HILBERT - SAMUEL POLYNOMIALS OF A PROPER MORPHISM

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Introduction

Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces or algebraic schemes and F a coherent sheaf on X flat over Y, or let $f : X \longrightarrow Y$ be a differentiable family of compact complex manifolds and F a sheaf on X, locally free of finite rank. For any point $y \in Y$ we can consider the fiber $X_y = (f^{-1}(y), \mathcal{O}_X/\hat{m}_y\mathcal{O}_X | f^{-1}(y))$ and the sheaf $F_y = F/\hat{m}_y F$, which is (in fact its restriction) a coherent sheaf on X_y. For any positive integer n one can also consider the n-infinitesimal fiber $X_y^{(n)} = (f^{-1}(y), \mathcal{O}_X/\hat{m}_y^{n+1}\mathcal{O}_X | f^{-1}(y))$ and, correspondingly, the sheaf $F_y^{(n)} = F/\hat{m}_y^{n+1}F$.

The aim of this work is the study of the cohomology $H^q(X_y^{(n)}, F_y^{(n)})$ $(= H^q(X, F/\hat{m}_y^{n+1}F))$ when $n \longrightarrow \infty$, in connection with the direct image sheaves $R^q f_*(F)$. Precisely, we study the functions $n \longrightarrow \sum (-1)^q \dim H^q(X_y^{(n)}, F_y^{(n)})$, $n \longrightarrow \dim H^q(X_y^{(n)}, F_y^{(n)})$ when $n \longrightarrow \infty$ (in accordance with the Hilbert function associated to a module we will make a shift of the argument n to $n+1$ and in the algebraic case, of course, we will put length instead of dim).

The results are of three types. First of all there are existence statements: one finds polynomials equal to these functions for large n (for the first function F need not be flat), secondly semicontinuity statements with respect to y and thirdly continuity statements (i.e. necessary and sufficient conditions such that the polynomials do not depend on y).

Following the results of [10], [11], [15], from some conditions about the function $y \longrightarrow \dim H^q(X_y, F_y)$ (i.e. "initial" condition $n = 0$) one gets informations concerning $R^q f_*(F)$ and the whole system $H^q(X_y^{(n)}, F_y^{(n)})$.

In our work, on the contrary, from some conditions about the polynomial associated to the function $n \longrightarrow \dim H^q(X_y^{(n)}, F_y^{(n)})$ (these stand for "asimptotic" condition $n \longrightarrow \infty$) we get informations concerning $H^q(X_y, F_y)$ (and also concerning $R^q f_*(F)$ and about the whole system $H^q(X_y^{(n)}, F_y^{(n)})$).

The paper ends with an analogue of the comparison theorem [10], [11] for a differentiable family of compact complex manifolds [15] .

Some of results of this work were announced in a Note in C.R. Acad. Sc. Paris, t.282 (26 janvier 1976).

I. The analytic case and the algebraic case

1. If A is a local noetherian ring, m its maximal ideal and M an A-module of finite type, then we denote by $P(M) = P_A(M)$ the associated Hilbert-Samuel polynomial, i.e. the polynomial associated to the Hilbert function $n \longrightarrow H_A(M)(n) = \text{length}_A(M/m^nM)$. When A is an analytic (or a formal) algebra over the complex field $\mathbb{C}$ and M is an A-module of finite length, then $\text{length}_A M = \dim_{\mathbb{C}} M$ (where M is considered as a vector space over $\mathbb{C}$ by means of the morphism $\mathbb{C} \longrightarrow A$) and in this case we prefer to write $\dim_{\mathbb{C}}$ (or $\dim$) instead of length_A.

Lemma 1. Let Y be a complex space and $\mathcal{O}$ its structural sheaf. Then:

(i) For every coherent $\mathcal{O}$-module G, the Hilbert function $n \longrightarrow \dim(G_y/m_y^nG_y)$ is upper semicontinuous with respect to y, i.e. for every $y \in Y$ there exists a neighbourhood V such that $\dim(G_{y'}/m_{y'}^nG_{y'}) \leq \dim(G_y/m_y^nG_y)$ for any n and any $y' \in V$.

(ii) If Y is reduced and the Hilbert-Samuel polynomial $P(\mathcal{O}_y)$ does not depend on y, then Y is nonsingular.

(iii) If Y is a connected complex manifold and G is a coherent sheaf on it, then G is locally free iff the Hilbert-Samuel polynomial $P(G_y)$ does not depend on y.

<u>Proof</u>. (i) The problem is local on Y. By a suitable embeding we can assume that Y is a connected manifold. We will consider Nagata $\mathcal{O}$-algebra $\mathcal{O}' = \mathcal{O} \oplus G$. The fibers $\mathcal{O}'_y$ are local rings with maximal ideals $m'_y = m_y \oplus G_y$. For all integers $n \geq 1$,

$$\mathcal{O}'_y/m'^n_y \simeq (\mathcal{O}_y \oplus G_y)/(m^n_y \oplus m^{n-1}_y G_y) \simeq (\mathcal{O}_y/m^n_y) \oplus (G_y/m^{n-1}_y G_y).$$

Since $P(\mathcal{O}_y)$ does not depend on y and $(Y,\mathcal{O}')$ is a complex space [6], then, in order to prove (i), we can suppose $G = \mathcal{O}$. In this case the statement is proved in [16] (ch.I, §4): by using the algebra of principal parts of order $n-1$ of Y one shows the semicontinuity for a fixed integer n and afterwards one uses the property of the family of Hilbert functions $(n \longrightarrow \dim(\mathcal{O}_y/m^n_y))_y$ of being locally finite on Y.

To prove (ii) and (iii) we will use the following statements:

- "Let A be a local noetherian ring. If the Hilbert-Samuel polynomial of A coincides with the Hilbert-Samuel polynomial of a regular local ring, then A is regular".

- "Let A be a regular local ring and M an A-module of finite type. If the Hilbert-Samuel polynomial of M is an entire multiple of the Hilbert-Samuel polynomial of A, then M is free."

One proves these statements by induction on Krull dimension and using superficial elements in the sense of Samuel and Nagata [17] ; see for details [2].

(ii) It suffices to note that there exist points $y \in Y$ such that $\mathcal{O}_y$ is a regular ring and to apply the first statement.

(iii) It suffices to note that there exist points $y \in Y$ such that G_y is a free $\mathcal{O}_y$-module and to apply the second statement.

2. Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces, F an analytic coherent sheaf on X and $y \in Y$. We denote by m_y both the maximal ideal of $\mathcal{O}_{Y,y}$ and the ideal-sheaf given by this on Y; we denote by $\hat{m}_y$ the ideal-sheaf $f^*(m_y)\mathcal{O}_X$. Consider the analytic fiber $X_y = (f^{-1}(y), \mathcal{O}_X/\hat{m}_y\mathcal{O}_X|f^{-1}(y))$ and the sheaf $F_y = F/\hat{m}_yF$, which is (in fact its restriction) an analytic coherent sheaf on X_y. Consider also the infinitesimal fibers $X_y^{(n)} = (f^{-1}(y), \mathcal{O}_X/\hat{m}_y^{n+1}\mathcal{O}_X|f^{-1}(y))$ and correspondingly the sheaves $F_y^{(n)} = F/\hat{m}_y^{n+1}F$ $(X_y^{(o)} = X_y$ and $F_y^{(o)} = F_y)$.

By Grauert's coherence theorem [10], for every $q \geqslant 0$ and $n \geqslant 0$, the $\mathcal{O}_y$-module

$$R^q f_*(F/\hat{m}_y^{n+1}F)_y \simeq H^q(X, F/\hat{m}_y^{n+1}F) \simeq H^q(X_y^{(n)}, F_y^{(n)})$$

is of finite type and since it is anihilated by m_y^{n+1}, $\dim_{\mathbb{C}} H^q(X_y^{(n)}, F_y^{(n)}) < \infty$.

We shall write shortly $n \gg 0$ instead of "n is large sufficiently".

<u>Theorem 1</u>. Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces, F an analytic coherent sheaf on X and $y \in Y$.

(i) The function of Hilbert type

$$n \longrightarrow H(F,f,y)(n) = \sum_q (-1)^q \dim H^q(X_y^{(n-1)}, F_y^{(n-1)})$$

is polynomial, i.e. there exists a polynomial $P(F,f,y)$ such that

$$P(F,f,y)(n) = H(F,f,y)(n) \quad (= \sum_q (-1)^q \dim H^q(X, F/\hat{m}_y^n F))$$

for $n \gg 0$; moreover $\deg P(F,f,y) \leqslant \dim_y Y$ (existence statement).

(ii) Assume F flat over Y. Then:

(a) $P(F,f,y) = \chi(X_y, F_y)\cdot P(\mathcal{O}_y)$, where $\chi(X_y, F_y) = \sum_q (-1)^q \dim H^q(X_y, F_y)$ is the Euler-Poincaré characteristic of the sheaf F_y and $P(\mathcal{O}_y)$ is the Hilbert-Samuel polynomial of the ring $\mathcal{O}_y$; particularly, $P(F,f,y)$ is locally constant with respect to y when Y is a complex manifold.

(b) If $\chi(X_y, F_y) > 0$ (respectively $\chi(X_y, F_y) < 0$), then there exists a neighbourhood V of y such that $H(F,f,y')(n) \leq H(F,f,y)(n)$ (respectively $H(F,f,y')(n) \geq H(F,f,y)(n)$) for all n and all $y' \in V$ (semicontinuity property).

(c) When Y is a reduced complex space and $\chi(X_y, F_y) \neq 0$ then $P(F,f,y') = P(F,f,y)$ for y' in a neighbourhood of y iff Y is nonsingular in y (continuity property).

(iii) If f is a finite morphism and $P(F,f,y) = \chi(X_y, F_y) \cdot P(\mathcal{O}_y)$, then F is flat over Y in the points of X_y.

Proof. (i) By [20] (Ch.II,B,1) it suffices to prove that the difference function ΔH, $\Delta H(n) = H(n+1) - H(n)$, is polynomial. From the exact sequence

$$0 \longrightarrow \hat{m}_y^n F/\hat{m}_y^{n+1} F \longrightarrow F/\hat{m}_y^{n+1} F \longrightarrow F/\hat{m}_y^n F \longrightarrow 0$$

one obtains the exact sequence

$$\ldots \longrightarrow H^{q-1}(X, F/\hat{m}_y^n F) \longrightarrow H^q(X, \hat{m}_y^n F/\hat{m}_y^{n+1} F) \longrightarrow H^q(X, F/\hat{m}_y^{n+1} F) \longrightarrow \ldots$$

therefore

$$\Delta H(n) = \sum_q (-1)^q \dim H^q(X, \hat{m}_y^n F/\hat{m}_y^{n+1} F).$$

Now it is sufficient to show that, for a fixed $q \geq 0$, the function $n \longrightarrow \dim H^q(X, \hat{m}_y^n F/\hat{m}_y^{n+1} F)$ is polynomial. The problem is local on Y; replacing Y by a neighbourhood of y, we can find sections $t_1, \ldots, t_N \in \Gamma(Y, \mathcal{O}_Y)$ such that their germs in y generate the ideal m_y.. $F^* = \bigoplus_{n=o}^{\infty} (\hat{m}_y^n F/\hat{m}_y^{n+1} F)$ has a natural structure of $\mathcal{O}_X[T_1, \ldots, T_N]$-module by setting $T_i \longrightarrow t_i$. Moreover, applying the theorems A and B for Stein compacts and the noetherianity theorem of Frisch [9], one can prove that F^* is a coherent $\mathcal{O}_X[T_1, \ldots, T_N]$-module (the argument is the same as in [1], Lemma 2.4). By theorem 1 from [1],

$$R^q f_*(F^*)_y \simeq \bigoplus_{n=o}^{\infty} H^q(X, \hat{m}_y^n F/\hat{m}_y^{n+1} F)$$

is $\mathcal{O}_y [T_1, \ldots, T_N]$-module of finite type. It follows, from the very definition of the graded structure, that

$E = \bigoplus_{n=o}^{\infty} H^q(X, \mathfrak{m}_y^n F/\mathfrak{m}_y^{n+1} F)$ is an $A = \bigoplus_{n=o}^{\infty} (m_y^n/m_y^{n+1})$ -module of finite type. Now, our last function is a genuine Hilbert function, hence it is polynomial ([20], Ch.II, Theórème 2). To see that $\deg P(F,f,y) \leq \dim_y Y = \dim \mathcal{O}_y$, we will look on the associated polynomial of the function $n \longrightarrow \dim H^q(X, \mathfrak{m}_y^n F/\mathfrak{m}_y^{n+1} F)$. There exists a surjective morphism $L \longrightarrow E \longrightarrow 0$, homogeneous of zero degree, where L is a finite direct sum of modules, each of them isomorphic with A or with a translation of A. Since $\dim E_n \leq \dim L_n$ and we look only for the degree, the conclusion follows from the fact that the Hilbert function associated to the graded ring A $(n \longrightarrow \dim(m_y^n/m_y^{n+1}))$ is the difference function of the Hilbert-Samuel function $n \longrightarrow \dim(\mathcal{O}_y/m_y^n)$ of the ring $\mathcal{O}_y$.

(ii) (a) There exist the isomorphisms:

$$\mathfrak{m}_y^n F/\mathfrak{m}_y^{n+1} F \simeq (F/\mathfrak{m}_y F) \otimes_{\mathbb{C}} (m_y^n/m_y^{n+1})$$

(we have identified $\mathbb{C}$ and m_y^n/m_y^{n+1} with the constant sheaves defined by them), hence the isomorphisms:

$$H^q(X, \mathfrak{m}_y^n F/\mathfrak{m}_y^{n+1} F) \simeq H^q(X_y, F_y) \otimes_{\mathbb{C}} (m_y^n/m_y^{n+1}).$$

Then we deduce that $\Delta H = \chi(X_y, F_y) \cdot \Delta H(\mathcal{O}_y)$. In this case we have immediately that $H(F,f,y)$ is polynomial; moreover $P(F,f,y) = \chi(X_y, F_y) \cdot P(\mathcal{O}_y)$ + const. , but for the desired equality we have still to work. By shrinking eventually Y around y, there exists a bounded complex $L^\bullet$ of free $\mathcal{O}_Y$-modules of finite type which satisfies the following property: for any coherent $\mathcal{O}_Y$-module M there are isomorphisms

$$R^\bullet f_*(F \otimes_{\mathcal{O}_X} f^*(M)) \simeq H^\bullet(L^\bullet \otimes_{\mathcal{O}_Y} M).$$

This result (in a more general form) is stated in [14] (Bemerkung 4.4.1); a little weaker assertion is proved in [18], but the complex used there satisfies the above isomorphisms (one can find details in [3]; we will show in part II how that works in the differential Kodaira-Spencer's case.

We consider the coherent sheaves concentrated in y with stalks $\mathcal{O}_y/m_y^n$. The cohomology groupes of the complex $L_y^\bullet/m_y^n L_y^\bullet$ coincide with

$$R^\bullet f_*(F/\hat{m}_y^n F)_y \simeq H^\bullet(X, F/\hat{m}_y^n F) ,$$

hence

$$\sum_q (-1)^q \dim H^q(X, F/\hat{m}_y^n F) = \sum_q (-1)^q \dim H^q(L_y^\bullet/m_y^n L_y^\bullet) =$$

$$= \sum_q (-1)^q \dim(L_y^q/m_y^n L_y^q) = \sum_q (-1)^q \operatorname{rank}(L^q)\cdot\dim(\mathcal{O}_y/m_y^n) .$$

$$\sum_q (-1)^q \operatorname{rank}(L^q) = \sum_q (-1)^q \dim(L^q/m_y L^q) =$$

$$= \sum_q (-1)^q \dim H^q(L_y^\bullet/m_y L_y^\bullet) = \sum_q (-1)^q \dim H^q(X_y, F_y) = \chi(X_y, F_y)$$

and (a) follows.

(b) and (c) follow by means of the equality $H(F,f,y) = \chi(X_y, F_y)\cdot H(\mathcal{O}_y)$ and also on the fact that the Euler-Poincare' characteristic $\chi(X_y, F_y)$ is locally constant on Y using the lemma 1 ((i) and (ii)).

(iii) The fiber X_y is a finite set and $f_*(F)_y \simeq \bigoplus_{x\in X_y} F_x$. It sufficies to show that $f_*(F)_y$ is a free $\mathcal{O}_y$-module. Since $R^q f_* = 0$ for $q \geq 1$, $H = H(F,f,y)$ is reduced to

$$H(n) = \dim H^0(X, F/\hat{m}_y^n F) = \sum_{x\in X_y} \dim(F_x/m_y^n F_x) = \dim(f_*(F)_y/m_y^n f_*(F)_y)$$

and also $\chi(X_y, F_y) = \dim(f_*(F)_y/m_y f_*(F)_y)$. Our hypothesis becomes:

$$\dim(f_*(F)_y/m_y^n f_*(F)_y) = \dim(f_*(F)_y/m_y f_*(F)_y)\cdot\dim(\mathcal{O}_y/m_y^n)$$

for $n \gg 0$. Let us consider a surjective morphism $\mathcal{O}_y^r \xrightarrow{\theta} f_*(F)_y$, where $r = \dim(f_*(F)_y/m_y f_*(F)_y)$. We have the equality of Hilbert-Samuel polynomials $P(\mathcal{O}_y^r) = P(f_*(F)_y)$. From [20], Ch.II, Proposition 10, the associated polynomial of the $\mathcal{O}_y$-module Ker θ is zero, hence Ker $\theta = 0$.

3. We preserve the above notations and we will suppose in addition that F is f-flat. Using Grauert's comparison theorem and the complex $L^{\bullet}$ from the theorem of Kiehl-Verdier and Schneider [14] [18], one can transpose without difficulty to the analytic case the cohomological formalism developed in [11] for the algebraic case (see [3] for details). Let $q \geqslant 0$ be an integer and $y \in Y$. We say following the terminology from [11] that F is q-cohomologically flat in y if $R^q f_*(F)_y$ is a free $\mathcal{O}_y$-module and the map

$$R^q f_*(F)_y \ (= H^q(X_y, F) \ ) \longrightarrow R^q f_*(F/\hat{m}_y F)_y \ (= H^q(X_y, F_y) \ )$$

is surjective. The following assertion holds:

"F is q-cohomologically flat in y iff the canonical maps

$$R^{q-1} f_*(F)_y \longrightarrow H^{q-1}(X_y, F_y), \quad R^q f_*(F)_y \longrightarrow H^q(X_y, F_y)$$

are surjections; in this case there are natural isomorphisms

$$R^{q-1} f_*(F)_y / m_y^n R^{q-1} f_*(F)_y \simeq H^{q-1}(X, F/\hat{m}_y^n F)$$

$$R^q f_*(F)_y / m_y^n R^q f_*(F)_y \simeq H^q(X, F/\hat{m}_y^n F) \text{ ".}$$

Let $L^{\bullet}$ be a complex, in a neighbourhood of y, given by the theorem of Kiehl-Verdier and Schneider. The following assertion holds:

"F is q-cohomologically flat in y iff the sheaves $\mathrm{Coker}(L^{q-1} \longrightarrow L^q)$, $\mathrm{Coker}(L^q \longrightarrow L^{q+1})$ are free in a neighbourhood of y ".

<u>Theorem 2</u>. Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces, F an analytic coherent sheaf on X, flat over Y and $q \geqslant 0$ an integer.

(i) For any $y \in Y$ the function

$$n \longrightarrow \dim H^q(X_y^{(n-1)}, F_y^{(n-1)}) \ (= \dim H^q(X, F/\hat{m}_y^n F) \ )$$

is polynomial of degree $\leqslant \dim_y Y$.

(ii) Suppose Y nonsingular. For any $y \in Y$ there exists a neighbourhood V such that

$$\dim H^q(X^{(n)}_{y'},F^{(n)}_{y'}) \leq \dim H^q(X^{(n)}_y,F^{(n)}_y) ,$$

for all n and any $y' \in V$.

(iii) Suppose Y nonsingular. Then F is q-cohomologically flat in y iff the polynomial associated to the function $n \longrightarrow \dim H^q(X^{(n-1)}_{y'},F^{(n-1)}_{y'})$ is independent of y' in a neighbourhood of y.

<u>Proof</u>. The problem is local on Y, hence we may suppose the complex $L^{\bullet}$ defined on whole Y. For any $y \in Y$ and any integers n, p the following isomorphism holds:

$$H^p(X,F/\hat{m}^n_y F) \simeq H^p(L^{\bullet}_y/m^n_y L^{\bullet}_y) .$$

(i) Let $d^p : L^p \longrightarrow L^{p+1}$ be the differentials and $d^p_y : L^p_y \longrightarrow L^{p+1}_y$, $d^p_y(n) : L^p_y/m^n_y L^p_y \longrightarrow L^{p+1}_y/m^n_y L^{p+1}_y$ the differentials induced by d^p. The complex $L^{\bullet}$ yields the exact sequences:

$$L^{q-1}_y/m^n_y L^{q-1}_y \xrightarrow{d^{q-1}_y(n)} L^q_y/m^n_y L^q_y \longrightarrow \text{Coker } d^{q-1}_y/m^n_y\text{Coker } d^{q-1}_y \longrightarrow 0$$

$$L^q_y/m^n_y L^q_y \xrightarrow{d^q_y(n)} L^{q+1}_y/m^n_y L^{q+1}_y \longrightarrow \text{Coker } d^q_y/m^n_y\text{Coker } d^q_y \longrightarrow 0$$

Therefore, using the additivity, it follows:

$$\dim H^q(L^{\bullet}_y/m^n_y L^{\bullet}_y) = \dim(\text{Ker } d^q_y(n)) - \dim(\text{Im } d^{q-1}_y(n)) =$$
$$= \dim(L^q_y/m^n_y L^q_y) - \dim(\text{Im } d^q_y(n)) - \dim(\text{Im } d^{q-1}_y(n)) =$$
$$= - \dim(L^{q+1}_y/m^n_y L^{q+1}_y) + \dim(\text{Coker } d^{q-1}_y(n)) + \dim(\text{Coker } d^q_y(n)) =$$
$$= - \dim(L^{q+1}_y/m^n_y L^{q+1}_y) + \dim((\text{Coker } d^{q-1})_y/m^n_y(\text{Coker } d^{q-1})_y) +$$
$$+ \dim((\text{Coker } d^q)_y/m^n_y(\text{Coker } d^q)_y) .$$

Thus the function $n \longrightarrow \dim H^q(X,F/\hat{m}^n_y F)$ is polynomial and its associated polynomial equals:

$$- P(L^{q+1}_y) + P((\text{Coker } d^{q-1})_y) + P((\text{Coker } d^q)_y) .$$

(ii) If Y is a complex manifold, then $P(L^{q+1}_y)$ is independent of y on connected components and we apply the lemma 1 (i).

(iii) If F is q-cohomologically flat in y, then Coker d^{q-1} and Coker d^q are free in a neighbourhood of y and the conclusion follows. Conversely, by the hypothesis and taking into account (i) it follows that $P((\text{Coker } d^{q-1} \oplus \text{Coker } d^q)_{y'})$ does not depend on y' in a neighbourhood of y. Lemma 1 (iii) implies that Coker $d^{q-1} \oplus$ Coker d^q is free in a neighbourhood of y; the sheaves Coker d^{q-1} and Coker d^q enjoy the same property and then F follows q-cohomologically flat in y.

Let us give some consequences of the theorems 1 and 2.

Corollary 1. Let Y be a reduced complex space and $f: X \longrightarrow Y$ an analytic family of Riemann surfaces of genus $g \neq 1$ [5] . If

$$\sum_q (-1)^q \dim H^q(X_y^{(n)}, \mathcal{O}_{X_y^{(n)}}) = \sum_q (-1)^q \dim H^q(X_{y'}^{(n)}, \mathcal{O}_{X_{y'}^{(n)}})$$

for any $y, y' \in Y$ and $n \gg 0$, then Y is nonsingular.

We only note that $\chi(X_y, \mathcal{O}_y) \neq 0$ for any $y \in Y$.

Corollary 2. Let $f : X \longrightarrow Y$ be an analytic family of compact complex manifolds in the sense of Kodaira-Spencer [15] (i.e. a smooth proper morphism of complex manifolds), F a locally free sheaf of finite rank on X and $q \geq 0$ an integer. Suppose $\dim H^q(X_y^{(n)}, F_y^{(n)}) = \dim H^q(X_{y'}^{(n)}, F_{y'}^{(n)})$ for any $y, y' \in Y$ and $n \gg 0$. Then $R^q f_*(F)$ is locally free sheaf of finite rank on Y.

Corollary 3. Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces, F an analytic coherent sheaf on X flat over Y, y a nonsingular point of Y and $q \geq 0$ an integer. Suppose $H^q(X_y^{(n)}, F_y^{(n)}) = 0$ for $n \gg 0$. Then $H^q(X_y^{(n)}, F_y^{(n)}) = 0$ for all n; in particular, $H^q(X_y, F_y) = 0$ and also $R^q f_*(F)_y = 0$.

The fact that $R^q f_*(F)_y = 0$ can be deduced straightforwardly from the hypothesis by comparison theorem.

The following two corollaries are for case $n \longrightarrow \infty$ the analogues of the results stated in [11] (Corollaire 4.6.5), [5] (Exposé 15) for case n = 0; the proof can be done similar to n=0

<u>Corollary 4</u>. Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces and $y \in Y$ a nonsingular point such that $H^1(X_y^{(n)}, \mathcal{O}_{X_y^{(n)}}) = 0$ for $n \gg 0$. If F and G are invertible sheaves on X such that the sheaves $F_y^{(n)}$ and $G_y^{(n)}$ are isomorphic for $n \gg$ then there exists a neighbourhood V of y with the property: $F|f^{-1}(V) \simeq G|f^{-1}(V)$.

<u>Corollary 5</u>. Let $f : X \longrightarrow Y$ be a proper flat morphism of complex spaces, y a nonsingular point of Y and L an invertible sheaf on X. If $H^1(X_y^{(n)}, L_y^{(n)}) = 0$ and $L_y^{(n)}$ is very ample for $n \gg$ then there exists a neighbourhood V of y such that $L|f^{-1}(V)$ is very ample with respect to the morphism $f^{-1}(V) \longrightarrow V$.

<u>Corollary 6</u>. Let $f : X \longrightarrow Y$ be an analytic family of compact complex manifolds, $y \in Y$ and $q \geqslant 0$ an integer. If $H^q(X_y^{(n)}, \Omega_{X_y^{(n)}}) = 0$, for $n \gg 0$, then $H^q(X_y, \Omega_{X_y}) = 0$ (Ω denotes the sheaf of differential forms).

For the proof we only consider the sheaf $\Omega_{X/Y}$ of relativ differential forms, which is locally free of finite rank on X, and note that $(\Omega_{X/Y})_y^{(n)} \simeq \Omega_{X_y^{(n)}}$; see [5] .

<u>Remark</u>. We do not know if in the statements (ii) and (iii of the theorem 2 one can change the hypothesis that the space Y is nonsingular with the hypothesis that it is reduced. Particula ly, we do not know if the following statement is still true:

"Let $f : X \longrightarrow Y$ be a proper morphism of complex spaces, F an analytic coherent sheaf on X, flat over Y, $y \in Y$ and $q \geqslant 0$ an integer. Suppose Y reduced in y and $H^q(X_y^{(n)}, F_y^{(n)}) = 0$ for $n \gg 0$. Then $H^q(X_y, F_y) = 0$ ".

For instance, in order to have a substitute of the statement (iii) in the case "Y reduced" we would require an assertion like the following:

"Let Y be a reduced complex space and G an analytic coherent sheaf on Y. Suppose that there exists an integer $e \geqslant 0$ such that $P(G_y) = e \cdot P(O_y)$ for any $y \in Y$. Then G is locally free".

We ignore if this is true.

4. Analogous statement can be established in the algebraic case (then we will use [4] instead of [16]).

For example, one has the following:

Theorem 3. Let $f : X \longrightarrow Y$ be a proper morphism of locally noetherian schemes and F a coherent sheaf on X.

(i) For any $y \in Y$, the function

$$n \longrightarrow \sum_q (-1)^q \text{ length } H^q(X, F/\hat{m}_y^n F)$$

is polynomial. Moreover, if F is f-flat, then the functions $n \longrightarrow \text{length } H^q(X, F/\hat{m}_y^n F)$ are also polynomials.

Suppose Y is a scheme of finite type over an algebraically closed field and F is f-flat. Denote by Y' the set of closed points of Y. Then:

(ii) (a) If $\chi(X_y, F_y) > 0$ (respectively $\chi(X_y, F_y) < 0$) for $y \in Y'$, then the function $n \longrightarrow \sum_q (-1)^q \text{ length } H^q(X, F/\hat{m}_y^n F)$ is upper (respectively lower) semicontinuous with respect to $y \in Y'$.

(b) If Y is nonsingular, then the function $n \longrightarrow \text{length } H^q(X, F/\hat{m}_y^n F)$ is upper semicontinuous with respect to $y \in Y'$.

(iii) (a) If Y is reduced, $\chi(X_y, F_y) \neq 0$ and the polynomial associated to the function $n \longrightarrow \sum_q (-1)^q \text{ length } H^q(X, F/\hat{m}_y^n F)$ does not depend on $y \in Y'$, then Y is nonsingular.

(b) Suppose Y nonsingular and let $q \geqslant 0$ be an integer. Then F is q-cohomologically flat in the points of Y' iff the polynomial associated to the function $n \longrightarrow \text{length } H^q(X, F/\hat{m}_y^n F)$ is locally constant with respect to $y \in Y'$.

II. The differential Kodaira-Spencer's case

1. In order to study Hilbert-Samuel polynomials for differentiable families of compact complex manifolds [15], in this section we must extend some cohomological facts from the algebraic and analytic case to this case. This can be done in the general context of relative analytical spaces [8], [12],[13], but we restrict ourselves to this classical case.

Let Y be a differentiable manifold and E the sheaf of germs of C^{∞}-functions, with complex values. Let $f : X \longrightarrow Y$ be a differentiable family of compact complex manifolds and F a sheaf on X, locally free of finite rank. Frequentely we shall use facts and notations from [7], [8], [18].

Let y_o be a point of Y. By shrinking Y around y_o, we can find a real number r_{*}, $r_{*} \leq r \leq 1$, and a finite number of relative carts

$$j_k : U_k \longrightarrow D_k(1) \times Y , \qquad 0 \leq k \leq k_{*} ,$$

such that: for any $r \leq 1$ and any open set V of Y if we note $U_k(r,V) = j_k^{-1}(D_k(r) \times V)$ and $U(r,V) = (U_k(r,V))_{0 \leq k \leq k_{*}}$, $U(r,V)$ is an aciclic covering relative to F of $f^{-1}(V)$ ($D_k(r)$ is open polydisc). We have $H^{\cdot}(f^{-1}(V),F) \simeq H^{\cdot}(C^{\cdot}(U(r,V),F))$, where $C^{\cdot}$ is the Čech complex of alternate cochains. The correspondence $V \longrightarrow C^{\cdot}(U(r,V),F)$ defines a complex of sheaves on Y, $C^{\cdot}(U(r),F)$, and its cohomology identifies with $R^{\cdot}f_{*}(F)$. Let Δ_n be the set $\{(k_o,\ldots,k_n) \mid 0 \leq k_o < \ldots < k_n \leq k_{*}\}$. For $\alpha = (k_o,\ldots,k_n)$ we define $U_{\alpha}(r,V) = \bigcap_{\nu=0}^{n} U_{k_\nu}(r,V)$ and $D_{\alpha}(r) = \prod_{\nu=0}^{n} D_{k_\nu}(r)$. The immersions j_k define an immersion $j_{\alpha} : U_{\alpha}(r,V) \longrightarrow D_{\alpha}(r) \times V$. For $\alpha \subset \beta$ one gets canonical projections $\pi_{\alpha\beta} : D_{\beta}(r) \times V \longrightarrow D_{\alpha}(r) \times V$. One obtains thus an atlas U, in the sense of [7],[8]. By a refinement of U and by shrinking eventually Y around y_o, for a real number r_{**}, $r_{*} < r_{**} < 1$, we find for the link system of sheaves $j_{*}(F)$ on U a resolution

$$\dots \longrightarrow R^k \longrightarrow R^{k-1} \longrightarrow \dots \longrightarrow R^1 \longrightarrow R^o \longrightarrow j_*(F) \longrightarrow 0$$

with free systems of finite rank. For a link system G on U one denotes by $C^\cdot(r,V;G)$ the associated Čech complex, of components:

$$C^n(r,V;G) = \prod_{\alpha\in\Delta_n} \Gamma(D_\alpha(r)\times V, G_\alpha) .$$

For each r, $r_* \leq r \leq r_{**}$, and any open set $V \subset Y$ consider the double complex $(C^l(r,V;R^k))_{l,k}$ and denote by $C^\cdot(r,V)$ the associated simple complex. By $C^\cdot(r)$ we denote the complex of E-modules, $V \to C^\cdot(r,V)$. The morphism $R^o \to j_*(F)$ defines a complex morphism

$$C^\cdot(r,V) \longrightarrow C^\cdot(r,V;j_*(F)) \simeq C^\cdot(U(r,V),F) ,$$

in fact an quasi-isomorphism. In particular we get that

$$H^\cdot(C^\cdot(r)) \xrightarrow{\sim} H^\cdot(C^\cdot(U(r),F)) \simeq R^\cdot f_*(F) .$$

For an E-module M, we write for simplicity $F\otimes_E M$ instead of $F\otimes_{O_X} f^*(M)$. If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of topological Fréchet E-modules of finite presentation, then the sequence

$$0 \longrightarrow F\otimes_E M' \longrightarrow F\otimes_E M \longrightarrow F\otimes_E M'' \longrightarrow 0$$

is exact. Indeed, F is locally free and the problem is local on X and Y: one can assume $F = O_X$ and f a suitable projection. Easyly the conclusion follows by nuclearity arguments.

Consider the complex $C^\cdot(U(r),F\otimes_E M)$ defined by $V \longrightarrow C^\cdot(U(r,V),F\otimes_E M)$. The morphisms $H^\cdot(U(r,V),F\otimes_E M) \longrightarrow H^\cdot(f^{-1}(V),F\otimes_E M)$ give the morphisms $H^\cdot(C^\cdot(U(r),F\otimes_E M)) \longrightarrow R^\cdot f_*(F\otimes_E M)$. We claim that these are isomorphisms provided that M is a pseudocoherent Fréchet E-module. For this, it suffices to show that $H^q(U_\alpha(r,V),F\otimes_E M) = 0$ for V sufficiently small and $q \geq 1$. Let d be the topological dimension of X; one can suppose $d < \infty$. On sufficiently small open sets V consider exact sequences of the form:

$E^{n_{d+2}} \longrightarrow E^{n_{d+1}} \longrightarrow \dots \longrightarrow E^{r_o} \longrightarrow M \longrightarrow 0$. Then, on the open sets $f^{-1}(V)$ we will obtain exact sequences

$$F^{n_d} \longrightarrow \dots \longrightarrow F^{n_o} \longrightarrow F\otimes_E M \longrightarrow 0$$

and easyly we get what we need as $H^q(U_\alpha(r,V),F) = 0$ for $q \geq 1$ and $H^q = 0$ for $q > d$. Now, from the proof of the finiteness theorem [7], [8], by shrinking eventually again Y around y_0, we can find a bounded complex of free sheaves of finite rank on Y together with an (-1)-quasi-isomorphism $L^\cdot \longrightarrow C^\cdot(r)$, for a suitable r, $r_* < r < r_{**}$. For an E-module M we obtain a complex morphism:

$$L^\cdot \otimes_E M \longrightarrow C^\cdot(r) \otimes_E M \longrightarrow C^\cdot(U(r),F) \otimes_E M .$$

By composing with the morphism $C^\cdot(U(r),F) \otimes_E M \longrightarrow C^\cdot(U(r),F \otimes_E M)$ obtained from the morphism

$$\Gamma(U_\alpha(r,V),F) \otimes_{E(V)} M(V) \to \Gamma(U_\alpha(r,V),F) \otimes_{O_X(U_\alpha(r,V))} \Gamma(U_\alpha(r,V),f^*(M)) \to$$

$$\to \Gamma(U_\alpha(r,V),F \otimes_E M)$$

we get a morphism

$$L^\cdot \otimes_E M \longrightarrow C^\cdot(U(r),F \otimes_E M) ,$$

which is functorial in M. If M is a pseudocoherent Fréchet E-module then, by passing to the cohomology, we get morphisms:

$$(*) \qquad H^q(L^\cdot \otimes_E M) \longrightarrow R^q f_*(F \otimes_E M) .$$

By decreasing induction on $q \geq 0$ we show that these morphisms, together with the corresponding morphisms obtained by the replacing of Y by open subsets, are isomorphisms. For a sufficiently large q this is obviously true. Suppose that the morphisms (*) are isomorphisms in dimension $>q$. Let M be a pseudocoherent Fréchet E-module defined on Y (or on an open set of Y). The problem is local on Y, so we can assume that there exists an exact sequence $0 \longrightarrow N \longrightarrow E^{n_0} \longrightarrow M \longrightarrow 0$.

We get an exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^\cdot \otimes_E N & \longrightarrow & L^\cdot \otimes_E E^{n_0} & \longrightarrow & L^\cdot \otimes_E M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^\cdot(U(r),F \otimes_E N) & \longrightarrow & C^\cdot(U(r),F \otimes_E E^{n_0}) & \longrightarrow & C^\cdot(U(r),F \otimes_E M) & \longrightarrow & 0 \end{array}$$

(for the exactity of the second row we require the exact sequence $0 \longrightarrow F \otimes_E N \longrightarrow F \otimes_E E^{n_0} \longrightarrow F \otimes_E M \longrightarrow 0$ and the fact that $H^\cdot(U_\alpha(r,V),F \otimes_E N) = 0$ for small V) .

We pass this diagram to cohomology. As (*) are isomor-

5-lemma. Remark also that (∗) are functorial in M and agree with the long exact sequences associated to short exact sequences $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ of pseudocoherent Fréchet E-modules.

We have proved the following:

Theorem 4. Let $f : X \longrightarrow Y$ be a differentiable family of compact complex manifolds and F a locally free sheaf of finite rank on X. Then there exists, locally on Y, a complex $L^{\cdot}$ of locally free E-modules of finite rank with the following property: for any pseudocoherent Fréchet E-module M the following isomorphisms take place

$$H^q(L^{\cdot} \otimes_E M) \simeq R^q f_*(F \otimes_E M) \qquad (\text{for } q \geqslant 0) ,$$

functorial in M and compatible with short exact sequences.

Remark. With the notations of [8], [18], the previous arguments should give the following general statement:

"Let $(Y, \mathcal{O}_Y)$ be a mFS-ringed space of type (J) and $f : X \longrightarrow Y$ a relative analytical space, proper over Y. Let F be a f-pseudocoherent $\mathcal{O}_X$-module, transflat over Y. Then there exists, locally on Y, a complex $L^{\cdot}$ of free $\mathcal{O}_Y$-modules of finite rank with the following property: for any pseudocoherent Fréchet $\mathcal{O}_Y$-module M there exist natural isomorphisms

$$H^q(L^{\cdot} \otimes_{\mathcal{O}_Y} M) \simeq R^q f_*(F \otimes_{\mathcal{O}_Y} M) \qquad (\text{for } q \geqslant 0) ,$$

functorial in M and compatible with the short exact sequences". (Of course, it should modify a little the definition of "transflat" such that if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of pseudocoherent Fréchet sheaves on an open set of Y, then the sequence $0 \rightarrow F \otimes_{\mathcal{O}_Y} M' \rightarrow F \otimes_{\mathcal{O}_Y} M \rightarrow F \otimes_{\mathcal{O}_Y} M'' \rightarrow 0$ will be exact).

We come back to our case. An E-module M is said to be of "analytic nature" if for any $y \in Y$ there exist an affine neighbourhood V and a coherent $\mathcal{O}_V$-module N ($\mathcal{O}_V$ is the sheaf of germs of complex valued analytic functions on V) such that $N \otimes_{\mathcal{O}_V} E_V \simeq M|V$. M is a Fréchet E-module. To prove that one considers (locally) finite presentations for the coherent sheaves N and uses the following result of Malgrange [21] Ch.VI, Corrolaire 1.5 :

"If $E^p(V) \longrightarrow E^q(V)$ is the morphism given by an analytical matrix, then its image is closed".

The sheaves N are O-pseudocoherent and, by another result of Malgrange [21] Ch.VI, Corrolaire 1.3, E is O-flat, therefore M is pseudocoherent. If y is a point of Y and $n \geq 1$ is an integer, then the E-module M concentrated in y and of fiber E_y/m_y^n is of "analytic nature", hence Fréchet and pseudocoherent.

Thus we get the following:

<u>Corollary 1</u>. For any M of "analytic nature",

$$H^q(L^\bullet \otimes_E M) \simeq R^q f_*(F \otimes_E M) \qquad \text{(for } q \geq 0\text{)} .$$

Particularly, $H^q(L_y^\bullet/m_y^n L_y^\bullet) \simeq H^q(X,F/\hat{m}_y^n F)$ for any $q \geq 0$, $n \geq 1$.

<u>Corollary 2</u>. Let $q \geq 0$ and y be fixed. The following assertions are equivalent:

(1) For any pseudocoherent Fréchet E-module M, the natural morphism $R^q f_*(F)_y \otimes_{E_y} M_y \longrightarrow R^q f_*(F \otimes_{E_Y} M)_y$ is isomorphism.

(2) The natural morphism

$R^q f_*(F)_y (= H^q(X_y,F)) \longrightarrow R^q f_*(F/\hat{m}_y F)_y (= H^q(X_y,F_y))$ is surjective.

(3) $\mathrm{Coker}(L^q \longrightarrow L^{q+1})$ is free in y.

Particularly, if these conditions are fulfiled, then

$$R^q f_*(F)_y/m_y^n R^q f_*(F)_y \simeq H^q(X,F/\hat{m}_y^n F) .$$

<u>Proof</u>. (1) $\Rightarrow$(2) is clear since the sheaf M concentrated in y and of fiber $M_y = E_y/m_y$ is Fréchet and pseudocoherent.

(2) $\Rightarrow$(3) Let C be $\mathrm{Coker}(L^q \longrightarrow L^{q+1})$ and $r = \dim(C_y/m_y C_y)$. Using convenient basis for vector spaces obtained $\mathrm{mod}(m_y)$ we can construct surjective morphisms $E_y^r \longrightarrow C_y$, $L_y^{q+1} \longrightarrow E_y^r$, which can be inserted in the exact commutative diagram:

$$\begin{array}{ccccccccc} & & L_y^q & \longrightarrow & L_y^{q+1} & \longrightarrow & C_y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & E_y^r & \longrightarrow & C_y & \longrightarrow & 0 \end{array}$$

where K is the kernel of $E_y^r \longrightarrow C_y$. The first vertical arrow is

$K = m_y K$, hence $K = 0$ and C_y is free. Let α be an element of K. As the induced morphism $E_y^r / m_y E_y^r \longrightarrow C_y / m_y C_y$ is bijective, $\alpha \in m_y E_y^r$. We lift α to $\beta \in m_y L_y^{q+1}$. There exists $\gamma \in L_y^q$ such that $d^q(\gamma) = \beta$. The image of γ in $L_y^q / m_y L_y^q$ is a cocycle. By hypothesis, there exist $\delta \in L_y^q$ and $\eta \in L_y^{q-1}$ such that $d^q(\delta) = 0$ and the element $(\gamma - \delta) - d^{q-1}(\eta) \in m_y L_y^q$. The image of this element by the map $L_y^q \longrightarrow K$ lies in $m_y K$ and is just α.

(3) $\Rightarrow$(1) The sheaves Im d^q, Ker d^q are also free in y. By the theorem we have $R^q f_*(F) \simeq H^q(L^\bullet)$, $R^q f_*(F \otimes_{\mathcal{O}_Y} M) \simeq H^q(L^\bullet \otimes_{\mathcal{O}_Y} M)$ and we conclude in the standard way: writing suitable short exact sequences, tensoring by $\otimes_{\mathcal{O}_Y} M$ etc....

Using the language from the algebraic case [11], we say that F is q-cohomologically flat in y if the conditions of the corollary 2 are fulfiled for q-1 and q. When this occurs in a point y, the condition (3) shows that this holds in the neighbouring points.

<u>Corollary 3</u>. F is q-cohomologically flat in y iff $R^q f_*(F)$ is free in y and $H^q(X_y, F) \longrightarrow H^q(X_y, F_y)$ is surjective.

<u>Proof</u>. One use the exact sequence:

$$0 \longrightarrow R^q f_*(F) \longrightarrow \text{Coker } d^{q-1} \longrightarrow L^{q+1} \longrightarrow \text{Coker } d^q \longrightarrow 0 .$$

2. Let y be a point of Y. Let us denote by $\bar{E}_y = \varprojlim_n (E_y / m_y^n)$ the completion of E_y in the m_y-adic topology. $\bar{E}_y$ is a ring of formal power series and the canonical morphism $E_y \longrightarrow \bar{E}_y$ is surjective (the Theorem of E. Borel [21], Ch. IV, Remarque 3.5). The kernel of this morphism is $\bigcap_n m_y^n$ and coincides with the ideal m_y^∞ of the germs of the flat functions in y. For any E_y-module M we remark that $M/m_y^n M \simeq \bar{M}/m_y^n \bar{M}$ for all n. Here $\bar{M} = M \otimes_{E_y} \bar{E}_y = M/m_y^\infty M$ and the image of m_y in $\bar{E}_y$ is denoted also by m_y. Particularly, it follows that if G is an E-module of finite type, then the function $n \dashrightarrow \dim(G_y / m_y^n G_y)$ is polynomial, the associated poly-

$K = m_y K$, hence $K = 0$ and C_y is free. Let α be an element of K. As the induced morphism $E_y^r/m_y E_y^r \longrightarrow C_y/m_y C_y$ is bijective, $\alpha \in m_y E_y^r$. We lift α to $\beta \in m_y L_y^{q+1}$. There exists $\gamma \in L_y^q$ such that $d^q(\gamma) = \beta$. The image of γ in $L_y^q/m_y L_y^q$ is a cocycle. By hypothesis, there exist $\delta \in L_y^q$ and $\eta \in L_y^{q-1}$ such that $d^q(\delta) = 0$ and the element $(\gamma - \delta) - d^{q-1}(\eta) \in m_y L_y^q$. The image of this element by the map $L_y^q \longrightarrow K$ lies in $m_y K$ and is just α.

(3) $\Rightarrow$(1) The sheaves $\operatorname{Im} d^q$, $\operatorname{Ker} d^q$ are also free in y. By the theorem we have $R^q f_*(F) \simeq H^q(L^\cdot)$, $R^q f_*(F \otimes_{\mathcal{O}_Y} M) \simeq H^q(L^\cdot \otimes_{\mathcal{O}_Y} M)$ and we conclude in the standard way: writing suitable short exact sequences, tensoring by $\otimes_{\mathcal{O}_Y} M$ etc....

Using the language from the algebraic case [11], we say that F is q-cohomologically flat in y if the conditions of the corollary 2 are fulfiled for q-1 and q. When this occurs in a point y, the condition (3) shows that this holds in the neighbouring points.

<u>Corollary 3</u>. F is q-cohomologically flat in y iff $R^q f_*(F)$ is free in y and $H^q(X_y, F) \longrightarrow H^q(X_y, F_y)$ is surjective.

<u>Proof</u>. One use the exact sequence:

$$0 \longrightarrow R^q f_*(F) \longrightarrow \operatorname{Coker} d^{q-1} \longrightarrow L^{q+1} \longrightarrow \operatorname{Coker} d^q \longrightarrow 0 .$$

2. Let y be a point of Y. Let us denote by $\overline{E}_y = \varprojlim_n (E_y/m_y^n)$ the completion of E_y in the m_y-adic topology. $\overline{E}_y$ is a ring of formal power series and the canonical morphism $E_y \longrightarrow \overline{E}_y$ is surjective (the Theorem of E. Borel [21], Ch.IV, Remarque 3.5) . The kernel of this morphism is $\bigcap_n m_y^n$ and coincides with the ideal m_y^∞ of the germs of the flat functions in y. For any E_y-module M we remark that $M/m_y^n M \simeq \overline{M}/m_y^n \overline{M}$ for all n. Here $\overline{M} = M \otimes_{E_y} \overline{E}_y = M/m_y^\infty M$ and the image of m_y in $\overline{E}_y$ is denoted also by m_y . Particularly, it follows that if G is an E-module of finite type, then the function $n \longrightarrow \dim(G_y/m_y^n G_y)$ is polynomial, the associated poly-

coordinates $y_1,\dots,y_m$, $x_1,\dots,x_s$. The natural bijection $Y \simeq Y \times \{0\}$, where $\{0\}$ is the origin of the space of coordinates $x_1,\dots,x_s$, and the correspondence

$$f \in E(y,x) \longrightarrow (f(y,o), \frac{\partial f}{\partial x_1}(y,o), \dots, \frac{\partial f}{\partial x_s}(y,o))$$

induce an isomorphism of ringed spaces

$$(Y, E_Y \oplus E_Y^s) \simeq (Y \times \{0\}, E(y,x)/(x_i x_j) | Y \times \{0\})$$

($(x_i x_j)$ is the ideal sheaf of $E(y,x)$ generated by the products $x_i x_j$). Each φ_i can be written $\varphi_i = (\varphi_{ij}(y))_{1 \leq j \leq s}$ and we put $\overline{\varphi}_i(y,x) = \sum_{j=1}^{s} \varphi_{ij}(y) \cdot x_j$. One verifies without difficulty the isomorphism:

$$(Y, E' = E_Y \oplus G) \simeq (Y \times \{0\}, E(y,x)/(x_i x_j, \overline{\varphi}_i(y,x)) | Y \times \{0\}) .$$

We identify Y with $Y \times \{0\}$ and E' with $E(y,x)/(x_i x_j, \overline{\varphi}_i(y,x))$.

Consider new coordinates $z_1,\dots,z_m$, $t_1,\dots,t_s$ and the ringed space (the "product" $(Y,E') \times (Y,E')$)

$$(Y \times Y, E(y,x,z,t)/(x_i x_j, \overline{\varphi}_i(y,x), t_i t_j, \overline{\varphi}_i(z,t))$$

(with common conventions and misuses of notation ...) .

If we denote by A its structural sheaf and by I the ideal generated by the classes of the differences $z_i - y_i$, $t_j - x_j$, $1 \leq i \leq m$, $1 \leq j \leq s$, (I is the "diagonal ideal"), then the sheaf A/I^{n+1} restricted to Y by the diagonal morphism $Y \longrightarrow Y \times Y$ and considered as E'-module by means of the coordinates $y_1,\dots,y_m$, $x_1,\dots,x_s$ is the desired E'-module $P^{(n)}$.

(ii) We need only the fact that G is of finite type. We show that $\dim(G_y/m_y G_y)$ does not depend on y and G will follow locally free [18], [19]. There exist points y_o such that G_{y_o} is free (for example, we take a point y_o where $\dim(G_{y_o}/m_{y_o} G_{y_o})$ is minimal). Therefore $P(G_y) = P(\overline{G}_y)$ equals the Hilbert-Samuel polynomial of a free module of finite type over the regular ring $\overline{E}_y$, for any $y \in Y$. By [2], $\overline{G}_y$ will be free over $\overline{E}_y$. The rank of $\overline{G}_y$ is independent of y since $P(G_y)$ is independent of y , hence $\dim(G_y/m_y G_y) = \dim(\overline{G}_y/m_y \overline{G}_y) = \mathrm{rank}(G_y)$ is constant.

Since we lack a full semicontinuity property, we can not obtain the next proposition as a consequence of the theorem (like in algebraic and analytic case).

Proposition. Let $f : X \longrightarrow Y$ be a differentiable family of compact complex manifolds, F a locally free sheaf of finite rank on X, y a point of Y and $q \geq 0$ an integer. Suppose $H^q(X_y^{(n)}, F_y^{(n)}) = 0$ for $n \gg 0$. Then $H^q(X_y, F_y) = 0$.

Proof. By shrinking Y to a neighbourhood of y, there exists a complex $L^\bullet$ as in theorem 4. From the hypothesis we deduce that the sequences

$$L_y^{q-1}/m_y^n L_y^{q-1} \xrightarrow{d_y^{q-1}(n)} L_y^q/m_y^n L_y^q \xrightarrow{d_y^q(n)} L_y^{q+1}/m_y^n L_y^{q+1}$$

are exact for $n \gg 0$. As $L_y^\bullet/m_y^n L_y^\bullet \simeq \bar{L}_y^\bullet/m_y^n \bar{L}_y^\bullet$, it follows that the sequences

$$\bar{L}_y^{q-1}/m_y^n \bar{L}_y^{q-1} \xrightarrow{\bar{d}_y^{q-1}(n)} \bar{L}_y^q/m_y^n \bar{L}_y^q \xrightarrow{\bar{d}_y^q(n)} \bar{L}_y^{q+1}/m_y^n \bar{L}_y^{q+1}$$

are exact for $n \gg 0$, hence the equality of Hilbert-Samuel polynomials

$P(\text{Coker } \bar{d}_y^{q-1} \oplus \text{Coker } \bar{d}_y^q) = P(\text{Coker } \bar{d}_y^{q-1}) + P(\text{Coker } \bar{d}_y^q) = P(\bar{L}_y^{q+1})$.

By [2], $\text{Coker } \bar{d}_y^{q-1} \oplus \text{Coker } \bar{d}_y^q$ is free $\bar{E}_y$-module, therefore $\text{Coker } \bar{d}_y^{q-1}$ and $\text{Coker } \bar{d}_y^q$ are free. Clearly, the sequence

$$\bar{L}_y^{q-1}/m_y \bar{L}_y^{q-1} \longrightarrow \bar{L}_y^q/m_y \bar{L}_y^q \longrightarrow \bar{L}_y^{q+1}/m_y \bar{L}_y^{q+1}$$

follows exact, hence the sequence

$$L_y^{q-1}/m_y L_y^{q-1} \longrightarrow L_y^q/m_y L_y^q \longrightarrow L_y^{q+1}/m_y L_y^{q+1}$$

is exact and one concludes.

As in the part I, one obtains:

Corollary 1. If $H^q(X_y^{(n)}, \Omega_{X_y^{(n)}}) = 0$ for $n \gg 0$, then $H^q(X_y, \Omega_{X_y}) = 0$.

Corollary 2. Let F, G be invertible sheaves on X such that $F_y^{(n)} \simeq G_y^{(n)}$ for $n \gg 0$. If $H^1(X_y^{(n)}, \mathcal{O}_{X_y^{(n)}}) = 0$ for $n \gg 0$, then there exists a neighbourhood V of y such that $F|f^{-1}(V) \simeq G|f^{-1}(V)$

Remark. Under the conditions of the theorem 4 one can even find, locally on Y, a bounded complex $L^{\cdot}$ such that $H^q(L^{\cdot}\otimes M) \simeq R^q f_{*}(F\otimes M)$ for any integer q .

Using this we deduce that the Hilbert-Samuel polynomial associated to the function $n \longrightarrow \sum_q (-1)^q \dim H^q(X,F/\hat{m}_y^n F)$ coincides with $\chi(X_y,F_y)\cdot P(E_y)$, hence it is locally constant on Y

3. This section is devoted to the analogue of the Grauert's and Grothendieck's comparison theorems for the differentiable families of compact complex manifolds. We preserve the previous notations. $R^q f_{*}(F)$ are the cohomology objects of a pseudocoherent complex of E-modules: this is the analogue of the Grauert's coherence theorem in this case [8], [12], [13]. Locally, using suitable resolutions with free sheaves of finite rank and taking on the space C^{∞} the usual Fréchet topology, we get on each $R^q f_{*}(F)$ a structure of topological E-module. From the open mapping theorem, it results that this topology does not depend on the chosen resolutions, particularly it can be calculated with the complexes $L^{\cdot}$ given by the theorem 4. For every open set V the topology of $\Gamma(V,R^q f_{*}(F)) \simeq H^q(f^{-1}(V),F)$ is not, generally, separated.

Theorem 6. Let $f : X \longrightarrow Y$ be a differentiable family of compact complex manifolds, F a locally free sheaf of finite rank on X, y a point of Y and $q \geqslant 0$ an integer. Then the natural morphism

$$R^q f_{*}(F)_y/m_y^{\infty} R^q f_{*}(F)_y \longrightarrow \varprojlim_n (R^q f_{*}(F)_y/m_y^n R^q f_{*}(F)_y)$$

is bijective and the natural morphism

$$\varprojlim_n (R^q f_{*}(F)_y/m_y^n R^q f_{*}(F)_y) \longrightarrow \varprojlim_n H^q(X,F/\hat{m}_y^n F)$$

is injective. If the topological E_y-module $R^{q+1} f_{*}(F)$ is Fréchet, then the last morphism is also bijective.

Proof. Let $\theta : R^q f_{*}(F)_y \longrightarrow \varprojlim_n H^q(X,F/\hat{m}_y^n F)$ be the canoni morphism. We show that its kernel equals $m_y^\infty R^q f_*(F)_y$. An inclusion is obvious. Let α be, with the above notations, an element of Ker d_y^q such that, if $\dot{\alpha}$ is the image in $R^q f_*(F)_y$ = Ker d_y^q/Im d_y^{q-1}, then $\sigma(\dot{\alpha}) = 0$. As $H^q(X,F/\hat{m}_y^n F) \simeq H^q(L_y^\bullet/m_y^n L_y^\bullet) \simeq H^q(L_y^\bullet/m_y^n L_y^\bullet)$, by [11] (Proposition 7.4.7) $\varprojlim_n H^q(X,F/\hat{m}_y^n F)$ identifies with the completion of the $\bar{E}_y$-module Ker $\bar{d}_y^q$/Im $\bar{d}_y^{q-1}$ in the topology given by the maximal ideal. Since $\bar{E}_y$ is complete, one obtains finally the isomorphisms

$$\varprojlim_n H^q(X,F/\hat{m}_y^n F) \simeq \text{Ker } \bar{d}_y^q/\text{Im } \bar{d}_y^{q-1} .$$

As $\theta(\dot{\alpha}) = 0$, there exists $\beta \in L_y^{q-1}$ such that the element $\gamma =$ $= \alpha - d_y^{q-1}(\beta)$ lies in $m_y^\infty L_y^q$. We consider the natural Fréchet topology on the sections of L^q over an open set; Ker d^q is a closed submodule of L^q. By lifting α, β, γ in a neighbourhood and since $\gamma \in$ Ker d_y^q, it follows by [21] (Ch.V, Prop. 2.3) that $\gamma \in m_y^\infty$ Ker d_y^q, hence $\dot{\alpha} \in m_y^\infty R^q f_*(F)_y$.

In this way, the canonical morphism

$$R^q f_*(F)_y/m_y^\infty R^q f_*(F)_y \longrightarrow \varprojlim_n H^q(X,F/\hat{m}_y^n F)$$

is injective. $M = R^q f_*(F)_y/m_y^\infty R^q f_*(F)_y$ follows an $\bar{E}_y = E_y/m_y^\infty$-module of finite type, hence coincides with its completion. As $M/m_y^n M$ identifies with $R^q f_*(F)_y/m_y^n R^q f_*(F)_y$, the first two statements of the theorem follow easily.

Now assume $R^{q+1} f_*(F)$ is Fréchet; then, for every open set V of Y the image of the map $L^q(V) \xrightarrow{d^q} L^{q+1}(V)$ is closed. Let ξ be an element of $\varprojlim_n H^q(X,F/\hat{m}_y^n F) \simeq$ Ker $\bar{d}_y^q$/Im $\bar{d}_y^{q-1}$. Let l be, in a neighbourhood V of y, an element of $L^q(V)$ such that its image in Ker $\bar{d}_y^q$/Im $\bar{d}_y^{q-1}$ is ξ. One has $d^q(l) \in (m_y^\infty L^{q+1}(V)) \cap (\text{Im } d^q(V))$. Again, by [21], $d^q(l)$ is in m_y^∞ Im $d^q(V)$. Let $\sum \varphi_i l_i$ be in $m_y^\infty L^q(V)$ such that $d^q(\sum \varphi_i l_i) = d^q(l)$. The element $l - \sum \varphi_i l_i$ lies in Ker $d^q(V)$ and its image in Ker d_y^q/Im $d_y^{q-1} \simeq R^q f_*(F)_y$ gives an element η. One has $\theta(\eta) = \xi$, thus θ is surjective and the proof over.

Corollary 1. Assume that $\varprojlim_n H^q(X, F/\hat{m}_y^n F) = 0$. Then $R^q f_*(F)_y = m_y^\infty R^q f_*(F)_y$. Moreover, if we know that $R^q f_*(F)_y$ is an E_y-module of finite type, then we get $R^q f_*(F)_y = 0$.

Corollary 2. Assume $R^{q+1} f_*(F) = 0$. Then the canonical morphism

$$\varprojlim_n (R^q f_*(F)_y / m_y^n R^q f_*(F)_y) \longrightarrow \varprojlim_n H^q(X, F/\hat{m}_y^n F)$$

is bijective.

Corollary 3. Assume that $R^{q+1} f_*(F)$ is Fréchet (for example, equals to zero). Then there exists an integer n_o such that

$$\mathrm{Im}(R^q f_*(F)_y \longrightarrow H^q(X_y, F_y)) = \mathrm{Im}(H^q(X, F/\hat{m}_y^{n_o} F) \longrightarrow H^q(X_y, F_y)) .$$

Proof. Since $H^q(X_y, F_y)$ is finite dimensional there exists an integer n_o such that

$$\mathrm{Im}(H^q(X, F/\hat{m}_y^n F) \to H^q(X_y, F_y)) = \mathrm{Im}(H^q(X, F/\hat{m}_y^{n_o} F) \to H^q(X_y, F_y))$$

for $n \geqslant n_o$. Again, there exists an integer $n_1 \geqslant n_o$ such that

$$\mathrm{Im}(H^q(X, F/\hat{m}_y^n F) \to H^q(X, F/\hat{m}_y^{n_o} F)) = \mathrm{Im}(H^q(X, F/\hat{m}_y^{n_1} F) \to H^q(X, F/\hat{m}_y^{n_o} F))$$

for $n \geqslant n_1$ and so on.

Let $\eta = \xi_o$ be an element from $\mathrm{Im}(H^q(X, F/\hat{m}_y^{n_o} F) \longrightarrow H^q(X_y, F_y))$. We take a preimage $\eta_1 \in H^q(X, F/\hat{m}_y^{n_1} F)$ of ξ_o and let ξ_1 be the image of η_1 in $H^q(X, F/\hat{m}_y^{n_o} F)$. The image of ξ_1 in $H^q(X_y, F_y)$ is ξ_o. Let $\eta_2 \in H^q(X, F/\hat{m}_y^{n_2} F)$ be a preimage of ξ_1 and let ξ_2 be the image of η_2 in $H^q(X, F/\hat{m}_y^{n_1} F)$. The image of ξ_2 in $H^q(X, F/\hat{m}_y^{n_o} F)$ is ξ_1. We go on and obtain an element in $\varprojlim_n H^q(X, F/\hat{m}_y^n F)$. By the theorem it has a preimage in $R^q f_*(F)_y$. The image of this in $H^q(X_y, F_y)$ is η.

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