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# STABILIZATION BY LINEAR FEEDBACK OF LINEAR DISCRETE STOCHASTIC SYSTEMS

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# STABILIZATION BY LINEAR FEEDBACK OF LINEAR DISCRETE STOCHASTIC SYSTEMS

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Abstract. Consider a linear discrete system and assume the matrices defining it are stochastically perturbed. Conditions on variances are obtained that allow the system to be stabilized by linear feedback and if such conditions are fulfilled a reasonable construction for the stabilizing feedback is recommended.

## 1. Introduction

Consider a linear discrete system of the form $x_{n+1} = \left[A + C_n(\omega)\right] x_n + \left[B + D_n(\omega)\right] u_n$

where $C_n$ and $D_n$ are random perturbations with zero mean; we shall assume some independence properties and look for a linear feedback which insures a stable behavior. We shall see that in order such feedback to exist the variances of the elements of $C_n$ and $D_n$ must satisfy some limitations, and in this case we shall recommend a construction for the stabilizing feedback.

The final result we obtain is the following. Let $Q \geq 0$ such that $(A; Q)$ is completely controllable and let $R > 0$; let $(\mu, \nu)$ be such that the equation

$$P = Q + A^* PA + \mu I(\mathrm{Tr}P) - A^* PB\left[R + B^* PB + \nu I(\mathrm{Tr}P)\right]^{-1} B^* PA$$

has a solution $P > 0$. Choose

$$L = -\left[R + B^* PB + \nu I(\mathrm{Tr}P)\right]^{-1} B^* PA;$$

Then, for $u_n = Lx_n$ the zero solution of the system is mean square exponentially stable for all random perturbations $C_n$, $D_n$ with zero mean value, with independent elements and such that the variances of the elements of $C_n$ are not larger than $\mu$ and the variances of the elements of $D_n$ are not larger than $\nu$.

Descriptions for the set $\mathcal{M}$ of allowable pairs $(\mu, \nu)$ are given from which it follows that if A is $\ell \times \ell$ then $\mu \leq \frac{1}{\ell}$ and that if the spectral radius

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of A is larger than $\Upsilon$, $\vartheta$ cannot be larger than a number specified in Th. 8 below.

To get these results we have to study the general linear- quadratic optimization problem on infinite interval for stochastic systems. For such studies we have to mention monographs by Åström [1], Kushner [2], [3], Khasminskii [4], Wonham [5]. The linear-quadratic problem for the case of an infinite interval has been considered for Itô equations in [5], [9], [10], [11] and for the discrete case by Zabczyk in [7], [8], [6].

## 2. Preliminaries

Let $\{\Omega, \mathcal{K}, \mathcal{P}\}$ be a probability field. If x is a random variable and $\mathcal{F} \subset \mathcal{K}$ is a $\sigma$- algebra of subsets of $\Omega$ by Ex we denote the mean value and by $E[x \mid \mathcal{F}](\cdot)$, the conditional mean of x with respect to $\mathcal{F}$. We denote by $L^2(\Omega)$ the set of random variables x with $Ex^2 < \infty$. If A is a matrix (or a vector) $A^*$ means the transposed.

If $A_k(\omega)$, $B_k(\omega)$ are *random* matrices we say that $\{A_k(\cdot), B_k(\cdot), k \geq 0\}$ are independent if $\mathcal{P}\{\omega; A_k(\omega) \in \Gamma_k, B_k(\omega) \in \hat{\Gamma}_k, k \geq 0\} = \prod_{k \geq 0} \mathcal{P}\{\omega; A_k(\omega) \in \Gamma_k\} \mathcal{P}\{\omega; B_k(\omega) \in \hat{\Gamma}_k\}$ for all sets of matrices $\Gamma_k$, $\hat{\Gamma}_k$.

We say that the elements of the matrices $A_k(\omega)$, $B_k(\omega)$ are independent if

$$\mathcal{P}\{\omega; a_{ig;k}(\omega) \leq \alpha_{ig;k}, b_{p,q;k}(\omega) \leq \beta_{p,q;k}, k \geq 0\} =$$

$$= \prod_{k \geq 0} \prod_{i,g} \prod_{p,q} \mathcal{P}\{\omega; a_{ig;k}(\omega) \leq \alpha_{ig;k}\} \mathcal{P}\{\omega; b_{p,q;k}(\omega) \leq \beta_{p,q;k}\}$$

We shall consider linear discrete stochastic systems of the form

$$(1) \qquad x_{n+1} = A_n(\omega) x_n + B_n(\omega) u_n, \qquad n \geq 0$$

Where $A_n(\omega)$ are $l \times l$ matrices with random elements and $B_n(\omega)$ are $l \times r$ matrices.

We associate the performances

$$(2) \quad I_k(x, \varphi) = E \sum_{n=k}^{\infty} \left[ x_n^*(\omega) Q_n x_n(\omega) + u_n^*(\omega) R_n u_n(\omega) \right]$$

where $Q_n \geq 0$, $R_n > 0$, and $x_n(\omega)$ are defined by

$$x_{n+1} = A_n(\omega) x_n + B_n(\omega) \varphi_{n,k}(x_n), \qquad x_k = x$$

with $\varphi_{n,k} : R^{\ell} \longrightarrow R^{r}$ Borel measurable, and $u_n(\omega) = \varphi_{n,k}(x_n(\omega))$.

We denote by $H_{k,x}$ the set of functions $\varphi$ such that $x_n(\cdot)$ and $u_n(\cdot)$ described above satisfy $|x_n| \in L^2(\Omega)$, $|u_n| \in L^2(\Omega)$; we denote by $H^{o}_{k,x}$ the set of functions $\varphi$ in $H_{k,x}$ such that $I_k(x, \varphi) < \infty$

### 3. The general linear - quadratic problem

We shall obtain some general results concerning the problem of minimizing (2) for the class of control described above.

Theorem 1. Assume $A_n(\cdot)$, $B_n(\cdot)$ have elements in $L^2(\Omega)$ and $\{A_n(\cdot), B_n(\cdot), n \geq 0\}$ are independent.

The following statements are equivalent : (i) for all k and x $H^{o}_{k,x} \neq \emptyset$ (ii) for all k and x there exist $\tilde{\varphi} \in H^{o}_{k,x}$ such that $I_k(x, \tilde{\varphi}) = \inf_{\varphi \in H^{o}_{k,x}} I_k(x, \varphi)$

Proof. We have to prove only (i)$\Longrightarrow$(ii) , that is we have to show that if admissible controls exist then an optimal control will exist. Let $H_{k,x,N}$ be the set of functions as in $H_{k,x}$ but restricted to $n < N$; let

$$I_{k,N}(x, \varphi) = E \sum_{n=k}^{N-1} \left[ x_n^*(\omega) \, Q_n x_n(\omega) + u_n^*(\omega) \, R_n u_n(\omega) \right]$$

where $x_n$, $u_n$ are associated to $\varphi \in H_{k,x,N}$ in the way described above. If $\varphi \in H_{k,x}$ then it can be restricted in a natural way to a function in $H_{k,x,N}$ and if $\varphi \in H^{o}_{k,x}$ we have $I_{k,N}(x,\varphi) \leq I_k(x, \varphi) < \infty$ for all $N \geq k$. The problem of minimizing $I_{k,N}$ for $\varphi \in H_{k,x,N}$ is solved [2] , [12], [13]; let $\tilde{\varphi}^N$ the corresponding optimal control. Take $\varphi \in H_{k,x,N+1}$ ; we may consider it in $H_{k,x,N}$ and

$$I_{k,N}(x, \tilde{\varphi}^N) \leq I_{k,N}(x, \varphi) \leq I_{k,N+1}(x, \varphi) ;$$

since this inequality is true for all $\varphi$ , we deduce that $I_{k,N}(x, \tilde{\varphi}^N) \leq I_{k,N+1}(x, \tilde{\varphi}^{N+1})$.

Take now $\hat{\varphi} \in H^{o}_{k,x}$, that exists according to (i)

We have

$$I_{k,N}(x, \tilde{\varphi}^N) \leq I_{k,N}(x, \hat{\varphi}) \leq I_k(x, \hat{\varphi}) < \infty .$$

The monotone increasing sequence $\{ I_{k,N}(x, \tilde{\varphi}^N) \}_N$ is thus bounded, hence it has a finite limit. From [12] , [13] it is known that $I_{k,N}(x, \tilde{\varphi}^N)$ is a quadratic form in x, $I_{k,N}(x, \tilde{\varphi}^N) = x^* P_{k,N} x$. It follows from the above reasoning that $\lim_{N\to\infty} P_{k,N} = P_k$ exists. Since $I_{k,N}(x, \tilde{\varphi}^N) \geqslant x^* Q_k x$ it follows that $P_{k,N} \geqslant Q_k$ for all $N \geqslant k$, hence $P_k \geqslant Q_k$. From [12], [13] it is also known that $P_{k,N}$ satisfy $P_{k,N} = Q_k + E(A_k^* P_{k+1,N} A_k) -$

$$- E(A_k^* P_{k+1,N} B_k)(R_k + E(B_k^* P_{k+1,N} B_n))^{-1} E(B_k^* P_{k+1,N} A_k)$$

If we take the limit for $N \to \infty$ we get

$$P_k = Q_k + E(A_k^* P_{k+1} A_k) - E(A_k^* P_{k+1} B_k)(R_k + E(B_k^* P_{k+1} B_k))^{-1} E(B_k^* P_{k+1} A_k)$$

Let now $\tilde{\varphi}_k(x) = L_k x, \quad L_k = -(R_k + E(B_k^* P_{k+1} B_k))^{-1} E(B_k^* P_{k+1} A_k)$

Since the elements of $A_n(\cdot)$ and $B_n(\cdot)$ are in $L^2(\Omega)$ it follows that $\tilde{\varphi}_k \in H_{k,x}$. We show that
$I_k(x, \tilde{\varphi}_k) = x^* P_k x$, hence $\tilde{\varphi}_k \in H^0_{k,x}$ and moreover $x^* P_k x \leq I_k(x, \varphi)$ for all $\varphi \in H^0_{k,x}$ hence $\tilde{\varphi}_k$ is optimal. To do that let $\varphi \in H^0_{k,x}$, $x_n$, $u_n$ the corresponding sequences; by using the equation for $P_k$ (which is true for all k) and the independence assumption we deduce by computation that

$$E(x_n^* Q_n x_n + u_n^* R_n u_n) = E(x_n^* P_n x_n) - E(x_{n+1}^* P_{n+1} x_{n+1}) + E[(u_n^* - x_n^* L_n^*)(R_n + E(B_n^* P_{n+1} B_n))(u_n - L_n x_n)]$$

hence

$$I_{k,N}(x, \varphi) = x^* P_k x - E(x_N^* P_N x_N) + E \sum_{n=k}^{N-1} [(u_n^* - x_n^* L_n^*)(R_n + E(B_n^* P_{n+1} B_n))(u_n - L_n x_n)].$$

It follows

$$I_{k,N}(x, \tilde{\varphi}^N) \leq I_{k,N}(x, \tilde{\varphi}_k) = x^* P_k x - E(\tilde{x}_N^* P_N \tilde{x}_N) \leq x^* P_k x$$ and taking limits for $N \to \infty$

$$x^* P_k x \leq I_k(x, \tilde{\varphi}_k) \leq x^* P_k x$$

hence $I_k(x, \tilde{\varphi}_k) = x^* P_k x$. On the other hand for any other control

$I_{k,N}(x, \varphi) \geqslant I_{k,N}(x, \tilde{\varphi}^N)$, hence $I_k(x, \varphi) \geqslant x^* P_k x$ and the theorem is proved.

## 4. A particular case

In the following we shall consider the special case

$A_n(\omega) = A + C_n(\omega)$ ; $B_n(\omega) = B + D_n(\omega)$

where $\{C_n, D_n, n \geq 0\}$ are independent with zero mean and $E(c_{ij;n} \cdot c_{p,q;n})$, $E(d_{ij;n}\, d_{p,q;n})$ are finite and do not depend on n. Moreover we shall take $Q_n = Q$, $R_n = R$.

Theorem 2. Under the above assumptions the following statements are equivalent :

(i) $H^{o}_{o,x} \neq \emptyset$ for all $x \in R^{\ell}$

(ii) $I_o(x,\varphi)$ has a minimum on $H^{o}_{o,x}$

(iii) The equation

$$(3)\quad P = Q + A^* PA + E(C_o^* PC_o) - A^* PB\,(R + B^* PB + E(D_o^* PD_o))^{-1} B^* PA$$

has a solution $P \geq 0$.

Proof We have $P_{N,N} = 0$ and

$$P_{k,N} = Q + E\left[(A + C_k)^* P_{k+1,N}(A + C_k)\right] - E\left[(A+C_k)^* P_{k+1,N}(B+D_k)\right]\Big[R + E\left((B+D_k)^* P_{k+1,N}(B+D_k)\right)\Big]^{-1} E\left[(B+D_k)^* P_{k+1,N}(A+C_k)\right]$$

The hypotheses on $C_k$ and $D_k$ allow us to write

$$P_{k,N} = Q + A^* P_{k+1,N} A + E(C_o^* P_{k+1,N} C_o) - A^* P_{k+1,N} B(R + B^* P_{k+1,N} B + E(D_o^* P_{k+1,N} D_o))^{-1} B^* P_{k+1,N} A.$$

Denote $P_{N-k,N} = S_k$ and remark

$$S_{k+1} = Q + A^* S_k A + E(C_o^* S_k C_o) - A^* S_k B(R + B^* S_k B + E(D_o^* S_k D_o))^{-1} B^* S_k A,$$

$$S_o = 0$$

From i) it follows that $P_{o,N} \leq P_{o,N+1} \leq I_o(x, \hat{\varphi})$

hence $\lim_{N\to\infty} P_{o,N} = \lim_{N\to\infty} S_N$ exists and is finite ; denoting this limit by S it follows

$S \geqslant Q \geqslant 0$ and S is a solution of (3); in this way (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Take $L = -(R + B^* SB + E(D_o^* SD_o))^{-1} B^* SA$, $\tilde{\varphi}(x) = Lx$; repeat all computations as in the general case to see that $\tilde{\varphi}$ is optimal, hence (iii) $\Longrightarrow$ (ii).

## 5. Stabilizability

Under the same assumptions as above we study now conditions for the existence of L such that with $u_n = Lx_n$ we obtain mean square exponential stability. We start with a general stability result.

Lemma 1. Consider the system

(4) $y_{n+1} = [A + C_n(\omega)] y_n$

where $\{C_n(\cdot),\ n \geqslant 0\}$ are independent, have zero mean and $E(C_n^* C_n)$ do not depend on n.

If $\lim_{n\to\infty} E|y_n|^2 = 0$ for all initial conditions $x \in R^l$ then the zero solution of (4) is mean square exponentially stable.

Proof. Let $Z_n = E(y_n y_n^*)$, $Z_o = xx^*$. We have

$$Z_{n+1} = E[(Ay_n + C_n y_n)(y_n^* A^* + y_n^* C_n^*)] = AZ_n A^* + D(Z_n)$$

Let T be the linear mapping defined by

$$TZ = AZA^* + D(Z)$$

We may write $Z_{n+1} = TZ_n$, hence $Z_n = T^n Z_o$. By assumption $T^n Z_o \longrightarrow 0$ for $Z_o = xx^*$; if $Z \geqslant 0$ then $Z = \sum_{j=1}^{l} \lambda_j e_j e_j^*$ hence $T^n Z_o \longrightarrow 0$ for all $Z_o \geqslant 0$. But if $Z_o$ is symmetric it can be written as difference of semipositive matrices hence $T^n Z_o \longrightarrow 0$ for all symmetric $Z_o$, and the lemma is proved.

Theorem 3. Under the same assumptions as in theorem 2, let $(A^*, Q)$ be completely controllable. Then the following statements are equivalent :

(i) There exists $L_o$ such that the zero solution of

$$x_{n+1} = (A + C_n(\omega) + (B + D_n(\omega)) L_o) x_n$$

is mean square exponentially stable;

(ii) Equation (3) has a solution $P > 0$.

Proof. If (i) is true, the function $\varphi$ defined by $\varphi(x) = L_o x$ belongs to $H^o_{o,x}$ for all $x$, hence there exists a solution $P \geqslant Q$ of (3) ; from the proof of Th. 2 we have

$$x^* P_{o,N} x \leq x^* P x - E(\tilde{x}^*_N P \tilde{x}_N)$$

$$\lim_{N \to \infty} x^* P_{o,N} x = x^* P x = I_o(x, \tilde{\varphi}), \quad \text{where}$$

$$\tilde{\varphi}(x) = Lx, \quad L = -(R + B^* PB + E(D_o^* PD_o))^{-1} B^* PA$$

We prove now that $P > 0$. Assume $y \neq 0$, $y \in R^\ell$, $y^* Py = 0$; it follows $I_o(y, \tilde{\varphi}) = 0$, hence $y^* Qy + y^* L^* RLy = 0$ hence $y^* Qy = 0$, $Ly = 0$; from here and $y^* Py = 0$ equation (3) leads to $y^* A^* PA y + y^* E(C_o^* P C_o) y = 0$ hence $y^* A^* PA y = 0$; $Ay$ has the same properties as $y$ hence $y^* (A^*)^k QA^k y = 0$ and with $y \neq 0$ we contradict the controllability assumption. In this way (i)$\Longrightarrow$(ii).

From $E(\tilde{x}^*_N P\tilde{x}_N) \leq x^* Px - x^* P_{o,N} x$ it follows that

$$\lim_{N \to \infty} E(\tilde{x}^*_N P \tilde{x}_N) = 0 \text{ and with } P > 0 \text{ we reduce } \lim_{N \to \infty} E|\tilde{x}_N|^2 = 0,$$

we may apply lemma 1 and (ii)$\Longrightarrow$(i) follows.

Theorem 4. Under the assumptions of theorem 3 if (3) has a solution $P \geqslant 0$ then :

(i) $P > 0$

(ii) $P$ is the unique semipositive solution of (3)

(iii) $\min_{\varphi \in H^o_{o,x}} I_o(x, \varphi) = x^* Px = I_o(x, \tilde{\varphi})$

where $\tilde{\varphi}(x) = Lx$, $L = -(R + B^* PB + E(D_o^* PD_o))^{-1} B^* PA$

(iv) The zero solution of

$$x_{n+1} = \left[ A + C_n(\omega) + (B + D_n(\omega)) L \right] x_n$$

is mean square exponentially stable.

Proof. The only statement to be proved is (ii).

Let $I_N(x, \varphi) = E(x_N^* P x_N) + I_{o,N}(x, \varphi)$

It is known [12], [13] that $I_N$ has a minimum given by $x^* S_{o,N} x$, where

$$S_{k,N} = Q + A^* S_{k+1,N} A + E(C_o^* S_{k+1,N} C_o) -$$

$$- A^* S_{k+1,N} B (R + B^* S_{k+1,N} B + E(D_o^* S_{k+1,N} D_o))^{-1} B^* S_{k+1,N} A, \quad S_{N,N} = P.$$

Since P is a solution of (3) we deduce that $S_{N-1,N} = P$ and then, step by step, we get $S_{o,N} = P$. We have further $x^* P_{o,N} x \leq I_{o,N}(x, \hat{\varphi}) \leq$ $I_N(x, \hat{\varphi}) = x^* S_{o,N} x = x^* P x \leq E(\tilde{x}_N^* P \tilde{x}_N) + I_{o,N}(x, \tilde{\varphi})$.

But $E|\tilde{x}_N|^2 \longrightarrow 0$ and $I_{o,N}(x, \tilde{\varphi}) \longrightarrow x^* S x$,

$P_{o,N} \longrightarrow S$, hence

$x^* S x \leq x^* P x \leq x^* S x$ ; finally $P = S$.

## 6. A stability result

Consider the system (4) with the assumptions on $C_n$ stated above and

(5) $Z_{n+1} = \left[ A + B_n(\omega) \right] Z_n$

Lemma 2. If the zero solution of (4) is mean square exponentially stable then there exists $\varepsilon_o > 0$ such that if $E(B_n) = 0$, if the elements of $B_n$ are independent and $E(b_{ij;n}^2) \leq E(c_{ij;0}^2) + \varepsilon_o$ for all i, j, n then the zero solution of (5) is also mean square exponentially stable.

Proof. From the assumption on (4) it follows [14] that there exists $H > 0$ such that

$$E\left[(A + C_0)^* H (A + C_0)\right] - H = -I$$

Let $V(x) = x^* Hx$; for $n \geq k$ we have

$$E\left[V(Z_{n+1}(\cdot)) \mid Z_n(\cdot)\right](\omega) - V(Z_n(\omega)) = Z_n^*(\omega)\left\{E\left[(A + B_n(\omega))^* H (A + B_n(\omega)\right] - H\right\} Z_n(\omega) = -Z_n^*(\omega)(I - \Delta_n) Z_n(\omega) =$$

$$= -Z_n^*(\omega)\left\{(1 - \varepsilon_1) I + (\varepsilon_1 I - \Delta_n)\right\} Z_n(\omega)$$

where $\Delta_n$ is a diagonal matrix with elements

$$\sum_j h_{jj} \left( E(b^2_{ji;n}) - E(c^2_{ji;0}) \right).$$

Let $\varepsilon_1 \in (0, 1)$, $0 < \varepsilon_0 \leq \frac{\varepsilon_1}{\operatorname{Tr} H}$ ; if $E(b^2_{ij;n}) \leq E(c^2_{ij;0}) + \varepsilon_0$

then $\varepsilon_1 I - \Delta_m > 0$, hence

$$E\left[V(Z_{n+1}(\cdot)) \mid Z_n(\cdot)\right](\omega) - V(Z_n(\omega)) \leq (1 - \varepsilon_1) |Z_n(\omega)|^2,$$

$H > 0$ and the mean square asymptotic stability follows.

Theorem 5. Under the same assumptions as in theorems 3 and 4 assume the equation

$$P = Q + A^* PA + \Delta_1(P, M) - A^* PB (R + B^* PB + \Delta_2(P, N))^{-1} B^* PA$$

has a solution $P \geq 0$ ; here $\Delta_1(P, M)$, $\Delta_2(P, N)$ are diagonal matrices with elements

$$\sum_s p_{ss} m_{si}, \quad \sum_s p_{ss} n_{sj} \quad \text{respectively}.$$

Let $L = -(R + B^* PB + \Delta_2(P, N))^{-1} B^* PA$

Then the zero solution of the system

$$x_{n+1} = (A + C_n(\omega) + (B + D_n(\omega)) L) x_n$$

is mean square exponentially stable for all random perturbations $C_n$, $D_n$ with zero mean, with independent elements, and $E(c^2_{ij;n}) \leq m_{ij}$, $E(d^2_{ij;n}) \leq n_{ij}$.

Proof. Consider random perturbations for which the variances are e equal to $m_{ij}$, $n_{ij}$ respectively. Denote them $\hat{C}_n$, $\hat{D}_n$. From theorem 4 it follows mean square exponential stability of the system corresponding to $\hat{C}_n$, $\hat{D}_n$; we apply then lemma 2 to the random perturbations $C_n + D_n L$ and $\hat{C}_n + \hat{D}_n L$. For applications might be useful the special case $m_{ij} = \mu$, $n_{ij} = \nu$; in this case the equation that gives the solution to the stabilization problem is

$$P = Q + A^* PA + \mu I(\mathrm{Tr}\,P) - A^* PB(R + B^* PB + \nu I(\mathrm{Tr}\,P))^{-1} B^* PA$$

It is now natural to study the set $\mathcal{M}$ of pairs $(\mu, \nu)$ $\mu \geq 0, \nu \geq 0$ for which this equation has solutions $P \geq 0$.

Theorem 6. Assume $(A, B)$ completely controllable and all assumption of theorems 3 and 4. Then (i) $(0, 0) \in \mathcal{M}$, (ii) if $(\mu, \nu) \in \mathcal{M}$ there exists $\varepsilon > 0$ such that $[0, \mu + \varepsilon) \times [0, \nu + \varepsilon) \subset \mathcal{M}$.

Proof. First assertion is wellknown and the second follows from lemma 2 and theorems 3 and 4.

Let now $X_1 = \{\mu \in [0, \infty);\ (\mu, 0) \in \mathcal{M}\}$

$X_2 = \{\nu \in [0, \infty);\ (0, \nu) \in \mathcal{M}\}$, $X_3 = \{\mu \in [0, \infty);\ (\mu, \mu) \in \mathcal{M}\}$

$\tilde{\mu}_1 = \sup X_1$, $\tilde{\mu}_2 = \sup X_2$, $\tilde{\mu}_3 = \sup X_3$

From theorem 6 $\tilde{\mu}_i > 0$, $i = 1, 2, 3$, $X_1 = [0, \tilde{\mu}_1)$,
$X_2 = [0, \tilde{\mu}_2)$, $X_3 = [0, \tilde{\mu}_3)$, $\tilde{\mu}_3 \leq \tilde{\mu}_1$, $\tilde{\mu}_3 \leq \tilde{\mu}_2$
$X_3 \times X_3 \subset \mathcal{M} \subset X_1 \times X_2$

From theorem 3 it follows that $\tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3$ do not depend on Q and R.

Theorem 7. Under the assumptions of theorem 6 $(\mu, \nu) \in \mathcal{M}$ iff

$$\inf_{M \in \mathcal{M}_0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \mathrm{Tr} \left\{ \mu \left[ \left( e^{-it} I - (A + BM)^* \right)^{-1} \left( e^{it} I - (A + BM) \right)^{-1} \right] + \right. \right.$$

$$\left. + \nu \left[ \left( e^{-it} I - (A + BM)^* \right)^{-1} M^* M \left( e^{it} I - (A + BM) \right)^{-1} \right] \right\} dt < 1$$

where $i = \sqrt{-1}$ and $\mathcal{M}_0$ is the set of all matrices M such that $\rho(A+BM) < 1$ ( $\rho(A)$ means the spectral radius of A, i.e. $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$, $\sigma(A)$ being the set of eigenvalues of A).

Proof. Let $(\mu, \nu) \in \mathcal{N}_0$, $\hat{C}_n$, $\hat{D}_n$ random perturbations as in the proof of Theorem 5 with $E(\hat{c}^2_{ij,n}) = \mu$, $E(\hat{d}^2_{ij,n}) = \nu$. From Theorem 4 it follows that a matrix M will exist such that the zero solution of

$$x_{n+1} = (A + \hat{C}_n(\omega) + (B + \hat{D}_n(\omega))\, M)\, x_n$$

is mean square exponentially stable.

From [14] it follows the existence of $H > 0$ such that

$$E\left[ (A^* + \hat{C}_n^* + M^* B^* + M^* \hat{D}_n^*) H (A + \hat{C}_n + BM + \hat{D}_n M) \right] - H = -I,$$

hence

$$(A+BM)^* H (A+BM) - H + \mu(\mathrm{Tr}\, H) I + \nu (\mathrm{Tr} H) M^* M = -I$$

Since $H > 0$ it follows that $M \in \mathcal{M}_0$.

Denote by $S_1(M)$ the solution of

$$(A+BM)^* S_1 (A+BM) - S_1 = -I$$

and by $S_2(M)$ the solution of

$$(A+BM)^* S_2 (A+BM) - S_2 = -M^* M.$$

We have $S_1(M) > 0$, $S_2(M) \geq 0$ and

$$-H + \mu (\mathrm{Tr} H) S_1(M) + \nu (\mathrm{Tr} H) S_2(M) = -S_1(M)$$

hence

$$H = S_1(M) + \left[ \mu S_1(M) + \nu S_2(M) \right] (\mathrm{Tr} H).$$

We deduce

$$\mathrm{Tr} H = (\mathrm{Tr} S_1(M)) + (\mathrm{Tr} H) \left[ \mu (\mathrm{Tr} S_1(M)) + \nu (\mathrm{Tr} S_2(M)) \right]$$

hence

$$(\mathrm{Tr} H) \left[ 1 - \mu (\mathrm{Tr} S_1(M)) - \nu (\mathrm{Tr}\, S_2(M)) \right] = \mathrm{Tr} S_1(M)$$

Since $H > 0$, $S_1(M) > 0$, it follows that

$$\mu(\operatorname{Tr} S_1(M)) + \nu(\operatorname{Tr} S_2(M)) < 1$$

But it is known that

$$S_1(M) = \sum_{n=0}^{\infty} \left[(A+BM)^*\right]^n (A+BM)^n$$

$$S_2(M) = \sum_{n=0}^{\infty} \left[(A+BM)^*\right]^n M^*M (A+BM)^n$$

By using Parseval's equality we deduce that

$S_2(M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[e^{-it} I - (A+BM)^*\right]^{-1} M^*M \left(e^{it} I - (A+BM)\right)^{-1} dt$ and the corresponding formula for $S_1(M)$ and the inequality in the statement is proved. Assume now this inequality : there exists $M \in \mathcal{M}$ such that $\mu \operatorname{Tr} S_1(M) + \nu \operatorname{Tr} S_2(M) < 1$.

Let

$$H = S_1(M) + \frac{\operatorname{Tr} S_1(M)}{1 - \mu \operatorname{Tr} S_1(M) - \nu \operatorname{Tr} S_2(M)} \left[\mu S_1(M) + \nu S_2(M)\right]$$

Since $S_1(M) > 0$, $S_2(M) \geq 0$ it follows that $H > 0$.

By using the Liapunov function defined by H we obtain [14] the mean square exponential stability of the system associated to the random perturbations $\hat{C}$, $\hat{D}$, hence $(\mu, \nu) \in \mathcal{M}$.

<u>Remark.</u> If $\rho(A) < 1$ then $0 \in \mathcal{M}$, hence $\tilde{\mu}_2 = \infty$

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<u>Theorem 8.</u> <u>Under the assumptions of Theorem 6 we have</u>

(i) $\inf_{M \in \mathcal{M}} \operatorname{Tr} S_1(M) \geq \ell$ and $\tilde{\mu}_1 = \dfrac{1}{\inf_{M \in \mathcal{M}} \operatorname{Tr} S_1(M)}$

(ii) $\tilde{\mu}_3 = \dfrac{1}{\ell}$ iff $A = 0$

(iii) If $\rho(A) > 1$ then $\inf_{M \in \mathcal{M}} \operatorname{Tr} S_2(M) > 0$, $\tilde{\mu}_2 = \dfrac{1}{\inf_{M \in \mathcal{M}} \operatorname{Tr} S_2(M)}$

<u>Proof</u>

If $M \in \mathcal{M}$ then

$$0 \leq (A+BM)^* S_1(M) (A+BM) = S_1(M) - I$$

Hence $S_1(M) \geq I$, $\operatorname{Tr} S_1(M) \geq \operatorname{Tr} I = \ell$ and thus by Theorem 7, (i) follows.

Suppose that $\tilde{\mu}_3 = \dfrac{1}{\ell}$

Since $\beta = \inf_{M \in \mathcal{M}} \{ \mathrm{Tr}\, S_1(M) + \mathrm{Tr}\, S_2(M) \} \geq \inf_{M \in \mathcal{M}} \mathrm{Tr} S_1(M) \geq \ell$

by Theorem 7 we have $\tilde{\mu}_3 = \frac{1}{\beta}$ . Hence $\beta = \ell$

There exist $M_n \in \mathcal{M}$ such that $\lim_{n\to\infty} S_1(M_n) = P_1$, $\lim_{n\to\infty} S_2(M_n) = P_2$, $\mathrm{Tr}\,(P_1 + P_2) = \beta = \ell$

From

$$0 \leq M_n^* M_n \leq M_n^* M_n + (A+BM_n)^* S_2(M_n)(A+BM_n) = S_2(M_n)$$

it follows that the sequence $M_n$ is bounded. Hence, there exists a convergent subsequence $M_{n_k}$ . Let $M_0$ be its limit.

We have

$$M_0^* M_0 + (A+BM_0)^* P_2 (A + BM_0) = P_2$$

$$I + (A+BM_0)^* P_1 (A+BM_0) = P_1$$

From the second equality it follows that $P_1 \geq I$ and since $\mathrm{Tr}\,(P_1 + P_2) = \mathrm{Tr}\, I$, $P_1 > 0$, $P_2 \geq 0$ we can prove easily that $P_1 + P_2 = I$, $P_1 = I$, $P_2 = 0$.

From the above relations it follows that $A = 0$.

Suppose that $A = 0$. Then $0 \in \mathcal{M}_0$, $S_1(0) = I$, $S_2(0) = 0$.

Hence $\beta \leq \ell$. But $\beta = \frac{1}{\tilde{\mu}_3}$ and $\tilde{\mu}_3 \leq \tilde{\mu}_1 \leq \frac{1}{\ell}$ . Thus $\beta = \ell$ and (ii) follows.

Suppose that $\rho(A) > 1$ and $\inf S_2(M) = 0$.

Using the same reasoning in the proof of (ii) we get the matrices $P_2$ and $M_0$ such that $P_2 \geq 0$, $\mathrm{Tr}\, P_2 = 0$, $\rho(A+BM_0) \leq 1$ and $(A+BM_0)^* P_2 (A+BM_0) - P_2 = - M_0^* M_0$

Hence $P_2 = 0$, $M_0 = 0$, $\rho(A) \leq 1$, and thus we get a contradiction.

<u>Remark</u> . From Theorem 8 it follows that $\tilde{\mu}_1 \leq \frac{1}{\ell}$ and if $\rho(A) > 1$ then $\tilde{\mu}_3 \leq \frac{\tilde{\mu}_1 \cdot \tilde{\mu}_2}{\tilde{\mu}_1 + \tilde{\mu}_2} < \min \{ \tilde{\mu}_1, \tilde{\mu}_2 \}$

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