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ABSTRACT BEURLING SPACES OF CLASS (Mp) AND ULTRADISTRIBUTION SEMI – GROUPS

by

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ABSTRACT BEURLING SPACES OF CLASS (Mp) AND ULTRADISTRIBUTION SEMI – GROUPS

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Abstract Beurling spaces of class (M_p) and ultradistribution semi-groups.

by Ioana Cioranescu.

In this work we associate to the generator of a "regular" Beurling ultradistribution semi-group of class (M_p) in a Banach space X a Fréchet space $X^{(M_p)}$ which is dense in X and on which the ultradistribution semi-group coincide with a locally-equicontinuous semi-group of operators in $\mathcal{L}(X^{(M_p)})$. We extend so to ultradistribution semi-groups a well-known result of T.Ushijima [16] concearning distribution semi-groups.

The notion of distribution semi-group was introduced by L.J.Lions in [11] and that of ultradistribution semi-group by J.Chazarain in [3] and [4].

T.Ushijima [16] (see also [7]) established that a distribution semi-group coincide on a dense Fréchet subspace Y of the Banach space X with a locally equicontinuous semi-group of operators in $\mathcal{L}(Y)$, which is in addition infinitely differentiable on $(0,+\infty)$.

The purpose of this work is to extend to ultradistribution semi-groups the above result of T.Ushijima. We define the notion of "reg larity" of an ultradistribution semi-group, which is automaticall verified by distribution semi-groups and fails in the case of ul distribution semi-groups because of specific problems arrised ultradistributions with support $\{0\}$. Corresponding to a regular Beurling ultradistribution semi-group of class (M_p) we defin dense subspace of X as a projective limit of some Banach which the considered ultradistribution semi-group coir locally equicontinuous semi-group of operators satisfy ness condition of Beurling type. This is the ultradis logue of the mentioned result of T.Ushijima.

On the other hand, R. Beals [1], [2] and Ju. I. Ljiubic [12], looked for conditions on operators such that the abstract Cauchy problem is solvable in a dense subspace. In [2] an abstract Gevrey space is constructed, corresponding to the particular case of the sequence $M_p = p^{pd}$, $d > 1$, as an inductive limit of some Banach spaces; in the general case no intrinsec characterisation of the space of "initial conditions" is given.

The ultradistributional point of view and the consideration of abstract Beurling spaces of class (M_p) permit us to precise and generalise results from [1], [2] and [12]. Moreover our abstract Beurling spaces are Fréchet spaces with families of norms implying a certain smoothness proprty of the solution of the abstract Cauchy problem.

Finally I want to thank Dr. L. Zsidó, who proved Lemma 3.2.

§ 1. Preliminaries.

For ultradistributions we use the notations and results from [8].

Let $\{M_p\}_{p\geq 0}$ be a sequence of positive numbers with the properties:

(M.1) (logarithmic convexity)

$$M_p^2 \leq M_{p-1}M_{p+1} \quad , \quad p=1,2,\ldots;$$

(M.2) (stability under ultradifferential operators) There are positive constants A and $H\geq 1$ such that

$$M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q} \quad , \quad p=0,1,\ldots;$$

(M.3) (strong non-quasi-analyticity) There is a positive constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Ap\frac{M_p}{M_{p+1}} \quad , \quad p=1,2,\ldots .$$

An example of such a sequence is the Gevrey sequence $M_p=p^{pd}$, $d>1$.

We denote by $m_p = M_p/M_{p-1}$, $p=1,2,\ldots$; then (M.1) is equivalent to saying that the sequence $\{m_p\}_{p\geq 1}$ is increasing.

(M.3) implies the non-quasi-analyticity of the sequence $\{M_p\}_{p\geq 0}$, that is

(M.3)' $\quad \displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < +\infty$

or equivalently $\displaystyle\sum_{p=1}^{\infty} 1/m_p < +\infty$.

We shall suppose $M_0=1$ and define the associated function on $(0,+\infty)$

$$M(t)= \sup_{p\geq 0} \ln \frac{t^p}{M_p} .$$

Then M(t) is increasing, vanishes for sufficiently small $t>0$ and

(1.1) $\quad \displaystyle\int_1^{\infty} \frac{M(t)}{t^2}\,dt<\infty; \; t^k \leq \text{const.}\, e^{M(t)} \quad , \quad \frac{M(t)}{t^k} \xrightarrow[t\to\infty]{} 0 \quad , \; k\in N$

Moreover (M.2) is equivalent to saying that

(1.2) $\quad e^{2M(t)} \leq Ae^{M(Ht)}$.

Inductively, (M.2) implies

(1.3) $\quad e^{(k+1)M(t)} \leq A^k e^{M(H^k t)}$, $\; k\in N$.

Let $h>0$ and $K\subset R$ a compact set; we put

$$\mathcal{D}_K^{\{M_p\},h} = \left\{ \varphi \in C_0^\infty(K);\ \sup_{\substack{p\geq 0\\ t\in K}} \frac{h^p|\varphi^{(p)}(t)|}{M_p} < +\infty \right\}$$

$$\mathcal{E}_K^{\{M_p\},h} = \left\{ \varphi \in C^\infty(K);\ \sup_{\substack{p\geq 0\\ t\in K}} \frac{h^p|\varphi^{(p)}(t)|}{M_p} < +\infty \right\}$$

For an open set $\Omega\subset\mathbb{R}$, we denote by

$$\mathcal{D}_\Omega^{(M_p)} = \varprojlim_{h\to\infty} \varinjlim_{K\Subset\Omega} \mathcal{D}_K^{\{M_p\},h} \quad \text{and} \quad \mathcal{E}_\Omega^{(M_p)} = \varprojlim_{h\to\infty} \varprojlim_{K\Subset\Omega} \mathcal{E}_K^{\{M_p\},h}$$

Then the dual $\mathcal{D}'^{(M_p)}$ of the space $\mathcal{D}_R^{(M_p)}$ is the space of Beurling ultradistributions of class (M_p) and the dual $\mathcal{E}'^{(M_p)}$ of the space $\mathcal{E}_R^{(M_p)}$ is the space of Beurling ultradistributions of class (M_p) with compact support(we omit to write Ω when $\Omega=\mathbb{R}$).

Further we put

$$\mathcal{D}_0^{(M_p)} = \mathcal{D}_{(0,\infty)}^{(M_p)} \quad \text{and} \quad \mathcal{D}_-^{(M_p)} = \mathcal{E}^{(M_p)} \cap \mathcal{D}_- ,$$

where $\mathcal{D}_-$ is the space of infinitely differentiable functions with the support bounded above.

We define the Fourier-Laplace transform for $\varphi\in\mathcal{D}^{(M_p)}$ by $\tilde{\varphi}(z)=\int_{\mathbb{R}}\varphi(t)e^{zt}dt$ and for $T\in\mathcal{E}'^{(M_p)}$ by $\tilde{T}(z)=T(e^{zt}),z\in\mathbb{C}$.

Then we have the following Paley-Wiener type theorem [5], [15] :

<u>Theorem 1.1.</u> i) <u>Let</u> $\varphi\in\mathcal{D}_K^{(M_p)}$ <u>;then for any</u> $L>0$ <u>there is a constant</u> $C>0$ <u>such that</u>

$$(1.4) \qquad |\tilde{\varphi}(z)| \leq Ce^{-M(L|z|)+H_K(z)}, z\in\mathbb{C},$$

<u>where</u> $H_K(z)=\sup_{t\in K}(t\,\mathrm{Re}\,z)$.

ii) <u>Let</u> $T\in\mathcal{E}'^{(M_p)}$,supp$T\subset[-a,a]$; <u>then there are positive constants</u> L <u>and</u> C <u>such that</u>

$$(1.5) \qquad |\tilde{T}(t)| \leq Ce^{M(L|t|)}, \quad t\in\mathbb{R},$$

<u>and</u> $T(z)$ <u>is of exponential type</u> $\leq a$ <u>on</u> $\mathbb{C}$.

This theorem has also a converse,that is the relations (1.4) and (1.5) completely characterise Fourier-Laplace transforms of elements in $\mathcal{D}^{(M_p)}$ or $\mathcal{E}'^{(M_p)}$ (in the above facts $\{M_p\}$ satisfies (M.1) and (M.3)')

If X is a Banach space,we shall denote by $\mathcal{D}^{(M_p)}(X)$ and $\mathcal{E}^{(M_p)}(X)$ the corresponding X-valued ultradifferentiable function spaces and we call X-valued ultradistributions of class (M_p) and of Beurling type the elements of the space $\mathcal{L}(\mathcal{D}^{(M_p)};X)$, endowed with the topology of the uniform convergence on bounded sets.

Definition 1.2. An operator of the form $P(D)=\sum_{p=0}^{\infty} a_p D^p, a_p \in \mathbb{C}, D={}^d/_{dt}$ whose coefficients satisfy :

there are positive constants l and c such that

$$|a_p| \leq c.l^p/M_p \quad , p=0,1,\ldots,$$

is called an ultradifferential operator of class (M_p).

Every ultradifferential operator is a linear, continuous operator from $\mathcal{D}^{(M_p)}$ in $\mathcal{D}^{(M_p)}$ and from $\mathcal{E}^{(M_p)}$ in $\mathcal{E}^{(M_p)}$, preserving the support, if the sequence $\{M_p\}$ satisfies (M.1), (M.2) and (M.3)'.

Let us denote by

$$\omega(z) = \prod_{p=1}^{\infty}\left(1+\frac{iz}{m_p}\right) ;$$

this entire function of exponential type zero [8] will play an important role in the present work. In fact, if the sequence $\{M_p\}$ satisfies (M.3), by Proposition 4.6 from [8], the operators

$$\omega(lD) = \prod_{p=1}^{\infty}\left(1+\frac{ilD}{m_p}\right) \quad , l\in\mathbb{C}$$

are ultradifferential operators of class (M_p) and their action is given by

$$\widetilde{\omega(lD)\varphi}(z) = \omega(-lz)\,\tilde{\varphi}(z) .$$

Moreover (Proposition 4.5, [8]) there are constants $l_0 \geq 1$ and $c_0>0$ such that

(1.6) $$|\omega(z)| \leq c_0 e^{M(l_0|z|)} \quad , z\in\mathbb{C} .$$

For each $n\in\mathbb{N}$ let us put

$$\omega^n(z) = \sum_{p=0}^{\infty} a_{n,p} z^p \quad ;$$

then by (1.6) and (1.3) we have

$$|\omega^n(z)| \leq c_0 A^{n-1} e^{M(l_0 H^{n-1}|z|)} .$$

Using the Cauchy integral formula to compute the coefficients $a_{n,p}$, we get:

(1.7) $$|a_{n,p}| \leq \text{const.}\,(l_0 H^{n-1})^p/M_p \quad ; p=0,1,\ldots$$

So the operators $\omega^n(lD), l \in \mathbb{C}, n \in \mathbb{N}$, are ultradifferential of class (M_p), as long as condition (M.3) is fullfilled.

Let further X be a Banach space and A a closed densely defined operator; we denote by D(A) the domain of A and endowe it with the graph norm. Then $A \in \mathcal{L}(D(A);X)$. Let us define $D(A^\infty) = \bigcap_{n=0}^{\infty} D(A^n)$ and $\|x\|_n = \sum_{j=0}^{n} \|A^j x\|$, for $x \in D(A^\infty)$. By the closedness of A, the set $D(A^\infty)$ can be considered as a Fréchet space with the topology determined by the norms $\|.\|_n$. Denote this space by Y and the closure of $D(A^\infty)$ in $\|\cdot\|_n$ by Y_n. Then by [16], Proposition 1.2 and 1.3 we have

<u>Proposition 1.3</u>. <u>If</u> $\rho(A) \neq 0$, <u>then</u> :

i) A^n <u>coincides with the closure of</u> $(A|Y)^n$ <u>in</u> X;

ii) $Y_n = D(A^n)$ <u>as topological spaces, if</u> $D(A^n)$ <u>has the graph topology</u>;

iii) $Y = \varprojlim_{n\to\infty} Y_n$.

For the closed and densely defined operator A and $x_0 \in X$ the abstract Cauchy problem (ACP) is to find an X-valued function $u(t), t \geq 0$, such that $u(0) = x_0$, $u(t) \in D(A)$ for $t > 0$, $u(t)$ is strongly differentiable for $t > 0$ and verifies : $u'(t) = Au(t)$.

The study of the (ACP) is closely connected with the theory of continuous semi-groups of operators, the theory of distribution semi-groups and ultradistribution semi-groups.

For semi-groups in a locally convex space Z we use the notations and definitions from [9].

So the semi group $\{U_t\}_{t\geq 0}$ is called equicontinuous on $[0,s] \subset [0,+\infty)$ if for every continuous semi-norm p on Z, there is a continuous semi-norm q such that

$$p(U_t x) \leq q(x) , \quad t \in [0,s] , \; x \in Z .$$

$\{U_t\}_{t\geq 0}$ is locally equicontinuous if it is equicontinuous on each interval $[0,s]$, $s > 0$.

We recall that if Z is barelled, then every continuous semi group $\{U_t\}_{t\geq 0}$ of linear, continuous operators on Z is locally-equicontinuous.

L.J.Lions studied the (ACP) in the space of operator valued distributions defining the notion of distribution semi-group [11].

J.Chazarain in [3] and [4] extended the results of L.J.Lions to ultradistributions.So he introduced the

Definition 1.4. Let A be a closed and densely defined operator in a Banach space X and $\{M_p\}_{p\geq 0}$ a sequence of positive numbers verifying (M.1),(M.2) and (M.3)'; $A\in C^{(M_p)}$ if there is $\mathcal{E}\in\mathcal{L}(\mathcal{D}_-^{(M_p)'};\mathcal{L}(X;D(A)))$, $\operatorname{supp}\mathcal{E}\subset[0,+\infty)$ such that

$$(1.8)\qquad (d/dt - A)*\mathcal{E}=\delta_0\otimes I_X\ ;\ \mathcal{E}*(d/dt - A)=\delta_0\otimes I_{D(A)}\ .$$

$\mathcal{E}$ is called the (M_p)-ultradistribution semi-group generated by A.

Let $l>0$;we say that a region in $\mathbb{C}$ is (M_p)-logarithmic of type l if it has the form

$$\Lambda_l=\{\,z;\ \operatorname{Re}z\geq aM(l|\operatorname{Im}z|)+b\,\}$$

for some $a,b\in R$.

Then we have the following spectral characterisation of generators of (M_p)-ultradistribution semi-groups (when (M.1),(M.2),(M.3)'hold)

Theorem 1.5.[4] $A\in C^{(M_p)}$ if and only if there is a constant $l\geq 1$ and an (M_p)-logarithmic region of type l where $R(z;A)$ exists and satisfies for each $\varepsilon>0$

$$(1.9)\qquad \|R(z;A)\|=O(e^{M(l|z|)+\varepsilon\operatorname{Re}z})\ .$$

The relation between $\mathcal{E}$ and $R(z;A)$ is given by

$$(1.10)\qquad \mathcal{E}(\varphi)=\frac{1}{2\pi i}\int_{\Gamma_l}\tilde{\varphi}(z)R(z;A)dz\quad,\forall\varphi\in\mathcal{D}^{(M_p)}\ ,$$

where Γ_l is the boundary of Λ_l .

If $\mathcal{E}$ is an (M_p)-ultradistribution semi-group and $x\in X$,we denote by $\mathcal{E}x$ the X-valued ultradistribution defined by

$$\mathcal{E}x(\varphi)=\mathcal{E}(\varphi)x\quad,\forall\ \varphi\in\mathcal{D}^{(M_p)}\ ;$$

then $\mathcal{E}x$ verifies :

$$(1.11)\qquad d/dt\,\mathcal{E}x - A\mathcal{E}x=\delta_0\otimes x\ .$$

Let us finally put $\mathcal{E}^*(\varphi)=[\mathcal{E}(\varphi)]^*$, $\varphi\in\mathcal{D}^{(M_p)}$; then if $A\in C^{(M_p)}$,the adjoint operator A^* is the generator of the dual ultradistribution semi-group $\mathcal{E}^*$ (we note that $D(A^*)$ is X-dense in X^*)

§ 2. Regular (M_p)-ultradistribution semi-groups.

Let X be a Banach space, $\{M_p\}_{p\geq 0}$ a sequence of positive numbers satisfying (M.1), (M.2) and (M.3)' and A a closed, densely defined operator in $C^{(M_p)}$; if $\mathcal{E}$ is the (M_p)-ultradistribution generated by A, we put

$$\mathcal{R}_{\mathcal{E}} = \{\sum_{j=0}^{n_j} \mathcal{E}(\varphi_j)x_j \ ; \ \varphi_j \in \mathcal{D}_0^{(M_p)}, \ x_j \in X, n_j \in \mathbb{N}\}$$

$$\mathcal{N}_{\mathcal{E}} = \{ x \in X \ ; \ \mathcal{E}(\varphi)x = 0, \forall \varphi \in \mathcal{D}_0^{(M_p)}\}$$

Contrary to the case of distribution semi-groups, the relations

$$\overline{\mathcal{R}_{\mathcal{E}}} = X \quad \text{and} \quad \mathcal{N}_{\mathcal{E}} = \{0\}$$

aren't evident and therefore we put

Definition 2.1. An (M_p)-ultradistribution semi-group is called regular if $\overline{\mathcal{R}_{\mathcal{E}}} = X$ and $\mathcal{N}_{\mathcal{E}} = \{0\}$.

Let us further define

$$\mathcal{L}_A = \left\{ x \in X; \begin{array}{l} z \longrightarrow R(z;A)x \text{ has an entire extension } z \longrightarrow S_z x \\ \text{of exponential type zero such that for } t \in \mathbb{R} \\ \|S_t x\| = O(e^{M(L|t|)}), \text{for some positive constant } L. \end{array} \right\}$$

Lemma 2.2. Let $A \in C^{(M_p)}$ and $\mathcal{E}$ the (M_p)-ultradistribution semi-group generated by A; then for every $z \in \mathbb{C}$ we have

$$S_z \mathcal{L}_A \subset \mathcal{L}_A \cap D(A) \quad \text{and} \quad \mathcal{L}_A = \mathcal{N}_{\mathcal{E}} .$$

Proof. If $z \in \rho(A)$, then it is claear that $S_z \mathcal{L}_A \subset \mathcal{L}_A$. Let $x \in \mathcal{L}_A$ and $x^* \in X^*$; $x^* = 0$ on $\mathcal{L}_A$; then $z \longrightarrow \langle x^*, S_z x\rangle$ is an entire function which vanishes on $\rho(A)$, so that $\langle x^*, S_z x\rangle \equiv 0$. By a simple consequence of the Hahn-Banach theorem, we obtain the fact that $S_z x \in \mathcal{L}_A$ for each $z \in \mathbb{C}$.

Let further $x^* \in D(A^*)$; for $z \in \rho(A)$, we have

$$\langle A^* x^*, S_z x\rangle = \langle x^*, zS_z x - x\rangle \tag{2.1}$$

and as both functions in this equality are entire, (2.1) holds for every $z \in \mathbb{C}$. But this implies

$$S_z x \in D(A^{**}) = D(A) \quad \text{and} \quad AS_z x = zS_z x - x, \quad z \in \mathbb{C}. \tag{2.2}$$

Thus the first part of the Lemma is proved.

Let now be $x \in \mathcal{N}_{\mathcal{E}}$; then the ultradistribution $\mathcal{E}x$ has the support

$\{0\}$ and by (1.11) it satisfies

$$(z - A)\widetilde{\mathcal{E}x}(z) = x , \quad z \in \mathbb{C} .$$

Hence putting $S_z x = \widetilde{\mathcal{E}} x(z), z \in \mathbb{C}$, using Theorem 1.1. we get $x \in \mathcal{L}_A$.

Further let be $x \in \mathcal{L}_A$; by the converse of Theorem 1.1., there is $T_x \in \mathcal{L}(\mathcal{E}^{(M_p)}; X)$, $\mathrm{supp}T_x = \{0\}$, such that $T_x(z) = S_z x$. By (2.2) we have

$$(z-A)S_z x = (z-A)T_x(z) = x ,$$

so that

$$({}^{d}/_{dt} - A)T_x = x .$$

But $\mathcal{E}x$ is the unique solution of the above equation, so that $\mathcal{E}x = T_x$. This implies $\mathcal{E}(\varphi)z = 0, \forall \varphi \in \mathcal{D}_0^{(M_p)}$, hence $x \in \mathcal{N}_{\mathcal{E}}$.

q.e.d.

Proposition 2.3. An (M_p)-ultradistribution semi-group is regular if and only if $\mathcal{L}_A = \mathcal{L}_{A^*} = \{0\}$.

Proof. As $\mathcal{E}^*$ is an (M_p)-ultradistribution semi-group too, by the above Lemma $\mathcal{L}_{A^*} = \mathcal{N}_{\mathcal{E}^*}$.

The proof is complete if we remark that $\mathcal{N}_{\mathcal{E}^*} = \{0\}$ if and only if $\overline{\mathcal{R}_{\mathcal{E}}} = X$.

q.e.d.

Next we shall give some examples of regular (M_p)-ultradistribution semi-groups.

Proposition 2.4. Let $A \in C^{(M_p)}$ such that for $z \in \rho(A)$, $R(z;A)$ and $R(z;A)^*$ have no nontrivial invariant subspace on which they are quasinilpotent; then A is the generator of a regular (M_p)-ultradistribution semi-group.

Proof. We shall verify that $\mathcal{L}_A = \mathcal{L}_{A^*} = \{0\}$.

Let us suppose $\mathcal{L}_A \neq \{0\}$ and $x \in \mathcal{L}_A$; from the resolvent equation results

$$S_\lambda x - S_z x = (z-\lambda)S_\lambda S_z x \quad , z, \lambda \in \mathbb{C} .$$

Let $z \in \rho(A)$; S_z^{-1} exists and satisfies on $\mathcal{L}_A$

(2.3) $$S_z^{-1}S_\lambda = I + (z-\lambda)S_\lambda \quad , \lambda \in \mathbb{C} .$$

For $\lambda \neq 0$ we put

$$(\lambda - S_z)^{-1} = \lambda^{-1} S_z^{-1} S_{z-\lambda^{-1}} \quad .$$

Then using (2.3) we get

$$(\lambda - S_z)^{-1}(\lambda - S_z) = (\lambda - S_z)(\lambda - S_z)^{-1} = I$$

so that $R(z;A)\big|_{\mathcal{L}_A} = S_z$ is quasinilpotent, which is impossible.

By a similar argument we obtain $\mathcal{L}_{A^*} = \{0\}$.

q.e.d.

Remark 2.5. The condition

" $R(z;A)^*$ has no nontrivial invariant subspaces on which it is quasinilpotent"

was required in [11] for the density of initial conditions of the (ACP) for a class of operators whose resolvents exist at least in a half plane and satisfy certains estimations.

Another example is furnished by

Proposition 2.6. Let A be a closed densely defined operator such that $R(z;A)$ exists in an (M_p)-logarithmic region of type 1, Λ_ℓ and satisfies

(2.4) $$\| R(z;A) \| \leq \text{const.}(1+|z|)^N \quad , \quad \text{for an } N \in \mathbb{N} \; .$$

Then A generates a regular (M_p)-ultradistribution semi-group.

Proof. For $x \in D(A^{N+2})$, we have

(2.5) $$R(z;A)x = \sum_{j=0}^{N+1} \frac{A^j x}{z^{j+1}} + \frac{R(z;A)A^{N+2}x}{z^{N+2}} \quad .$$

Let $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}_0^{(M_p)}$, $\varphi_k \xrightarrow[k\to\infty]{} \delta_0$ in $\mathcal{E}'^{(M_p)}$; then $\tilde{\varphi}_k(z) \xrightarrow[k\to\infty]{} 1$ uniformly on bounded subsets in $\mathbb{C}$.

We can suppose $0 \notin \Lambda_\ell$. By (1.10) and (2.5), we have

(2.6) $$\mathcal{E}(\varphi_k)x = \sum_{j=0}^{N+1} \frac{1}{2\pi i}\int_{\Gamma_\ell} \frac{\tilde{\varphi}_k(z)A^j x}{z^{j+1}}\, dz + \frac{1}{2\pi i}\int_{\Gamma_\ell} \frac{\tilde{\varphi}_k(z)R(z;A)A^{N+2}x}{z^{N+2}}\, dz$$

Using the estimation (1.4) we can easily verify that $\tilde{\varphi}_k$, $k\in\mathbb{N}$, are holomorphic in $\complement\Lambda_1$ and $O(|z|^{-2})$ at ∞; so applying the Cauchy integral formula, we obtain

$$\int_{\Gamma_\ell} \frac{\tilde{\varphi}_k(z)A^j x}{z^{j+1}}\, dz = \left({}^{d}/_{dz}\right)^j \tilde{\varphi}_k(z)A^j x \Big|_{z=0} \quad , \; j=0,1, \ldots, N+1 \; .$$

Hence, when $k \to \infty$ the terms for $1 \leq j \leq N+1$ in (2.6) converge to zero and we get

$$\lim_{k\to\infty} \mathcal{E}(\varphi_k)x = x + \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{R(z;A)A^{N+2}x}{z^{N+2}}\, dz \quad .$$

But by (2.4), the integrand which is holomorphic in Λ_1, is $O(|z|^{-2})$ at ∞, so that the above integral vanishes.

Finally we obtain

$$\lim_{k\to\infty} \mathcal{E}(\varphi_k)x = x$$

and hence $\mathcal{R}_{\mathcal{E}}$ is dense in $D(A^{N+2})$. But $D(A^{N+2})$ is dense in X, so we get $\overline{\mathcal{R}_{\mathcal{E}}} = X$. With the mention that $D(A^*)$ is X-dense in X^*, the operator A^* verifies similar conditions with A, thus $\mathcal{R}_{\mathcal{E}^*}$ is X-dense in X^*; but this implies $\mathcal{N}_{\mathcal{E}} = \{0\}$.

q.e.d.

Remark 2.6. In [2] R.Beals considered operators for which $R(z;A)$ exist in the region

$$\operatorname{Re} z \geqslant \text{const.}\,|\operatorname{Im} z|^{\alpha} \quad \text{, for some } \alpha < 1$$

and satisfies there the estimation 2.2.

As for $M_p = p^{pd}$, $d = \alpha^{-1}$, $M(t) = t^{\alpha}$, it is clear that these operators are generators of regular (p^{pd})-ultradistribution semi-groups.

We next state the following basic property:

Lemma 2.7. *If $\mathcal{E}$ is an (M_p)-ultradistribution semi-group, then*

$$\mathcal{E}(\varphi * \psi) = \mathcal{E}(\varphi)\mathcal{E}(\psi) = \mathcal{E}(\psi)\mathcal{E}(\varphi) \quad , \forall\, \varphi, \psi \in \mathcal{D}_0^{(M_p)} .$$

Proof. Using the fact that $\widetilde{\varphi * \psi} = \tilde{\varphi}\tilde{\psi}$ and (1.10), we need only to verify

$$2\pi i \int_{\Gamma_\varepsilon} \tilde{\varphi}(z)\tilde{\psi}(z)R(z;A)dz = \int_{\Gamma_\varepsilon} \tilde{\varphi}(z)R(z;A)dz . \int_{\Gamma_\varepsilon} \tilde{\psi}(\lambda)R(\lambda;A)d\lambda \quad .$$

This verification is a routine exercise in the operational calculus, but since we shall use the argument several more times, we shall sketch it.

Let Γ_1^1 be the curve $\{z+1;\ z \in \Gamma_1\}$ and Λ_1^1 the region at the right of Γ_1^1; then $\Lambda_1^1 \subset \Lambda_1$.

If $\varphi \in \mathcal{D}_0^{(M_p)}$, one can easily verify that by (1.4) $\tilde{\varphi}$ is holomorphic in $\complement \Lambda_1^1$ and vanishes rapidly at ∞ in this region. Therefore Γ_1 can be replaced by Γ_1^1 in (1.10) and using the resolvent equation we have:

$$\int_{\Gamma_\ell}\tilde{\varphi}(z)R(z;A)dz.\int_{\Gamma_\ell}\tilde{\Psi}(\lambda)R(z;A)d\lambda=\int_{\Gamma_\ell}\tilde{\varphi}(z)R(z;A)dz.\int_{\Gamma_\ell^1}\tilde{\Psi}(\lambda)R(\lambda;A)d\lambda=$$

$$=\int_{\Gamma_\ell}\int_{\Gamma_\ell^1}\tilde{\varphi}(z)\tilde{\Psi}(\lambda).\frac{R(z;A)-R(\lambda;A)}{\lambda-z}\,dzd\lambda=$$

$$=\int_{\Gamma_\ell}\tilde{\varphi}(z)R(z;A)\left[\int_{\Gamma_\ell^1}\frac{\tilde{\Psi}(\lambda)}{\lambda-z}d\lambda\right]dz-\int_{\Gamma_\ell^1}\tilde{\Psi}(\lambda)R(\lambda;A)\left[\int_{\Gamma_\ell}\frac{\tilde{\varphi}(\lambda)}{\lambda-z}d\lambda\right]dz=$$

$$=2\pi i\int_{\Gamma_\ell}\tilde{\varphi}(z)\tilde{\Psi}(z)R(z;A)dz$$

since the first term in branches is the Cauchy integral formula for $\tilde{\Psi}$ and the second term vanishes by the same theorem for $\tilde{\varphi}$.

q.e.d.

By a sequence of regularisations we understand a sequence

$$\{\varphi_k\}_{k\in N}\subset\mathcal{D}_0^{(M_p)},\quad \varphi_k\xrightarrow[k\to\infty]{}\delta_0\ \text{ in } \mathcal{E}'^{(M_p)}.$$

Definition 2.8. Let $\mathcal{E}$ be a regular (M_p)-ultradistribution semi-group and $T\in\mathcal{E}'^{(M_p)}$, $\operatorname{supp}T\subset[0,\infty)$. We define the operator $\mathcal{E}(T)$ by:

$$D(\mathcal{E}(T))=\left\{x\in X;\ \begin{array}{l}\text{there is a sequence of regularisations }\{\varphi_k\}\\ \text{such that } \mathcal{E}(\varphi_k)x\xrightarrow[k\to\infty]{}x \text{ and } \mathcal{E}(T*\varphi_k)x\xrightarrow[k\to\infty]{}y\end{array}\right\}$$

and $\mathcal{E}(T)x=y$.

Using Lemma 2.7. and the fact that $\mathcal{N}_{\mathcal{E}}=\{0\}$ one can easily verify that y in the above definition is independent of the sequence $\{\varphi_k\}$.

In addition $\mathcal{E}(T)$ is densely defined because $\mathcal{R}_{\mathcal{E}}\subset D(\mathcal{E}(T))$ and it is preclosed.

We denote by $\overline{\mathcal{E}(T)}$ the closure of $\mathcal{E}(T)$ and endowe $D(\overline{\mathcal{E}(T)})$ with the graph norm. Then the closure of $\mathcal{R}_{\mathcal{E}}$ in this norm is $D(\overline{\mathcal{E}(T)})$.

Moreover we have the following properties whose verification is quite simple and similar to that for distribution semi-groups:

Proposition 2.9. i) $\overline{\mathcal{E}(T)}\,\mathcal{E}(\varphi)x=\mathcal{E}(T)\,\mathcal{E}(\varphi)x=\mathcal{E}(T*\varphi)x,\ x\in X,\ \varphi\in\mathcal{D}_0^{(M_p)}$.

ii) $\overline{\mathcal{E}(T)|_{\mathcal{R}_{\mathcal{E}}}}=\overline{\mathcal{E}(T)}$.

iii) $\overline{\mathcal{E}(T)}^n\,\mathcal{E}(\varphi)x=\mathcal{E}(\underbrace{T*T*\ldots*T}_{n})x,\quad x\in X,\ \varphi\in\mathcal{D}_0^{(M_p)},\ n\in N.$

iv) $\overline{\mathcal{E}(\delta)}=I;\ \overline{\mathcal{E}(-\delta')}=A$.

§ 3. Abstract Beurling spaces of class (M_p).

Let $\{M_p\}_{p\geq 0}$ be a sequence of positive numbers verifying (M.1), (M.2) and (M.3), $\omega(z) = \prod_{p=1}^{\infty}(1+\frac{iz}{m_p})$ for $m_p = M_p/M_{p-1}$, $p=1,2,\dots$ and let $\mathcal{E}$ a regular (M_p)-ultradistribution semi-group.

For every $n\in\mathbb{N}$, $\omega^n(-iD)\delta_0\in\mathcal{E}'^{(M_p)}$ and so we can consider the operators $\mathcal{E}(\omega^n(-iD)\delta_0)$ which is densely defined and closed. Moreover we have the

Proposition 3.1. *If $\mathcal{E}$ is a regular (M_p)-ultradistribution semi-group, then for n sufficiently large, the operator $\mathcal{E}(\omega^n(-iD)\delta_0)$ is invertible.*

In order to prove this proposition we need the following

Lemma 3.2. *Let $l>0$ and Λ_l an (M_p)-logarithmic region of type l, $m_1\in\Lambda_l$; then there are positive constants C and $\delta\leq 1$ such that*

$$|\omega(iz)|\geq C|\omega(|z|)|^{\delta}, \quad z\in \complement\Lambda_l.$$

Proof. As $\complement\Lambda_l$ is simply connected and $\omega'(iz)/\omega(iz)$ is holomorphic in $\complement\Lambda_l$, the function $f(z) = \ln\omega(iz)$ is well defined in $\complement\Lambda_l$ and $f'(z) = \omega'(iz)/\omega(iz)$. Moreover

(3.1) $$e^{\mathrm{Re} f(z)} = |\omega(iz)| \quad , z\in\complement\Lambda_l.$$

Let us verify that in $\complement\Lambda_l$

(3.2) $$\lim_{|z|\to\infty}|f'(z)| = 0 .$$

Indeed, $f'(z) = \sum_{p=0}^{\infty} 1/z-m_p$ and putting $z=|z|e^{i\theta_z}$, we have

(3.3) $$|z-m_p|^2 = (|z|\cos\theta_z-m_p)^2+|z|^2\sin^2\theta_z\geq m_p^2\sin^2\theta_z .$$

Further we remark that by (1.1), for $z\in\complement\Lambda_l$

(3.4) $$|tg\,\theta_z|\longrightarrow +\infty \quad ,\text{when } |\mathrm{Im} z|\longrightarrow +\infty$$

so that there is a $\theta_0>0$ such that $\theta_z\geq\theta_0$ for $z\in\complement\Lambda_l$ and $|z|$ large enough.

Let $\varepsilon>0$ and $p_0\in\mathbb{N}$ such that $\frac{1}{\sin\theta_0}\sum_{p=p_0}^{\infty}\frac{1}{m_p}<\frac{\varepsilon}{2}$; then by (3.3) and (3.4) we obtain

$$|f'(z)|\leq \sum_{p=0}^{\infty}\frac{1}{|z-m_p|}+\frac{1}{\sin\theta_0}\sum_{p=p_0}^{\infty}\frac{1}{m_p}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

if $z\in\complement\Lambda_1$ and $|z|$ is sufficiently large. Hence (3.2) holds.

As for $\mathrm{Re}z\leq 0$, the inequality of the lemma is satisfied for $C=\delta=1$, let us consider $z\in\complement\Lambda_1$ with $\mathrm{Re}z>0$. We have

$$f(z)=f(i\mathrm{Im}z)+\int_{[i\mathrm{Im}z;z]} f'(\lambda)d\lambda$$

so that

$$|f(i\mathrm{Im}z)-f(z)|\leq \sup_{\lambda\in[i\mathrm{Im}z;z]}|f'(\lambda)|\,\mathrm{Re}z \quad .$$

Hence

$$\mathrm{Re}f(i\mathrm{Im}z)-\mathrm{Re}f(z)\leq \sup_{\lambda\in[i\mathrm{Im}z;z]}|f'(\lambda)|.(aM(1|\mathrm{Im}z|)+b)$$

(a and b are the positive constants in the definition of the region Λ_l)

By 3.2) we can find a constant C' such that $\sup_{\lambda\in[i\mathrm{Im}z;z]}|f'(\lambda)|\leq C'$ and $1-aC'>0$, if $z\in\complement\Lambda_1$ and $|z|$ is large.

Using now (3.1) and the evidents facts: $|\omega(t)|=|\omega(|t|)|$, $e^{M(t)}\leq|\omega(t)|$, $t\in R$, we get

$$\left|\frac{\omega(|\mathrm{Im}z|)}{\omega(iz)}\right|\leq e^{bC'}.\omega(1|z|)^{aC'}$$

But by (3.4) there is a constant $L'>0$ such that for $z\in\complement\Lambda_1$, $|z|$ sufficiently large, holds

$$|\omega(L'|z|)|\leq|\omega(|\mathrm{Im}z|)|$$

so that finally we get

$$|\omega(iz)|\geq e^{-bC'}.|\omega(\gamma|z|)|^{1-aC'}\quad,\quad \gamma=\min(1,L')\ .$$

If $\gamma\geq 1$, it is obvious that

$$|\omega(\gamma|z|)|^{1-aC'}\geq|\omega(|z|)|^{1-aC'}\ ,$$

so that we can take $\delta=1-aC'$.

If $\gamma<1$, then a simple computation shows

$$|\omega(\gamma|z|)|^{1-aC'}\geq|\omega(|z|)|^{\gamma^2(1-aC')}$$

so we can take $\delta=\gamma^2(1-aC')$.

q.e.d.

<u>Remark</u> 3.3. From this lemma, by a routine computation follows that the function $z\longrightarrow 1/\omega(iz)$ is of exponential type zero in $\complement\Lambda_1$. Moreover this function vanishes rapidly at ∞ in $\complement\Lambda_1$.

<u>Proof of Proposition</u> 3.1. As $A \in C^{(M_p)}$, by (1.9) and (1.2), for $z \in \Gamma_1$ and $\varepsilon = 1/2a$, we have

$$\|R(z;A)\| \leq \text{const.}\, e^{3/2\, M(l|z|)} \leq \tag{3.5}$$

$$\leq \text{const.}\, e^{2M(l|z|)}/z^k \leq \text{const.}\, e^{M(Hl|z|)}/z^k \quad , k \in \mathbb{N}.$$

We still remark that translating A by a convenient multiple of the identity, we can suppose $0 < b < m_1$ in the definition of Λ_1.

Let further $\alpha \geq 1$; by (1.6) and the Bernoulli inequality, we get

$$|\omega(\alpha|z|)| \leq \omega(-i\alpha|z|) \leq \omega(-i|z|)^{\alpha} \leq c_0 e^{\alpha M(l_0|z|)} \leq c_0 |\omega(l_0|z|)|^{\alpha}$$

so that

$$|\omega(|z|)|^{\alpha} \geq c_0^{-1} |\omega(\alpha l^{-1}|z|)| \quad , \quad z \in \mathbb{C} . \tag{3.6}$$

For $n \in \mathbb{N}$, we put

$$D_n = \frac{1}{2\pi i}\int_{\Gamma_\ell} \frac{R(z;A)}{\omega^n(iz)}\, dz \; ; \tag{3.7}$$

then for $n > Hl_0 l\,\delta^{-1}$, $D_n \in \mathcal{L}(X)$. Indeed, as H, l and l_0 are ≥ 1, $n\delta \geq 1$ and using Lemma 3.2. and the inequalities (3.5) and (3.6), we can estimate the integrand in (3.7) as follows

$$\frac{\|R(z;A)\|}{|\omega^n(iz)|} \leq \text{const.} \frac{e^{M(Hl|z|)}}{|z|^2 |\omega^{n\delta}(|z|)|} \leq \text{const.} \frac{\omega(Hl\; z\;)}{|z|^2 |\omega(n\delta l_0^{-1}|z|)|} \leq \frac{\text{const.}}{|z|^2} \tag{3.8}$$

Hence if $n > Hl_c l\,\delta^{-1}$, the integral (3.7) converges and defines a bounded operator on X.

Let $\varphi \in \mathcal{D}_0^{(M_p)}$; we have

$$\mathcal{E}(\varphi) = \frac{1}{2\pi i}\int_{\Gamma_\ell} \tilde{\varphi}(z) R(z;A)\,dz = \frac{1}{2\pi i}\int_{\Gamma_\ell} \tilde{\psi}(z) . \frac{R(z;A)}{\omega^n(iz)}\, dz \quad ,$$

where $\psi = \omega^n(-iD)\varphi \in \mathcal{D}_0^{(M_p)}$.

As the function $z \longrightarrow 1/\omega^n(iz)$ is holomorphic and vanishes rapidly at ∞ in $\complement\Lambda_1$ (see the Remark 3.3.), using a similar argument as in Lemma 2.7., we can prove that

$$\int_{\Gamma_\ell} \tilde{\psi}(z) . \frac{R(z;A)}{\omega^n(iz)}\, dz = \int_{\Gamma_\ell} \frac{R(z;A)}{\omega^n(iz)}\, dz \, . \int_{\Gamma_\ell} \tilde{\psi}(\lambda) . R(\lambda;A)\, d\lambda =$$

$$= \int_{\Gamma_\ell} \psi(\lambda) . R(\lambda;A)\, d\lambda \, . \int_{\Gamma_\ell} \frac{R(z;A)}{\omega^n(iz)}\, dz \quad .$$

Thus we get

$$(3.9)\qquad \mathcal{E}(\varphi) = D_n \mathcal{E}(\omega^n(-iD)\varphi) = \mathcal{E}(\omega^n(-iD)\varphi)D_n\ ,\ \varphi\in\mathcal{D}_0^{(M_p)},$$

and this implies

$$\mathcal{R}_{\mathcal{E}} \subset \mathrm{Im} D_n$$

where by Im we denote the range of an operator.

Analogously it follows that for the operators D_n^*, $\mathcal{R}_{\mathcal{E}^*} \subset \mathrm{Im} D_n^*$.

Hence D_n is injective and his range is dense in X, that is D_n^{-1} exists and is a closed and densely defined operator.

We shall prove that $D_n^{-1} = \overline{\mathcal{E}(\omega^n(-iD)\delta_0)}$, if $n > Hl_0 l\delta^{-1}$.

Indeed, the first equality of (3.9) gives

$$(3.10)\qquad D_n \overline{\mathcal{E}(\omega^n(-iD)\delta_0)}x = x,\qquad \forall\ x\in D(\overline{\mathcal{E}(\omega^n(-iD)\delta_0)}) .$$

By the second equality of (3.9), we have

$$\mathcal{E}(\omega^n(-iD)\delta_0)\mathcal{E}(\varphi)D_n = \mathcal{E}(\varphi),\quad \forall\ \varphi\in\mathcal{D}_0^{(M_p)},$$

and as we can prove by a routine exercise that $\mathcal{E}(\varphi)$ and D_n commute, we get for $\varphi\in\mathcal{D}_0^{(M_p)}$

$$D_n\mathcal{E}(\varphi)X \subset D(\overline{\mathcal{E}(\omega^n(-iD)\delta_0)})\quad\text{and}\quad \overline{\mathcal{E}(\omega^n(-iD)\delta_0)}D_n\mathcal{E}(\varphi) = \mathcal{E}(\varphi)$$

But $\overline{\mathcal{R}_{\mathcal{E}}} = X$ and D_n is continuous so that finally we obtain

$$(3.11)\qquad \overline{\mathcal{E}(\omega^n(-iD)\delta_0)D_n} = I$$

(3.10) and (3.11) imply

$$(3.12)\qquad \overline{\mathcal{E}(\omega^n(-iD)\delta_0)}^{-1} = D_n\qquad ,n > Hl_0 l\delta^{-1} .$$

q.e.d.

For $n_0 = [Hl_0 l\delta^{-1}]+1$ we denote by $B = \overline{\mathcal{E}(\omega^{n_0}(-iD)\delta)}$.

<u>Corollary</u> 3.4. <u>For every</u> $n\in N$, $B^n = \overline{\mathcal{E}(\omega^{n_0 n}(-iD)\delta)}$ <u>and</u>

$$B^{-n} = \frac{1}{2\pi i}\int_{\Gamma_\ell} \frac{R(z;A)}{\omega^{n_0 n}(iz)}\,dz = \left[\frac{1}{2\pi i}\int_{\Gamma_\ell}\frac{R(z;A)}{\omega^{n_0}(iz)}\,dz\right]^n .$$

<u>Proof</u>. By the above proposition, we have

$$D_{n_0}^{-1} = \overline{\mathcal{E}(\omega^{n_0}(-iD)\delta_0)}\quad\text{and}\quad D_{n_0 n}^{-1} = \overline{\mathcal{E}(\omega^{n_0 n}(-iD)\delta_0)} .$$

But we can prove using similar arguments as in the proof of Lemma 2.7. that $D_{n_0 n} = D_{n_0}^n$, hence $D_{n_0}^{-n} = D_{n_0 n}^{-1}$ and this implies the statement.

q.e.d.

Remark 3.5. If $A \in C^{(M_p)}$ and $R(z;A)$ is majorized by a polynomial, then $\int_{\Gamma_\ell} \frac{R(z;A)}{\omega(iz)}\,dz$ is convergent, so that in this case $n_0 = 1$ and $B = \mathcal{E}(\omega(-iD)\delta)$.

For each $n \in N$ we endowe the subspace $D(B^n)$ with the graph topology and we put

$$Y = \bigcap_{n=1}^{\infty} D(B^n)$$

Then Y is a Fréchet space with the topology determined by the system of norms $\left\{ \|x\|_n = \sum_{j=0}^{n} \|B^j x\| \right\}_{n \in N}$.

Denote by Y_n the closure of Y in the norm $\|x\|_n$; then by Proposition 1.3., we have

i) $Y_{n+1} \subset Y_n$ and $Y = \varprojlim_{n \to \infty} Y_n$;

ii) $\overline{B^n|_Y} = B^n$;

iii) $\overline{\mathcal{R}_{\mathcal{E}}}^Y = Y$ and $\overline{Y_n}^Y = D(B^n)$.

(where the notation $\overline{\mathcal{R}_{\mathcal{E}}}^Y$ means the closure of $\mathcal{R}_{\mathcal{E}}$ in the topology of Y)

The Fréchet space Y will play an important role in the present theory and therefore we are further looking for an intrinsec characterisation for it.

Let A be a closed and densely defined operator in X and $\{M_p\}_{p \geq 0}$ a sequence of positive numbers verifying (M.1), (M.2) and (M.3); for $h > 0$ we denote by

$$X_h^{\{M_p\}} = \left\{ x \in D(A^\infty) \; ; \; \|x\|_h^{\{M_p\}} = \sup_{p \geq 0} \frac{h^p \|A^p x\|}{M_p} < +\infty \right\}$$

$X_h^{\{M_p\}}$ is complete in the norm $\|\cdot\|_h^{\{M_p\}}$ and if $0 < h < h' < +\infty$, then

$$X_{h'}^{\{M_p\}} \subset X_h^{\{M_p\}}.$$

Definition 3.6. The space

$$X^{(M_p)} = \varprojlim_{h \to \infty} X_h^{M_p}$$

endowed with the corresponding projective topology is called the abstract Beurling space of class (M_p) associated to A.

It is clear that if X is the space of continuous functions on an interval $K \subset R$ and $A = d/dx$, then $X^{(M_p)} = \mathcal{D}_K^{(M_p)}$.

Proposition 3.7. The restriction $A|X^{(M_p)}$ is continuous from $X^{(M_p)}$ in $X^{(M_p)}$.

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Proof. We shall prove first that for h'/h sufficiently small, holds

$$AX_h^{\{M_p\}} \subset X_h^{\{M_p\}} .$$

Indeed, (M.2) implies $M_p/M_{p-1} \leq AM_1 H^p$ and thus for $x \in X_h^{\{M_p\}}$, we have

$$\frac{h'^{\,p-1} \| A^{p-1}(Ax) \|}{M_{p-1}} = \frac{1}{h'} \cdot \frac{h^p \| A^p x \|}{M_p} \cdot \left(\frac{h'}{h}\right)^p \cdot \frac{M_p}{M_{p-1}} \leq \frac{AM_1}{h'} \cdot \left(\frac{h'}{h} H\right)^p \| x \|_h^{\{M_p\}}$$

So that if $h'H/h < 1$, the above inequality implies

$$\| Ax \|_{h'}^{\{M_p\}} \leq \text{const.} \| x \|_h^{\{M_p\}} \tag{3.13}$$

so that $Ax \in X_{h'}^{\{M_p\}}$.

By the definition of the topology on $X^{(M_p)}$ we get from (3.13) the cotinuity of A restricted to $X^{(M_p)}$.

q.e.d.

For X and A arbitrary it is not clear that $X^{(M_p)} \neq \{0\}$ but if A is the generator of a regular (M_p)-ultradistribution semi-group then we shall see that the associated abstract Beurling space of class (M_p) is even dense in X.

Theorem 3.8. If A is the generator of a regular (M_p)-ultradistribution semi-group, then

$$X^{(M_p)} = \varprojlim_{n \to \infty} D(B^n) .$$

Proof. The assertion results if we prove

$$D(B^n) \subset X_h^{\{M_p\}} \qquad \text{for } n \geq h/Hl + 2 \tag{3.14}$$

and

$$X_h^{\{M_p\}} \subset D(B^n) \qquad \text{for } h \geq 2l_0 H^{n_0 n} \tag{3.15}$$

and the inclusion mappings are continuous.

Let $n \geq h/Hl + 2$ and $y \in D(B^n)$; then $y = B^{-n}x$ for some $x \in X$.

From $AR(z;A) = zR(z;A) - I$ and using Corollary 3.4. we get :

$$AB^{-n} = \frac{1}{2\pi i}\int_{\Gamma_e} \frac{zR(z;A)}{\omega^{n_0 n}(iz)}\,dz - \frac{1}{2\pi i}\int_{\Gamma_e} \frac{dz}{\omega^{n_0 n}(iz)} = \frac{1}{2\pi i}\int_{\Gamma_e} \frac{zR(z;A)}{\omega^{n_0 n}(iz)}\,dz .$$

Inductively we have

$$A^p B^{-n} = \frac{1}{2\pi i}\int_{\Gamma_e} \frac{z^p R(z;A)}{\omega^{n_0 n}(iz)}\,dz , \quad p \in \mathbb{N}.$$

Thus

$$\frac{h^p\|A^py\|}{M_p} = \frac{h^p\|A^pB^{-n}x\|}{M_p} \le \frac{1}{2\pi}\int_{\Gamma_\ell} \frac{h^p|z|^p}{M_p} \cdot \frac{\|R(z;A)\|}{|\omega^{n_0n}(iz)|} \cdot \|x\|dz \quad .$$

Let us estimate the integrand in the above integral using Lemma 3.2. (3.6) and (3.8) :

$$\frac{h^p|z|^p\|R(z;A)\|}{M_p|\omega^{n_0n}(iz)|} \le \frac{e^{M(h|z|)}}{|\omega^{(n-1)n_0}(iz)|} \cdot \frac{\|R(z;A)\|}{|\omega^{n_0}(iz)|} \le$$

$$\le \frac{\text{const.}}{|z|^2} \frac{|\omega(h|z|)|}{|\omega((n-1)n_0l_0^{-1}\delta|z|)|} \le \frac{\text{const.}}{|z|^2}$$

since $h/(n-1)n_0l_0^{-1}\delta = h/(n-1)([Hl_0l\delta^{-1}]+1)l_0^{-1}\delta \le h/(n-1)Hl \le 1$

Hence $y \in X_h^{\{M_p\}}$ and moreover

$$\|y\|_h^{\{M_p\}} \le \text{const.}\|x\| = \text{const.}\|B^nB^{-n}x\| \le \text{const.}\|y\|_n$$

and so (3.14) is proved.

Let further $h \ge 2l_0H^{n_0n}$ and $y \in X_h^{\{M_p\}}$; we are looking for an $x \in X$ such that $y = B^{-n}x$.

Let $\omega^{n_0n}(iz) = \sum_{p=0}^{\infty} a_{n,p}z^p$; then , by (1.7) we get

$$|a_{n,p}| \le \text{const.} \frac{(l_0H^{n_0n})^p}{M_p} .$$

so that

$$|a_{n,p}|\,\|A^py\| \le \text{const.} \frac{h^p\|A^py\|}{M_p}\left(\frac{l_0H^{n_0n}}{h}\right)^p \le \text{const.}\|y\|_h^{\{M_p\}}.\left(\frac{1}{2}\right)^p$$

Putting $x = \sum_{p=0}^{\infty} a_{n,p}A^py$, x is well defined and verifies

(3.16) $$\|x\| \le \text{const.}\|y\|_h^{\{M_p\}} \quad .$$

Finally

$$B^{-2n}x = \frac{1}{2\pi i}\sum_{p=0}^{\infty} a_{n,p}\int_{\Gamma_\ell} \frac{R(z;A)A^py}{\omega^{2n_0n}(iz)}dz = \frac{1}{2\pi i}\sum_{p=0}^{\infty} a_{n,p}\int_{\Gamma_\ell}\frac{z^pR(z;A)y}{\omega^{2n_0n}(iz)}dz =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_\ell}\frac{\omega^{n_0n}(iz)R(z;A)}{\omega^{2n_0n}(iz)}dz = \frac{1}{2\pi i}\int_{\Gamma_\ell}\frac{R(z;A)y}{\omega^{n_0n}(iz)}dz = B^{-n}y$$

so that $y = B^nB^{-2n}x = B^{-n}x$.

Now from (3.16) we get

$$\|y\|_n = \|B^nB^{-n}x\| = \|x\| \le \text{const.}\|y\|_h^{\{M_p\}} .$$

q.e.d.

As $\overline{\mathcal{R}_\mathcal{E}}^Y = Y$, it is clear that

<u>Corollary</u> 3.9 $\mathcal{R}_\mathcal{E} \subset X^{(M_p)}$, $\overline{\mathcal{R}_\mathcal{E}}^X{}^{(M_p)} = X^{(M_p)}$ and $\overline{X^{(M_p)}} = X$.

Ultradistribution semi-groups and the Abstract Cauchy Problem.

In this paragraph we establish our main result, namely the connection between ultradistribution semi-groups and semi-groups of operators on a Fréchet space.

Theorem 4.1. Let A be the generator of a regular (M_p)-ultradistribution semi-group $\mathcal{E}$; then the restriction of A to the abstract Beurling space $X^{(M_p)}$ generates a locally equi-continuous semi-group $\{U_t\}_{t\geq 0}$ on $X^{(M_p)}$; moreover the function $t \longrightarrow U_t x$ is in $\mathcal{E}_{(0,+\infty)}^{(M_p)}(X)$ for every $x \in X$.

Proof. We define $\{U_t\}_{t\geq 0}$ on $\mathcal{R}_{\mathcal{E}} \subset X^{(M_p)}$ by

$$U_t \mathcal{E}(\varphi)x = \mathcal{E}(\tau_t \varphi)x, \quad x \in X, \varphi \in \mathcal{D}_0^{(M_p)}, t \geq 0,$$

where $(\tau_t \varphi)(s) = \varphi(s-t)$.

Next we ramark that for $t \geq 0$ and $n \geq H^{at-1}+1$

$$E_t^{n_o n} = \frac{1}{2\pi i}\int_{\Gamma_e} \frac{e^{tz} R(z;A)}{\omega^{n_o n}(iz)} dz \tag{4.1}$$

is a bounded operator on X.

Indeed, using Lemma 3.2., (3.6) and (3.8) as in the proof of Proposition 3.1., we have

$$\frac{e^{tRez}\|R(z;A)\|}{|\omega^{n_o n}(iz)|} \leq \text{const.} \frac{e^{atM(l|z|)+bt}}{|\omega^{(n-1)n_o}(iz)|} \cdot \frac{\|R(z;A)\|}{|\omega^{n_o}(iz)|}$$

$$\leq e^{bt} \cdot \frac{\text{const.}}{|z|^2} \frac{|\omega(H^{at} l|z|)|}{|\omega((n-1)n_o l_o^{-1} \delta|z|)|} \leq \frac{\text{const.}}{|z|^2} e^{bt}$$

since $H^{at} l / (n-1) n_o l_o^{-1} \delta = H^{at} l / (n-1)([Hl_o l \delta^{-1}]+1) l_o^{-1} \delta \leq H^{at-1} / n-1 \leq 1$

Let now $m \in \mathbb{N}, x \in X$ and $\varphi \in \mathcal{D}_0^{(M_p)}$; we have

$$\| U_t \mathcal{E}(\varphi)x\|_m = \| B^m \mathcal{E}(\tau_t \varphi)x\| = \| \mathcal{E}(\omega^{mn_o}(-iD)\tau_t \varphi)x\| .$$

But for $n \geq H^{at-1}+1$

$$\mathcal{E}(\omega^{mn_o}(-iD)\tau_t\varphi)x = \frac{1}{2\pi i}\int_{\Gamma_e} e^{tz}\omega^{mn_o}(iz)\tilde{\varphi}(z)R(z;A)dz =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_e} e^{tz}\omega^{(m+n)n_o}(iz)\tilde{\varphi}(z) \cdot \frac{R(z;A)x}{\omega^{n_o n}(iz)} dz =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_e} \frac{e^{tz}R(z;A)}{\omega^{n}{}_{o}{}^{n}(iz)}\,dz.\frac{1}{2\pi i}\int_{\Gamma_e}\omega^{(n+m)n_o}(iz)\tilde{\varphi}(z)R(z;A)x\,dz =$$

$$= E_t^{n_o n}\,\mathcal{E}(\omega^{(n+m)n_o}(-iD)\varphi)x = E_t^{n_o n}B^{n+m}\,\mathcal{E}(\varphi)x$$

so that

(4.2) $$\|U_t\mathcal{E}(\varphi)x\|_m = \|E_t^{n_o n}B^{m+n}\mathcal{E}(\varphi)x\| \leq$$

$$\leq \|E_t^{n_o n}\|\cdot\|B^{n+m}\mathcal{E}(\varphi)x\| = \|E_t^{n_o n}\|\,\|\mathcal{E}(\varphi)x\|_{n+m}\ .$$

Hence U_t is continuous on $\mathcal{R}_{\mathcal{E}}$ in the topology induced by $X^{(M_p)}$ and by the Corollary 3.9., U_t can be extended continuously to all $X^{(M_p)}$.

The semi-group property of the family of operators $\{U_t\}_{t\geq 0}$ results from the definition and one can easily verify that the generator of $\{U_t\}_{t\geq 0}$ is the restriction of A to $X^{(M_p)}$.

Moreover, as $\lim\limits_{t\to 0} E_t^{n_o n} = B^{-n}$, using (4.2), for $x\in X$, $\varphi\in\mathcal{D}_o^{(M_p)}$ and $m\in N$ we get

$$\lim_{t\to 0}\|U_t\mathcal{E}(\varphi)x - \mathcal{E}(\varphi)x\|_m = \lim_{t\to 0}\|E_t^{n_o n}B^{n+m}\mathcal{E}(\varphi)x - B^m\mathcal{E}(\varphi)x\| = 0$$

and this implies the continuity in 0 of the function $t\to U_t x$, $x\in X^{(M_p)}$.

The locally equicontinuity of the semi-group $\{U_t\}_{t\geq 0}$ results from the general result of Komura mentioned in the first section but it also results directly from (4.2) if we remark that for $s>0$ and $n\geq H^{as-1}+1$ there is a constant C(independent of $t\leq s$) such that $\|E_t^{n_o n}\|\leq C$, $\forall t\in[0,s]$.

By Theorem 3.8. the topologies defined by the norms families $\{\|\cdot\|_n\}_{n\in N}$ and $\{\|\cdot\|_h^{\{M_p\}}\}_{h>0}$ are equivalent and therefore the locally equicontinuity of $\{U_t\}_{t\geq 0}$ implies that for every $h>0, x\in X^{(M_p)}$ and $K\subset(0,+\infty)$ we have

$$\sup_{\substack{t\in K\\ p\geq 0}}\frac{h^p\|U_t^{(p)}x\|}{M_p} = \sup_{\substack{t\in K\\ p\geq 0}}\frac{h^p\|A^pU_tx\|}{M_p} \leq \sup_{t\in K}\|U_tx\|_h^{\{M_p\}} < +\infty$$

so that the function $t\to U_t x$ is in $\mathcal{E}^{(M_p)}_{(0,+\infty)}$, $\forall x\in X^{(M_p)}$.

q.e.d.

We have the following natural consequence for differential equations:

Corollary 4.2. *Let A be the generator of a regular ultradistribution* semi-group, $X^{(M_p)}$ the associated abstract Beurling space, $x_0 \in X^{(M_p)}$, $f:[0,+\infty) \to X^{(M_p)}$ a continuous function; then there is a unique continuous differentiable function $u:[0,+\infty) \to X^{(M_p)}$ such that

$$u(0) = x \quad , \; u'(t) = Au(t) + f(t)$$

which is given by

$$u(t) = U_t x_0 + \int_0^t U_{t-s} f(s)ds$$

where $\{U_t\}_{t\geq 0}$ is the semi-group in $X^{(M_p)}$ generated by the restriction of A to $X^{(M_p)}$.

The proof beeing standard we omit it (se for details [2] or [7]).

Corollary 4.3. Let A be a closed and densely defined operator in X such that $R(z;A)$ exists for

$$\operatorname{Re} z \geq c|\operatorname{Im} z|^{\alpha} \quad , \qquad 0 < \alpha < 1$$

and satisfies the estimate

$$\| R(z;A) \| \leq \text{const.}(1+|z|)^N \quad \text{for some } N \in \mathbb{N}.$$

Let $d = \alpha^{-1}$ and $X^{(p^{pd})}$ the associated abstract-(p^{pd}) space; then A restricted to $X^{(p^{pd})}$ generates a locally-equicontinuous semi-group in $X^{(p^{pd})}$.

This last result appears in a slight different form in [2] , where the associated abstract space is the inductive limit, when $h \to 0$, of the spaces $X_h^{\{p^{pd}\}}$. So our results are in a certain sense a generalisation of [2].

Remark 4.4. In [1] R.Beals studied the abstract Cauchy problem for the following class of operators:

Let Ψ be the space of all continuous, nonnegative and concave functions ψ on $[0,+\infty)$, such that

$$\lim_{t\to+\infty} \psi(t) = +\infty \; ; \quad \lim_{t\to\infty} \frac{\psi(t)}{t} = 0 \quad ; \quad \int_1^{\infty} \frac{\psi(t)}{t^2}\, dt = \infty$$

Let A be a closed and densly defined operator in X such that

$$(4.3) \qquad R(z;A) \text{ exists for } \operatorname{Re} z \geq \psi(|\operatorname{Im} z|), \text{ for some } \psi \in \Psi \text{ and}$$
$$\| R(z;A) \| \leq \text{const.}(1+|z|)^N \quad , \quad N \in \mathbb{N} .$$

Then R.Beals established that for A the (ACP) has a solution for every x in a dense subspace of X, but he hasn't obtain an intrinsec characterisation of this subspace.

Let us remark that if $\psi \in \Psi$ then ψ is increasing. Indeed, if $t \leq u \leq s$, we have

$$(4.4) \qquad (s-u)\psi(t) + (u-t)\psi(s) \leq (s-t)\psi(u)$$

and dividing (4.4) by s and letting $s \to \infty$, we get $\psi(t) \leq \psi(u)$.

Further we recall the following result which is an immediate consequence of Roumieu's Lemma 2, Cap. II, §1 [14] (see also [13]) and of Körner's results from [10], §6, Kap. II :

<u>Lemma</u> 4.5. <u>Let</u> $f:[0,+\infty) \to (0,+\infty)$ <u>an increasing function such that</u>

$$\int_1^{+\infty} \frac{\ln f(t)}{t^2}\, dt < +\infty.$$

<u>There exist a sequence of positive numbers</u> $\{M_p\}_{p\geq 0}$ <u>satisfying</u> (M.1), (M.2) <u>and</u> (M.3)' <u>and a positive constant</u> $h>0$ <u>such that</u>

$$(4.5) \qquad f(t) \leq \text{const.}\, e^{M(ht)} \quad , \quad t \geq 0 .$$

Thus if A satisfies the conditions (4.3), by the above Lemma R(z;A) exists in an (M_p)-logarithmic region and is majorised there by a polynomial and then Proposition 2.6. implies that A generates a regular (M_p)-ultradistribution semi-group.

If in addition for the function ψ the sequence $\{M_p\}_{p\geq 0}$ given by Lemma 4.5. satisfies the stronger condition (M.3), then we are in the conditions of Theorem 4.1. and the space on which the (ACP) for A has a unique solution is $X^{(M_p)}$.

Condition (M.3) on the sequence $\{M_p\}_{p\geq 0}$ is necessary as far as we work in the space of Beurling ultradistributions of class (M_p). In an other work with L.Zsido we shall study the (ACP) in spaces of ω-ultradistributions ; this frame will permit us to remouve the condition (M.3) and also to enlarge the class Ψ of R.Beals.

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