ON THE CHOQUET AND BISHOP - DE LEEUW THEOREMS

by

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In the theory of integral representation of points in compact convex sets (Choquet theory), there are two fundamental theorems, which assert that the maximal Radon probability measures on such (sub) sets (of Hausdorff locally convex topological real vector spaces) are pseudoconcentrated on their sets of extreme points. Maximality is meant here either in the sense of the Bishop-de Leeuw preorder relation, to which the Bishop-de Leeuw theorem belongs (see [3] , theorem 5.3), or in the sense of the Choquet order relation, to which the Choquet theorem belongs (see [5] ; [7] , theorem T32 ; [8] , Ch.4). An immediate consequence of these two theorems is the Choquet - Bishop - de Leeuw theorem, which yields the representability of any point in the given compact convex set by a boundary integral (see [8] , Ch.4).

Chronologically, Choquet first proved a theorem of this kind, for metrizable compact convex sets, in 1956 (see [4]). In 1959 Bishop and de Leeuw, by using a certain preorder relation (which will be recalled in what follows) in the set of all Radon probability measures, on arbitrary compact convex subsets of Hausdorff locally convex topological real vector spaces, proved that any such measure, which is maximal for the considered preorder relation, is pseudoconcentrated on the set of all extreme points of the given compact convex set.

By establishing the existence of such maximal measures, they thus obtained an extension of Choquet's theorem (see [3] , theorem 5.6).

In 1960, Choquet, by using another order relation, showed that any maximal measure (with respect to his order relation)is pseudoconcentrated on the set of all extreme points of the given compact convex set. Again, by proving the existence of such maximal measures, Choquet obtained another proof of Bishop's and de Leeuw's extension of Choquet's theorem, to the effect that any point of the given compact convex set K is represented by a Radon probability measure, which is pseudoconcentrated on the set of all extreme points of K. This result is known as the Choquet - Bishop - de Leeuw theorem (see [8] ,Ch.4).

Of course, both the Bishop-de Leeuw theorem and the Choquet theorem are stronger than the Choquet - Bishop - de Leeuw theorem, in that they assert that any maximal measure, either with respect to the Bishop - de Leeuw preorder relation, or with respect to the Choquet order relation, is pseudoconcentrated on the set of extreme points.

Since the two (pre) order relations are, in general, different (see [9] , p.289), these two theorems are different.

The Bishop - de Leeuw preorder relation is very useful in connection with the central and irreducible disintegration of the representations of C^* - algebras (see [9] , [10] , [11] , [12]). Howeover, for the orthogonal Radon probability measures, defined on the state space of a C^* - algebra, the two order relations coincide (see [9] , theorem 13).

The aim of this Note is to give a generalization of the theorems of Choquet and Bishop - de Leeuw. In this manner we shall obtain a new and unitary proof for both the theorems of Bishop - de Leeuw and Choquet. This will be achieved by means of a slight extension of H.Bauer's Minimum Principle.

1. Let E be a Hausdorff locally convex topological real vector space and $K \subset E$ a compact convex subset of E.

By $C(K ; \mathbb{R})$ we shall denote the algebra of all continuous real functions, defined on K; by $A(K ; \mathbb{R})$ we shall denote the real vector space of all continuous affine real functions, defined on K ; by $S(K ; \mathbb{R})$ we denote the convex sup-cone of all convex continuous real functions, which are defined on K. It is well knovon that $S(K ; \mathbb{R}) - S(K ; \mathbb{R})$ is a vector sublattice of $C(K ; \mathbb{R})$, uniformly dense in virtue of the Stone approximation theorem (see [8],Ch.4). It is obvious that $h \in A(K ; \mathbb{R}) \Rightarrow h^2 \in S(K ; \mathbb{R})$.

Let $\mathcal{M}^1_+(K)$ be the convex set of all Radon probability measures on K ; for any $\mu \in \mathcal{M}^1_+(K)$ there exists a uniquely determined $b(\mu) \in K$, such that

$$h(b(\mu)) = \int_K h(x)d\mu(x), \qquad \forall \quad h \in A(K ; \mathbb{R});$$

$b(\mu)$ is called the <u>barycenter</u> of μ (see [8] , Ch.1).

Two measures $\mu, \nu \in \mathcal{M}^1_+(K)$ are said to be <u>equivalent</u> if $b(\mu) = b(\nu)$. One denotes this relation by $\mu \sim \nu$. A point $x \in K$ is said to be <u>represented</u> by $\mu \in \mathcal{M}^1_+(K)$ if $x = b(\mu)$. It is obvious that any $x \in K$ is represented by the Dirac measure ε_x at x.

The fundamental theorems of Choquet and Bishop - de Leeuw give existential solutions to the following problem : given $x \in K$, represent it by a measure $\mu \in \mathcal{M}^1_+(K)$, whose support be as close to ex K, as possible (here ex K denotes the set of all extreme points of K).

We consider the following relations on $\mathcal{M}_+^1(K)$:

a) $\mu \prec \nu$ iff $\mu(f) \leq \nu(f)$, for any $f \in S(K;\mathbb{R})$;

b) $\mu \ll \nu$ iff $\mu(h^2) \leq \nu(h^2)$, for any $h \in A(K;\mathbb{R})$.

The first is due to Choquet (see [5] , [7] , [8]), and the second to Bishop and de Leeuw (see [3]).

Obviously, $\mu \prec \nu$ implies $\mu \ll \nu$, for any $\mu, \nu \in \mathcal{M}_+^1(K)$. Both relations are reflexive and transitive ; the first is also antisymmetric, hence an order relation ; this follows from the fact that "$\mu \prec \nu$ and $\nu \prec \mu$" implies $\mu(f) = \nu(f)$, for any $f \in S(K;\mathbb{R})$. Consequently, the same equality holds for any $f \in S(K;\mathbb{R}) - S(K;\mathbb{R})$. Stone's approximation theorem now implies that $\mu = \nu$.

Since $A(K;\mathbb{R}) = -A(K;\mathbb{R}) \subset S(K;\mathbb{R})$, it is easy to see that $\mu \prec \nu \Rightarrow \mu \sim \nu$. Since $\pm 1 \in A(K;\mathbb{R})$, an elementary argument shows that $\mu \ll \nu \Rightarrow \mu \sim \nu$.

An element $\mu \in \mathcal{M}_+^1(K)$ is maximal for the Choquet order relation if $\nu \in \mathcal{M}_+^1(K)$ and $\mu \prec \nu$ implies $\mu = \nu$. Similarly, an element $\mu \in \mathcal{M}_+^1(K)$ is maximal for the Bishop - de Leeuw preorder relation if $\nu \in \mathcal{M}_+^1(K)$ and $\mu \ll \nu$ implies $\mu(h^2) = \nu(h^2)$, $\forall\, h \in A(K;\mathbb{R})$.

In order to get the generalization of the theorems of Choquet and Bishop - de Leeuw, we have in view, we shall consider an arbitrary subset $S \subset S(K;\mathbb{R})$, such that

$$\{h^2 \,;\, h \in A(K;\mathbb{R})\} \subset S.$$

We shall define a preorder relation in $\mathcal{M}_+^1(K)$ by defining

$$\mu \blacktriangleleft \nu \iff \mu(f) \leq \nu(f),\ \forall f \in S,\ \forall \mu, \nu \in \mathcal{M}_+^1(K).$$

Of course, the relation is reflexive and transitive, and $\mu \prec \nu \Rightarrow \mu \lhd \nu$, $\mu \lhd \nu \Rightarrow \mu \ll \nu$, $\forall \mu, \nu \in \mathcal{M}_+^1(K)$. It follows that $\mu \lhd \nu \Rightarrow \mu \sim \nu$, $\forall\ \mu, \nu \in \mathcal{M}_+^1(K)$. A measure $\mu \in \mathcal{M}_+^1(K)$ is <u>maximal</u> for the preorder relation $\lhd$ iff $\nu \in \mathcal{M}_+^1(K)$ and $\mu \lhd \nu \Rightarrow \mu(f) = \nu(f)$, $\forall f \in S$. In this case we shall say that μ is $(\lhd)$ - <u>maximal</u>.

2. Let $\mathcal{B}(K)$ be the σ- algebra of all Borel measurable subsets of K and $\mathcal{B}_o(K)$ the σ- algebra of all Baire measurable subsets of K. We recall that $\mathcal{B}(K)$ is the σ- algebra generated by all open (or, equivalently, closed) subsets of K, whereas $\mathcal{B}_o(K)$ is the smallest σ- algebra of subsets of K, such that all functions $f \in C(K ; \mathbb{R})$ be measurable. Obviously, one has the inclusion $\mathcal{B}_o(K) \subset \mathcal{B}(K)$. If K is metrizable, then the equality holds.

A measure $\mu \in \mathcal{M}_+^1(K)$ is said to be <u>pseudoconcentrated</u> on ex K if

$$(*)\quad \mu(U) = 0, \text{ for any } U \in \mathcal{B}_o(K), \text{ such that } U \cap (\text{ex } K) = \emptyset.$$

Let $\mathcal{A}_o(\text{ex } K)$ be the σ- algebra of all subsets of ex K, which are traces on ex K of the Baire measurable subsets of K

$$\mathcal{A}_o(\text{ex } K) = \left\{ U \cap (\text{ex } K);\ U \in \mathcal{B}_o(K) \right\}.$$

If the measure $\mu \in \mathcal{M}_+^1(K)$ is pseudoconcentrated on ex K, then one can define a probability measure $\check{\mu}$ on $\mathcal{A}_o(\text{ex } K)$ by the formula

$$\check{\mu}(U \cap (\text{ex } K)) = \mu(U), \quad \forall\ U \in \mathcal{B}_o(K).$$

Property (*) ensures the correctness of the definition.

It is easy to prove now that if $f : K \to \mathbb{R}$ is a Baire measurable function (i.e., if it is $\mathcal{B}_0(K)$ - measurable), then $f \mid \mathrm{ex}\, K$ is $\mathcal{A}_0(\mathrm{ex}\, K)$ - measurable. The importance of measures which are pseudoconcentrated on ex K is shown by the following.

<u>Lemma 1.</u> Let $\mu \in \mathcal{M}_+^1(K)$ be pseudoconcentrated on ex K and $f : K \to \mathbb{R}$ a bounded Baire measurable function. Then

$$\int_K f \, d\mu = \int_{\mathrm{ex}\, K} f \, d\check{\mu}.$$

<u>Proof.</u> For any $\varepsilon > 0$ one can find a partition $\{U_1, U_2, \ldots, U_n\}$ of K, consisting of Baire measurable subsets of K, and real numbers $a_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$\left| f(x) - \sum_{i=1}^{n} a_i \chi_{U_i}(x) \right| < \varepsilon \quad , \quad \forall \; x \in K.$$

We then have

$$\left| \int_K f \, d\mu - \sum_{i=1}^{n} a_i \mu(U_i) \right| < \varepsilon.$$

and

$$\left| \int_{\mathrm{ex}\, K} f \, d\check{\mu} - \sum_{i=1}^{n} a_i \check{\mu}(U_i \cap (\mathrm{ex}\, K)) \right| < \varepsilon.$$

By taking into account the definition of $\check{\mu}$ we now easily get that

$$\left| \int_K f \, d\mu - \int_{\mathrm{ex}\, K} f \, d\check{\mu} \right| < 2\varepsilon,$$

for any $\varepsilon > 0$. The lemma is proved.

A simple compactness argument shows that the ($\lhd$) - ordered set $\mathcal{M}^1_+(K)$ is inductive. Hence Zorn's lemma implies that for any $\mu \in \mathcal{M}^1_+(K)$ there exists a ($\lhd$) - maximal measure $\nu \in \mathcal{M}^1_+(K)$, such that $\mu \lhd \nu$. In particular, this is true for the order relation $\prec$ (see [8], Ch.4), as well as for the pre-order relation $\ll$ (see [3], p.307).

3. The following lemma is a slight extension of proposition 4.2. from [8] (which is identical with part b) of the lemma). The proof is adapted from that of proposition 4.2. from [8].

We first recall that for any $f \in C(K ; \mathbb{R})$ one defines its <u>upper semicontinuous concave hull</u> $\bar{f}$ by

$$\bar{f}(x) = \inf_{\substack{h \in A(K ; \mathbb{R}) \\ h \geq f}} h(x) .$$

It is easy to prove that (see [8], Ch.3) :

a) $\bar{f}$ is concave, bounded and upper semicontinuous, for any $f \in C(K ; \mathbb{R})$;

b) $f \leq \bar{f}$ and $\bar{f} = f \Leftrightarrow f$ is concave, $\forall f \in C(K ; \mathbb{R})$;

c) if $f, g \in C(K ; \mathbb{R})$, then $(f+g)^- \leq \bar{f} + \bar{g}$ and $|\bar{f} - \bar{g}| \leq \leq \|f - g\|$; $(f+g)^- = \bar{f} + g$, for $g \in A(K ; \mathbb{R})$; $(r f)^- = = r\bar{f}$, for any $r \in \mathbb{R}_+$ and $f \in C(K ; \mathbb{R})$;

d) if $f \leq f'$, where f' is concave and upper semicontinuous, then $\bar{f} \leq f'$.

<u>Lemma 2.</u> a) If $\mu \in \mathcal{M}^1_+(K)$ is any ($\lhd$) - maximal measure then $\mu(\bar{f}) = \mu(f)$, for any $f \in S$; in particular, $\mu((h^2)^-) =$

$= \mu(h^2), \forall h \in A(K ; \mathbb{R})$.

b) If $\mu \in \mathcal{M}_+^1(K)$ is any $(\prec)$ - maximal measure, then $\mu(\bar{f}) = \mu(f)$, for any $f \in C(K ; \mathbb{R})$.

<u>Proof.</u> Let $f_o \in S$. We define a real linear functional $L : \mathbb{R}f_o \to \mathbb{R}$ by the formula

$$L(\alpha f_o) = a\,\mu(\bar{f}_o) \quad , \quad a \in \mathbb{R}.$$

We also define a positively homogeneons sublinear functional $p : C(K ; \mathbb{R}) \longrightarrow \mathbb{R}$ by the formula

$$p(f) = \mu(\bar{f}), \qquad f \in C(K ; \mathbb{R}).$$

For $a \in \mathbb{R}$, $a \geq 0$, we obviously have that $L(a f_o) = p(a f_o)$. For $a < 0$, from

$$0 = (a f_o - a f_o)^- \leq (a f_o)^- + (-a f_o)^- = (a f_o)^- - a \bar{f}_o,$$

we infer that

$$L(af_o) = a\,\mu(\bar{f}_o) = \mu(a\bar{f}_o) \leq \mu((af_o)^-) = p(a f_o).$$

The Hahn - Banach theorem now implies that there exists a linear functional $L' : C(K ; \mathbb{R}) \to \mathbb{R}$, such that

$$L' \mid \mathbb{R} f_o = L \text{ and } L'(f) \leq p(f), \forall f \in C(K ; \mathbb{R}).$$

If $g \in C(K ; \mathbb{R})$, $g \leq 0$, we have $\bar{g} \leq 0$ and, therefore,

$$L'(g) \leq p(g) = \mu(\bar{g}) \leq 0.$$

This shows that $L' \geqslant 0$ as a linear functional on $C(K;\mathbb{R})$ and, therefore, there exists a positive Radon measure ν on K, such that

$$L'(f) = \nu(f), \quad \forall \; f \in C(K;\mathbb{R}).$$

If $f \in C(K;\mathbb{R})$ is convex, then $-f$ is concave and, therefore, $(-f)^- = -f$; it follows that

$$\mu(f) = -\mu(-f) = -\mu((-f)^-) = -p(-f) \leqslant -L'(-f) =$$
$$= L'(f) = \nu(f),$$

and, consequently, we have that $\mu \prec \nu$; hence we also have that $\mu \lhd \nu$, i.e, $\mu(f) \leqslant \nu(f)$, $\forall$ $f \in S$.

a) If μ is ($\lhd$)-maximal, we infer that $\mu(f) = \nu(f)$, $\forall$ $f \in S$. In particular, we have

$$\mu(f_o) = \nu(f_o) = L'(f_o) = L(f_o) = \mu(\bar{f}_o).$$

b) If μ is ($\prec$)- maximal, then $\mu = \nu$. It follows that

$$\mu(f_o) = \nu(f_o) = L'(f_o) = L(f_o) = \mu(\bar{f}_o).$$

The lemma is proved.

For any $f \in C(K;\mathbb{R})$ let us denote $S_f = \{x \in K; \bar{f}(x) = f(x)\}$. Obviously, S_f is a G_δ - subset of K.

<u>Proposition 1.</u> (H.Bauer) For any $x \in K$ we have

$$x \in \operatorname{ex} K \iff \text{there is a unique } \mu \in \mathcal{M}^1_+(K)$$
$$\text{such that } b(\mu) = x.$$

Proof. (see [8] , Ch.1). If $x \in \operatorname{ex} K$ and $\mu \in \mathcal{M}^1_+(K)$ is such that $b(\mu) = x$, we have $\operatorname{supp}\mu = \{x\}$. Indeed it is sufficient to prove that $\mu(D) = 0$ for any compact set $D \subset K \setminus \{x\}$. If this is not true, then there exists a compact set $D \subset K \setminus \{x\}$ and a point $y \in D$, such that $\mu(D \cap U) > 0$ for any neighbourhood U of y. Let us choose a compact convex set U, such that $x \notin U$ and define $K_0 = \overline{co}(D \cap U)$. Then $x \notin K_0$ and $\mu(K_0) > 0$. Let us define $\mu_1 = (\mu(K_0))^{-1}(\chi_{K_0}\mu)$. We obviously have $b(\mu_1) \in K_0$. If $\mu(K_0) = 1$, then $\mu_1 = \mu$, hence $x = b(\mu) = b(\mu_1) \in K_0$, a contradiction . Consequently, we have $\mu(K_0) < 1$. Let $\mu_2 = (1 - \mu(K_0))^{-1}(\chi_{\complement K_0}\mu)$. Then $b(\mu_2) \in K$ and $\mu = \mu(K_1)\mu_1 + (1 - \mu(K_1))\mu_2$; this implies that

$$x=b(\mu) = \mu(K_1)\, b(\mu_1) + (1-\mu(K_1))\, b(\mu_2),$$

where $b(\mu_1) \neq x$, a contradiction. Consequently, $\mu = \varepsilon_x$.

Conversely, if $x \in K \setminus \operatorname{ex} K$, then there exist $x_1, x_2 \in K$, such that $x = \frac{1}{2}(x_1 + x_2)$ and $x_1, x_2 \neq x$. We then have

$\varepsilon_x \sim \frac{1}{2}(\varepsilon_{x_1} + \varepsilon_{x_2})$ and $\varepsilon_x \neq \frac{1}{2}(\varepsilon_{x_1} + \varepsilon_{x_2})$. The proposition is proved.

Proposition 2. (see [8] , proposition 3.1). For any $f \in C(K\,;\mathbb{R})$ and any $x \in K$ we have

$$\overline{f}(x) = \sup \left\{ \int_K f(x)\,d\mu(x) \;;\; \mu \in \mathcal{E}_x \right\}.$$

In particular, if $x \in \operatorname{ex} K$, then $\overline{f}(x) = f(x)$.

<u>Proof.</u> Let us define $f'(x) = \sup \left\{ \int_K f(x)\, d\mu(x) \; ; \mu \sim \varepsilon_x \right\}$. Then we obviously have $f(x) \leq f'(x)$, $x \in K$, and f' is easily shown to be concave. In order to prove that it is upper semicontinuous, let $r \in \mathbb{R}$ and $(x_\alpha)_{\alpha \in I}$ be a convergent net in K, $x = \lim_{\alpha \in I} x_\alpha$, such that $f'(x_\alpha) \geq r, \forall \alpha \in I$. For any $\varepsilon > 0$ and any $\alpha \in I$ we can choose a $\mu_\alpha \in \mathcal{M}^1_+(K)$, such that $\mu_\alpha \sim \varepsilon_{x_\alpha}$ and $r - \varepsilon \leq f'(x_\alpha) - \varepsilon < \mu_\alpha(f)$, $\forall \alpha \in I$. Since the set $\mathcal{M}^1_+(K)$ is compact for the vague topology, we can choose a convergent subnet $(\mu_{\alpha(\beta)})_{\beta \in J}$, $\lim_{\beta \in J} \mu_{\alpha(\beta)} = \mu$. We obviously have $b(\mu) = x$ and $r - \varepsilon \leq \mu(f) \leq f'(x)$, for any $\varepsilon > 0$; hence, $r \leq f'(x)$, and this shows that f' is upper semicontinuous. Consequently, the inequality $f \leq f'$ implies that $\bar{f} \leq \bar{f'} = f'$. On the other hand, we have (for any $\mu \in \mathcal{M}^1_+(K)$, $\mu \sim \varepsilon_x$),

$$\int_K f(x)\, d\mu(x) \leq \int_K \bar{f}(x)\, d\mu(x) \leq \bar{f}(x),$$

and this implies that $f'(x) \leq \bar{f}(x)$, $x \in K$.

<u>Corollary 1.</u> a) For any $f \in C(K ; \mathbb{R})$ we have $S_f \supset \mathrm{ex}\, K$.

b) If $H \subset A(K ; \mathbb{R})$ is a total set, then

$$\bigcap_{h \in H} S_{h^2} = \mathrm{ex}\, K.$$

c) If K is metrizable, then there exists a sequence

$(h_n)_{n \geq 0}$, $h_n \in A(K ; \mathbb{R})$, $\|h_n\| \leq 1$, $\forall n \geq 0$, such that

$$\bigcap_{n=0}^{\infty} S_{h_n^2} = \mathrm{ex}\, K.$$

Consequently, if K is metrizable, then ex K is a G_δ -

subset of K.

Proof. a) is an immediate consequence of proposition 2 b) Let us remark that if H is a total subset of $A(K ; \mathbb{R})$, then H separates the points of K. Let now $x \in \bigcap_{h \in H} S_{h^2}$. If $x \notin ex K$, then there exist $x_1, x_2 \in K$, $x_1, x_2 \neq x$, such that $x = \frac{1}{2}(x_1 + x_2)$. Since the set H separates the points of K, there exists a $h_o \in H$, such that $h_o(x_1) \neq h_o(x)$. Then, for any $h \in A(K ; \mathbb{R})$ we have

$$h \geqslant h_o^2 \Rightarrow h\,(x) \geqslant |h_o(x_1) - h_o(x)|^2 + h_o^2(x).$$

Consequently, we have $\overline{h}(x) \geqslant |h_o(x_1) - h_o(x)|^2 + h_o^2(x) > h_o^2(x)$, and this implies that $x \notin S_{h_o^2}$.

c) If K is metrizable, then $C(K ; \mathbb{R})$ is separable. Consequently, $A(K ; \mathbb{R})$ is separable and, therefore, in $\{h \in A(K ; \mathbb{R});\ \|h\| \leq 1\}$ there exists a countable, dense subset H, which, obviously, is total in $A(K ; \mathbb{R})$. Since any S_f is a G_δ - subset in K, in this case ex K is also a G_δ - subset.

Corollary 2. a) For any ($\prec$) - maximal measure $\mu \in \mathcal{M}_+^1(K)$, and any $f \in S$ we have $\mu(S_f) = 1$; in particular, $\mu(S_{h^2}) = 1$, $\forall h \in A(K ; \mathbb{R})$.

b) If K is metrizable, and $\mu \in \mathcal{M}_+^1(K)$ is any ($\prec$) - maximal measure, then $\mu(ex\ K) = 1$.

Proof. Assertion a) is an immediate consequence of lemma 2 and of the definition of the set S_f, $f \in C(K ; \mathbb{R})$. Assertion b) is an immediate consequence of assertion a) and of corollary 1.

4. An important result in Convex Analysis is the following " Minimum Principle " of H.Bauer (see [2]), lemma 1).

<u>Theorem 1</u> (H.Bauer). Let $f_1, f_2 : K \to \mathbb{R}$ be semicontinuous functions, with f_1 convex and f_2 concave.
Then

$$f_1(x) \leq f_2(x), \forall x \in \text{ex } K \Rightarrow f_1(x) \leq f_2(x), \forall x \in K .$$

(Here any kind of semicontinuity is allowed for the two functions ; moreover, one does not assume that these have the same kind of semicontinuity, i.e., any combination is allowed).

We refer to [2] for the proof of this theorem. We shall need a slight, but partial, extension of this theorem, which will play the main role in our generalization of Choquet's and Bishop's de Leeuw's theorems.

<u>Theorem 2.</u> Let $f : K \to \mathbb{R}$ be a concave semicontinuous function. Then

$$f(x) > 0, \forall x \in \text{ex } K \Rightarrow f(x) > 0, \forall x \in K.$$

<u>Proof.</u> a) Let us assume that f is lower semicontinuous. Then $m = \inf f(x)$ is attained on K, i.e., there exists an $x_o \in K$, such that $f(x_o) = m$. Let $K_o = \{ x \in K \; ; \; f(x) = m \}$. Since f is lower semicontinuous, K_o is a (non-empty) compact subset of K. Let $K_1 = \overline{co}(K_o)$. Milman's theorem implies that $ex K_1 \subset K_o$. Let $x_1 \in$ ex K_1 ; then $f(x_1) = m$. If $x', x'' \in K$, are such that $x', x'' \neq x_1$, $x_1 = \frac{1}{2}(x' + x'')$, then, since f is concave, we have

$$m = f(x_1) \geq \frac{1}{2}(f(x') + f(x'')) \geq m ,$$

and, therefore, $f(x') = f(x'') = m$. Hence $x', x'' \in K_o \subset K_1$.

Consequently, since $x_1 \in \mathrm{ex}\, K_1$, we have $x' = x'' = x_1$. This shows that $x_1 \in \mathrm{ex}\, K$, and $m = f(x_1) > 0$; therefore, we have $f(x) \geq m > 0$, for any $x \in K$.

b') Let us now assume that f is upper semicontinuous, and let $x_o \in K$. Then we have $f = \bar{f}$ and, therefore, we have

$$f(x_o) = \inf_{\substack{h \geq f \\ h \in A(K\,;\,\mathbb{R})}} h(x_o).$$

For any $n \in N^*$ we can choose a $h_n \in A(K\,;\,\mathbb{R})$, such that $h_n \geq f$ on K and

$$f(x_o) + \frac{1}{n} > h_n(x_o)\ , \qquad n \geq 1.$$

Let $f_o = \inf_{n \geq 1} h_n$. Then $f_o : K \longrightarrow \mathbb{R}$ is a Baire measurable, concave, upper semicontinuous function, such that

$$f_o(x) \geq f(x) > 0, \text{ for any } x \in \mathrm{ex}\, K$$

and

$$f_o(x_o) = f(x_o).$$

b") For any $r \in \mathbb{R}$ let us define

$$K(r) = \{x \in K\ ; \quad f_o(x) < r\} \quad .$$

For any $r \in \mathbb{R}$, the set $K(r)$ is Baire measurable and therefore, there exists a countable subset $H_r \subset A(K\,;\,R)$, such that $K(r)$ belongs to the smallest σ- algebra Σ_r of subsets of K, such that all functions in H_r are Σ_r- measurable.

Let $H = \bigcup_{r \in Q} H_r$. Then H is a comutable subset of $A(K\ ;\ R)$. Let Σ be the smallest σ- algebra of subsets of K, such that all functions in H be Σ- measurable. Then we have $\Sigma_r \subset \Sigma$, $\forall r \in Q$, and, therefore, $K(r) \in \Sigma$, $\forall r \in Q$.

For any $r \in \mathbb{R}$ and any sequence $(r_n)_{n \geq 0}$, such that $r_n \in Q$, $r_n \uparrow r$, we have

$$K(r) = \bigcup_{n \geq 0} K(r_n),$$

and this implies that $K(r) \in \Sigma$, for any $r \in \mathbb{R}$.

Consequently, the function f_0 is Σ - measurable.

Let $(g_n)_{n \geq 0}$ be an ennumeration of the functions in H.

b"'). Let us now define a continuous affine mapping $g : K \rightarrow R^{\mathbb{N}}$ by the formula

$$g(x) = (g_n(x))_{n \geq 0}, \quad x \in K.$$

Then $g(K) \subset \mathbb{R}^{\mathbb{N}}$ is a metrizable compact convex subset of $R^{\mathbb{N}}$. Let us now prove that if $x, y \in K$ and $g(x) = g(y)$, then $f_0(x) = f_0(y)$.

Indeed, let us define

$$L(x) = \left\{ z \in K\ ;\ f_0(z) = f_0(x) \right\}.$$

Then $L(x) \in \Sigma$. Since Σ is the smallest σ- algebra of subset of K, such that all the functions g_n, $n \geq 0$, be measurable, we have

$$z_1 \in K,\ z_0 \in L(x),\ g_n(z_1) = g_n(z_0), \forall n \geq 0 \Rightarrow z_1 \in L(x).$$

Consequently, since $x \in L(x)$, we have $y \in L(x)$, i.e., $f_o(y) = f_o(x)$.

b^{iv}) From b''') we conclude that by the formula

$$k\,(g(x)) = f_o(x), \qquad x \in K,$$

we correctly define a function $k : g(K) \to \mathbb{R}$.

It is obvious that this function is concave on $g(K)$.

Let now $y_1 \in \text{ex}\, g(K)$. Then there exists an $x_1 \in \text{ex}\, K$, such that $g(x_1) = y_1$ and, therefore, we have

$$k(y_1) = k(g(x_1)) = f_o(x_1) > 0, \ \forall\, y_1 \in \text{ex}\, g(K).$$

From

$$\{y \in g(K);\ k(y) < r\} = \{g(x);\ \ x \in K,\ f_o(x) < r\} = \\ = g(\{x\ ;\ \ x \in K,\ f_o(x) < r\}),\ \forall r \in \mathbb{R},$$

and from the fact that the mapping g is open, we infer that the function k is upper semicontinuous on $g(K)$.

b^{v}) Let now $\mu_o \in \mathcal{M}_+^1(g(K))$ be any $(\prec)$ - maximal or $(\ll)$ - maximal measure, which represents the point $g(x_o)$. We then have (by taking into account corollary 2 to proposition 2, and also lemma 1)

$$f_o(x_o) = k(g(x_o)) \geq \int_{g(K)} k(y)\, d\,\mu_o(y) = \int_{\text{ex}\, g(K)} k(y)\, d\,\mu_o(y) > 0\,.$$

The theorem is proved.

5. In this section we shall give our generalization of the Bishop- de Leeuw and Choquet theorems.

Let $D \subset K$ be a Baire measurable subset ; it is easy prove that there exists a sequence $(h_n)_{n\in\mathbb{N}}$, $h_n \in A(K;\mathbb{R})$, $n \in \mathbb{N}$, such that $\|h_n\| \leq 1$, $\forall n \in \mathbb{N}$, and $x \in D$, $y \in K$, $h_n(x) = h_n(y)$, $\forall n \in \mathbb{N} \Rightarrow y \in D$.

Let us define $D_o = D \cap (\bigcap_{n=0}^{\infty} S_{h_n^2})$.

<u>Proposition 3.</u> For any $x_o \in D_o \setminus (\text{ex } K)$ there exist $x_1, x_2 \in D_o$ such that $x_1 \neq x_o \neq x_2$ and $x_o = \frac{1}{2}(x_1 + x_2)$.

<u>Proof.</u> Since $x_o \notin \text{ex } K$, there exist $x_1, x_2 \in K$, such that $x_1 \neq x_o \neq x_2$ and $x_o = \frac{1}{2}(x_1+x_2)$. Let us prove that $x_1, x_2 \in D_o$.

Indeed, we have

$$\overline{h_n^2}(x_o) = h_n^2(x_o), \quad \forall n \in \mathbb{N},$$

and

$$\overline{h_n^2}(\lambda x_1 + (1-\lambda)x_2) \geq h_n^2(\lambda x_1 + (1-\lambda)x_2),$$

for any $n \in \mathbb{N}$, and any $\lambda \in [0, 1]$.

Since $\overline{h_n^2}$ is concave and h_n^2 is convex, we infer that

$$(*) \quad \overline{h_n^2}(\lambda x_1 + (1-\lambda)x_2) = h_n^2(\lambda x_1 + (1-\lambda)x_2),$$

for any $n \in \mathbb{N}$, and any $\lambda \in [0, 1]$. We then infer that the mapping

$$\lambda \mapsto h_n^2(\lambda x_1 + (1-\lambda)x_2), \quad \lambda \in [0, 1],$$

is affine, for any $n \in \mathbb{N}$. Since it is the square of an affine function, we deduce that the function

$$\lambda \longmapsto h_n(\lambda x_1 + (1-\lambda)x_2), \quad \lambda \in [0, 1],$$

is constant on $[0, 1]$, for any $n \in \mathbb{N}$. Consequently, we have

$$h_n(x_1) = h_n(x_o) = h_n(x_2), \quad \forall n \in \mathbb{N},$$

and this implies that

$$x_1, x_2 \in D.$$

On the other hand, from (*) we infer that

$$x_1, x_2 \in \bigcap_{n=0}^{\infty} S_{h_n^2} .$$

The proposition is proved.

Theorem 3. a) Any ($\blacktriangleleft$) - maximal measure $\mu \in \mathcal{M}_+^1(K)$ is pseudoconcentrated on ex K. In particular, we have

b) (Choquet) Any ($\prec$) - maximal measure $\mu \in \mathcal{M}_+^1(K)$ is pseudoconcentrated on ex K.

c) (Bishop-de Leeuw) Any ($\ll$) - maximal measure $\mu \in \mathcal{M}_+^1(K)$ is pseudoconcentrated on ex K.

Proof. a) It will be sufficient to prove that for any compact, Baire measurable set $D \subset K$, such that

$$D \cap (\mathrm{ex}\, K) = \emptyset ,$$

we have $\mu(D) = 0$. Indeed, let us consider the set D_o from the preceding proposition. We shall prove that $D_o = \emptyset$, i.e.,

$$D \subset \bigcup_{n=0}^{\infty} (\complement S_{n_n^2}).$$

Corollary 2 to proposition 2 will then imply that $\mu(D) = 0$.

Let us assume that $D_o \neq \emptyset$. From the equality

$$\overline{co}(D_o) = \overline{co}(\overline{D}_o),$$

and from the Milman theorem we then infer that

$$ex(\overline{co}(D_o)) \subset \overline{D}_o \subset D,$$

because D is compact. Let us consider the function

$$\varphi = \sum_{n=0}^{\infty} \frac{1}{2^n} (\overline{h_n^2} - h_n^2),$$

where $h_n, n \in \mathbb{N}$, are the functions already used in proposition 3. It is obvious that φ is concave, finite and upper semicontinuous on K, hence on $\overline{co}(D_o)$.

If $\varphi(x) > 0$, for any $x \in ex(\overline{co}(D_o))$, then, with theorem 2, we would infer that $\varphi(x) > 0$, for any $x \in D_o$, a contradiction. Consequently, there exists an $x_o \in ex(\overline{co}(D_o))$, such that $\varphi(x_o) = 0$; hence, we have

$$x_o \in D_o \subset D.$$

From $D \cap (ex\ K) = \emptyset$, we infer that

$$x_o \notin ex\ K ;$$

proposition 3 now implies that there exist $x_1, x_2 \in D_o \subset \subset \overline{co}(D_o)$, such that

$$x_1 \neq x_o \neq x_2 \quad \text{and} \quad x_o = \frac{1}{2}(x_1 + x_2),$$

thus contradicting the extremality of x_o in $\overline{co}(D_o)$.

The theorem is proved.

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