi)$ is

<u>Proposition 3.3.</u> For every ρ, δ there is K with the following property: let be $u \in S'(C^n)$ and suppose that there are c, C such that for every (ρ,δ) test function of order $(c, |\tau| + 2n + 1)$, $y \in \operatorname{supp} e \Rightarrow \|eu\|_{\tau} \leqslant C\exp-(\beta\varphi(y) + H_B(\operatorname{Im} y) + (b+K)\ln(1 + |y|))$. Then u is in $H^{\tau}_{\beta,B,b}(C^n,-\varphi)$.

Moreover, when $\delta = 0$, then we may take $K = 0$.

Proof. Choose $g \in C_0^{\infty}(C^n)$. There is c' such that for every $y \in C^n$ we can find (ρ,δ) test functions e_i of order $(c, |\tau| + 2n + 1)$, $i=1,\dots,[c'(1+|y|)^{2n\delta}]$ ($[C]$ is the integer part of C) such that $\sum e_i \equiv 1$ on $\operatorname{supp} g(z - y)$. We now write $\|g(z-y)u(z)\|_{\tau} \leqslant \sum_i \|e_i(z)g(z-y)u(z)\|_{\tau} \leqslant \tilde{C}\sum_i \|e_i u\|_{\tau}$, with $\tilde{C} = \int |\hat{g}(\zeta)|(1+|\zeta|)^{\tau}\,d\zeta d\bar{\zeta}$.

2. One of the main aims of this paragraph, is to study the solvability of the system $\bar{\partial}$ (with the notations from complex function theory) in the spaces $H^{\tau}_{\beta,B,b}$. This problem can be reduced

to the case when τ is large.

Proposition 3.4. For every $u \in [H^{\tau}_{\beta,B,b}(C^n,-\varphi)]_{(0,k)}$, $\bar\partial u = 0$ there is $v \in [H^{\tau+1}_{\beta,B,b}(C^n,-\varphi)]_{(0,k-1)}$ and $w \in [H^{\tau+1}_{\beta,B,b}(C^n,-\varphi)]_{(0,k)}$ such that $u = w + \bar\partial v$.

Here $X_{(0,q)}$ is the space of (0,q) forms with coefficients in X.

Proof. For every $\lambda \in Z^n + i\,Z^n$, choose e_λ, e'_λ with the following properties:

- e_λ is a partition of unity in C^n, and $e_\lambda e'_\lambda = e_\lambda$.
- $\operatorname{supp} e_\lambda \subset \{z;\ |z_i - \lambda_i| \leq 1\}$, $\operatorname{supp} e'_\lambda \subset \{z;\ |z_i - \lambda_i| \leq 2\}$.
- $|D^\gamma e_\lambda| + |D^\gamma e'_\lambda| \leq C$, for $|\gamma| \leq |\tau| + 2n + 1$.

From the hypothesis it follows that $\|e_\lambda u\|_\tau \leq C \exp-(\beta\varphi(\lambda) + H_B(\operatorname{Im}\lambda) + b(1+|\lambda|))$. We may then find $v_\lambda \in [H^{\tau+1}(C^n)]_{(0,k-1)}$ such that $e_\lambda \bar\partial v_\lambda = e_\lambda u$ and such that $\|e_\lambda v_\lambda\|_{\tau+1} \leq C\ \|e'_\lambda u\|_\tau$. The lemma now follows, if we take $v = \sum e_\lambda v_\lambda$ and $w = -\sum v_\lambda \wedge \bar\partial e_\lambda$.

Further, we need the following result, which is a consequence of the results from L. Hörmander [3] (cf. O. Liess [1]).

Proposition 3.5. There are constants C, γ, K with the following property: suppose $u \in [L_2^{loc}(C^n)]_{(0,k)}$ satisfies $\bar\partial u = 0$ and $\int |u(z)|^2 \exp-2(\beta\varphi(z) + H_B(\operatorname{Im} z) + b \ln(1+|z|))\, dz\, d\bar z \leq 1$.

Then there is $v \in [L_2^{loc}(C^n)]_{(0,k-1)}$ such that $\bar\partial v = u$ and such that

$$\int |v(z)|^2 \exp-2(2\beta\varphi(z) + H_B(\operatorname{Im} z) + \gamma\beta|\operatorname{Im} z| + (b+K)\ln(1+|z|))\, dz\, d\bar z \leq C.$$

In the next proposition, we denote $|\mu|_- = \min(\mu, 0)$ and for $B \subset R^n$ a compact convex set and $\varkappa \in R_+$, $B + \varkappa = \{x \in R^n,\ x = x^1 + x^2,\ x^1 \in B, |x^2| \leq \varkappa\}$.

Proposition 3.6. There are constants γ, K with the following property:

Suppose $u \in [H^{|\tau|_-}_{2\beta,B+\gamma\beta,b+K}(C^n,-\varphi)]_{(0,k)}$ satisfies $\bar\partial u = 0$ when $k < n$ and $u(\exp i\langle x,z\rangle) = 0$ for $x \in B + \gamma\beta$, when $k=n$.

Then there is $v \in [H^{|z+1|-}_{B,B,b}(C^n,-\varphi)]_{(0,k-1)}$ such that $\bar\partial v = u$.

Remark 3.7. For γ', K' sufficiently great, the condition for $k = n$ implies that $u(h) = 0$ for all entire $h \in A(C^n)$ which satisfy $|h(z)| \leq C \exp(2B\varphi(z)+(B+(\gamma'/2)|\mathrm{Im}\, z| +(b+K'/2)\ln(1+|z|))$. This follows from I. Hörmander [4] (cf. also O. Liess [1]).

Proof of proposition 3.6. For γ, K sufficiently great suppose that $u \in [H_{2B,B+\gamma B,b+K+n+1}(C^n,-\varphi)]_{(0,k)}$ satisfies the hypothesis from the proposition. We may apply lemma 3.4. several times and conclude, that the proposition follows, if we can prove it, when $u \in [H^{n+1}_{2B,B+\gamma B,b+K+n+1}(C^n,-\varphi)]_{(0,k)}$.

In view of Sobolevs immersion lemma, it follows that the coefficients u_J of the form u satisfy

$$|u_J(z)| \leq C \exp -(2B\varphi(z) +H_B(\mathrm{Im}\, z) + B|\mathrm{Im}\, z|+(b+K+n+1)\ln(1+|z|))$$

and therefore that

$$\int |u_J(z)|^2 \exp -2(2B\varphi(z)+H_B(\mathrm{Im}\, z)+ B|\mathrm{Im}\, z|+(b+K)\ln(1+|z|))dz\, d\bar z \leq C.$$

Dualizing proposition 3.5, it follows, in view of the remark, and for γ, K sufficiently great, that there is $v \in [L_2^{loc}(C^n)]_{(0,k-1)}$ such that $\int |v|^2 \exp 2(B\varphi(z)+H_B(\mathrm{Im}\, z) + b\ln(1+|z|))\, dz\, d\bar z \leq C$. This implies $v \in [H^0_{B,B,b}(C^n,-\varphi)]_{(0,k-1)}$.

Theorem 3.8. Consider $p = (p_{ij})$ an $s \times m$ matrix of polynomials. There are constants $\varkappa$, γ, K with the following property: Suppose $u \in [H^{|z|-}_{\varkappa B,B+\gamma B,b+K}(C^n,-\varphi)]^s_{(0,k)}$ satisfies $\bar\partial\, {}^tp\, u = 0$, when $k < n$ and ${}^tp\, u(\exp i\langle x,z\rangle) = 0$ for $x \in B+\gamma B$, when $k=n$. Then there is $v \in [H^{|z+1|-}_{B,B,b}(C^n,-\varphi)]^s_{(0,k-1)}$ such that $\bar\partial\, {}^tpv = {}^tpu$.

Proof. The proof is by induction in (what we call by slight abuse of laguage) the cohomological dimension of p. We say that p is of cohomological dimension r, if there are matrices $q_1, \dots, q_r$ such that

$$0 \to P^{m_1} \xrightarrow{q_1} P^{m_2} \xrightarrow{q_2} P^{m_3} \xrightarrow{q_3} \cdots \to P^{m_r} \xrightarrow{q_r} P^m \xrightarrow{p} P^s \qquad (5).$$

By a wellknown theorem of Hilbert, every p is of cohomological dimension smaller then n.

a) Suppose that p is of cohomological dimension 0. We apply first proposition 3.6 and write ${}^t p\, u = \bar\partial v'$, with suitable v'. It therefore remains to solve the equation $v' = {}^t p\, v$, with v in $[H^{\tau'}_{B,B,b}(C^n, -\varphi)]^s$. This can be done by first localizing, and the applying theorem 1.2.

b) We may now assume that theorem 3.8 is proved already for the matrices $q_1, \ldots, q_r$ in (3), and want to prove it for p. This can be done by drawing a suitable diagram. We prefer here the direct argument. In fact, we first solve, with proposition 3.6, ${}^t p\, u = \bar\partial\, v'$ for suitable v'. Then $\bar\partial\, {}^t q_r\, v' = 0$, and therefore the induction hypothesis shows that ${}^t q_r v' = \bar\partial\, {}^t q_r v''$, with v'' satisfying suitable estimates. We can now apply theorem 1.2 as in the above, and conclude that $v' - \bar\partial v'' = {}^t p\, v$, with some estimable v. Obviously, ${}^t p\, u = \bar\partial\, {}^t p\, v$.

3. We conclude the paragraph with the proofs of proposition 1.12 and of theorem 2.4. We prepare these proofs with a lemma.

Lemma 3.9. Consider $V = \{z;\ t^q + \sum_{j<q} c_j(z')t^j = 0\}$, $a > 0$, $\nu > 0$. Then there are constants $\rho > 0$, $\delta \geq 0$, $c > 0$, $c' > 0$, with the following property: for every $z^o \in C^n$ there are :

- (ρ, δ) test functions e'_i, $i = 1, \ldots, [c'(1 + |z'_o|)^{2n\delta}]$
- a map $\tilde q : \{1, \ldots, [c'(1 + |z'_o|)^{2n\delta}]\} \longrightarrow \{1, 2, \ldots, q\}$
- functions $e''_{ij} \in C^\infty_o(C)$, $i = 1, \ldots, [c'(1 + |z'_o|)^{2n\delta}]$, $j = 1, \ldots, \tilde q(i)$

such that

- $\sum_i (e'_i)^2 = 1$, on $\{z';\ |z_i - z^o_i| \leq \varepsilon, i = 1, \ldots, n-1\}$,
- $|D^\gamma_{t,\bar t} e''_{ij}| \leq c$, for $|\gamma| \leq |z| + 2n + 1$,
- the diameter of supp e''_{ij} is less then ν,
- if $p(z', t) = 0$ and $(z', t) \in$ supp $e'_i \times$ supp e''_{ij}, then $e''_{ij}(\tau) = 1$ for $|\tau - t| \leq \nu/8q$,
- $\sum_{i,j} (e'_i)^2 (e''_{ij})^2 = 1$ in a (ρ, δ) neighborhood of V, when $|z_i - z^o_i| \leq a$.

Proof. We apply lemma 5.4, part I, and obtain (ρ,δ) such that if $z', \tilde{z}' \in C^{n-1}$ satisfy $|z' - \tilde{z}'| \leq \rho(1 + |z'|)^{-\delta}$, then the roots $t_i(z'), t_i(\tilde{z}')$ of $t \to p(z',t) = 0, t \to p(\tilde{z}',t) = 0$ can be labbeled in such a way that $|t_j(z') - t_j(\tilde{z}')| \leq \nu/8q$.

Let us now fix z' and consider $t_1,\ldots,t_{q'}, q' \leq q$ the distinct values of the solutions of $t \to p(z',t) = 0$, and denote $J_i(z')$, $i=1,\ldots,q'$ the sets $J_i(z') = \{z \in C, |z - t_i| < \nu/2q\}$. Further, we perform a partition in the set $\{1,\ldots,q'\}$: we write

$\{1,\ldots,q'\} = I^1 \cup I^2 \cup \ldots \cup I^{\tilde{q}(z')}$ such that

- $k \neq k', i \in I^k, i' \in I^{k'} \Rightarrow J_i(z') \cap J_{i'}(z') = \emptyset$
- if $i,i' \in I^k$, then there are $i_1,\ldots,i_\varkappa \in I^k$ such that $i_1 = i$, $i_\varkappa = i'$ and such that $J_{i_s}(z') \cap J_{i_{s+1}}(z') \neq \emptyset$.

Finally denote $J'_j(z')$, $j=1,\ldots,\tilde{q}(z')$ the sets $J'_j(z') = \bigcup_{i \in I^j} J_i(z')$.

The following properties are then easily verified:

- the diameter of any J'_j is smaller then ν,
- $j \neq j' \Rightarrow J'_j(z') \cap J'_{j'}(z') = \emptyset$,
- if $|\tilde{z}' - z'| \leq \rho(1 + |z'|)^{-\delta}$ and if $p(\tilde{z}',\tilde{t}) = 0$, then there is j such that $\{z; |z - \tilde{t}| \leq \nu/8q\} \subset \bigcup_{i \in I^j} \{z; |z - t_i(z')| < \nu/4q\}$.

The lemma now follows, if we choose spheres U_i, $i=1,\ldots,[c'(1 + |z'_0|)^{2n\delta}]$, of diameter $\rho(1 + |z'_0|)^{-\delta}$ such that the spheres U'_i with the same center, with diameter $\rho/2\,(1 + |z'_0|)^{-\delta}$ form a covering for $\{z'; |z_s - z_s^0| \leq a, s=1,\ldots,n-1\}$. If the centers of these spheres are z^i, then we define $\tilde{q}(i) = \tilde{q}(z^i)$. It is now easy to construct partitions of unity with the required properties. (It suffices to choose ρ small compared with ν).

Proof of proposition 1.12. We may suppose that V is of pure codimension one and thus that it is of the form from lemma 3.9. We apply the lemma, and solve for all i, j the equations $\partial/\partial\bar{t}\, v_{ij} = e'_i e''_{ij} u$, such that $\| e'_i e''_{ij} v_{ij} \|_{\varkappa+1} \leq C'(1 + |z^0|)^{K'}$.

We assume here that $v_{ij} = 0$ if $e_i' e_{ij}'' u = 0$, and set $v = \sum_{i,j} e_i' e_{ij}'' v_{ij}$. At points (z',t) where e_{ij}'' is identically one close to t, we have $\partial/\partial\bar{t}\, e_i' e_{ij}'' v_{ij} = (e_i')^2 (e_{ij}'')^2 u$. The proposition therefore follows if ρ is small compared with ν.

Proof of theorem 2.4. When $V = C^n$, the theorem follows from proposition 1.12. When $V \neq C^n$, we will use induction in n. This induction will be similar to the induction in n in the proof of theorem 2.1.

First we choose $p \in I(V)$ of form $p = t^q + \sum_{j<q} c_j(z')t^j$, without multiple factors, such that the dicriminant Δ of p with respect to t is not in $I(V)$. Further, we may suppose, that the support of u is in a set with property (W) which is contained in $\{|z_i - z_i^o| \leq a\}$ (with the aid of lemma 3.9 we can write the given u as a sum of not more then $[qc'(1 + |z^o|)^{2n\delta}]$ terms with this property).

Let us denote π the operator constructed in § 1 for ρ and $\alpha = 1$. $Q u = 0$ for all $Q \in I(V)$ implies therefore that $\pi u = \sum {}^t\delta^k v^k$, for (W^k, δ^k), $k=1,\ldots,\mu$, Noetherian operators which appear in § 4, part I, and $v^k \in S'(W^k, |z_i - z_i^o| \leq a + \nu/3)$, all v^k estimable. As was pointed out in § 4, part I, we may suppose that W^k is the projection on C^{n-1} of V (we are dealing here with the global situation). For every v^k we now apply the induction hypothesis, and obtain $\tilde{v}^k \in S'(W^k, |z_i - z_i^o| \leq a + 2\nu/3)$, all estimable, such that $\pi u - \sum {}^t\delta^k \tilde{v}_k = \sum_{j<n} \bar{\partial}_j w_j$ and such that the $\tilde{v}_k$ vanish in a (ρ,δ) neighborhood of $\Delta(z') = 0$ and in a (ρ,δ) neighborhood of the profection on C^{n-1} of W.

Let us now consider $w = u - \eta(\sum {}^t\delta^k v_k)$. (Since the support of v_k avoids a (ρ,δ) neighborhood of $\Delta(z') = 0$, we do not need factors Δ^σ here.) It follows that $w(h) = 0$ for any h which is analytic near supp w. The same is then true if we conside $w' = u - \tilde{w}$

where $\widetilde{w}$ is the distribution $\eta(\sum {}^t\delta^k \widetilde{v}^k)$ on the set ... of property (W) considered above, and $\widetilde{w} = 0$ outside that set. The theorem now follows dualizing wellknown properties of concerning the solvability of the $\bar\partial$ system on convex sets.

§ 4. The proof of the fundamental principle.

1. Consider $\Omega \subset R^n$ a convex domain, $B \subset \Omega$ a convex compact set, and $f \in C^{\infty}(\Omega)$. If h is an entire function on C^n which satisfies

$|h(z)| \leq C \exp (H_B(\mathrm{Im}\ z) + b \ln(1 + |z|))$ (1)

then $h = \hat{v}$ for some $v \in E'(B)$, and we can define $f(h) = v(f)$. We obtain for every b and B a linear functional on the space of all entire functions which satisfy for some C the inequality (1). Applying the Hahn-Banach theorem in an obvious way, we can find a Radon measure μ, defined on C^n, such that $\int d|\mu| < \infty$, and such that, with $u = \mu / \exp (H_B(\mathrm{Im}\ z) + b\ (1 + |z|))$, $f(h) = \int h(z)\ d\ u(z)$. For $h = \exp i\langle x,z\rangle$, this gives (if $b \geq 0$)

$f(x) = (\int \exp i\langle x,z\rangle\ du(z)) = u(\exp i\langle x,z\rangle)$, for $x \in B$ (2).

It follows from Sobolevs immersion lemma that $u \in H^{-n-1}_{0,B,b}(C^n,0)$. The representation (2), with $v \in H^{-n-1}_{0,B,b}(C^n,0)$ is nonunique.

If we had known that f is real analytic in Ω, then with the arguments from above, we could have shown, that there is β (which depends on f and B) and $u \in H^{z}_{\beta,B,b}(C^n,|z|)$ such that $f(x) = u(\exp i\langle x,z\rangle)$ for $x \in B$.

More generally, we will suppose that there is $\varphi : C^n \longrightarrow R_+$, with $|\varphi(z^1) - \varphi(z^2)| \leq |z^1 - z^2|$, τ, β and $u \in H^{\tau}_{\beta,B,b}(C^n,-\varphi)$ such that $f(x) = u(\exp i\langle x,z\rangle)$ for $x \in B$.

Theorem 4.1. (The fundamental principle). Consider

- $p = (p_{ij})$, an $s \times m$ system of polynomials,
- (V^k, ∂^k), $k=1,\ldots,\mu$ a collection of algebraic Noetherian operators associated with p,

- $\varphi : C^n \longrightarrow R_+$ such that $|\varphi(z^1) - \varphi(z^2)| \leq |z^1 - z^2|$,
- $f_1,\dots,f_m \in C^\infty(\Omega)$ and $\tau \in R$ such that for every compact B and every $b \geq 0$, there is β and $u_i \in H^\tau_{\beta,B,b}(C^n,-\varphi)$ such that $f_i(x) = u_i(\exp i\langle x,z\rangle)$ for $x \in B$ and $P(D_x)f = 0$.

Then for every $B, b \geq 0$, there are $\tau', \beta' > 0$ and $v^k \in H^{\tau'}_{\beta',B,b}(C^n,-\varphi)$ such that $f(x) = (\sum_k {}^t\partial^k v^k)(\exp i\langle x,z\rangle)$ for $x \in B$, and such that Q $I(V^k)$ implies $Q v^k = 0$.

Proof. Let B, b be given and denote γ, K the constants from theorem 3.8. When β is small enough, then $B+\gamma\beta$ is a compact in Ω. If we shrink β still further, we may suppose that there are $u_i \in H^\tau_{\beta,B+\gamma\beta,b+K}(C^n,-\varphi)$ for which (3) is valid, when $x \in B+\gamma\beta$. In view of theorem 3.8, there are $v_r \in H^{\tau'}_{\beta',B,b}(C^n,-\varphi)$, $\beta' > 0$, $r=1,\dots,n$ such that ${}^tp(z)\, u = \sum_r (\partial/\partial\bar z_r)\, {}^t v_r$. If we denote $w = u - \sum_r {}^t(\partial/\partial \bar z_r) v_r$, then still $f(x) = w(\exp i\langle x,z\rangle)$ for $x \in B$, and we now have ${}^tp(z)w=0$. The theorem now follows by localization from theorem 2.1.

2. Theorem 4.1 is not yet the fundamental principle, as this is stated in the introduction, since the v^k are not necessarily Radon measures. The fact (which should be of small interest), that we can also obtain a representation with Radon measures, follows from the following theorem.

<u>Theorem 4.2.</u> Consider V an irreducible algebraic variety, $W \subsetneq V$ a subvariety, and $u \in H^\tau_{\beta,B,b}(C^n,-\varphi)$ such that $Q(z)\, u = 0$ for all $Q \in I(V)$.

Then there are $\rho, \delta, \tau', \beta', \gamma, K, v \in H^{\tau'}_{\beta',B-\gamma\beta,b-K}(C^n,-\varphi)$ and $w_j \in H^{\tau'}_{\beta',B-\gamma\beta,b-K}(C^n,-\varphi)$ such that $Q \in I(V)$ implies $Q\, v = 0$, that $u = v + \sum_{j=1}^{n} (\partial/\partial\bar z_j)\, w_j$, and such that supp $v \subset_{(\rho,\delta)} \complement W$.

This follows from localization of theorem 2.4.

To return now to the discussion from before, we first see, that in theorem 4.1 we may suppose that the supports of the v^k stay away from a (ρ,δ) neighborhood of the singular part of the varieties V^k. We can now localize and then the result can be reduced to the case, when V^k is an affine subspace. For this situation the theorem is easy to prove. We leave the details to the reader.

Theorem 4.2 is related to the concept of "sufficient sets" introduced by L. Ehrenpreis [2]. In fact such theorems can be obtained from L. Hörmander [3].

§ 5. The solvability in $S'(C^n)$ of overdeterminated systems.

1. In this paragraph, we return to theorem 1.2. This theorem follows from

Theorem 5.1. Let p and q be matrices of polynomials such that $P^k \xrightarrow{q} P^m \xrightarrow{p} P^s$ is exact. Suppose that $u \in [S'(C^n)]^m$ satisfies ${}^t q\, u = 0$. Then there is $v \in [S'(C^n)]^s$ such that $u = {}^t p\, v$.

Proof of theorem 1.2. From theorem 5.1 we obtain (with the notations from theorem 1.2) that $[S'(C^n)]^t \xrightarrow{Q} [S'(C^n)]^r \xrightarrow{R} [S'(C^n)]^k$ is exact. Since S' is the dual of a Frechet space, this implies the following: for every seminorm $\varkappa$ in $S(C^n)$, there is a seminorm $\varkappa'$ in $S(C^n)$ and $C > 0$, such that if $Ru = 0$ and $|u(\varphi)| \leq C|\varphi|_\varkappa$ for all $\varphi \in S(C^n)$, then there is v with $u = Qv$ and such that $|v(\varphi)| \leq |\varphi|_{\varkappa'}$. Further, if $u = Qv$, and if $\varphi \in C_0^\infty(C^n)$, $\varphi \equiv 1$ in $|z_i - z_i^0| \leq a + \nu/2$, and $\varphi(z) = 0$ for $z \notin |z_i - z_i^0| \leq a + \nu$, then $\tilde{v} = \varphi v$ is a solution with suitable support of the equation $u = Q\tilde{v}$. Further, we may suppose v estimable, and therefore φv will satisfy an estimate of the desired type, if $|D^\gamma \varphi| \leq C$, for $|\gamma| \leq \sigma$, σ sufficiently great.

2. It is easy to see, that theorem 5.1 can be restated in the following way: the system $u = {}^t p\, v$ is solvable, if and only if for every $\lambda = (\lambda_1, \dots, \lambda_m)$ with $\lambda\, {}^t p = 0$ it follows that $\lambda u = 0$. If $\lambda\, {}^t p = 0$ implies $\lambda u = 0$, we will say that u satisfies the compatibility conditions of the system ${}^t p v = u$ (or else that the system ${}^t p\, v = u$ is compatible). In this formulation, theorem 5.1 is

related rather to p then to the pair p,q.

We state separately

Theorem 5.2. Suppose $p \in P$. Then for every $u \in S'(C^n)$ there is $v \in S'(C^n)$ such that $u = pv$.

Theorem 5.2 follows from results of L.Hörmander [1] and S.Lojasiewicz [1]. Theorem 5.1 for general m,s follows from B.Malgrange [4]. For the convenience of the reader, we show in this paragraph, that it is possible to reduce theorem 5.1 to theorem 5.2. This will be done by induction in n and s.

3) Induction in n. Thus consider a system of form

$$p_j\, v = u_j\ ,\ j=1,\dots,m\ ,\ p_j \in P \qquad (1).$$

We say that the system (1) has dimension r, if the dimensions of the irreducible components of $V = \{z \in C^n;\ p_1(z) = \dots = p_m(z) = 0\}$ are all smaller then r.

Proposition 5.3. Suppose that theorem 5.1 is proved in n-1 variables, and suppose that it is also proved in n variables, for systems of form (1), which are of dimension $r' < r$. Then it is true also for systems of form (1), when $\dim V = r$.

We prepare the proof of proposition 5.3 with some remarks and lemmas.

Suppose that for some u, (1) is compatible, and that $\dim V = r$. Let us choose p, α and coordinates (z',t), such that for some $f_i \in P$, $p^\alpha = \sum p_i f_i$, $p = t^q + \sum_{j<q} c_j(z')\, t^j$, such that p has no multiple factors and such that $\dim (V \cap \{z;\ \Delta(z') = 0\}) < r$. That such a polynomial exists, follows from lemma 1.10, part I. In fact, write $V = \bigcup V^i$, V^i the irreducible components of V, and let p^i be polynomials associated with $I(V^i)$ as in lemma 1.10, part I which are such that $j \neq i \Longrightarrow p^i \notin I(V^j)$. (Such polynomials can be found in the form $p^i = S^i + c(R^i)^2$, where S^i is associated with $I(V^i)$ by lemma 1.10, part I, c is a suitable number and $R^i \in I(V^i) \setminus \bigcup_{j\neq i} I(V^j)$). The product of all the p^i is then a polynomial with the desired properties.

Lemma 5.4. Consider $\widetilde{v} \in S'(C^n)$ such that $p^\alpha \widetilde{v} = \sum_{j=1}^{m} f_i u_i$. Then the system

$$p^\alpha w = 0, p_j w = u_j - p_j \widetilde{v}, j=1,\ldots,m, \tag{2}$$

is compatible. In particular, $p^\alpha (u_j - p_j v) = 0$.

Proof. Suppose λ, $\lambda_j \in P$ are such that $0 = \lambda p^\alpha + \sum \lambda_j p_j = \sum_j (\lambda f_j + \lambda_j) p_j$. Since (1) is compatible, this implies, $0 = \sum_j (\lambda f_j + \lambda_j) u_j = \lambda p^\alpha \widetilde{v} + \sum \lambda_j u_j = -\sum \lambda_j p_j \widetilde{v} + \sum \lambda_j u_j$. This gives $\sum \lambda_j (u_j - p_j \widetilde{v}) = 0$, which means that (2) is compatible.

We now want to use projection on C^{n-1} as we have used it in §2. This time, the distributions which we consider, are no more with compact support. Two possibilities are here at hand. Either, we use a partition of unity, to make all elements with compact support (then we have to write down estimates), or else, we extend the results from §1 to the spaces

$S'(W) = \{u \in S'(C^n); Q \in I(W) \text{ implies } Qu = 0\}$, and

$\widetilde{S}'(p, \alpha) = \{u \in S'(C^n);\ u(g) = 0 \text{ if } \rho < \alpha, p(z) = 0 \Rightarrow (\partial/\partial t)^\rho g(z) = 0\}$.

Here p is the one from before and W is an algebraic variety. For notational reasons, we prefer the second modality. The proof of proposition 1.10 now gives the following result: there are σ, μ and $w_{j\varkappa} \in \widetilde{S}'(p, \alpha)$, $\varkappa \leq \mu$, such that $\Delta^\sigma (u_j - p_j \widetilde{v}) = \sum_\varkappa (\partial/\partial \bar{t})^\varkappa w_{j\varkappa}$, and if σ' is great enough, then it follows from the unicity in proposition 1.10, that the systems

$$p^\alpha w_\varkappa = 0, p_j w_\varkappa = \Delta^{\sigma'} w_{j\varkappa} \tag{3$_\varkappa$}$$

are compatible.

In fact, if λ, λ_j are such that $\lambda p^\alpha + \sum \lambda_j p_j = 0$, then $\sum_\varkappa (\partial/\partial t)^\varkappa \sum_j \lambda_j w_{j\varkappa} = \sum \Delta^\sigma \lambda_j (u_j - p_j \widetilde{v}) = 0$, which implies $\Delta^{\sigma'} \sum \lambda_j w_{j\varkappa} = 0$ for all $\varkappa$.

Lemma 5.5. Suppose that we can solve $(3)_\varkappa$ for all $\varkappa$. Then proposition 5.3 follows.

Proof. If $w_\varkappa$ is a solution for $(3)_\varkappa$, then $\widetilde{w} = \sum (\partial/\partial \bar{t})^\varkappa w_\varkappa$ satisfies $p^\alpha \widetilde{w} = 0, p_j \widetilde{w} = \Delta^{\sigma'+\sigma}(u_j - p_j\widetilde{v})$. If we write $\widetilde{w} = \Delta^{\sigma'+\sigma} v$, then $p_j v = (u_j - p_j\widetilde{v}) + g_j$ with $\Delta^{\sigma'+\sigma} g_j = 0$. It therefore remains to find a solution $f \in S'(C^n)$ for $p_j f = g_j$. Apparently, we have not made much progress, but the important fact is here, that $\Delta^{\sigma'+\sigma} g_j = 0$.

To find f, denote $I \subset P$ the ideal generated by $p_1, \ldots, p_m$, and let $I = \bigcap I_i$ be the irreducible primary decomposition of I. Further, denote I' the intersection of those ideals in the decomposition which have $\dim I_i < r$, and I'' the intersection of the remaining ideals. Now choose $S \in I', S \notin I''$. We claim that

$$p_j f = g_j, j=1,\ldots,m, S\Delta^{\sigma'+\sigma} f = 0 \qquad (4)$$

is compatible, of dimension $< r$, and therefore solvable by the induction hypothesis. In fact, if $\sum \lambda_j p_j + \lambda S \Delta^{\sigma'+\sigma} = 0$, then $\lambda S \in I'$ and $\lambda S \Delta^{\sigma'+\sigma} \in I$. Since Δ is not in the radical of I'', it follows that $\lambda S \in I''$, which gives $\lambda S \in I$, i.e., $\lambda S = \sum \mu_j p_j$ for some $\mu_j \in P$. Therefore $(\lambda_j + \mu_j \Delta^{\sigma'+\sigma}) p_j = 0$, and then $\sum (\lambda_j + \mu_j \Delta^{\sigma'+\sigma}) g_j = 0$. In view of $\Delta^{\sigma'+\sigma} g_j = 0$, we arrive at $\sum \lambda_j g_j = 0$, and the compatibility of (4) is proved.

We have now reduced proposition 5.3 to the systems $(3)_\varkappa$.

Lemma 5.6. Suppose that $w_j \in \widetilde{S}'(p,\alpha)$ are such that the system $p^\alpha w = 0, p_j w = w_j$ is compatible. Suppose further, that theorem 5.1 is proved in n-1 variables. Then there is σ' and $\widetilde{w}$ such that $p^\alpha \widetilde{w} = 0, p_j \widetilde{w} = \Delta^{\sigma'} w_j$.

Proof. Denote π the (extended) operator constructed in §1. We want to study the system $\pi(p_j w) = \pi w_j$. According to lemma 1.5 this is a system of form

$$\sum_i p^j_{ki} h_i = \pi_k w_j, k = 0, \ldots, \alpha q-1, j=1,\ldots,m \qquad (5)$$

which is easily seen to be compatible. If we now choose a solution h of the system (5), then $\widetilde{w} = \eta(\Delta^{\sigma'} h_0, \ldots, \Delta^{\sigma'} h_{\alpha q-1})$ is a solution of $p^\alpha \widetilde{w} = 0, p_j \widetilde{w} = \Delta^{\sigma'} w_j$.

4) To complete the reduction of theorem 5.1 to theorem 5.2, it remains to perform an induction in s.

<u>Proposition 5.7.</u> Suppose theorem 5.1 is proved for $\sigma \times m$ matrices when $\sigma < s$. Then it is also true for $s \times m$ matrices.

Proof. Recall the notations from §4, part I. First we observe that the system ${}^tq'\,{}^tp_1\,v_1 = {}^tq'\,u$ is compatible. Indeed, $\lambda\,{}^tq'\,{}^tp_1 = 0$ implies $\lambda\,{}^tq'\,{}^tp = 0$, which gives $\lambda\,{}^tq'u = 0$. We may therefore apply the induction hypothesis, and obtain a solution v_1 of the system ${}^tq'\,{}^tp_1\,v_1 = {}^tq'u$. If we can find a solution v' of the system ${}^tp'\,v' = u - {}^tp_1 v_1$, then (v_1, v') is a solution for ${}^tpv = u$. To solve ${}^tp'v' = u - {}^tp_1 v_1$, we again use the induction hypothesis. In fact, the system ${}^tp'v' = u - {}^tp_1 v_1$ is compatible, since ${}^tq'(u - {}^tp_1 v_1) = 0$ was the defining equation for v_1.

One may use the results from this paragraph, to give a proof for the solvability of overdeterminated systems of partial differential equations.

Comments and remarks.

1.Historical comments. The fundamental principle has first been stated in L.Ehrenpreis [1] .Proofs were available only for m=s=1 then.At about the same time V.P.Palamodov [1] obtained a variant of the fundamental principle,also for the case m=s=1. Complete detailed proofs appeared in V.P.Palamodov [2] and L.Ehrenpreis [2] .Important results connected with the fundamental principle have been obtained by B.Malgrange [1,2] and L.Hörmander [3] .Recently a new proof has been given by J.E.Björck [1] .For m=s=1 cf. also C.A.Berenstein-M.Dostal [1] . The fundamental principle has been extended to hyperfunction solutions by Kaneko [1,2] (cf. also Oshima [1]).

2.Related with the fundamental principle is the concept of analytically uniform spaces,introduced by L.Ehrenpreis (cf. L.Ehrenpreis [2]).Only the proof of the fundamental principle in L.Ehrenpreis [2] is in the framework of analytically uniform spaces.The present one is formulated such that it works in weak analytically uniform spaces,in the sense of O.Liess [1] . If one wants to obtain the variant from L.Ehrenpreis [2] one may apply the technique from B.A.Taylor [1] .

3.Theorem 2.1,part I,is not true in the C^∞ set-up.This follows from wellknown examples of operators with no solutions for the equation L(x,D)u=0 (examples of H.Lewy and L.Nirenberg). The theorem breaks also down in the analytic case,when we suppose that the implication in it holds at just one point.The condition (L) from V.P.Palamodov [3] (in which the case when $\sum_j L_{ij} u_j = 0$ for all i implies $u_j = 0$ is treated in another context),may however be sufficient to assure that theorem 2.1 remains valid, when we only know that the implication holds at some point.

4. With the arguments from §4, part I, we can prove the following proposition:

<u>Proposition.</u> Let $p_{ij}(z)$ be a $s \times m$ matrix with entries in O_z, (V^k, ∂^k), $k=1,\dots,\mu$ an associated collection of Noetherian operators and consider, in variables $y = (y_1,\dots,y_d)$ a prime ideal I in $\tilde{O}$, $\tilde{O}$ the germs of holomorphic functions in $y_1,\dots,y_d$ at $0 \in C^d$. Further let $r_1,\dots,r_k$ be generators for I and denote W the analytic variety of common zeros of $r_1,\dots,r_k$. Then $h(z,y) \in O^s_{z,0}$ is of form $h_i = \sum_j p_{ij}g_j + \sum_\nu r_\nu h_{i\nu}$, for some $g \in O^m_{z,0}, h_{i\nu} \in O_{z,0}$, if and only if $(V^k \times W, \partial^k)h = 0$ for $k=1,\dots,\mu$.

This proposition has the following corollary

<u>Corollary.</u> Let $V \subset C^n$ be an analytic variety, which is irreducibl at $z^o \in C^n$. Suppose that $I_{z^o}(V)$ is generated by $f_1,\dots,f_m \in O_{z^o}$. Now we identify C^n with R^{2n}, and consider h, a real analytic function in the 2n real variables Re z, Im z. Suppose that h vanishes on V. Then there are real analytic functions $h_1,\dots,h_{2m}$ such that $h = \sum_{i \leq m} (h_i(\text{Re } z, \text{Im } z)f_i(z) + h_{m+i}(\text{Re } z, \text{Im } z)\bar{f}_i(z))$.

This problem appears in B. Malgrange [3].

5. One may ask, what conditions should be imposed on $f \in [C^\infty(C^n)]^s$ in order to assure that the system $f = pg$ has a solution $g \in [C^\infty(C^n)]^m$. Such conditions are easy to obtain from the form in which theorem 4.2, part I, appears in this paper (and which is somewhat stronger then necessary in other proofs of the fundamental principle).

Let us mention first two lemmas:

<u>Lemma.</u> Suppose V is an irreducible analytic variety defined near zero. Then $f \in F_o$ is in $I^F_o(V)$ if and only if, for every q there is C_q such that $|\sum_{|\beta| \leq q} f_\beta z^\beta| \leq C_q |z|^{q+1}$ for $z \in V$.

This follows, e.g. combining results from B. Malgrange [1], with the corollary from above (it should be anyway easy to prove).

Now consider $h \in C^\infty_{z^o}$, $C^\infty_{z^o}$ the germs of functions from $C^\infty(C^n)$

at z^o, and suppose that h is defined for $|z - z^o| < \eta$. If $h = \sum_{\alpha, \beta} h_{\alpha\beta}(z - \tilde{z})^\alpha (\bar{z} - \bar{\tilde{z}})^\beta$ is the Taylor expansion (written in variables $z, \bar{z}$ instead of Re z, Im z) of h at $\tilde{z}$, $|z^o - \tilde{z}| < \eta$ then we denote $h_{\beta z} = \sum_\alpha h_{\alpha\beta}(z - \tilde{z})^\alpha$. Using the preceding lemma, we now obtain, by Taylor expansion, applied first in $\bar{z}$, and then in z:

Lemma. $h_{\beta z}$ is in $I_z^F(V)$ for all z, $|z - z^o| < \eta$, and all β, if and only if $(\partial/\partial\bar{z})^\beta h$ vanishes on V for all β.

We can now prove the following theorem:

Theorem. Let (V^k, ∂^k), $k=1,\ldots,\mu$ be a collection of Noetherian operators associated with the matrix p. Then $pg = f$, $f \in [C^\infty(C^n)]^s$ has a solution $g \in [C^\infty(C^n)]^m$, if and only if $(\partial/\partial\bar{z})^\beta \partial^k f$ vanishes on V^k for all β and all k

Proof. Only the "if" part is nontrivial. To prove it, we first observe, that $p[C^\infty(C^n)]^m$ is closed in $[C^\infty(C^n)]^s$. This is a result of B. Malgrange (cf. B. Malgrange [4]). We can now apply a theorem of H. Whitney (cf. B. Malgrange [4], ch. II), to reduce our problem to the solvability of the system $pg = f$, when f and g are formal power series of the form $f = \sum f_{\alpha\beta}(z - \tilde{z})^\alpha (\bar{z} - \bar{\tilde{z}})^\beta$, $g = \sum g_{\alpha\beta}(z - \tilde{z})^\alpha (\bar{z} - \bar{\tilde{z}})^\beta$. This gives $f_{\beta z} = pg_{\beta z}$, with the notations from above, and the theorem therefore follows from theorem 4.2, part I, in view of the preceding lemma.

6. As is only natural, I have been influenced in many arguments by the existing proofs of the fundamental principle. I have not tried to mention these (and other) influences explicitely. There have been external influences as well. One example is, that I have added §5, part I, only when I became aware of J.E. Björcks paper [1]. At that time however, the present paper was almost finished, and I have not tried to use the new ideas from J.E. Björck [1] here.

7. §3, part I, and §1, §3, part II, have been prepared in another context, for the paper O. Liess [2]. For part I, cf. also O. Liess [3].

References

Berenstein C.A., Dostal M. - [1] Analytically uniform spaces and their applications to convolution operators. Springer, Lecture Notes, 256.

Björck, J.E. [1] The fundamental principle. Lecture from the Summer school at Grebbestad, June 1975. Stockholm University 1975, preprint Nr. §9.

Bourbaki, N. [1] Algebre commutative, ch. 1 Modules plats. Hermann ed., Paris, 1961

Ehrenpreis, L. [1] The fundamental principle for linear constant coefficient partial differential equations. Proc. Intern. Symp. Lin. Spaces, Jerusalem 1960. Jerusalem 1961.

[2] Fourier Analysis in Several Complex Variables, Wiley-Interscience, New-York, 1970.

Grothendieck, A. [1] Elements de geometrie algebrique. Etude local de schemas et des morphismes de schemas, IHES, 1967

Hörmander, L. [1] On the division of distributions by polynimials, Ark. Mat. 3, 1958, 555-568.

[2] Linear partial differential operators, Springer, 1963

[3] An introduction to Complex Analysis in Several Complex Variables. D. Van Nostrand Comp. 1966.

[4] Convolution equations in convex domains, Invent. Math. 4 1968, 306-318.

Kaneko [1] A representation of hyperfunctions by measures, and some of its applications. J. Fac. Sci. Tokyo, Sect. I A, 19, 321-357, 1972

[2] Fund. principle and ext. of solutions of lin. diff. equations with constant coefficients. Hyperfunctions and Pseudodifferential equations. Proc. of a Conference at Katata, 1971, Lecture Notes, 287, 1973 422-134

Liess, O. [1] Intersection theorems for weak analytically uniform classes of functions. To appear.

[2] Solvability conditions for a system pg = f in formal power series, holomorphic germs and C^∞ functions. Proc. of a conference on analytic geometry Bucarest, 1976, To appear.

[3] teză de doctorat, Bucarest, June 1975.

Lojasiewicz, S. [1] Sur le probleme de la division, Studia Math. 18, Nr 1, 1959, 87-136.

Malgrange,B [1] Existence et approximation des solutions des equations aux derivees partielles,et des equations de convolution.Ann.Inst.Fourier 6,1956 271-355

[2] Sur les systemes differentielles a coefficients constants.Col.CN.R,S.1963

[3] Sur les fonctions differentiables et les ensembles analytiques.Bull.Soc.Math.France,91,1963,113-127

[4] Ideals of differentiable functions,Oxford University press,1966

[5] Comologie de Spencer,Universite de Paris,1966 (rotoprint)

Palamodov,V.P. [1] On the general form of solutions of the homogeneous equations with constant coefficients (in Russian) D.A.N.137,Nr.4,774-777 (1961)

[2] Linear differential operators with constant coefficients (in Russian),Moscow,1967 (also Springer Verlag,Grundlehren,Bd 168,1970)

[3] Differential operators in the class of convergent power series and the Weierstrass preparation theorem (in Russian).Funk.Anal.i ego pril.1968, t2:3,58-70

Oshima [1] A proof of Ehrenpreis' fundamental principle in hyperfunctions.Proc.Jap.Acad. vol L Nr.1

Taylor,B.A. [1] A seminorm topology for some (DF) spaces of entire functions.Duke Math.J.vol.38:2,1971,379-387

[2] Analytically uniform spaces of infinitely differentiable functions.Comm.Pure Appl.Math. 24,1971,39-51.