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THIS PREPRINT CONTAINS TWO NOTES :

1) *A sufficient condition that a transitive algebra be equal to B (H), by C. Peligrad*

2) *Invariant subspaces of von Neumann algebras II by C. Peligrad*

BUCUREȘTI

A SUFFICIENT CONDITION THAT A TRANSITIVE ALGEBRA BE EQUAL TO B (H)

by

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Let H be a complex Hilbert space, and let B(H) be the algebra of all bounded linear operators on H. For $B \subset B(H)$ we denote by Lat B, the set of all closed subspaces of H, invariant under every $b \in B$. An algebra $A \subset B(H)$ is called transitive if it is weakly closed, $1 \in A$, and $\mathrm{Lat}A = \{(0), H\}$.

A linear transformation $T: \mathcal{D}_T \longmapsto H$ ($\mathcal{D}_T \subset H$) is a graph transformation for A if there exist $n \in \mathbb{N}$ and linear transsformations $\{T_i\}_{i=1}^{n-2}$ defined on $\mathcal{D}_T$, such that $\{x \oplus Tx \oplus T_1 x \oplus \cdots \oplus T_{n-2}x \mid x \in \mathcal{D}_T\} \in \mathrm{Lat}\, A^{(n)}$ (where $A^{(n)} = \{a \oplus a \oplus \cdots \oplus a \mid a \in A\}$).

The transitive algebra problem is the following:

If A is a transitive algebra on H, must A be equal to B(H)? Sufficient conditions for a transitive algebra to be equal to B(H) have been obtained by several authors (see [2]). All such results are based on the following:

Arveson's Lemma [1]. Let $A \subset B(H)$ be a transitive algebra. If every graph transformation for A is a multiple of the identity operator, then $A = B(H)$.

In [2] the following question is set: if the only closed densely defined ~~operator~~ operators which commute with a transitive algecra are the multplies of the identity, does it follow that the only graph transformations for A are the multplies of identity? Equivalently if $\mathrm{Lat}\, A^{(2)} = \mathrm{Lat}\, B(H)^{(2)}$ does it follow that $A = B(H)$? The Corollary below answers this question.

Proposition. Let $A \subset B(H)$ be a transitive algecra. Then every densely defined, linear operator T on H which commutes with A is pre-closed.

Proof. It is obvious that since A is transitive, $A^* = \{a^* \mid a \in A\}$ is also transitive. Let T be a densely defined linear operator on H which commutes with A. Then T^* exists. Now, we show that $\mathcal{D}_{T^*}$ is invariant under A^*. Indeed, let $x \in \mathcal{D}_{T^*}$, $y \in \mathcal{D}_T$, $a \in A$. We have:

$$\langle Ty, a^* x \rangle = \langle aTy, x \rangle = \langle Tay, x \rangle = \langle ay, T^* x \rangle =$$
$$= \langle y, a^* T^* x \rangle .$$

Therefore, $a^* x \in \mathcal{D}_{T^*}$ for every $a^* \in A^*$. Since A^* is transitive, it follows $\overline{\mathcal{D}_{T^*}} = H$ and hence T^* is densely defined, whence $T \subset T^{**}$ and so T is preclosed.

Corollary. Let $A \subset B(H)$ be a transitive algebra.

If every closed, densely defined operator which commutes with A is a multiple of the identity, then $A = B(H)$.

Proof. The result follows from Arveson's Lemma and the Proposition above.

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INVARIANT SUBSPACES OF VON NEUMANN
ALGEBRAS II

by

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May 1977

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In what follows, H denotes a complex Hilbert space. The algebra of all bounded operators on H is denoted by B(H). For $A \subset B(H)$, A' is the commutant of A, and A'' the bicommutant. Also, we denote by M_A, the von Neumann algebra generated by A. For every $n \in \mathbb{N}$, we denote $H^{(n)} = \bigoplus_{i=1}^{n} H_i$ where $H_i = H$ for every i, $1 \leq i \leq n$, and $A^{(n)} = \{a \oplus a \oplus \ldots \oplus a \mid a \in A\}$. The collection of all closed linear subspaces of H invariant under A(i.e. invariant under every $a \in A$) is denote by Lat A. A weakly closed algebra $A \subset B(H)$ is reductive ([6]) if $1 \in A$, and $\mathrm{Lat}_A = \mathrm{Lat}\, M_A$. A linear subspace $K \subset H$ is para-closed ([2]) if there exist a Hilbert space H_0 and a bounded linear operator $Q: H_0 \longmapsto H$ such that $QH_0 = K$. The collection of all para-closed subspaces of H, invarint under A is denoted $\mathrm{Lat}_{1/2}A$. A weakly closed algebra $A \subset B(H)$ will be called para-reductive if $1 \in A$, and $\mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}M_A$. In this paper (Section 2) we show that if A is a para-reductive algebra such that M_A has the property (P) (i.e. for every $x \in B(H)$ we have $\overline{Co}^{Wo}\{u^* xu \mid u \in M_A, \text{unitary}\} \cap M'_A \neq \emptyset$ ([7])), then $A = M_A$. It is known that all discrete von Neumann algebras have the property (P). Thus, Theorem 2.1 below, together with [9] Théorème 1 or [5] Corollary 1.3 show that if A is a para-reductive algebra such that M_A is either discrete or continuous, then $A = M_A$. In order to prove this result, in Section 1 we give some new results on para-closed operators.

Finally, in Section 3 we prove a result, announced (without proof) in [5]

<u>1. Paraclosed operators</u>. Let H_1, H be Hilbert spaces. A linear transformation $S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) is <u>para-closed</u>([2]) if its graph $\Gamma_S = \{\xi \oplus S\xi \mid \xi \in \mathcal{D}_S\}$ is a para-closed subspace of $H_1 \oplus H$. A linear transformation

$S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) is semi-closed ([1]) if there exist a Hilbert space H_2 and two closed operators $S_1: \mathcal{D}_S \longmapsto H_2$ $S_2: \mathcal{D}_{S_2} \longmapsto H$ ($\mathcal{D}_{S_2} \subset H_2$) such that $S=S_2S_1$. The following Proposition shows that these two notions are equivalent:

1.1. Proposition. Let H_1, H be Hilbert spaces, and let $S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) be a linear transformation. The following are equivalent.

a) S is para-closed

b) S is semi-closed

Proof. a) $\Longrightarrow$ b). Since S is para-closed, its graph Γ_S is a para-closed subspace of H_1 H. By the definition of paraclosed subspaces, there exist a Hilbert space H_2 and a bounded linear operator $Q: H_2 \longmapsto H_1 \oplus H$ such that $QH_2 = \Gamma_S$ We may suppose that Q is injective, since otherwise we replace H_0 by $(\ker Q)^{\perp}$. Then, for every $\xi \in \mathcal{D}_S$ there exists a unique $S_1(\xi) \in H_2$ such that $QS_1(\xi) = \xi \oplus S(\xi)$. Obviously, S_1 is a linear transformation $S_1: \mathcal{D}_S \longmapsto H_2$. Moreover, S_1 is a closed operator with $\mathcal{D}_{S_1} = \mathcal{D}_S$. Indeed, let $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{D}_S$ be a sequence such that $\lim_{n\to\infty} \xi_n = \xi_0$ and $\lim S_1(\xi_n) = \eta_0$. Since Q is a continous, it follows that $\lim QS_1(\xi_n) = Q(\eta_0)$. On the other hand $QS_1(\xi_n) = \xi_n \oplus S(\xi_n)$ and hence $Q(\eta_0) = \xi_0 \oplus S\xi_0$. It follows that $\xi_0 \in \mathcal{D}_S$, and $S_1(\xi_0) = \eta_0$. Therefore S_1 is closed and $\mathcal{D}_{S_1} = \mathcal{D}_S$. If p_H is the projection of $H_1 \oplus H$ onto H, then $S_2 = p_H Q$ is a bounded operator, and $S = S_2 S_1$, whence S is semiclosed.

b) $\Longrightarrow$ a). Let H_2 be a Hilbert space and let $S_1: \mathcal{D}_S \longmapsto H_2$, $S_2: \mathcal{D}_{S_2} \longrightarrow H$ ($\mathcal{D}_{S_2} \subset H_2$) be closed operators such that $S=S_2S_1$. Let $H_0 = H_1 \oplus H_2 \oplus H$, and $H_3 = \{\xi \oplus S_1\xi \oplus S\xi \mid \xi \in \mathcal{D}_S\}$. If q is the projection of H_0 onto $H_1 \oplus \{0\} \oplus H$, we see that $qp_{H_3} H_0 = \Gamma_S$ and therefore

S is para-closed, which completes the proof of Proposition.

In [1] it is proved that if S and T are semi-closed operators, then so are S+T and ST (whenever the latter are defined). Therefore

1.2. Corollary. If S and T are paraclosed operators, then so are S+T and ST (whenever the latter are defined). We need also the following:

1.3. Proposition [2]. Let $S: \mathcal{D}_S \longmapsto H(\mathcal{D}_S \subset H_1)$ be a para-closed operator. If $\mathcal{D}_S$ is closed, then S is continuous.

2. Para-reductive algebras.

2.1. Theorem. Let $A \subset B(H)$ be a para-reductive algebra such that M_A has the property (P). Then $A=M_A$.

For the proof of this Theorem we need some lemmas. The next Lemma is due to D. Voiculescu [10]. We include his proof for the convenience of the reader.

2.2. Lemma. Let Z be a commutative von Neumann algebra, an let A be a para-reductive algebra such that $Z \subset A \subset Z'$ and $M_A = Z'$. Then $A' = Z$.

Proof. Since $Z \subset A \subset Z'$, it follows that $Z \subset A' \subset Z'$. So, to prove the Lemma, we must show that $A' \subset Z$. Let $t \in A'$. Then, by ([8], Proposition 6.4), there exists $z \in Z$, such that for every projection $p \in Z$, the element $(z-t)p$ has no inverse in the algebra $Z'p$. Obviously $\ker(z-t) \in \mathrm{Lat}A \subset \mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}Z'$, and therefore the projection p_o onto $\ker(z-t)$ is in Z. The element $(z-t)(1-p_o)$ of $Z'(1-p_o)$ is injective. We show that it is equal to zero. Indeed, if this is not true, then

$0 \neq \mathrm{Range}\,[(z-t)(1-p_o)] \in \mathrm{Lat}_{1/2}A(1-p_o) = \mathrm{Lat}_{1/2}Z'(1-p_o)$. By ([9] Théorème 2) it follows that there exists a positive $z_o \in Z(1-p_o)$ such that $\mathrm{Range}\,[(z-t)(1-p_o)] = \mathrm{Range}\, z_o$. By the spectral theorem, it follows that there exists a spectral projection $0 \neq p \leq 1-p_o$ of such that $pH \subset \mathrm{Range}\, z_o =$

= Range $[(z-t)(1-p_o)]$. Then $(z-t)p$ is invertible in $Z'p$ which is impossible. Therefore $(z-t)(1-p_o)=0$, and hence $t=z\in Z$.

<u>2.3. Lemma.</u> Let $A\subset B(H)$ be a para-reductive algebra. Then $A'=M'_A$

<u>Proof.</u> Since A is para-reductive, it follows that A is reductive. Let Z be the center of M_A. According to ([3] Corollary 1) we have $Z\subset A''$. Let $p\in Z$ be the projection such that pM_A is abelian and of infinite uniform multiplicity. Then, by ([3] Theorem 3) it follows that $((1-p)A'')'=(1-p)M'_A$ Thus, since $((1-p)A'')'=(1-p)A'$, we have $(1-p)A'=(1-p)M'_A$. By the preceding Lemma, it follows also $pA'=pM'_A$ and hence $A'=M'_A$.

The following Lemma is a consequence of ([5], Corollary 1.3).

<u>2.4. Lemma.</u> Let $A\subset B(H)$ be a para-reductive algebra. Suppose that every para-closed, densely defined operator that commutes with A, commutes also with M_A. Then $A=M_A$

<u>Proof. of Theorem 2.1.</u> We shall verify the hypothesis of Lemma 2.4. By Lemma 2.3 we have $A'=M'_A$. Let $S:\mathcal{D}_S\longmapsto H$ ($\mathcal{D}_S\subset H$) be a para-closed densely defined operator, that commutes with A. It is easy to see that $\mathcal{D}_S\in \mathrm{Lat}_{1/2}A=\mathrm{Lat}_{1/2}M_A$. By ([9] Théorème 2) there exists $m'\in M'_A$, $m'\geq 0$ such that $\mathcal{D}_S=m'H$. Since S is para-closed and m' continuous, by Corollary 1.2, it follows that Sm' is a para-closed operator. Since $\mathcal{D}_{Sm'}=H$, by Proposition 1.3 we have Sm' continuous. Since S commutes with A and $m'\in M'_A=A'$, it follows that $Sm'\in A'=M'_A$.

Now, we show that S commutes with M_A. Let $m\in M_A$. Then we have:

$$mSm'=Sm'm=Smm'$$

Therefore $mS=Sm$ on $m'H=\mathcal{D}_S$ and the Theorem is proved.

2.5.Corollary. Let $A\subset B(H)$ be a para-reductive algebra such that M_A is a discrete von Neumann algebra. Then $A=M_A$

Proof. Since discrete von Neumann algebras have the property (P), the corollary follows imediately from Theorem 2.1.

The following Corollary generalizes ([6] Corollary 8.5) to the case of von Neumann algebras.

2.6.Corollary. Let $A\subset B(H)$ be a weakly closed algebra, such that M_A is a discrete von Neumann algebra. If every linear subsopace of H, invariant under A, is invariant under M_A, then $A=M_A$.

Remark. If M_A is a continous von Neumann algebra, the preceding Corollary follows from ([9] Théorème 1) or ([5] ,Corollary 1.3).

3.Reductive algebras. In ([5] Theorem 2.2) is proved the following:

3.1.Theorem. Let $A\subset B(H)$ an algebra with the following properties:

a) $A^{(2)}$ is reductive

b) $A^{(2)}$ contains a von Neumann algebra $N^{(2)}$ with the property (P) and having finite commutant.

Then $A=M_A$

At the end of the same paper, we claimed (without proof) that the following improvement of this theorem can be given (see the proof below):

3.2.Theorem. Let $A\subset B(H)$ be an algebra with the following properties:

a) $A^{(2)}$ is reductive

b) A contains a von Neumann algebra N having finite commutant

Then $A=M_A$

We recall that a linear transformation $T: \mathcal{D}_T \longmapsto H$, ($\mathcal{D}_T \subset H$) is a graph transformation for A([6]) if there exist $n \in N$ and linear transformations $\{T_i\}_{i=1}^{n-2}$ defined on $\mathcal{D}_T$ such that. $\{\xi \oplus T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi \mid \xi \in \mathcal{D}_T\} \in \operatorname{Lat} A^{(n)}$.

To prove the Theorem 3.2 we need the following:

<u>3.3. Lemma.</u> Let $N \subset B(H)$ be a von Neumann algebra, having finite commutant. Then, every densely defined graph transformation for N , is pre-closed.

<u>Proof.</u> Let T be a densely defined graph transformation of N and let $\{T_i\}_{i=1}^{n-2}$ be linear transformations defined on $\mathcal{D}_T$ such that $\{\xi \oplus T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi \mid \xi \in \mathcal{D}_T\} \in \operatorname{Lat} H^{(n)}$. If Δ_{n-1} is the diagonal of $H^{(n-1)}$, then it is easy to see that the transformation : $\tilde{T}: (\mathcal{D}_T^{(n-1)} \cap \Delta_{n-1}) \oplus \Delta_{n-1}^{\perp} \longmapsto H^{(n-1)}$ defined by:

$$\tilde{T}(\xi \oplus \cdots \oplus \xi) = T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi \quad \text{if } \xi \in \mathcal{D}_T$$

$$T(\xi_1 \oplus \cdots \oplus \xi_{n-1}) = 0 \oplus 0 \oplus \cdots \oplus 0 \quad \text{if } \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_{n-1} \in \Delta_{n-1}^{\perp}$$

is a densely defined, closed operator affiliated to $N^{(n-1)'}$. Since N' is finite, it follows that $N^{(n-1)'}$ is a finite von Neumann algebra. Let p be the projection of $H^{2(n-1)}$ onto its n-th component. Then $p \in N^{(n-1)'}$. Since T is affiliated to $N^{(n-1)'}$, then according to ([4] Theorem XV, p.119) we obtain that pT is preclosed, whence T is preclosed.

<u>The proof of Theorem 3.2</u>, is similar with the proof of Theorem 2.2. [5] applying Lemma 3.3 instead of Lemma 2.3 [5]

<u>3.4. Corollary.</u> Let $N, M \subset B(H)$ be two type II von Neumann algebras such that $N \subset M$ and N' is finite . Then every reductive algebra A with $N \subset A \subset M$ is a von Neumann algebra.

<u>Proof.</u> Since M is a type II von Neumann algebra, M'

is also a type II von Neumann algebra. Therefore, there exist $p_1, p_2 \in M'$ orthogonal projections which are equivalent and $p_1+p_2=1$. A standard argument shows that the algebras A and $(p_1 A p)^{(2)}$ are unitarily equivalent.

Since A is reductive, it follows that $(p_1 A p_1)^{(2)}$ is reductive. The von Neumann algebra $p_1 N p_1$ has finite commutant because $p_1 \in M' \subset N'$. Therefore the algebra $p_1 A p_1$ satisfies the hypothesis of Theorem 3.2. By this theorem $p_1 A p_1$ is a von Neumann algebra and hence A is a von Neumann algebra.

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INVARIANT SUBSPACES OF VON NEUMANN ALGEBRAS II

by

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May 1977

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In what follows, H denotes a complex Hilbert space. The algebra of all bounded operators on H is denoted by B(H). For $A \subset B(H)$, A' is the commutant of A, and A'' the bicommutant. Also, we denote by M_A, the von Neumann algebra generated by A. For every $n \in \mathbb{N}$, we denote $H^{(n)} = \bigoplus_{i=1}^{n} H_i$ where $H_i = H$ for every i, $1 \leq i \leq n$, and $A^{(n)} = \{a \oplus a \oplus \ldots \oplus a \mid a \in A\}$. The collection of all closed linear subspaces of H invariant under A(i.e. invariant under every $a \in A$) is denote by Lat A. A weakly closed algebra $A \subset B(H)$ is reductive ([6]) if $1 \in A$, and $\mathrm{Lat}_A = \mathrm{Lat}\ M_A$. A linear subspace $K \subset H$ is para-closed ([2]) if there exist a Hilbert space H_0 and a bounded linear operator $Q: H_0 \longmapsto H$ such that $QH_0 = K$. The collection of all para-closed subspaces of H, invarint under A is denoted $\mathrm{Lat}_{1/2}A$. A weakly closed algebra $A \subset B(H)$ will be called para-reductive if $1 \in A$, and $\mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}M_A$. In this paper (Section 2) we show that if A is a para-reductive algebra such that M_A has the property (P) (i.e. for every $x \in B(H)$ we have $\overline{co}^{wo}\{u^* x u \mid u \in M_A, \text{unitary}\} \cap M'_A \neq \emptyset$ ([7])), then $A = M_A$ It is known that all discrete von Neumann algebras have the property (P). Thus, Theorem 2.1 below, together with [9] Théorème 1 or [5] Corollary 1.3 show that if A is a para-reductive algebra such that M_A is either discrete or continuous, then $A = M_A$. In order to prove this result, in Section 1 we give some new results on para-closed operators.

Finally, in Section 3 we prove a result, announced (without proof) in [5]

<u>1. Paraclosed operators.</u> Let H_1, H be Hilbert spaces. A linear transformation $S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) is <u>para-closed</u> ([2]) if its graph $\Gamma_S = \{\xi \oplus S\xi \mid \xi \in \mathcal{D}_S\}$ is a para-closed subspace of $H_1 \oplus H$. A linear transformation

$S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) is semi-closed ([1]) if there exist a Hilbert space H_2 and two closed operators $S_1: \mathcal{D}_S \longmapsto H_2$ $S_2: \mathcal{D}_{S_2} \longmapsto H$ ($\mathcal{D}_{S_2} \subset H_2$) such that $S=S_2S_1$. The following Proposition shows that these two notions are equivalent:

1.1. Proposition. Let H_1, H be Hilbert spaces, and let $S: \mathcal{D}_S \longmapsto H$ ($\mathcal{D}_S \subset H_1$) be a linear transformation. The following are equivalent.

a) S is para-closed

b) S is semi-closed

Proof. a) $\Longrightarrow$ b). Since S is para-closed, its graph Γ_S is a para-closed subspace of H_1 H. By the definition of paraclosed subspaces, there exist a Hilbert space H_2 and a bounded linear operator $Q: H_2 \longmapsto H_1 \oplus H$ such that $QH_2 = \Gamma_S$ We may suppose that Q is injective, since otherwise we replace H_0 by $(\ker Q)^{\perp}$. Then, for every $\xi \in \mathcal{D}_S$ there exists a unique $S_1(\xi) \in H_2$ such that $QS_1(\xi) = \xi \oplus S(\xi)$. Obviously, S_1 is a linear transformation $S_1: \mathcal{D}_S \longmapsto H_2$. Moreover, S_1 is a closed operator with $\mathcal{D}_{S_1} = \mathcal{D}_S$. Indeed, let $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{D}_S$ be a sequence such that $\lim_{n\to\infty} \xi_n = \xi_0$ and $\lim S_1(\xi_n) = \eta_0$ Since Q is a continous, it follows that $\lim QS_1(\xi_n) = Q(\eta_0)$. On the other hand $QS_1(\xi_n) = \xi_n \oplus S(\xi_n)$ and hence $Q(\eta_0) = \xi_0 \oplus S\xi_0$. It follows that $\xi_0 \in \mathcal{D}_S$, and $S_1(\xi_0) = \eta_0$. Therefore S_1 is closed and $\mathcal{D}_{S_1} = \mathcal{D}_S$. If p_H is the projection of $H_1 \oplus H$ onto H, then $S_2 = p_H Q$ is a bounded operator, and $S = S_2 S_1$, whence S is semiclosed.

b) $\Longrightarrow$ a). Let H_2 be a Hilbert space and let $S_1: \mathcal{D}_S \longmapsto H_2$, $S_2: \mathcal{D}_{S_2} \longrightarrow H$ ($\mathcal{D}_{S_2} \subset H_2$) be closed operators such that $S=S_2S_1$. Let $H_0 = H_1 \oplus H_2 \oplus H$, and $H_3 = \{\xi \oplus S_1\xi \oplus S\xi \mid \xi \in \mathcal{D}_S\}$. If q is the projection of H_0 onto $H_1 \oplus \{0\} \oplus H$, we see that $qp_{H_3} H_0 = \Gamma_S$ and therefore

S is para-closed, which completes the proof of Proposition. In [1] it is proved that if S and T are semi-closed operators, then so are S+T and ST (whenever the latter are defined). Therefore

1.2. Corollary. If S and T are paraclosed operators, then so are S+T and ST (whenever the latter are defined). We need also the following:

1.3. Proposition [2]. Let $S: \mathcal{D}_S \longmapsto H(\mathcal{D}_S \subset H_1)$ be a paraclosed operator. If $\mathcal{D}_S$ is closed, then S is continuous.

2. Para-reductive algebras.

2.1. Theorem. Let $A \subset B(H)$ be a para-reductive algebra such that M_A has the property (P). Then $A=M_A$.

For the proof of this Theorem we need some lemmas. The next Lemma is due to D. Voiculescu [10]. We include his proof for the convenience of the reader.

2.2. Lemma. Let Z be a commutative von Neumann algebra, an let A be a para-reductive algebra such that $Z \subset A \subset Z'$ and $M_A=Z'$. Then $A'=Z$.

Proof. Since $Z \subset A \subset Z'$, it follows that $Z \subset A' \subset Z'$. So, to prove the Lemma, we must show that $A' \subset Z$. Let $t \in A'$. Then, by ([8], Proposition 6.4), there exists $z \in Z$, such that for every projection $p \in Z$, the element $(z-t)p$ has no inverse in the algebra $Z'p$. Obviously $\ker(z-t) \in \mathrm{Lat}A \subset \mathrm{Lat}_{1/2}A = \mathrm{Lat}_{1/2}Z'$, and therefore the projection p_0 onto $\ker(z-t)$ is in Z. The element $(z-t)(1-p_0)$ of $Z'(1-p_0)$ is injective. We show that it is equal to zero. Indeed, if this is not true, then

$0 \neq \mathrm{Range}\,[(z-t)(1-p_0)] \in \mathrm{Lat}_{1/2}A(1-p_0) = \mathrm{Lat}_{1/2}Z'(1-p_0)$. By ([9] Théorème 2) it follows that there exists a positive $z_0 \in Z(1-p_0)$ such that $\mathrm{Range}\,[(z-t)(1-p_0)] = \mathrm{Range}\,z_0$. By the spectral theorem, it follows that there exists a spectral projection $0 \neq p \leq 1-p_0$ of such that $pH \subset \mathrm{Range}\,z_0 =$

= Range $[(z-t)(1-p_o)]$. Then $(z-t)p$ is invertible in $Z'p$ which is impossible. Therefore $(z-t)(1-p_o)=0$, and hence $t=z\in Z$.

2.3. Lemma. Let $A\subset B(H)$ be a para-reductive algebra. Then $A'=M'_A$

Proof. Since A is parareductive, it follows that A is reductive. Let Z be the center of M_A. According to ([3] Corollary 1) we have $Z\subset A''$. Let $p\in Z$ be the projection such that pM_A is abelian and of infinite uniform multiplicity. Then, by ([3] Theorem 3) it follows that $((1-p)A'')'=(1-p)M'_A$ Thus, since $((1-p)A'')'=(1-p)A'$, we have $(1-p)A'=(1-p)M'_A$. By the preceding Lemma, it follows also $pA'=pM'_A$ and hence $A'=M'_A$.

The following Lemma is a consequence of ([5], Corollary 1.3).

2.4. Lemma. Let $A\subset B(H)$ be a para-reductive algebra. Suppose that every para-closed, densely defined operator that commutes with A, commutes also with M_A. Then $A=M_A$

Proof. of Theorem 2.1. We shall verify the hypothesis of Lemma 2.4. By Lemma 2.3 we have $A'=M'_A$. Let $S: \mathcal{D}_S\longmapsto H$ ($\mathcal{D}_S\subset H$) be a para-closed densely defined operator, that commutes with A. It is easy to see that $\mathcal{D}_S\in \mathrm{Lat}_{1/2}A=\mathrm{Lat}_{1/2}M_A$. By ([9] Théorème 2) there exists $m'\in M'_A$, $m'\geq 0$ such that $\mathcal{D}_S=m'H$. Since S is para-closed and m' continuous, by Corollary 1.2, it follows that Sm' is a para-closed operator. Since $\mathcal{D}_{Sm'}=H$, by Proposition 1.3 we have Sm' continous. Since S commutes with A and $m'\in M'_A=A'$, it follows that $Sm'\in A'=M_A'$.

Now, we show that S commutes with M_A. Let $m\in M_A$. Then we have:

$$mSm'=Sm'm=Smm'$$

Therefore $mS=Sm$ on $m'H=\mathcal{D}_S$ and the Theorem is proved.

2.5.Corollary. Let $A\subset B(H)$ be a para-reductive algebra such that M_A is a discrete von Neumann algebra. Then $A=M_A$

Proof. Since discrete von Neumann algebras have the property (P), the corollary follows imediately from Theorem 2.1.

The following Corollary generalizes ([6] Corollary 8.5) to the case of von Neumann algebras.

2.6.Corollary. Let $A\subset B(H)$ be a weakly closed algebra, such that M_A is a discrete von Neumann algebra. If every linear subscpace of H, invariant under A, is invariant under M_A, then $A=M_A$.

Remark. If M_A is a continous von Neumann algebra, the preceding Corollary follows from ([9] Théorème 1) or ([5], Corollary 1.3).

3.Reductive algebras. In ([5] Theorem 2.2) is proved the following:

3.1.Theorem. Let $A\subset B(H)$ an algebra with the following properties:

a) $A^{(2)}$ is reductive

b) $A^{(2)}$ contains a von Neumann algebra $N^{(2)}$ with the property (P) and having finite commutant.

Then $A=M_A$

At the end of the same paper, we claimed (without proof) that the following improvement of this theorem can be given (see the proof below):

3.2.Theorem. Let $A\subset B(H)$ be an algebra with the following properties:

a) $A^{(2)}$ is reductive

b) A contains a von Neumann algebra N having finite commutant

Then $A=M_A$

We recall that a linear transformation $T: \mathcal{D}_T \longmapsto H$, ($\mathcal{D}_T \subset H$) is a graph transformation for A([6]) if there exist $n \in N$ and linear transformations $\{T_i\}_{i=1}^{n-2}$ defined on $\mathcal{D}_T$ such that. $\{\{\xi \oplus T\xi\} \oplus T_1\xi\} \oplus \cdots \oplus T_{n-2}\xi\} \mid \xi \in \mathcal{D}_T\} \in \operatorname{Lat} A^{(n)}$.

To prove the Theorem 3.2 we need the following:

<u>3.3.Lemma.</u> Let $N \subset B(H)$ be a von Neumann algebra, having finite commutant. Then, every densely defined graph transformation for N , is pre-closed.

<u>Proof.</u> Let T be a densely defined graph transformation of N and let $\{T_i\}_{i=1}^{n-2}$ be linear transformations defined on $\mathcal{D}_T$ such that $\{\{\xi \oplus T\xi\} \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi\} \mid \xi \in \mathcal{D}_T\} \in \operatorname{Lat} N^{(n)}$. If Δ_{n-1} is the diagonal of $H^{(n-1)}$, then it is easy to see that the transformation : $\tilde{T}: (\mathcal{D}_T^{(n-1)} \cap \Delta_{n-1}) \oplus \Delta_{n-1}^{\perp} \longmapsto H^{(n-1)}$ defined by:

$$\tilde{T}(\xi \oplus \cdots \oplus \xi) = T\xi \oplus T_1\xi \oplus \cdots \oplus T_{n-2}\xi \quad \text{if } \xi \in \mathcal{D}_T$$

$$T(\xi_1 \oplus \cdots \oplus \xi_{n-1}) = 0 \oplus 0 \oplus \cdots \oplus 0 \quad \text{if } \xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_{n-1} \in \Delta_{n-1}^{\perp}$$

is a densely defined, closed operator affiliated to $N^{(n-1)'}$. Since N' is finite, it follows that $N^{(n-1)'}$ is a finite von Neumann algebra. Let p be the projection of $H^{2(n-1)}$ onto its n-th component. Then $p \in N^{(n-1)'}$. Since T is affiliated to $N^{(n-1)'}$, then according to ([4] Theorem XV, p.119) we obtain that pT is preclosed, whence T is preclosed.

<u>The proof of Theorem 3.2</u>, is similar with the proof of Theorem 2.2. [5] applying Lemma 3.3 instead of Lemma 2.3 [5]

<u>3.4.Corollary.</u> Let $N, M \subset B(H)$ be two type II von Neumann algebras such that $N \subset M$ and N' is finite . Then every reductive algebra A with $N \subset A \subset M$ is a von Neumann algebra.

<u>Proof.</u> Since M is a type II von Neumann algebra, M'

is also a type II von Neumann algebra. Therefore, there exist $p_1, p_2 \in M'$ orthogonal projections which are equivalent and $p_1+p_2=1$. A standard argument shows that the algebras A and $(p_1Ap)^{(2)}$ are unitarily equivalent.

Since A is reductive, it follows that $(p_1Ap_1)^{(2)}$ is reductive. The von Neumann algebra p_1Np_1 has finite commutant because $p_1 \in M' \subset N'$. Therefore the algebra p_1Ap_1 satisfies the hypothesis of Theorem 3.2. By this theorem p_1Ap_1 is a von Neumann algebra and hence A is a von Neumann algebra.

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