INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
STIINȚIFICĂ ȘI TEHNICĂ

QUANTIZATION AND PROJECTIVE REPRESENTATIONS OF SOLVABLE LIE GROUPS

by

HENRI MOSCOVICI AND ANDREI VERONA

PREPRINT SERIES IN MATHEMATICS
No. 8/1976

BUCUREȘTI

QUANTIZATION AND PROJECTIVE REPRESENTATIONS OF SOLVABLE LIE GROUPS

by

Henri Moscovici *) and Andrei Verona *)

May 1976

*) Institutul de Matematică, Str. Academiei 14, Bucureşti

QUANTIZATION AND PROJECTIVE REPRESENTATIONS OF SOLVABLE LIE GROUPS

Henri Moscovici and Andrei Verona

Introduction

Refining the Kirillov orbit method for the construction of the irreducible unitary representations of a nilpotent Lie group, Kostant has developed a geometric quantization theory of obtaining unitary representations for an arbitrary Lie group from symplectic manifolds on which the group acts as a transitive group of symplectic automorphisms. When applied to orbits of the coadjoint representations, which possess a canonical symplectic structure, Kostant' quantization procedure goes along way towards constructing, in many significant cases, all (or "almost" all) the irreducible unitary representations of the given group. This method was particulary succesful for solvable Lie groups, in which case it provides both a geometric criterion for being of type I and, in that case, a complete description of the unitary dual.

Besides the orbits of the coadjoint representation there are other symplectic homogeneous spaces for a Lie group G, for instance those which correspond to non-coboundary 2-cocycles of its Lie algebra $\mathfrak{g}$. In fact, when G is connected and simply connected, all the simply connected symplectic homogeneous G-spaces arise in a canonical way from 2-cocycles in $Z^2(\mathfrak{g})$ (cf. B.-Y. Chu, Symplectic homogeneous spaces, Trans.Amer.Math.Soc.197 (1974),145-159).

Although we can not afford to go into details here, it must be said that, by Kostant's method, a unitary representation of G can be obtained from a symplectic homogeneous space X only under the additional assumption that X is a Hamiltonian G-space ([K], §5) and this is the case when and only when X covers an orbit of the coadjoint representation of G.

We have found that when the quantization procedure is applied to a general symplectic homogeneous G-space one can still obtain a representation of G, which is no longer unitary but a projective one. This remark allows us to construct irreducible projective representations of a solvable Lie group G starting from integral 2-cocycles on its Lie algebra $\mathfrak{g}$ (Theorem 5.4.1) and to classify some of them in terms of the orbits of G in $Z^2(\mathfrak{g})$ (Theorem 5.4.5). In the special case of a nilpotent (or, more generally, exponential) Lie group, our construction provides a complete classification of the projective dual (Corollary 5.4.6).

Now let us describe in a few words our construction of projective representations. Assume G is a connected and simply connected Lie group. First, to each cocycle $\omega \in Z^2(\mathfrak{g})$ we associate a strongly symplectic homogeneous G-space $(X_\omega, \theta_\omega)$, namely the orbit through 0 in $\mathfrak{g}^*$ under the affine action of G corresponding to ω (cf. the paper of Chu quoted above). Then, after choosing a polarization $\mathfrak{h}$ of $\mathfrak{g}$ at ω, we attach to each line bundle with connection and Hermitian structure (L,α) over X_ω, with curvature form θ_ω, a projective representation $\pi(L,\alpha;\mathfrak{h})$ of G whose equivalence class $\pi_{\ell,\mathfrak{h}}$ depends only on the equivalence class ℓ of (L,α). Under no additional hypothesis we have nothing to say about $\pi_{\ell,\mathfrak{h}}$; it may be or may *not* be irreducible Even worse it may happen that $\pi_{\ell,\mathfrak{h}}=0$. However, if G is assumed to be solvable, the results of Auslander and Kostant [A-K] allow us to conclude that $\pi_{\ell,\mathfrak{h}}$ is irreducible and independent of the choice of the polarization. Furthermore, G acts naturally on the set of all such isomorphism classes of line bundle with connection over symplectic homogeneous G-spaces of the form $(X_\omega, \theta_\omega)$ with ω running over $Z^2(\mathfrak{g})$, and the map $\ell \longmapsto \pi_\ell = \pi_{\ell,\mathfrak{h}}$ is constant on the orbits of G. This construction is particularly fruitful in the case when G is of projective type I (see p. 20), when it yields a complete parametrization of all equivalence classes of irreducible projective representations of G. If G is exponential, then it is of projective type I and the above parametrization is realized by the orbits of G in $Z^2(\mathfrak{g})$. It should be mentioned that, in principle, our method of constructing irreducible projective representations works whenever it is applied to a class of Lie groups which is closed under central extensions by R and for which the Kirillov-Kostant method of obtaining irreducible unitary representations works.

The material in this paper is organized as follows. Section 1 contains some facts about extensions of Lie groups and algebras we will need later. Section 2 deals with the relationship between projective representations and group extensions. Section 3 is devoted to the study of the symplectic homogeneous space associated to a 2-cocycle. The concept of polarization for a 2-cocycle is discussed in Section 4. The construction of a projective representation of a Lie group G by quantizing a symplectic homogeneous G-space is given in Section 5. This section also contains the statements of the main results, while their proofs are given in the final section.

0. <u>Notational conventions</u>. In order to prevent misunderstandings we list below some notations we will use in the paper, which might not be standard.

0.1. T stands for the group of complex numbers of modulus 1.

0.2. The complexification of a real vector space V is denoted V_C, while the complexification of a linear (or multilinear) map λ is denoted by the same symbol λ, without adding the subscript C.

0.3. If a Lie group G with Lie algebra $\mathfrak{g}$ acts (smoothly) on the left on a manifold X, we denote by $L_X(g)$ the diffeomorphism of X defined by $g\in G$; sometimes we shall write simply $g\cdot u$ instead of $L_X(g)u$, for $u\in X$.

The smooth vector field on X determined by $x\in\mathfrak{g}$ is denoted by $r_X(x)$; recall that

$$r_X(x)_u(f) = \frac{d}{dt}(f(\exp(-tx)\cdot u))\Big|_{t=0}, \quad f\in C^\infty(X),\ u\in X.$$

When $X = G$ with the natural left G-action, we shall omit the subsctipt G in the above notation. Further we shall write $R(g)$ for the wright translation by $g^{-1}\in G$.

For $g\in G$, $I(g)$ denotes the inner automorphism $L(g)\circ R(g)$.

0.4. If $\mathfrak{g}$ is the Lie algebra of the Lie group G, $\mathfrak{g}^*$ denotes the real dual vector space of $\mathfrak{g}$, $\langle\,,\rangle : \mathfrak{g}^*\times\mathfrak{g} \longrightarrow R$ the canonical pairing, Ad^*, ad^* the coadjoint representations of G and $\mathfrak{g}$ on $\mathfrak{g}^*$.

0.5. If $\mathfrak{g}$ is a Lie algebra, $\iota(x)$ and $\mathcal{L}(x)$ will denote the interior product and the Lie derivative with respect to $x\in\mathfrak{g}$.

0.6. If L is a smooth vector bundle over X, $\Gamma(X,L)$ stands for the vector space of all its smooth sections.

1. Extensions of Lie groups and algebras

1.1. Let G and K be two connected Lie groups. By an extension of G by K we shall mean an exact sequence

$$(M,p): \qquad 1 \longrightarrow K \xrightarrow{\ i\ } M \xrightarrow{\ p\ } G \longrightarrow 1$$

where M is a separable locally compact group and i, p are continuous homomorphisms.

Let us add some comments on this definition.

(1.1.1.) K being separable, $i : K \longrightarrow Ker(p)$ is in fact a homeomorphism. Owing on this remark we shall identify K to $Ker(p)$ via i, viewing i as the inclusion map.

(1.1.2) Since M is separable, the canonically induced map $M/K \longrightarrow G$ is a homeomorphism too. This and the connectedness of G and K ensure us that M is also connected.

(1.1.3) Moreover, M admits a (unique) structure of a Lie group whose underlying topology is the original one. Indeed, to see this it suffices to note that M is without small subgroups and then to apply [M-Z, Theorem, p.169].

When in addition K is central in M, the extension (M,p) is called <u>central</u>.

The set of all ~~extensions (resp.~~ central extensions of G by K will be denoted ~~Ext(G,K) (resp.~~ $\mathcal{E}xt_o(G,K)$. Further, we denote by ~~Ext(G,K) (resp.~~ $\mathrm{Ext}_o(G,K)$ the factor set of $\mathcal{E}xt_o(G,K)$ ~~(resp. $\mathcal{E}xt_o(G,K)$)~~ with respect to the usual equivalence relation. The equivalence class of an extension (M,p) will be denoted $[M,p]$.

1.2. From now on G will be always assumed not only connected but also simply connected.

Let $(M,p) \in \mathcal{E}xt_o(G,K)$. Since $\pi_1(G) = 0 = \pi_2(G)$, the homotopy exact sequence of the fibration $M \xrightarrow{p} G$ shows that the inclusion map $K \hookrightarrow M$ induces an isomorphism between $\pi_1(K)$ and $\pi_1(M)$; therefore M is simply connected if and only if K is so.

Now let L be the simply connected covering group of K, with $p_K : L \longrightarrow K$ the corresponding projection. We identify $\pi_1(K)$ to $\mathrm{Ker}(p_k)$.

Given $(M,p) \in \mathcal{E}xt_o(G,L)$ we shall define $(M_\#, p_\#) \in \mathcal{E}xt_o(G,K)$ as follows: $M_\# = M/\pi_1(K)$ and and $p_\# : M_\# \longrightarrow G$ is canonically induced by $p : M \longrightarrow G$. The map $(M,p) \longmapsto (M_\#, p_\#)$ from $\mathcal{E}xt_o(G,L)$ to $\mathcal{E}xt_o(G,K)$ will be denoted by $\mathcal{C}ov_\#$, while the induced map from $\mathrm{Ext}_o(G,L)$ to $\mathrm{Ext}_o(G,K)$ will be denoted by $\mathrm{Cov}_\#$.

Conversely, for $(N,q) \in \mathcal{E}xt_o(G,K)$ let us define $(N^\#, q^\#) \in \mathcal{E}xt_o(G,L)$ in the following way: $N^\#$ is the simply connected covering group of

N and $q^{\#} = q \circ p_N$, where $p_N : N^{\#} \longrightarrow N$ is the covering homomorphism. It is easy to check that $\mathrm{Ker}(q^{\#})$ is isomorphic to L, hence, after identifying them, $(N^{\#}, q^{\#})$ becomes really an extension of G by L. The map thus defined $(N,q) \longmapsto (N^{\#}, q^{\#})$ from $\mathcal{E}xt_0(G,K)$ to $\mathcal{E}xt_0(G,L)$ is denoted by $\mathcal{C}ov^{\#}$ and the induced map from $\mathrm{Ext}_0(G,K)$ to $\mathrm{Ext}_0(G,L)$ is denoted $\mathrm{Cov}^{\#}$.

To conclude this subsection we note that there is no problem in verifying that $\mathrm{Cov}_{\#}$ and $\mathrm{Cov}^{\#}$ are mutually inverse maps which put in a one-to-one correspondence ~~Ext(G,L) with Ext(G,K) and~~ $\mathrm{Ext}_0(G,L)$ with $\mathrm{Ext}_0(G,K)$.

1.3. Let $\mathfrak{k}$ and $\mathfrak{g}$ denote the Lie algebras of K and G respectively. An <u>extension</u> of $\mathfrak{g}$ by $\mathfrak{k}$ is an exact sequence of Lie algebras and Lie homomorphisms

$$(\mathfrak{m}, \gamma): \qquad 0 \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{m} \xrightarrow{\ \gamma\ } \mathfrak{g} \longrightarrow 0 .$$

When $\mathfrak{k}$ is contained in the center of $\mathfrak{m}$, the extension is called <u>central</u>.

The set of all extensions (resp. central extensions) of $\mathfrak{g}$ by $\mathfrak{k}$ will be denoted $\mathcal{e}xt(\mathfrak{g},\mathfrak{k})$ (resp. $\mathcal{e}xt_0(\mathfrak{g},\mathfrak{k})$) and the corresponding factor set relative to the usual equivalence relation will be denoted $\mathrm{ext}(\mathfrak{g},\mathfrak{k})$ (resp. $\mathrm{ext}_0(\mathfrak{g},\mathfrak{k})$). By $[\mathfrak{m},\gamma]$ we shall denote the equivalence class of the extension $(\mathfrak{m},\gamma)$.

There is a simple relationship between ~~Ext(G,K) and ext(g,k) (resp.~~ $\mathrm{Ext}_0(G,K)$ and $\mathrm{ext}_0(\mathfrak{g},\mathfrak{k})$ which we proceed now to describe. First of all let us remark that in view of the previous subsection there will be no loss of generality in assuming K simply connected.

Now given $(M,p) \in \mathcal{E}xt_0(G,K)$, by passing to Lie algebras we get an extension $(\mathfrak{m},\gamma) \in \mathcal{e}xt_0(\mathfrak{g},\mathfrak{k})$ which we denote $\mathcal{L}ie^{\#}(M,p)$. The map $\mathcal{L}ie^{\#}$: $\mathcal{E}xt_0(G,K) \longrightarrow \mathcal{e}xt_0(\mathfrak{g},\mathfrak{k})$ so defined induces a map $\mathrm{Lie}^{\#}$ from $\mathrm{Ext}_0(G,K)$ to $\mathrm{ext}_0(\mathfrak{g},\mathfrak{k})$.

Conversely, to each element $(\mathfrak{m},\gamma) \in \mathcal{e}xt_0(\mathfrak{g},\mathfrak{k})$ we associate an extension $(M,p) = \mathcal{L}ie_{\#}(\mathfrak{m},\gamma)$ of G by K as follows: M is the simply connected Lie group with Lie algebra $\mathfrak{m}$ and $K \longrightarrow M$, $p : M \longrightarrow G$ are the Lie homomorphisms whose differentials are $\mathfrak{k} \longrightarrow \mathfrak{m}$ and $\psi : \mathfrak{m} \longrightarrow \mathfrak{g}$, respectively. We have thus obtained a map $\mathcal{L}ie_{\#} : \mathcal{E}xt_0(\mathfrak{g}, \mathfrak{k}) \longrightarrow \mathcal{E}xt_0(G,K)$, which induces a map $\mathrm{Lie}_{\#}$ from $\mathrm{ext}_0(\mathfrak{g}, \mathfrak{k})$ to $\mathrm{Ext}_0(G,K)$.

It is just a trivial observation to remark that $\mathrm{Lie}_{\#}$ and $\mathrm{Lie}^{\#}$ are mutually inverse maps which ~~they~~ induce one-to-one correspondences between $\mathrm{ext}_0(\mathfrak{g}, \mathfrak{k})$ and $\mathrm{Ext}_0(G,K)$.

1.4. By $Z^2(\mathfrak{g})$ we shall denote as usually the vector space of all 2-cocycles on $\mathfrak{g}$ relative to the trivial action of $\mathfrak{g}$ on R.

For each $\omega \in Z^2(\mathfrak{g})$ one defines a central extension $(\mathfrak{m}_\omega, \psi_\omega)$ of $\mathfrak{g}$ by R in the following way: $\mathfrak{m}_\omega$ is the Lie algebra whose underlying vector space is $R \times \mathfrak{g}$, the bracket being given by the formula

$$[(r,x),(s,y)] = (-\omega(x,y), [x,y]), \quad r,s \in R, \ x,y \in \mathfrak{g};$$

the projection $\psi_\omega : \mathfrak{m}_\omega \to \mathfrak{g}$ is just the canonical projection of $R \times \mathfrak{g}$ onto $\mathfrak{g}$, and $R \longrightarrow \mathfrak{m}_\omega$ is the canonical injection of R into $R \times \mathfrak{g}$.

Now if $\omega' = \omega + d\lambda$, with $\lambda \in \mathfrak{g}^*$, then the Lie homomorphism $\varphi_\lambda : \mathfrak{m}_\omega \longrightarrow \mathfrak{m}_{\omega'}$ given by

$$\varphi_\lambda(r,x) = (r+\lambda(x), x), \quad r \in R, \ x \in \mathfrak{g},$$

establishes an equivalence of extensions between $(\mathfrak{m}_\omega, \psi_\omega)$ and $(\mathfrak{m}_{\omega'}, \psi_{\omega'})$. It follows that the assignement $\omega \in Z^2(\mathfrak{g}) \longmapsto [\mathfrak{m}_\omega, \psi_\omega] \in \mathrm{ext}_0(\mathfrak{g}, R)$ gives rise to a map $[\omega] \longmapsto [\mathfrak{m}_\omega, \psi_\omega]$ from $H^2(\mathfrak{g})$ to $\mathrm{ext}_0(\mathfrak{g}, R)$. It is well-known that this map is in fact a bijection.

2. Projective representations and group extensions

Given a separable Hilbert space H we denote by U(H) the group of all its unitary automorphisms, endowed with the strong operatorial topology. Further we denote by PU(H) the projective unitary group U(H)/T where the circle group T is viewed as the central closed subgroup of U(H) consisting of the scalar multiples of the identity operator Id, and we give PU(H) the quotient topology; $p_H : U(H) \dashrightarrow PU(H)$ stands for the canonical projection. It is convenient to regard an element of

PU(H) as an automorphism of the projective space PH associated to H.

By a <u>unitary</u> (resp. <u>projective</u>) <u>representation</u> of G in H we mean a continuous homomorphism of G in U(H) (resp. PU(H)). Recall that two projective representations $\pi_i : G \longrightarrow PU(H_i)$, i=1,2 are said to be <u>projectively equivalent</u> if there exists a unitary isomorphism $U : H_1 \longrightarrow H_2$ such that, if $\hat{U} : PH_1 \longrightarrow PH_2$ denotes the corresponding isomorphism of projective spaces, then $\hat{U}\circ\pi_1(g) = \pi_2(g)\circ\hat{U}$. The set of all equivalence classes of irreducible projective representations of G, which we call the projective dual of G, will be denoted G^{π}.

2.1. Let (M,p) be a central extension of the connected and simply connected Lie group G by R. A unitary representation $\rho : M \longrightarrow U(H)$ will be called <u>projectable</u> if $\rho(r)=e^{2\pi ir}\mathrm{Id}$. In this case there exists a unique projective representation $\hat{\rho} : G \longrightarrow PU(H)$ such that

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "e^{2\pi i ?}"'] & M \arrow[r, "p"] \arrow[d, "\rho"'] & G \arrow[r] \arrow[d, "\hat{\rho}"'] & 1 \\
1 \arrow[r] & T \arrow[r] & U(H) \arrow[r, "p_H"'] & PU(H) \arrow[r] & 1
\end{tikzcd}$$

commutes.

Now let $\rho_i : M \longrightarrow U(H_i)$, i=1,2 be projectable unitary representations such that $\hat{\rho}_1$ and $\hat{\rho}_2$ are projectively equivalent through the unitary isomorphism $U : H_1 \longrightarrow H_2$. For each $m \in M$ one has

$$(U\circ\rho_1(m))^{\wedge} = \hat{U}\circ\hat{\rho}_1(p(m)) = \hat{\rho}_2(p(m))\circ\hat{U} = (\rho_2(m)\circ U)^{\wedge},$$

hence there exists a unitary character $\chi : M \longrightarrow T$ such that

$$\rho_2(m) = \chi(m)\cdot U\circ\rho_1(m)\circ U^{-1}, \ m \in M.$$

This means that ρ_2 and $\chi\otimes\rho_1$ are unitarily equivalent representations of M.

2.2. We want now to attach to a given projective representation $\pi : G \longrightarrow PU(H)$ an extension $(M_\pi, p_\pi) \in \mathcal{E}xt_0(G,R)$ together with a unitary representation $\check{\pi} : M \longrightarrow U(H)$ and then to relate this construction to that discussed in 2.1.

Consider the topological subgroup N_π of $U(H)\times G$ consisting of those pairs (u,g) which satisfy $p_H(u) = \pi(g)$. Define $T \longrightarrow N_\pi$ to be

$t \longmapsto (t\cdot \mathrm{Id}, 1)$ and $q_\pi : N_\pi \longrightarrow G$ by $q_\pi(u,g) = g$. Clearly

$$(N_\pi, q_\pi): \quad 1 \longrightarrow T \longrightarrow N_\pi \xrightarrow{q_\pi} G \longrightarrow 1$$

is an exact sequence of topological groups and continuous homomorphisms. Moreover, q_π is an open map (since p_H is so), hence N_π/T is homeomorphic to G. It follows that N_π is a (separable) locally compact group. Therefore $(N_\pi, q_\pi) \in \mathcal{E}xt_0(G,T)$.

Let us now define $(M_\pi, p_\pi) \in \mathcal{E}xt_0(G,R)$ to be $\mathcal{C}ov^{\#}(N_\pi, q_\pi)$. The map $(u,g) \longmapsto u$ is a unitary representation of N_π, which when composed with the covering homomorphism $M_\pi \longrightarrow N_\pi$ gives rise to a unitary representation $\check{\pi} : M_\pi \longrightarrow U(H)$. It is an easy matter to see that $\varrho = \check{\pi}$ is projectable and that $\hat{\varrho} = \pi$.

Conversely, let $(M,p) \in \mathcal{E}xt_0(G,R)$ and $\varrho : M \longrightarrow U(H)$ be a projectable unitary representation. Put $\pi = \hat{\varrho}$ and consider, as above, the associated extensions $(N_\pi, q_\pi) \in \mathcal{E}xt_0(G,T)$, $(M_\pi, p_\pi) \in \mathcal{E}xt_0(G,R)$. Define now $\Psi : M \longrightarrow N_\pi$ by $\Psi(m) = (\varrho(m), p(m))$, and then form the diagram

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d, "e^{2\pi i ?}"'] & M \arrow[d, "\Psi"'] \arrow[dr, "p"] & & \\
 & & & G \arrow[r] & 1 \\
1 \arrow[r] & T \arrow[r] & N_\pi \arrow[ur, "q_\pi"'] & &
\end{tikzcd}$$

which obviously commutes. Since M is simply connected Ψ can be lifted to a Lie homomorphism $\Phi : M \longrightarrow M_\pi$ which makes the following diagram commutative:

$$\begin{tikzcd}
 & M \arrow[d, "\Phi"] \arrow[dr, "p"] & & \\
0 \arrow[r] & R \arrow[u] \arrow[r, phantom] \arrow[dr] & G \arrow[r] & 1 \\
 & M_\pi \arrow[ur, "p_\pi"'] & &
\end{tikzcd}$$

Actually Φ is an equivalence between the extensions (M,p) and (M_π, p_π). In addition one has $\varrho = \check{\pi} \circ \Phi$.

2.3. Let $\pi_i : G \longrightarrow PU(H_i)$, $i=1,2$ be projectively equivalent representations via the unitary isomorphism $U : H_1 \longrightarrow H_2$. It is not difficult to see that the isomorphism $\Psi_U : N_{\pi_1} \longrightarrow N_{\pi_2}$, defined by

$$\Psi_U(u,g) = (U\circ u \circ U^{-1}, g), \quad (u,g) \in N_{\pi_1}$$

establishes an equivalence of extensions between (N_{π_1}, q_{π_1}) and

(N_{π_2}, q_{π_2}). Then $\Phi_U : M_{\pi_1} \longrightarrow M_{\pi_2}$, the lifting of Ψ_U to the simply connected covering groups, establishes an equivalence of extensions between (M_{π_1}, p_{π_1}) and (M_{π_2}, p_{π_2}). Furthermore, $\check{\pi}_1$ and $\check{\pi}_2 \circ \Phi_U$ are unitarily equivalent representations.

2.4. Suppose now that $(M_i, p_i) \in \mathcal{E}xt_o(G,R)$, $i=1,2$ and that $\rho_i : M_i \longrightarrow U(H_i)$ are projectable unitary representations whose associated projective representations $\hat{\rho}_i : G \longrightarrow PU(H_i)$ are equivalent. Then, by combining 2.2 and 2.3, one can see without difficulty that there exists a Lie isomorphism $\Phi : M_1 \longrightarrow M_2$ which determines an equivalence of extensions between (M_1, p_1) and (M_2, p_2) and has further the property that ρ_1 and $\rho_2 \circ \Phi$ are unitarily equivalent representations.

3. The symplectic homogeneous space associated to a 2-cocycle

In this section ω will denote a fixed element in $Z^2(\mathfrak{g})$, $\mathfrak{g}$ being the Lie algebra of the connected and simply connected Lie group G.

3.1. To begin with, we shall define a representation τ_ω of $\mathfrak{g}$ on $R \times \mathfrak{g}^*$ by the formula

$$\tau_\omega(x)(r,\lambda) = (0, \mathrm{ad}^*(x)\lambda + r\iota(x)\omega), \quad x \in \mathfrak{g}, \ (r,\lambda) \in R \times \mathfrak{g}^*.$$

The group G being simply connected, there exists a unique representation T_ω of G on $R \times \mathfrak{g}^*$ whose differential is τ_ω. It is easy to see that

$$T_\omega(g)(0,\lambda) = (0, \mathrm{Ad}^*(g)\lambda), \ g \in G, \ \lambda \in \mathfrak{g}^*.$$

On the other hand we observe that T_ω must verify

$$T_\omega(g)(r,0) = (r\varkappa(g), rF_\omega(g)), \ g \in G, \ r \in R,$$

with $\varkappa : G \longrightarrow R-\{0\}$ and $F_\omega : G \longrightarrow \mathfrak{g}^*$ analytic. Since T_ω is a representation, one can see that $\varkappa$ is a Lie homomorphism and F_ω satisfies

$$(3.1.1) \qquad F_\omega(gh) = \mathrm{Ad}^*(g)F_\omega(h) + F_\omega(g), \ g,h \in G.$$

Now taking into account the fact that the differential of T_ω is τ_ω one deduces that $\varkappa$ is the trivial homomorphism and that F_ω has the expression

$$(3.1.2) \qquad F_\omega(\exp x) = \sum_{n=1}^{\infty} \frac{1}{n!}(\mathrm{ad}^*(x))^{n-1}(\iota(x)\omega), \quad x \in \mathfrak{g}.$$

Finally, one obtains

(3.1.3) $T_\omega(g)(r,\lambda) = (r, Ad^*(g)\lambda + rF_\omega(g)),\ g\in G,\ (r,\lambda)\in R\times \mathfrak{g}^*.$

Let us denote by Ad^* the natural representation of G on $Z^2(\mathfrak{g})$ too. Explicitely, $Ad^*(g)(\sigma)(x,y) = \sigma(Ad(g^{-1})x, Ad(g^{-1})y)$, for $g\in G$, $\sigma\in Z^2(\mathfrak{g})$, $x,y\in\mathfrak{g}$.

Starting from (3.1.2) one can easily see that

$$F_{Ad^*(g)\omega}(\exp x) = Ad^*(g)F_\omega(g^{-1}(\exp x)g),\ g\in G,\ x\in\mathfrak{g},$$

which, when combined with (3.1.1), leads to the formula

(3.1.4) $F_{Ad^*(g)\omega}(h) = Ad^*(g)F_\omega(g^{-1}hg),\ g,h\in G.$

Consider now the linear map $D_\omega : R\times\mathfrak{g}^* \longrightarrow Z^2(\mathfrak{g})$, given by $D_\omega(r,\lambda) = d\lambda + r\omega$. One has

$$D_\omega\circ \mathcal{T}_\omega(x) = \mathcal{L}(x)\circ D_\omega,\ x\in\mathfrak{g}.$$

As $x\longmapsto \mathcal{L}(x)$ from $\mathfrak{g}$ to $\mathfrak{gl}(Z^2(\mathfrak{g}))$ is the differential of the homomorphism $Ad^* : G \longrightarrow GL(Z^2(\mathfrak{g}))$, it follows that

(3.1.5) $D_\omega\circ T_\omega(g) = Ad^*(g)\circ D_\omega,\ g\in G,$

which is equivalent to

(3.1.6) $Ad^*(g)\omega - \omega = d(F_\omega(g)),\ g\in G.$

3.2. Let all notation be as above. In addition , let us denote: $G(\omega) = \{g\in G;\ F_\omega(g)=0\}$ and $\mathfrak{g}(\omega) = \{x\in\mathfrak{g};\ \iota(x)\omega = 0\}$.

The formula (3.1.1) implies that $G(\omega)$ is a (closed) subgroup of G and that $G(\omega) = \{g\in G;\ F_\omega(hg) = F_\omega(h)$ for any $h\in G\}$. Therefore F_ω factorizes through a map $f_\omega : G/G(\omega) \longrightarrow \mathfrak{g}^*$.

Since Lie $G(\omega) = \{x\in\mathfrak{g};\ \exp tx\in G(\omega)$ for any $t\in R\}$ and, on the other hand, $\exp tx\in G(\omega)$ for any $t\in R$ if and only if $\iota(x)\omega = 0$ (cf. (3.1.2)), one get

(3.2.1) $\mathrm{Lie}\ G(\omega) = \mathfrak{g}(\omega).$

Furthermore, we observe that, in view of (3.1.6), we have

(3.2.2) $Ad^*(g)\omega = \omega,\ g\in G(\omega).$

Now set $X_\omega = G/G(\omega)$ and let $q_\omega : G \longrightarrow X_\omega$ be the canonical projection. From the definition of $\mathfrak{g}(\omega)$, (3.2.1) and (3.2.2) we deduce that there exists a unique G-invariant closed 2-form θ_ω on X_ω such that $(q_\omega^*(\theta_\omega))_1 = \omega$, where $1 \in G$ is the unit element.

To simplify the notation, when no confusion can arise, we shall write shortly X, q, θ, F, f instead of $X_\omega, q_\omega, \theta_\omega, F_\omega, f_\omega$ respectively.

For each $x \in \mathfrak{g}$ let us define $f^x \in C^\infty(X_\omega)$ by

$$f^x(q(g)) = \langle F(g), x \rangle, \quad g \in G.$$

It is an easy matter to see that the vector field $r_X(x)$ defined as in 0.3 satisfies the following relations:

$$(3.2.3) \qquad L_X(g)_* r_X(x)_u = r_X(\mathrm{Ad}(g)x)_{g \cdot u}, \quad g \in G, \ u \in X;$$

$$(3.2.4) \qquad r_X(x)_{q(g)} = -q_*(L(g)_*(\mathrm{Ad}(g^{-1})x)), \quad g \in G.$$

3.2.1. <u>PROPOSITION</u>. $(X_\omega, \theta_\omega)$ <u>is a strongly symplectic G-space.</u>

<u>Proof</u>. Precisely, we shall prove that

$$(3.2.5) \qquad \iota(r_X(x))\theta = df^x, \quad x \in \mathfrak{g}.$$

To this end, let us first note that, since q^* is injective, it suffices to check that

$$q^*(\iota(r_X(x))\theta) = q^*(df^x), \quad x \in \mathfrak{g}.$$

Now let $g \in G$ and $x \in \mathfrak{g}$. One has, for any $y \in \mathfrak{g}$,

$$\langle (q^*(\iota(r_X(x))\theta))_g, L(g)_* y \rangle = \langle (\iota(r_X(x))\theta)_{q(g)}, L_X(g)_* \circ q_*(y) \rangle =$$

$$= \theta_{q(g)}(r_X(x)_{q(g)}, L_X(g)_* \circ q_*(y)) = (L_X(g^{-1})^* \theta_{q(1)})(r_X(x)_{q(g)}, L_X(g)_* \circ q_*(y)) =$$

$$= \theta_{q(1)}(L_X(g^{-1})_* r_X(x)_{q(g)}, q_*(y)) = \theta_{q(1)}(-q_*(\mathrm{Ad}(g^{-1})x), q_*(y)) =$$

$$= \omega(y, \mathrm{Ad}(g^{-1})x).$$

On the other hand,

$$\langle (q^*(df^x))_g, L(g)_* y \rangle = \langle d(f^x \circ q)_g, L(g)_* y \rangle =$$

$$= \frac{d}{dt} \langle F(g \cdot \exp ty), x \rangle \Big|_{t=0} = \frac{d}{dt} \langle \mathrm{Ad}^*(g) F(\exp ty), x \rangle \Big|_{t=0} +$$

$$+ \frac{d}{dt} \langle F(g), x \rangle \Big|_{t=0} = \langle \mathrm{Ad}^*(g)(\iota(y)\omega), x \rangle = \omega(y, \mathrm{Ad}^*(g^{-1})x).$$

This proves our assertion.

3.2.2. Remark. In contrast with the construction given in [K] for the case when ω is a coboundary, the map $x \longmapsto f^x$ is no more a Lie homomorphism with respect to the Poisson bracket on $C^\infty(X)$. However, it satisfies

$$(3.2.6) \qquad [f^x, f^y] = f^{[x,y]} - \omega(x,y), \quad x,y \in \mathfrak{g}.$$

3.3. We shall say that ω is an integral 2-cocycle if the cohomology class $[\theta_\omega] \in H^2(X_\omega, R)$ is integral. This integrality property is preserved by the action of G on $Z^2(\mathfrak{g})$. Indeed, if $\omega' = Ad^*(g)\omega$ with $g \in G$, then from (3.1.4) it follows that $G(\omega') = I(g)G(\omega)$, hence $I(g)$ induces a diffeomorphism $i(g) : X_\omega \longrightarrow X_{\omega'}$ such that $i(g)^* \theta_{\omega'} = \theta_\omega$. This means that $i(g)$ is an isomorphism of symplectic spaces. In particular it preserves the integrality condition.

The subspace of all integral 2-cocycle in $Z^2(\mathfrak{g})$ will be denoted $Z_0^2(\mathfrak{g})$.

4. Polarizations

4.1. A complex subalgebra $\mathfrak{h} \subset \mathfrak{g}_C$ will be called a polarization of $\mathfrak{g}$ at $\omega \in Z^2(\mathfrak{g})$ if it satisfies

(i) $\mathfrak{h}$ is a maximally isotropic subspace of $\mathfrak{g}_C$ relative to ω;

(ii) $\mathfrak{h} + \overline{\mathfrak{h}}$ is a Lie subalgebra of $\mathfrak{g}_C$;

(iii) $\mathfrak{h}$ is $Ad_G G(\omega)$-stable.

If in addition $\mathfrak{h}$ satisfies

(iv) $i\omega(x, \overline{x}) \leqslant 0$ for any $x \in \mathfrak{h}$,

then it is called a positive polarization.

By $\mathfrak{n}$ we denote the nil-radical of $\mathfrak{g}$. Set $G(\omega|\mathfrak{n}) = \{ g \in G;\ F_\omega(g)|\mathfrak{n} = 0 \}$. Now we shall say that the polarization $\mathfrak{h}$ is nil-admissible if it has the property:

(v) $\mathfrak{h} \cap \mathfrak{n}_C$ is a maximally isotropic subspace of $\mathfrak{n}_C$ relative to $\omega|\mathfrak{n}_C \times \mathfrak{n}_C$ and it is $Ad_G G(\omega|\mathfrak{n})$-stable.

From now on $\mathfrak{h}$ will be a fixed polarization of $\mathfrak{g}$ at ω. Consider the Lie subalgebras of $\mathfrak{g}$ given by: $\mathfrak{d} = \mathfrak{h}\cap\mathfrak{g}$ and $\mathfrak{e} = (\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{g}$. Then it is easily seen that

(4.1.1) $\mathfrak{g}(\omega)\subset\mathfrak{d}\subset\mathfrak{e}$;

(4.1.2) $\mathfrak{d}_C = \mathfrak{h}\cap\bar{\mathfrak{h}}$ and $\mathfrak{e}_C = \mathfrak{h}+\bar{\mathfrak{h}}$;

(4.1.3) $\mathfrak{d}$ is the orthogonal subspace of $\mathfrak{e}$ relative to ω and thus the canonically induced 2-form $\hat{\omega}$ on $\mathfrak{e}/\mathfrak{d}$ is non-degenerate.

Let D_o and E_o denote the connected Lie subgroups of G which correspond to the subalgebras $\mathfrak{d}$ and $\mathfrak{e}$ respectively. Since $\mathfrak{h}$ is stable under $Ad_G G(\omega)$, D_o and E_o are normalized by $G(\omega)$. It follows that $D = D_oG(\omega)$ and $E = E_oG(\omega)$ are subgroups of G.

With this notation, we shall say that $\mathfrak{h}$ is a <u>closed polarization</u> if $F_\omega(E)$ $(= F_\omega(E_o))$ is a closed set in $\mathfrak{g}^*$.

4.2. Returning now to the notation in 1.4, let $(\mathfrak{m}_\omega, \psi_\omega) \in ext_o(\mathfrak{g}, R)$ and let $(M_\omega, p_\omega) = \mathcal{L}ie_\#(\mathfrak{m}_\omega, \psi_\omega) \in \mathcal{E}xt_o(G,R)$ be the extensions associated to the fixed 2-cocycle ω. We shall identify $\mathfrak{m}_\omega^*$ to $R\times\mathfrak{g}^*$ in the obvious manner. Since $R\subset\mathfrak{m}_\omega$ is central, the coadjoint representation of $\mathfrak{m}_\omega$ factorizes through a representation of $\mathfrak{g}$ in $\mathfrak{m}_\omega^* = R\times\mathfrak{g}^*$, which is just τ_ω. It follows that one has

(4.2.1) $T_\omega(p_\omega(m)) = Ad^*(m)$, $m\in M_\omega$.

Define $\nu_\omega\in\mathfrak{m}_\omega^*$ as being $(1,0)\in R\times\mathfrak{g}^*$. The danger of confusion being out of question, we shall drop in the sequel the subscript ω in $\mathfrak{m}_\omega$, ψ_ω, ν_ω, M_ω, p_ω.

We will now list some facts which are verified in a straightforward way:

(4.2.2) $\psi^*(\omega) = d\nu$;

(4.2.3) $\mathfrak{h}\subset\mathfrak{g}_C$ is a polarization of $\mathfrak{g}$ at ω if and only if $\tilde{\mathfrak{h}} = \psi^{-1}(\mathfrak{h})\subset\mathfrak{m}_C$ is a polarization of $\mathfrak{m}$ at ν (in the sense of [A-K]);

(4.2.4) $\mathfrak{h}$ is a positive polarization at ω if and only if $\tilde{\mathfrak{h}}$ is a positive polarization at ν;

(4.2.5) let D,E be the groups associated to the polarization $\mathfrak{h}$ as above, and let $\tilde{D}$, $\tilde{E}$ denote the groups associated to $\tilde{\mathfrak{h}}$ as in [A-K, I.5]; then $\tilde{D} = p^{-1}(D)$ and $\tilde{E} = p^{-1}(E)$;

(4.2.6) the polarization $\mathfrak{h}$ is closed (resp. nil-admissible) if and only if $\tilde{\mathfrak{h}}$ satisfies the Pukanszky condition (resp. is strongly admissible).

These remarks together with Proposition I.5.1 and Proposition I.5.4 from [A-K] give the following consequences:

(4.2.7) with the above notation D is closed in G and D_o is its identity component;

(4.2.8) if $\mathfrak{h}$ is a closed polarization, then E is closed in G and E_o is its identity component.

Furthermore one has

(4.2.9) if $\mathfrak{h}$ is a closed polarization at ω, then $F_\omega(D)$ is a linear variety in $\mathfrak{g}^*$.

Indeed, from (4.2.1) and (3.1.3) we deduce that $\{Ad_M(\tilde{d})\vartheta\ ;\ \tilde{d}\in\tilde{D}\} = \{(1,F_\omega(d));\ d\in D\}\subset R\times\mathfrak{g}^*$, and our last claim is now a direct consequence of Proposition I.5.6 in [A-K].

Finally, let us observe that, in view of (4.2.6) and Proposition II.2.5 in [A-K], one has

(4.2.10) if G is solvable, any nil-admissible polarization of $\mathfrak{g}$ at ω is closed.

4.3. From now on we shall suppose that $\mathfrak{h}$ is a positive, closed polarization of $\mathfrak{g}$ at ω. Since the 2-form $\omega|e\times e$ is $Ad_E D$-invariant and $\iota(y)(\omega|e\times e) = 0$ for any $y\in\mathfrak{d}$, it induces an E-invariant 2-form on E/D which by (4.1.3) is non-degenerate. In particular there exists on E/D an E-invariant volume element. Therefore the modular function Δ_D of D coincides with the restriction to D of the modular function Δ_E of E. We pick now a strictly positive function $\beta\in C^\infty(G)$ such that $\beta(1) = 1$ and $\beta(ge) = \Delta_E(e)\Delta_G(e)^{-1}\beta(g)$ for any $g\in G$, $e\in E$. Then for each $g\in G$ we define $\beta_g\in C^\infty(G/D)$ by $\beta_g(q_D(a)) = \beta(g^{-1}a)\beta(a)^{-1}$, $a\in G$, where $q_D : G \longrightarrow G/D$ is the canonical projection.

4.3.1. LEMMA. If $x \in \mathfrak{h}$, $g, a \in G$, then $(q_D)_*(L(a)_* x)\beta_g = 0$.

Proof. Actually, the stated formula is valid even for $x \in \mathfrak{g}$. Indeed, one has

$$(q_D)_*(L(a)_* x)\beta_g = \frac{d}{dt}(\beta_g(q_D(a\cdot \exp tx)))\Big|_{t=0} =$$

$$= \frac{d}{dt}\left(\frac{\beta(g^{-1}a\cdot \exp tx)}{\beta(a\cdot \exp tx)}\right)\Big|_{t=0} = \frac{d}{dt}\left(\frac{\beta(g^{-1}a)}{\beta(a)}\right)\Big|_{t=0} = 0.$$

Before finishing this subsection, let us fix one more notation, for later use: μ will stand for the quasi-invariant measure on G/D which corresponds to β and to a choice of a left Haar measure on G. Recall that for any $f \in C_c(G/D)$ and $g \in G$ one has

$$(4.3.1) \quad \int_{G/D} \beta_g(u) f(g^{-1}u)\, d\mu(u) = \int_{G/D} f(u)\, d\mu(u).$$

5. Projective representations constructed by the quantization procedure

Throughout this section ω will be an integral cocycle in $Z^2(\mathfrak{g})$, $\mathfrak{h}$ will denote a closed, positive polarization of $\mathfrak{g}$ at ω, D, E will be the associated closed subgroups of G; $X = X_\omega$, $q = q_\omega$, $\Theta = \Theta_\omega$, $F = F_\omega$ have the same meaning as in 3.2.

5.1. As in [K], let $\mathcal{L}_c(X,\Theta)$ denote the set of all equivalence classes of complex line bundles with connection and invariant Hermitiam structure over X, with curvature form Θ. Since ω is assumed to be integral, $\mathcal{L}_c(X,\Theta)$ is non-void (cf. [K]). Pick $\ell \in \mathcal{L}_c(X,\Theta)$, then $(L,\alpha) \in \ell$ and let $pr_L : L \longrightarrow X$ denote the corresponding projection. Now let $\mathcal{D}_\ell(X)$ be the group of all diffeomorphisms of X which leave ℓ unchanged, and let $\mathcal{E}(L,\alpha)$ be the group of all diffeomorphisms of L which commute with the scalar multiplication and preserve both the connection form and the Hermitiam structure. Then, according to Theorem 1.13.1 in [K], one has the exact sequence of groups

$$(5.1.1) \qquad 1 \longrightarrow T \longrightarrow \mathcal{E}(L,\alpha) \longrightarrow \mathcal{D}_\ell(X) \longrightarrow 1$$

where the injection $T \longrightarrow \mathcal{E}(L,\alpha)$ is defined by the scalar action of T on L and the projection $\mathcal{E}(L,\alpha) \longrightarrow \mathcal{D}_\ell(X)$ is given by $e \longmapsto \hat{e}$, $\hat{e}$ denoting the unique diffeomorphism of X such that $pr_L \circ e = \hat{e} \circ pr_L$.

Consider now the projection $p_D : X = G/G(\omega) \longrightarrow G/D$ with fibre $D/G(\omega)$. Since F induces a diffeomorphism of $D/G(\omega)$ onto $F(D)$, (4.2.9) implies that $D/G(\omega)$ is connected and simply connected. Note also that θ vanishes on the fibres of p_D. These two remarks ensure us that the parallel transport along curves which are completely contained in the fibres of p_D depends only on their extremities. Thus for any two points $u,v \in X$ such that $p_D(u) = p_D(v)$ one can define unambiguously an isometry $P_{u,v} : L_u \longrightarrow L_v$, namely given by the parallel transport along any curve contained in $p_D^{-1}(p_D(u))$ with initial point u and end point v.

Define now an equivalence relation on L as follows: $a \sim b$ if $p_D(u) = p_D(v)$ and $P_{u,v}(a) = b$, where $u = pr_L(a)$, $v = pr_L(b)$. The corresponding quotient space will be denoted L/D, while $\tilde{p}_D : L \longrightarrow L/D$ will stand for the canonical projection. It is perfectly clear that pr_L factorizes through a map $pr_{L/D} : L/D \longrightarrow G/D$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\tilde{p}_D} & L/D \\ {\scriptstyle pr_L}\downarrow & & \downarrow{\scriptstyle pr_{L/D}} \\ X & \xrightarrow[p_D]{} & G/D \end{array}$$

commutes. Moreover $pr_{L/D} : L/D \longrightarrow G/D$ is a complex line bundle with Hermitian structure $\langle\,,\rangle$ inherited from the Hermitian structure of L.

5.2. Let all notation be as be as above. Consider the vector space $\Gamma_D(X,L,\alpha)$ consisting of all sections $s \in \Gamma(X,L)$ such that $\nabla_\xi s = 0$ for any tangent vector ξ with $(p_D)_* \xi = 0$. It is an easy matter to see that for any two points $u,v \in X$ with $p_D(u) = p_D(v)$ one has $P_{u,v}(s(u)) = s(v)$; hence s determines a section $s_D \in \Gamma(G/D, L/D)$.

5.2.1. <u>LEMMA</u>. <u>The map $s \longmapsto s_D$ establishes a one-to-one correspondence between $\Gamma_D(X,L,\alpha)$ and $\Gamma(G/D,L/D)$.</u>

The proof is straightforward, so we omit it.

5.3. Consider now the subspace $\mathcal{H}(L,\alpha;\mathfrak{h})$ of $\Gamma_D(X,L,\alpha)$ consisting of those sections $s\in\Gamma(X,L)$ which satisfy:

(5.3.1) $$\nabla_{q_*(L(g)_* x)} s = 0,\ x\in\mathfrak{h},\ g\in G;$$

(5.3.2) $$\int_{G/D}\| s_D(u)\|^2 d\mu(u) < \infty .$$

Take then the obvious scalar product on $\mathcal{H}(L,\alpha;\mathfrak{h})$ and let $H(L,\alpha;\mathfrak{h})$ be the associated Hilbert space.

We intend to define a projective representation of G on $H(L,\alpha;\mathfrak{h})$. To this end let us consider $g\in G$ and note that, in view of Remark 4.4.2 in [K] and our Proposition 3.2.1, $L_X(g)\in\mathcal{D}_\ell(X)$. Choose then an element $e\in\mathcal{E}(L,\alpha)$ such that $\hat{e} = L_X(g)$ (see 5.1) and define $\rho_e: H(L,\alpha;\mathfrak{h}) \longrightarrow H(L,\alpha;\mathfrak{h})$ by

$$\rho_e(s) = (\beta_g\circ p_D)^{1/2}(e\circ s\circ L_X(g)^{-1}),\ s\in\mathcal{H}(L,\alpha;\mathfrak{h}).$$

We shall prove that $\rho_e(s)$ satisfies (5.3.1) and

(5.3.3) $$\int_{G/D}\|(\rho_e(s))_D(u)\|^2 d\mu(u) = \int_{G/D}\| s_D(u)\|^2 d\mu(u) ,$$

which means that $\rho_e\in U(H(L,\alpha;\mathfrak{h}))$. Indeed, keeping in mind Lemma 4.3.1, we have

$$\nabla_{q_*(L(a)_* x)}\rho_e(s) = q_*(L(a)_* x)((\beta_g\circ p_D)^{1/2})\cdot(e\circ s\circ L_X(g)^{-1}) + $$
$$+ (\beta_g\circ p_D)^{1/2}\nabla_{q_*(L(a)_* x)}(e\circ s\circ L_X(g)^{-1}) = (q_D)_*(L(a)_* x)((\beta_g)^{1/2})\cdot$$
$$\cdot(e\circ s\circ L_X(g)^{-1}) + (\beta_g\circ p_D)^{1/2}(e\circ\nabla_{q_*(L(ga)_* x)} s) = 0 ,$$

for any $x\in\mathfrak{h}$ and $a\in G$. This proves that $\rho_e(s)$ satisfies (5.3.1). On the other hand, (5.3.3) is a direct consequence of (4.3.1).

Now if e' is another element of $\mathcal{E}(L,\alpha)$ such that $\widehat{e'} = L_X(g)$, then by (5.1.1) $e' = te$ with $t\in T$, whence $\rho_{e'} = t\rho_e$. Thus ρ_e determines a well-defined element in $PU(H(L,\alpha;\mathfrak{h}))$ which will be denoted $\pi(L,\alpha;\mathfrak{h})(g)$. One sees without any difficulty that $g\longmapsto\pi(L,\alpha;\mathfrak{h})(g)$ is a homomorphism of G in $PU(H(L,\alpha;\mathfrak{h}))$. Moreover, the following result, whose proof we defer until the next section, holds true.

5.3.1. PROPOSITION. $\pi(L,\alpha;\mathfrak{h})$ is a projective representation of G on $H(L,\alpha;\mathfrak{h})$.

Clearly if (L',α') is another representative in the class ℓ, then $\pi(L',\alpha';\mathfrak{h})$ is equivalent to $\pi(L,\alpha;\mathfrak{h})$. Owing on this fact we may denote the equivalence class of $\pi(L,\alpha;\mathfrak{h})$ by $\pi_{\ell,\mathfrak{h}}$.

5.4. In the remainder of this section we shall state our main results. The group G will be assumed in what follows not only connected and simply connected as before, but also solvable. The proofs of the results stated below will be given in Section 6.

5.4.1. THEOREM. Let $\omega\in Z_o^2(\mathfrak{g})$.

(i) The set of positive, nil-admissible polarizations of $\mathfrak{g}$ at ω is non-void.

(ii) For any positive, nil-admissible polarization $\mathfrak{h}$ of $\mathfrak{g}$ at ω and any $\ell = [(L,\alpha)]\in \mathcal{L}_c(X_\omega,\Theta_\omega)$ the projective representation $\pi(L,\alpha;\mathfrak{h})$ is irreducible.

(iii) If $\mathfrak{h}$ and $\mathfrak{h}'$ are two positive, nil-admissible polarizations of $\mathfrak{g}$ at ω and $\ell\in\mathcal{L}_c(X_\omega,\Theta_\omega)$, then $\pi_{\ell,\mathfrak{h}} = \pi_{\ell,\mathfrak{h}'}$.

The choice of the polarization $\mathfrak{h}$ being thus immaterial, we may denote the class $\pi_{\ell,\mathfrak{h}}\in G^{\Pi}$ simply by π_ℓ.

We shall define now an action of G on the set $\mathcal{L}_G = \bigcup_\omega \mathcal{L}_c(X_\omega,\Theta_\omega)$, ω running over $Z_o^2(\mathfrak{g})$, as follows. For $g\in G$, $\omega\in Z_o^2(\mathfrak{g})$ and $\ell = [(L,\alpha)]\in \mathcal{L}_c(X_\omega,\Theta_\omega)$ let $g^{-1}\ell$ denote the equivalence class of the pull-back of (L,α) by the diffeomorphism $i(g^{-1}):(X_{g\cdot\omega},\Theta_{g\cdot\omega})\longrightarrow(X_\omega,\Theta_\omega)$, where $g\cdot\omega$ stands for $Ad^*(g)\omega$ (see 3.3). It is an easy matter to see that $(g,\ell)\longmapsto g^{-1}\ell$ is a well-defined action of G on $\mathcal{L}_G$.

5.4.2. THEOREM. Let $\ell, \ell' \in \mathcal{L}_G$. Then $\pi_\ell = \pi_{\ell'}$ if and only if there exists $g \in G$ such that $\ell' = g^{-1}\ell$.

In other terms, the map $\ell \longmapsto \pi_\ell$ from $\mathcal{L}_G$ to the projective dual G^π of G factorizes through an injective map from $\mathcal{L}_G/G$ into G^π.

To state our next results we need some more definitions.

First we shall say that $(M,p) \in \mathcal{E}xt_o(G,R)$ is a type I extension if M is a group of type I. Now a cocycle $\omega \in Z^2(\mathfrak{g})$ (resp. a projective representation π of G) will be called a type I cocycle (resp. a type I projective representation) if the corresponding extension (M_ω, p_ω) (resp. (M_π, p_π)) is of type I. Let us remark that if $(M,p) \in \mathcal{E}xt_o(G,R)$ is a type I extension, the same is true for any other equivalent extension. Thus one can speak about type I equivalence classes of projective representations. The set of all such equivalence classes of irreducible projective representations of G will be denoted G_I^π. Also the set $Z_I^2(\mathfrak{g})$, consisting of all type I cocycles in $Z^2(\mathfrak{g})$, is G-stable and thus $\mathcal{L}_G^I = \bigcup_\omega \mathcal{L}_c(X_\omega, \theta_\omega)$, ω running over $Z_I^2(\mathfrak{g})$, is a G-stable subset of $\mathcal{L}_G$. The following result is a geometrical characterization of type I cocycles.

5.4.3. PROPOSITION. $\omega \in Z^2(\mathfrak{g})$ is of type I if and only if for any $\sigma \in Z^2(\mathfrak{g})$ of the form $\sigma = r\omega + d\lambda$, with $r \in R$ and $\lambda \in \mathfrak{g}^*$, the cohomology class of θ_σ in $H^2(X_\sigma, R)$ vanishes and $F_\sigma(G)$ $(=f_\sigma(X_\sigma))$ is a locally closed subset of $\mathfrak{g}^*$.

Now we shall say that G is of projective type I if each projective representation of G is of type I. Since it can be easily seen that any extension $(M,p) \in \mathcal{E}xt_o(G,R)$ is equivalent to an extension of the form (M_π, p_π) with π a projective representation of G, it follows that G is of projective type I if and only if any extension $(M,p) \in \mathcal{E}xt_o(G,R)$ is of type I. Clearly, if G is of projective type I then it is of type I (in the usual sense).

5.4.4. THEOREM. G is of projective type I if and only if for any $\omega \in Z^2(\mathfrak{g})$, the cohomology class $[\theta_\omega]$ in $H^2(X_\omega, R)$ vanishes and $F_\omega(G)$ $(= f_\omega(X_\omega))$ is a locally closed subset of $\mathfrak{g}^*$.

Returning to the problem of parametrizing the projective representations of G, we have

5.4.5. THEOREM. (i) $\ell \in \mathcal{L}^I_G$ if and only if $\pi_\ell \in G^\pi_I$.

(ii) The map $\ell \longmapsto \pi_\ell$ induces a bijection of $\mathcal{L}^I_G/G$ onto G^π_I.

(iii) If G is of projective type I, the map $\ell \longmapsto \pi_\ell$ induces a bijection of $\mathcal{L}_G/G$ onto G^π.

In the special case of exponential Lie groups, the above result provides a parametrization of the projective dual of G by the orbits of G in $Z^2(\mathfrak{g})$. More explicitely, we have

5.4.6. COROLLARY. Assume G exponential. Then G is of projective type I, and the set $\mathcal{L}_c(X_\omega, \theta_\omega)$ reduces to a single element $\ell(\omega)$ for any $\omega \in Z^2(\mathfrak{g})$. Furthermore the map $\omega \longmapsto \pi_{\ell(\omega)}$ from $Z^2(\mathfrak{g})$ to G^π factorizes through a bijection of $Z^2(\mathfrak{g})/G$ onto G^π.

6. Proofs

6.1. Returning for the moment to the general case when G is not necessarily solvable, let $\omega \in Z^2(\mathfrak{g})$, $\mathfrak{h}$ be a closed, positive polarization of $\mathfrak{g}$ at ω, and $X = X_\omega$, $q = q_\omega$, $\theta = \theta_\omega$, $F = F_\omega$, $\mathfrak{m} = \mathfrak{m}_\omega$, $M = M_\omega$, $p = p_\omega$, $\nu = \nu_\omega$, D, E, $\tilde{\mathfrak{h}}$, $\tilde{D}$, $\tilde{E}$ be the associated data (see 3.2, 4.1, 4.2). Set $Y = M/M(\nu)$, where $M(\nu) = \{ m \in M;\ Ad^*(m)\nu = \nu \}$

and let $\tilde{q} : M \longrightarrow Y$ be the canonical projection. Then $d\vartheta$ induces a M-invariant 2-form $\sigma = \sigma_\omega$ on Y such that (Y, σ) is a symplectic homogeneous M-space (see [K]). In view of (4.2.1) and (3.1.3), $M(\vartheta) = p^{-1}(G(\omega))$, so that $p : M \longrightarrow G$ induces an isomorphism $\tilde{p} : (Y, \sigma) \longrightarrow (X, \theta)$ of symplectic spaces which has the additional property

$$\tilde{p}(L_Y(m)u) = L_X(p(m))\tilde{p}(u), \quad m \in M, \ u \in Y.$$

Assume now ω integral and let $\ell = [(L, \alpha)] \in \mathcal{L}_c(X, \theta)$. Of course, (L, α) can be viewed also as a line bundle with connection and invariant Hermitian structure over Y, the projection being this time $\tilde{p}^{-1} \circ pr_L$; when regarded as such, its equivalence class in $\mathcal{L}_c(Y, \sigma)$ will be denoted $\tilde{\ell}$. According to Theorem 5.7.1 in [K], there exists a unique character $\eta = \eta_\ell : M(\vartheta) \longrightarrow T$ whose differential is $2\pi i \vartheta | \mathfrak{m}(\vartheta)$ ($\mathfrak{m}(\vartheta)$ being the Lie algebra of $M(\vartheta)$), such that $\tilde{\ell}$ is the equivalence class of the line bundle with connection and Hermitian structure $(\tilde{L}, \tilde{\alpha})$, where

(i) $\tilde{L} = M \times_\eta C$ is the line bundle over Y associated to the principal bundle $\tilde{q} : M \longrightarrow Y$ with structure group $M(\vartheta)$ and to the representation $\eta : M(\vartheta) \longrightarrow T = U(C)$. In more detail, $M \times_\eta C$ is obtained as the orbit space $(M \times C)/M(\vartheta)$, where the action of $M(\vartheta)$ on $M \times C$ is given by: $n \cdot (m, z) = (mn^{-1}, \eta(n)z)$, $n \in M(\vartheta)$, $(m, z) \in M \times C$; the projection $pr_{\tilde{L}} : \tilde{L} \longrightarrow Y$ is the map $[m, z] \longrightarrow \tilde{q}(m)$, where $[m, z]$ stands for the orbit of $M(\vartheta)$ through (m, z), while the linear and the Hermitian structure on $\tilde{L}$ are those determined by the usual ones on C.

(ii) $\tilde{\alpha}$ is the "push-down" of the 1-form $(\delta_\vartheta, \frac{1}{2\pi i}\frac{dz}{z})$ on $M \times (C - \{0\})$, δ_ϑ being the left invariant 1-form on M corresponding to $\vartheta \in \mathfrak{m}^*$.

As it is known (cf. [A-K]), η extends to a unique character $\chi = \chi_\ell$ of $\tilde{D}$ whose differential is $2\pi i \vartheta | \tilde{\mathfrak{d}}$. Now let $\varrho(\varkappa, \mathfrak{h})$ be the holomorphically induced unitary representation of M corresponding to the polarization $\tilde{\mathfrak{h}}$ and to the character $\varkappa$ (see [A-K]).

6.1.1. LEMMA. The unitary representation $\rho(\varkappa,\mathfrak{h})$ of M is projectable and the corresponding projective representation $\hat{\rho}(\varkappa,\mathfrak{h})$ of G is projectively equivalent to $\pi(L,\alpha;\mathfrak{h})$.

Proof. Recall first that the Hilbert space $H(\varkappa,\mathfrak{h})$ on which $\rho(\varkappa,\mathfrak{h})$ acts comes from the vector space $\mathcal{H}(\varkappa,\mathfrak{h})$ of all C^∞-functions $f : M \longrightarrow C$ satisfying

(i) $f(md) = \varkappa(d)^{-1}f(m)$, $m \in M$, $d \in \tilde{D}$;

(ii) $r(x)f + 2\pi i\langle\nu,x\rangle f = 0$, $x \in \tilde{\mathfrak{h}}$;

(iii) $\int_{M/\tilde{D}} |f(m)|^2 d\tilde{\mu}(m\tilde{D}) < \infty$,

where $\tilde{\mu}$ is the volume element on $M/\tilde{D}$ corresponding to μ via the isomorphism $M/\tilde{D} \xrightarrow{\sim} G/D$ induced by $p : M \longrightarrow G$ (cf. (4.2.5)). The action of $\rho(\varkappa,\mathfrak{h})$ on $\mathcal{H}(\varkappa,\mathfrak{h})$ is expressed by the formula

$$(\rho(\varkappa,\mathfrak{h})(m)f)(m') = ((\beta_{p(m)} \circ p_D \circ p)(m'))^{1/2} f(m^{-1}m'),\ m,m' \in M.$$

There is no problem in verifying that $\rho(\varkappa,\mathfrak{h})$ is projectable, so that we do not insist on this point. In order to prove the second assertion, let us note first that $(\tilde{L},\tilde{\alpha})$, when regarded as a line bundle with connection and Hermitian structure over X (the corresponding projection being $\tilde{p}\circ pr_{\tilde{L}}$), is obviously equivalent to (L,α). Thus, for our purposes, there is no loss of generality in assuming $(L,\alpha) = (\tilde{L},\tilde{\alpha})$ as line bundles over X. Now for each $f \in \mathcal{H}(\varkappa,\mathfrak{h})$ let $s_f : X \longrightarrow L$ be the section

$$s_f(u) = [\tilde{p}^{-1}(u), f(\tilde{p}^{-1}(u))],\ u \in X.$$

By a routine computation one checks that $s_f \in \mathcal{H}(L,\alpha;\mathfrak{h})$ and that the assignment $f \longmapsto s_f$ gives rise to a unitary isomorphism $U : H(\varkappa,\mathfrak{h}) \longrightarrow H(L,\alpha;\mathfrak{h})$. Let $g \in G$ and choose $m \in M$ such that $p(m) = g$; m defines an element $e_m \in \mathcal{E}(L,\alpha)$ by $e_m([m',z']) = [mm',z']$, and clearly

$$s_{\rho(\mathfrak{X},\mathfrak{h})(g)f} = (\beta_g \circ p_D)^{1/2}(e_m \circ s_f \circ L_X(g)^{-1}), \quad f \in \mathcal{H}(\mathfrak{X},\mathfrak{h}),$$

which means that

$$\hat{U} \circ \hat{\rho}(\mathfrak{X},\mathfrak{h})(g) = \pi(L,\alpha;\mathfrak{h})(g) \circ \hat{U} .$$

This proves both the fact that $\pi(L,\alpha;\mathfrak{h})$ is indeed a projective representation, as asserted in Proposition 5.3.1, and the claim of the present lemma.

6.2. In the remainder of this section G will be assumed solvable. All the notation remains as above.

6.2.1. <u>Proof of Theorem 5.4.1.</u> It suffices to combine (4.2.3), Lemma 6.1.1 and Theorems II.3.2, III.4.1 and IV.5.7 from [A-K].

6.2.2. <u>Proof of Theorem 5.4.2.</u> Let us begin by proving that if $\ell \in \mathcal{L}_c(X,\theta)$ with $X = X_\omega$, $\theta = \theta_\omega$, and $\ell' \in \mathcal{L}_c(X',\theta')$ with $X' = X_{\omega'}$, $\theta' = \theta_{\omega'}$, are related by the equality $\ell' = g^{-1}\ell$, then $\pi_\ell = \pi_{\ell'}$.

Note first that $\omega' = Ad^*(g)\omega$, therefore, after choosing a positive, nil-admissible polarization $\mathfrak{h}$ of $\mathfrak{g}$ at ω, $\mathfrak{h}' = Ad(g)\mathfrak{h}$ will be a polarization of the same type at ω'. Further pick $(L,\alpha) \in \ell$; then $(L',\alpha') = i(g^{-1})^*(L,\alpha) \in \ell'$. All the data concerning $\omega', \mathfrak{h}', \ell'$ will be denoted by the same symbol as those attached to ω, $\mathfrak{h}$, ℓ, but affected by a prime. This convention will be valid all over this section.

Put $\lambda = F(g)$; then, by (2.2.6), $\omega' - \omega = d\lambda$. Let $\gamma_\lambda : \mathfrak{M} \longrightarrow \mathfrak{M}$ and $\Phi_\lambda : M \longrightarrow M'$ be the corresponding Lie homomorphisms, as in 1.4. Now choose $m \in M$ such that $p(m) = g$ and let $\Lambda = \gamma_\lambda^* \vartheta' - Ad_M^*(m)\vartheta$. Then

$$d(\gamma_\lambda^* \vartheta') = \gamma_\lambda^* d\vartheta' = \gamma_\lambda^*(\psi'^* \omega') = \psi^* \omega'$$

while

$$d(Ad_M^*(m)\vartheta) = Ad_M^*(m)d\vartheta = Ad_M^*(m)\psi^*\omega = \psi^*(Ad_G^*(g)\omega) = \psi^*\omega',$$

so that $d\Lambda = 0$, which means that Λ is a character of $\mathfrak{M}$. From this reason $\mathfrak{h}$ is a polarization for $\gamma_\lambda^*\vartheta' = Ad_M^*(m)\vartheta + \Lambda$, too. Since

M is connected and simply connected, Λ gives rise to a character χ_Λ of M whose differential is $2\pi i\Lambda$.

Let now η and χ be the characters of $M(\vartheta)$ and $\tilde{D}$ respectively, associated to ℓ as in 6.1. Since $\vartheta' = (\mathrm{Ad}_M^*(m)\vartheta + \Lambda)\circ\varphi_\lambda$, whence $M'(\vartheta')$ $M'(\vartheta') = \Phi_\lambda(I(m)M(\vartheta))$, we can define a character η' of $M'(\vartheta')$ by $\eta' = (\chi_\Lambda\cdot(\chi\circ I(m^{-1})))\circ\Phi_\lambda^{-1}|M'(\vartheta')$. Consider now the diagram

$$\begin{array}{ccc}
M'\times C & \longrightarrow & M\times C \\
\downarrow{\scriptstyle \eta'} & & \downarrow{\scriptstyle \eta} \\
Y' = M'/M'(\vartheta') & \longrightarrow & M/M(\vartheta) \quad Y \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
X' = G/G(\omega') & \xrightarrow{i(g^{-1})} & G/G(\omega) = X
\end{array}$$

where the top horizontal arrow is given by $[m',z] \longmapsto [I(m^{-1})\circ\Phi_\lambda^{-1}(m'), \chi_\Lambda(m')z]$, the middle horizontal arrow is induced by $I(m^{-1})\circ\Phi_\lambda^{-1} : M' \longrightarrow M$ and the top vertical arrows are determined by projections onto the first factors followed by projections onto quotient spaces. The fact that this diagram commutes, which can be easily verified, together with the discussion about line bundles in 6.1, imply that, by composing the left vertical arrows, one obtains a line bundle with connection over X' which is equivalent to (L',α'). This means that η' is precisely the character associated to ℓ'.

Furthermore, since $\tilde{D}' = \Phi_\lambda(I(m)\tilde{D})$, one can see that χ', the extension of η' to $\tilde{D}'$, is given by the formula

$$\chi' = (\chi_\Lambda\cdot(\chi\circ I(m^{-1})))\circ\Phi_\lambda^{-1}|\tilde{D}' .$$

Denote for simplicity $\chi_m = \chi\circ I(m^{-1})|D'$. Now $\rho(\chi',\mathfrak{h}') = \rho((\chi_\Lambda\cdot\chi_m)\circ\Phi_\lambda^{-1}|\tilde{D}',\mathfrak{h}')$ is by [A-K, Proposition I.5.13] unitarily equivalent to $\rho((\chi_\Lambda|\tilde{D}')\cdot\chi_m,\mathfrak{h}')\circ\Phi_\lambda^{-1}$; consequently $\rho(\chi',\mathfrak{h}')\circ\Phi_\lambda$ is unitarily equivalent to $\rho((\chi_\Lambda|\tilde{D}')\cdot\chi_m,\mathfrak{h}')$. At this moment we observe that $\rho((\chi_\Lambda|\tilde{D}')\cdot\chi_m,\mathfrak{h}')$ is unitarily equivalent to $\rho(\chi_m,\mathfrak{h}')\otimes\chi_\Lambda$ through the unitary isomorphism $f \longmapsto \chi_\Lambda\cdot f$ of $H((\chi_\Lambda|\tilde{D}')\cdot\chi_m,\mathfrak{h}')$ onto $H(\chi_m,\mathfrak{h}')$. Using [A-K, (IV.2.2)], $\rho(\chi_m,\mathfrak{h}')$ is seen to be unitarily equivalent to $\rho(\chi,\mathfrak{h})$. Summing up these last remarks it follows that $\rho(\chi',\mathfrak{h}')\circ\Phi_\lambda$ is unitarily equivalent to $\rho(\chi,\mathfrak{h})\otimes\chi_\Lambda$. To conclude this part of the proof, that is to get $\pi_{\ell,\mathfrak{h}}=\pi_{\ell',\mathfrak{h}'}$, it is enough to invoke Lemma 6.1.1.

Assume now, conversely, that $\pi_\ell=\pi_{\ell'}$, that is $\pi(L,\alpha;\mathfrak{h})$ and $\pi(L',\alpha';\mathfrak{h}')$ are projectively equivalent, where $\omega,\omega'\in Z^2_0(\mathfrak{g})$, $\mathfrak{h}$ (resp. $\mathfrak{h}'$) is a positive, nil-admissible polarization of $\mathfrak{g}$ at ω (resp. ω') and $\ell=[(L,\alpha)]\in\mathcal{L}_c(X,\theta)$, $\ell'=[(L',\alpha')]\in\mathcal{L}_c(X',\theta')$. From Lemma 6.1.1 and 2.4 we deduce that there is an isomorphism $\Phi: M\longrightarrow M'$ with differential $\varphi:\mathfrak{m}\longrightarrow\mathfrak{m}'$ such that $\rho(\chi,\mathfrak{h})$ and $\rho(\chi',\mathfrak{h}')\circ\Phi$ are unitarily equivalent. But $\rho(\chi',\mathfrak{h}')\circ\Phi$ is unitarily equivalent to $\rho(\chi'\circ(\Phi|D),\varphi^{-1}(\mathfrak{h}))$, hence, by [A-K, Theorem IV.5.7], there exists $m\in M$ such that $\varphi^*\nu'=Ad^*_M(m)\nu$ and $\eta= =\eta'\circ((\Phi\circ I(m))|M(\nu))$. Now looking at the construction of the line bundle with connection associated to a character, which has been recalled in 6.1, it is easily seen that $\ell'=p(m)^{-1}\ell$.

6.2.3. <u>Proof of Proposition 5.4.3.</u> Let $\omega\in Z^2(\mathfrak{g})$ and $(\mathfrak{m}_\omega,\psi_\omega)\in \mathcal{E}xt_0(\mathfrak{g},R)$. In view of [A-K, Theorem V.3.2], ω is of type I if and only if any orbit of M_ω in $\mathfrak{m}^*_\omega$ under the coadjoint representation is locally closed, and the cohomology class of its canonical symplectic 2-form vanishes. Let us look more closely at such an orbit. To this end choose $(r,\lambda)\in R\times\mathfrak{g}^*=\mathfrak{m}^*_\omega$ and observe that, by (4.2.1) and (3.1.3) the orbit of M_ω through (r,λ) is exactly the set $\{(r, Ad^*(g)\lambda+rF_\omega(g))\in R\times\mathfrak{g}^*;\ g\in G\}$. On the other hand, using (3.1.2), one sees that

$$F_{r\omega+d\lambda}(g)=rF_\omega(g)+F_{d\lambda}(g)=rF_\omega(g)+Ad^*(g)\lambda-\lambda\,.$$

Thus the above orbit is obtained from $\{0\}\times F_{r\omega+d\lambda}(G)$ by translating with (r,λ). Hence it is locally closed in $\mathfrak{m}^*_\omega$ if and only if $F_{r\omega+d\lambda}(G)$ is locally closed in $\mathfrak{g}^*$. Furthermore one sees that the symplectic space $(X_{r\omega+d\lambda},\theta_{r\omega+d\lambda})$ is isomorphic to the symplectic space associated to the orbit of M_ω through (r,λ) in $\mathfrak{M}_\omega^*$. This completes the proof.

6.2.4. Proof of Theorem 5.4.4. Noting that any central central extension of $\mathfrak{g}$ by R is equivalent to an extension of the form $(\mathfrak{M}_\omega, \psi_\omega)$ with $\omega \in Z^2(\mathfrak{g})$, the theorem follows directly from Proposition 5.4.3.

6.2.5. Proof of Theorem 5.4.5. The first claim results from Lemma 6.1.1 and 2.2. To prove the second claim we only have to show that an arbitrary type I irreducible projective representation of G is equivalent to a projective representation of the form $\pi(L,\alpha;\mathfrak{h})$. To this end consider the "unitary lifting" $\rho = \check{\pi}$: $M = M_\pi \longrightarrow U(H)$ (cf. 2.2). In view of [A-K, Theorem 5.3.3], there exists $\vartheta \in \mathfrak{M}^*$ and a character η of $M(\vartheta)$ with differential $2\pi i \vartheta | \mathfrak{M}(\vartheta)$ such that, if we choose a positive, strongly admissible polarization $\tilde{\mathfrak{h}}$ of $\mathfrak{M}$ at ϑ and denote by $\tilde{D}$ its corresponding "D"-group and by χ the corresponding extension of η to $\tilde{D}$, then ρ is unitarily equivalent to $\mathrm{ind}_M(\eta,\tilde{\mathfrak{h}})$, the holomorphically induced representation of M associated to η and $\tilde{\mathfrak{h}}$ (see [A-K]). Since R is central in M, the very definition of $\mathrm{ind}_M(\eta,\tilde{\mathfrak{h}})$ ensures us that

$$\mathrm{ind}_M(\eta,\tilde{\mathfrak{h}})(r) = e^{2\pi i \langle \vartheta, r\rangle}\mathrm{Id}, \quad r \in R;$$

on the other hand, ρ being projectable, we have

$$\mathrm{ind}_M(\eta,\tilde{\mathfrak{h}})(r) = e^{2\pi i r}\mathrm{Id}, \quad r \in R.$$

It follows that $\langle \vartheta, r\rangle = r$ for any $r \in R$.

Now according to Section 1, we can assume $\mathfrak{M} = \mathfrak{M}_\omega$ for a suitable $\omega \in Z^2(\mathfrak{g})$. Define $\lambda \in \mathfrak{g}^*$ by $\langle \lambda, x\rangle = \langle \vartheta, (0,x)\rangle$ and $\omega' = \omega + d\lambda$. Further let $\varphi_\lambda : \mathfrak{M} \longrightarrow \mathfrak{M}' = \mathfrak{M}_{\omega'}$ be the isomorphism associated to λ. Clearly $\vartheta = \vartheta' \circ \varphi_\lambda$, where $\vartheta' = \vartheta_{\omega'}$. Then

$$\mathrm{ind}_M(\eta, \tilde{\mathfrak{h}}) \simeq \mathrm{ind}_{M'}(\eta', \tilde{\mathfrak{h}}') \circ \Phi_\lambda$$

where M' is the connected and simply connected Lie group with Lie algebra $\mathfrak{m}'$, $\Phi_\lambda : M \longrightarrow M'$ is the isomorphism whose differential is φ_λ, $\tilde{\mathfrak{h}}' = \varphi_\lambda(\tilde{\mathfrak{h}})$, $\eta' = \eta \circ (\Phi_\lambda^{-1} | M'(\gamma'))$. But $\mathrm{ind}_{M'}(\eta', \mathfrak{h}')$ is just the representation of M' we have denoted $\rho(\chi', \mathfrak{h}')$, where $\chi' = \chi \circ (\Phi_\lambda^{-1} | \tilde{D}')$ and $\mathfrak{h}' = \psi_{\omega'}(\tilde{\mathfrak{h}}')$. Letting (L', α') denote the line bundle with connection over $X_{\omega'}$ associated to η', one sees that $\pi(L', \alpha'; \mathfrak{h}')$ is projectively equivalent to π.

6.2.6. Proof of Corollary 5.4.6. Since a central extension by R of an exponential group is again an exponential group and since such a group, when connected and simply connected, is of type I, it follows that G is of projective type I. Let now $\omega \in Z^2(\mathfrak{g})$. Because $M_\omega(\gamma_\omega)$ is connected and simply connected, $\mathcal{L}_c(X_\omega, \theta_\omega)$ consists of only one element. (For more details concerning the exponential groups the reader is refered to [S]). The rest of the proof is merely a simple consequence of Theorem 5.4.5(iii).

References

[A-K] L. Auslander and B. Kostant, Polarization and unitary representations of solvable Lie groups, Inventiones Math. 14 (4), 255-354 (1971).

[K] B. Kostant, Quantization and unitary representations, Lectures Notes in Math. 170, 87-208 (1970).

[M-Z] D. Montgomery and L. Zippin, Topological Transformation Groups, Interscience, New York, 1955.

[S] P. Bernat et al., Représentations des groupes de Lie résolubles, Dunod, Paris, 1972.