FATOU AND SZEGÖ THEOREMS FOR OPERATOR VALUED FUNCTIONS

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Ion Suciu and Ilie Valuşescu

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1. Introduction

The celebrated Fatou and Szegö theorems play an important role in the study of non-normal operators on Hilbert spaces. Fatou theorem was the principal tool from the analytic function theory used by B. Sz.-Nagy and C. Foiaş [12] in construction on theirs functional calculus with functions in H^{∞}. In theirs functional model for contractions they used also, in decisive way, the variants of this theorem for vector or operator valued analytic functions. Szegö theorem and their implications in factorizations are also very intimately related with basic problems in operator theory, like structure of invariant subspaces, Jordan models, cyclicity, etc. The applications of the operatorial methods in prediction, cross also through ideas contained in this very important theorem.

Therefore it is not surprising that several efforts were made in order to obtain clear variants of these theorems for the operator valued functions (see for instance [12], [13], [3]).

In this paper, following the treatement given in [12] for the bounded (operator valued) analytic functions, we intend to point out and some how to overcame in a new way the difficulties which appear in the non bounded case.

After some necessary preliminaries given in Section 2, we prove in section 3 an analogous, for the non bounded case, of B. Sz.-Nagy and C. Foiaş Lemma on Fourier representation of operators which intertwine unilateral shifts (Lemma Q). Section 4 contains the results from [9] about factorization of semi- spectral measures by means of L^2-bounded analytic functions. We prove also that any L^2-contractive analytic function can be factorized into a contractive analytic function and an evaluation function [10]. These theorems are used in section 5 to obtain variants for Fatou and Szegö theorems for operator valued functions.

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2. Preliminaries

Let us recall the classical Fatou and Szegö theorems, in a particular case which will be convenient in understanding the variants which we propose for such type of theorems in operator valued case.

Denote by $\mathbb{T}$ the one-dimensional torus $\{z \in \mathbb{C};\ |z| = 1\}$ in the complex plane and by $\mathbb{D}$ the open unit disc $\{z \in \mathbb{C};\ |z| < 1\}$. By L^2 we denote the usual Hilbert Space of measurable complex valued functions v on $\mathbb{T}$ which are square integrable in modulus, with the norm

$$(2.1)\qquad \|v\|^2_{L^2} = \frac{1}{2\pi}\int_0^{2\pi} |v(e^{it})|^2\, dt$$

when dt is the one-dimensional Lebesque measure. By L^2_+ we denote the closed subspace of L^2 consisting from all function in L^2 whose negative Fourier coefficients are zero. Denote by H^2 the Hilbert space of all complex valued functions f on $\mathbb{D}$ which are analytic in $\mathbb{D}$ and verify

$$(2.2)\qquad \|f\|^2_{H^2} = \sup_{0 \le r < 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt < \infty .$$

The map

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \longrightarrow f_+(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int}$$

is an isometric isomorphism between H^2 and L^2_+ and we have

$$(2.3)\qquad \|f\|^2_{H^2} = \sum_{n=0}^{\infty} |a_n|^2 = \|f_+\|^2_{L^2_+} .$$

For a function $f \in H^2$ let $\tilde{f}(\lambda) = \int_0^{\lambda} f(z)\, dz$ be its primitive. Then $\tilde{f}$ is an Lipschitzian function on $\mathbb{D}$, thus it can be extended to an absolutely continuous function on $\mathbb{D}$. The restriction of this function to $\mathbb{T}$ gives rise to a complex valued finite Borel measure on $\mathbb{T}$ denoted by μ_f which is absolutely continuous with respect to Lebesque measure.

The variant of Fatou theorem to keep in mind is the following :

THEOREM F. <u>Let $f \in H^2$, f_+ be its correspondent in L^2_+ and μ_f be its primitive measure. Then</u>

(1) $d\mu_f = f_+\, dt$

(2) $f(re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, d\mu_f(s) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\, f_+(s)\, ds$

where $P_r(t)$ is the Poisson kernell

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}$$

(3) $f(z)$ tends to $f_+(e^{it})$ as z tends to e^{it} non-tangentially with respect to the unit circle at every point t such that

$$\frac{1}{2s}\int_{t-s}^{t+s} d\mu_f(\tau) = \frac{1}{2s}\int_{t-s}^{t+s} f_+(\tau)\,d\tau \longrightarrow f_+(t)$$

thus a.e.

If we consider instead of the spaces of the scalar valued functions L^2 and H^2 the similar spaces $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ of $\mathcal{E}$-valued functions, where $\mathcal{E}$ is a locally convex vector space (with suitable definition for the measurability, analyticity and square integrability), then we can look for the existence of measure μ_f and eventually for its derivative f_+ as in the Fatou theorem. In case $\mathcal{E}$ is a separable Hilbert space, we can transpose Theorem F with the same proof as in the scalar case, the isometric isomorphism between the Hilbert space $H^2(\mathcal{E})$ and $L^2_+(\mathcal{E})$ being also preserved. We are not interested in the generalisation of Fatou theorem along this line, for a larger class of locally convex vector spaces, because of two reasons : firstly, the conditions we must impose to $\mathcal{E}$ in order to obtain consistent Fatou theorems are of such type that permit the same proof as in the scalar case; secondly, the space (of the maximal interest for us) of linear bounded operators, both in the norm or strong topology, do not satisfies such a type of conditions.

These are the reasons why we shall study variants of Fatou theorem for operator valued functions with pure operator methods.

It is not surprising that these methods work better in the case of another famous theorem of classical function theory, namely the Szegö theorem. Let us recall Szegö theorem in a variant which contains Kolmogorov-Krein generalisations (cf. [5]).

THEOREM Sz. Let μ be a positive measure on $\mathbb{T}$ such that $\mu(\mathbb{T}) = 1$ and let $d\mu = \frac{1}{2\pi}\cdot h\,dt + d\mu_s$ be the Lebesque decomposition of μ with respect to Lebesque measure. Then

(1) $$\Delta = \inf_p \int_0^{2\pi} |1-p|^2\,d\mu = \inf_p \frac{1}{2\pi}\int_0^{2\pi} |1-p|^2\,h\,dt = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} \log h\,dt\right]$$

where the infimum is taken over all analytic polynomial p which wanish in origin.

(2) In order to exists a function $f \in H^2$ such that $|f_+|^2 = h$ it is necessary and suf-

ficient that $\log h \in L^1$ or equivalently $\Delta > 0$. In this case, there exist an outer function f in H^2 such that $|f_+|^2 = h$ and $\Delta = |f(0)|^2$.

We shall recognise parts of Theorem F and Theorem Sz. in the results we shall give in operator valued case. But the Fatou - Szegö problematic in general case is far to be elucidate, the nature of the obstructions being variate and mysterious.

3. Operator valued analytic functions

Let $\mathcal{E}$ and $\mathcal{F}$ be two separable Hilbert spaces. A function defined on $\mathbb{D}$ where values are bounded operators $\Theta(\lambda)$ from $\mathcal{E}$ to $\mathcal{F}$ will be called analytic provided it has a power series expansion

$$(3.1) \qquad \Theta(\lambda) = \sum_{n=0}^{\infty} \lambda^n \Theta_n \qquad \lambda \in \mathbb{D}$$

where Θ_n are bounded operators from $\mathcal{E}$ to $\mathcal{F}$. The series is supposed to be convergent weakly, strongly or in norm which amounts to the same for the power series. As in [12] we shall denote such a function by the triplet $\{\mathcal{E}, \mathcal{F}\ \Theta(\lambda)\}$.

We shall introduce the following three types of boundedness for operator valued analytic functions.

The analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ will be called bounded provided

$$(3.2) \qquad \|\Theta(\lambda)\| \leq M \qquad \lambda \in \mathbb{D}.$$

If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ verifies

$$(3.3) \qquad \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|\Theta(re^{it})\|^2 \, dt \leq M^2$$

then it will be called L^2-norm bounded analytic function.

If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ verifies

$$(3.4) \qquad \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \|\Theta(re^{it})\, a\|^2 \, dt \leq M^2 \|a\|_{\mathcal{E}}^2$$

for any $a \in \mathcal{E}$, then it will be called L^2-strongly bounded or shortly L^2-bounded analytic function.

It is easy to verify that (3.3) and (3.4) are respectively equivalent to

$$(3.3)' \qquad \sum_0^{\infty} \|\Theta_n\|^2 \leq M^2$$

and

$$(3.4)' \qquad \sum_0^{\infty} \|\Theta_n a\|_{\mathcal{F}}^2 \leq M^2 \|a\|_{\mathcal{E}}^2 \qquad a \in \mathcal{E}.$$

Let us remark that (3.4) and (3.4)' may be stated with $M = M(a)$ depending of $\underline{a}$ (which corresponds to the term "strongly bounded"). Indeed if we consider the convergent series

$$S(a) = \left(\sum_0^\infty \|\Theta_n a\|^2\right)^{1/2} \qquad (a \in \mathcal{E})$$

then $S_N(a) = \left(\sum_0^N \|\Theta_n a\|^2\right)^{1/2}$, is a continuous function on $\mathcal{E}$ and consequently $S(a) = \sup_N S_N(a)$ is a lower semi-continuous semi-norm on $\mathcal{E}$. It is known then that $S(a)$ is bounded i.e.

$$S(a) \leq M \|a\| \qquad (a \in \mathcal{E})$$

with M independent of a.

Clearly (3.2) $\Longrightarrow$ (3.3) $\Longrightarrow$ (3.4). If we consider the function $\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ defined as

$$\Theta(\lambda) a = \delta(\lambda) a \qquad (\lambda \in \mathbb{D},\ a \in \mathcal{E})$$

where $\delta(\lambda)$ is a scalar valued function from H^2 which is not bounded, then clearly $\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ verifies (3.3) but not (3.2).

Let now $\mathcal{E} = H^2$, $\mathcal{F} = \mathbb{C}$ and $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ the analytic function defined as :

$$\Theta(\lambda)\, h = h(\lambda) \qquad (\lambda \in \mathbb{D},\ h \in H^2).$$

For a fixed λ in $\mathbb{D}$ and any $h \in H^2$ of the form $h(z) = \sum_0^\infty c_k z^k$ we have :

$$\|\Theta(\lambda) h\|_{\mathcal{F}} = |h(\lambda)| \leq \sum_0^\infty |\lambda|^k |c_k| \leq \left(\sum_0^\infty |\lambda|^{2k}\right)^{1/2} \left(\sum_0^\infty |c_k|^2\right)^{1/2} \leq \frac{1}{\sqrt{1-|\lambda|^2}} \|h\|_{H^2}.$$

Thus $\Theta(\lambda)$ is a bounded operator from $\mathcal{E}$ into $\mathcal{F}$ and

$$\|\Theta(\lambda)\| \leq \frac{1}{\sqrt{1-|\lambda|^2}}.$$

If we put $\Theta_k h = c_k$ then clearly Θ_k is a bounded operator from $\mathcal{E}$ into $\mathcal{F}$ and

$$\Theta(\lambda) = \sum_0^\infty \lambda^k \Theta_k$$

For any h in H^2 we have :

$$\sum_0^\infty \|\Theta_k h\|^2 = \sum_0^\infty |c_k|^2 = \|h\|^2_{H^2}$$

thus $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ verifies (3.4)'. By the other way if we take $h(z) = \sum_1^\infty \frac{1}{k} z^k$, then clearly $h \in H^2$ and for $\underline{n}$ sufficiently large such that $\frac{1}{\sqrt{n}} \left(\sum_1^\infty \frac{1}{k^2}\right)^{1/2}$ we have

$$\|\Theta_n\| \geq \frac{\|\Theta_n h\|}{\|h\|} = \frac{1/n}{\left(\sum_1^\infty 1/k^2\right)^{1/2}} \geq \frac{1}{\sqrt{n}}.$$

It results that $\sum_0^\infty \|\Theta_n\|^2$ is divergent i.e. $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ does not vierifies (3.3).

Let now $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an L^2-bounded analytic function. We can define the operator V_Θ from $\mathcal{E}$ into $H^2(\mathcal{F})$ by

(3.5) $$(V_{\Theta} a)(\lambda) = \Theta(\lambda)\, a \qquad (a \in \mathcal{E}).$$

We have

$$\|V_{\Theta} a\|^2_{H^2(\mathcal{F})} = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \|(V_{\Theta} a)(re^{it})\|^2_{\mathcal{F}}\, dt = \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \|\Theta(re^{it})a\|^2\, dt \le$$
$$\le M^2 \|a\|^2 .$$

Thus V_{Θ} is bounded. Conversely, if V is a bounded operator from $\mathcal{E}$ into $H^2(\mathcal{F})$, then setting

$$\Theta(\lambda)a = (Va)(\lambda)$$

we obtain an L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $V_{\Theta} = V$.

Thus (3.5) establish on one-to-one correspondence between L^2-bounded analytic functions $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ and the bounded operators from $\mathcal{E}$ into $H^2(\mathcal{F})$.

Let us remark that we can consider V_{Θ} as the multiplication by the operator valued function $\Theta(\lambda)$ on the constant functions $\underline{a}$ from $H^2(\mathcal{E})$. Our next intention is to analyse the maximal multiplication operator on $H^2(\mathcal{E})$ generated by $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ and to give an intrinsic characterization of such operators. We shall obtain in Lemma Q the corresponding result for non bounded case of the B. Sz.-Nagy and C. Foiaş lemma of Fourier representation of the operators which intertwine unilateral shifts (cf.[12] pp. 195-198).

Let us call <u>evaluation operator</u> e_{λ} on $H^2(\mathcal{E})$ the operator defined for a fixed $\lambda \in \mathbb{D}$ by

$$e_{\lambda} h = h(\lambda).$$

We have already seen that e_{λ} is a bounded operator from $H^2(\mathcal{E})$ into $\mathcal{E}$ and

$$\|e_{\lambda} h\| \le \frac{1}{\sqrt{1-|\lambda|^2}}\, \|h\|_{H^2(\mathcal{E})}.$$

If no confusion will arises we shall denote with the same symbol e_{λ} the corresponding evaluation operator for different spaces $H^2(\mathcal{E})$.

Let Q be a linear operator defined on the subspace $D(Q)$ of $H^2(\mathcal{E})$ with values in $H^2(\mathcal{F})$. We say that Q <u>intertwines the evaluations</u> on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ if the following conditions hold :

(i) $\mathcal{E} \subset D(Q)$ and $Q|_{\mathcal{E}}$ is a bounded operator from $\mathcal{E}$ into $H^2(\mathcal{F})$.

(ii) $D(Q)$ contains any function $h \in H^2(\mathcal{E})$ for which the function h_Q defined as

$$h_Q(\lambda) = e_{\lambda} Q e_{\lambda} h \qquad (\lambda \in \mathbb{D})$$

belongs to $H^2(\mathcal{F})$.

(iii) For any $\lambda \in \mathbb{D}$ and $h \in D(Q)$ we have

$$e_{\lambda} Q h = e_{\lambda} Q e_{\lambda} h .$$

Any operator Q which intertwines the evaluations on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ is closed. Indeed, if $h_n \in D(Q)$ such that $h_n \longrightarrow h$ in $H^2(\mathcal{E})$ and $Qh_n \longrightarrow g$ in $H^2(\mathcal{F})$ then $e_\lambda Q e_\lambda h_n \longrightarrow e_\lambda Q e_\lambda h$ for any $\lambda \in \mathbb{D}$, because of (i) it results that $e_\lambda Q e_\lambda$ is bounded. By the other way from (iii) it results that $e_\lambda Q e_\lambda h_n = e_\lambda Q h_n \longrightarrow e_\lambda g$ in $H^2(\mathcal{F})$. Thus for any $\lambda \in \mathbb{D}$ we have $e_\lambda Q e_\lambda h = e_\lambda g$, i.e. $h_Q(\lambda) = e_\lambda Q e_\lambda h = e_\lambda g = g(\lambda)$.

From (ii) it results $h_Q \in D(Q)$ and $Qh = g$.

If Q intertwines the evaluations on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ then D(Q) contains any analytic ($\mathcal{E}$-valued) polynomial $p(z) = \sum_0^m z^k a$. Indeed, for such a p we have

$$p_Q(\lambda) = e_\lambda Q e_\lambda p = e_\lambda Q \sum_0^m \lambda^k a_k = e_\lambda \sum_0^m \lambda^k Q a_k = \sum_0^m \lambda^k e_\lambda (Qa_k) = \left(\sum_0^m z^k Qa_k\right)(\lambda).$$

Since clearly $\sum_0^m z^k Qa_k \in H^2(\mathcal{F})$ from (ii) it results $p \in D(Q)$.

Thus any operator Q which intertwines evaluations on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$, is a closed operator with dense domain. It results that Q is bounded on its domain D(Q) if and only if $D(Q) = H^2(\mathcal{E})$. In this case (iii) is equivalent to the fact that Q intertwines the shift operators on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ i.e. with :

$$zQh = Qzh \qquad h \in H^2(\mathcal{E}) \tag{3.6}$$

Indeed, if $h = \sum_{k=0}^\infty z^k a_k$, then using (3.6) we have :

$$e_\lambda Qh = (Qh)(\lambda) = \left(\sum_0^\infty Q(z^k a_k)\right)(\lambda) = \left(\sum_0^\infty z^k Qa_k\right)(\lambda) = \sum_0^\infty \lambda^k e_\lambda (Qa_k) =$$

$$= e_\lambda Q\left(\sum_0^\infty \lambda^k a_k\right) = e_\lambda Q e_\lambda h.$$

Thus (3.6) $\Longrightarrow$ (iii). Conversely, from (iii) it results

$$(zQh)(\lambda) = \lambda(Qh)(\lambda) = \lambda e_\lambda Qh = e_\lambda Q(\lambda h) = e_\lambda Q e_\lambda h = e_\lambda Q e_\lambda (zh) =$$

$$= e_\lambda Q(zh) = [Q(zh)](\lambda) .$$

LEMMA Q. There exists an one-to-one correspondence between L^2-bounded analytic functions $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ and the operators Q which intertwines the evaluations on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ given by

$$(Qh)(\lambda) = \Theta(\lambda)\, h(\lambda), \qquad \lambda \in \mathbb{D},\ h \in D(Q). \tag{3.7}$$

The L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded if and only if the corresponding operator Q is bounded.

Proof. Let $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an L^2-bounded analytic function. Denote by $D(\Theta)$ the subspace of $H^2(\mathcal{E})$ consisting from all function h in $H^2(\mathcal{E})$ for which the function $\lambda \longrightarrow \Theta(\lambda)h(\lambda)$ is in $H^2(\mathcal{F})$, and Q be the operator defined on $D(\Theta)$ by (3.7). Since $\Theta(\lambda)\, a = (V_\Theta a)(\lambda)$, we have $\mathcal{E} \subset D(\Theta)$ and $Qa = V_\Theta a$. Thus Q verifies (i). For any $h \in H^2(\mathcal{E})$ we have :

$$h_Q(\lambda) = e_\lambda Q e_\lambda h = e_\lambda (V_\Theta e_\lambda h) = \Theta(\lambda)\, h(\lambda).$$

Thus, if $h_Q \in H^2(\mathcal{F})$, we have $h \in D(\Theta) = D(Q)$ i.e. Q verifies (ii). For any $h \in D(\Theta)$ we have :

$$e_\lambda Q h = \Theta(\lambda) h(\lambda) = \Theta(\lambda) e_\lambda h = \left[Q(e_\lambda h)\right](\lambda) = e_\lambda Q e_\lambda h.$$

Thus Q verifies (iii). Hence Q intertwines the evaluations on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$.

Conversely, if Q verifies (i)-(iii), then if we put $V_\Theta = Q\,|_{\mathcal{E}}$ we obtain a bounded operator from $\mathcal{E}$ into $H^2(\mathcal{F})$ and we already seen that $\Theta(\lambda)a = (V_\Theta a)(\lambda)$ defines an L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$. Since for any $h \in H^2(\mathcal{E})$ we have

$$h_Q(\lambda) = e_\lambda Q e_\lambda h = e_\lambda (V_\Theta h) = (V_\Theta h)(\lambda) = \Theta(\lambda)\, h(\lambda),$$

it is clear that $D(\Theta) \subset D(Q)$. From (iii) it results that for any $h \in D(Q)$ we have

$$e_\lambda Q h = e_\lambda Q e_\lambda h = e_\lambda (V_\Theta h) = \Theta(\lambda) h(\lambda).$$

Thus $D(Q) = D(\Theta)$ and for any $h \in D(Q)$

$$(Qh)(\lambda) = \Theta(\lambda) h(\lambda) \qquad (\lambda \in \mathbb{D}).$$

Suppose now that $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded. Then for any $h \in D(Q)$ we have

$$\|Qh\|^2_{H^2(\mathcal{F})} = \sup_{0 \le r < 1} \frac{1}{2\pi}\int_0^{2\pi} \|\Theta(re^{it})\, h(re^{it})\|^2_{\mathcal{F}}\, dt \le \sup_{0 \le r < 1} \frac{1}{2\pi}\int_0^{2\pi} M^2 \|h(re^{it})\|^2_{\mathcal{E}} dt =$$

$$= M^2 \|h\|^2_{H^2(\mathcal{E})} .$$

Thus Q is a bounded operator.

Suppose now that Q is bounded. Then for all analytic scalar valued polynomial $\underline{p}$ and $\underline{a} \in \mathcal{E}$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^2 \, \|(V_\Theta a)(e^{it})\|^2 dt = \|p V_\Theta a\|^2_{H^2(\mathcal{F})} = \|pQa\|^2 = \|Qpa\|^2 \le$$

$$\le \|Q\|^2 \|pa\|^2_{H^2(\mathcal{E})} = \|Q\|^2 \frac{1}{2\pi}\int_0^{2\pi} |p(e^{it})|^2 \|a\|^2 dt.$$

It results that for any trigonometric polynomial p we have

$$\int_0^{2\pi} |p(e^{it})|^2 \|(V_\Theta a)(e^{it})\|^2 dt \le \int_0^{2\pi} |p(e^{it})|^2 \|a\|^2_{\mathcal{E}} \|Q\|^2\, dt$$

which implies

$$\|(V_\Theta a)(e^{it})\|_{\mathcal{F}} \le \|Q\| \|a\| \qquad \text{a.e.}$$

Using known properties of Poisson kernel we obtain

$$\|\Theta(\lambda)a\|_{\mathcal{F}} = \left\|\frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\,(V_\Theta a)(s)ds\right\| \le \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\,\|(V_\Theta a)(s)\|\, ds \le$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} P_r(t-s)\,\|Q\|\,\|a\|\,ds = \|Q\|\,\|a\|_{\mathcal{E}},$$

i.e. $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded.

The proof of the Lemma Q is complete.

We shall denote by Θ_+ the operator which intertwines the evaluation on $H^2(\mathcal{E})$ and $H^2(\mathcal{F})$ uniquely associated to the L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ as in Lemma Q.

Using the natural isomorphism between $H^2(\mathcal{E})$ and $L^2_+(\mathcal{E})$ we consider Θ_+ as an operator from $L^2_+(\mathcal{E})$ into $L^2_+(\mathcal{F})$. The operator Θ_+ is closed and its domain $D(\Theta_+)$ contains any analytic polynomial in $L^2_+(\mathcal{E})$.

Let now p be a trigonometric polynomial in $L^2(\mathcal{E})$. There exists an integer $n \geq 0$ such that $e^{int}p$ is an analytic polynomial. Let us define

$$\Theta p = e^{-int}\,\Theta_+(e^{int}p) \tag{3.8}$$

If $e^{int}p$ and $e^{imt}p$, $(n, m \geq 0)$, belong to $L^2_+(\mathcal{E})$ then if $n \geq m$ we have

$$\Theta_+(e^{int}p) = \Theta_+(e^{i(n-m)t}e^{imt}p) = e^{i(n-m)t}\,\Theta_+(e^{int}p)$$

because Θ_+ is an multiplication operator on $H^2(\mathcal{E})$. Then clearly (3.8) defines a linear operator Θ from the subspace of trigonometric polynomials in $L^2(\mathcal{E})$ into $L^2(\mathcal{F})$.

In case Θ_+ is bounded then clearly Θ is a bounded operator from $L^2(\mathcal{E})$ into $L^2(\mathcal{F})$ and $\Theta_+ = \Theta \,|\, L^2_+(\mathcal{E})$. If Θ_+ is non bounded then Θ is not in general closable. We shall see later that this problem is related to the existence of the boundary limit for the L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$.

A simple example which shows that Θ is not, in general, closable is the following: let $\mathcal{E} = H^2$, $\mathcal{F} = \mathbb{C}$ and $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ defined by

$$\Theta(\lambda)\,h = h(\lambda) \qquad h \in H^2.$$

Consider in $L^2(\mathcal{E})$ the sequence of polynomials

$$p_n = \frac{1}{\log n}\sum_{k=1}^{n} e^{-ikt}\,a_k \quad ,$$

were a_k is the function $\frac{1}{k}e^{ikt}$ over $\mathcal{E} = H^2$. Then $\|p_n\|^2 = \frac{1}{(\log n)^2}\sum_{k=1}^{n}\frac{1}{k^2} \longrightarrow 0$

Thus $p_n \longrightarrow 0$ in $L^2(\mathcal{E})$. But

$$\Theta\,p_n = e^{-int}\frac{1}{\log n}\sum_{k=1}^{n}(e^{i(n-k)t}a_k) = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}.$$

Thus $\Theta\,p_n \longrightarrow c$ (Euler's const.) $\neq 0$ i.e. Θ is not closable.

The L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is called <u>inner</u> if the attached operator Θ_+ is an isometry. Such a function is thus necessary bounded. The L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is called L^2-bounded outer function provided

(3.9) $$\bigvee_0^\infty z^n V_\Theta \mathcal{E} = H^2(\mathcal{F}).$$

If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an L^2-bounded outer function then for any $\lambda \in \mathbb{D}$ we have $\overline{\Theta(\lambda)\mathcal{E}} = \mathcal{F}$. If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is simultaneously inner and outer, then it is a unitary constant function.

4. Attached semi-spectral measures and factorizations

Recall that an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure is a map $\omega \to F(\omega)$ from the family $B(\mathbb{T})$ of all Borel subsets of $\mathbb{T}$ into $\mathcal{L}(\mathcal{E})$ such that for any $a \in \mathcal{E}$ the map $\omega \to (F(\omega)a, a)$ is a positive Borel measure on $\mathbb{T}$. If for any two Borel sets ω_1, ω_2 we have $F(\omega_1 \cap \omega_2) = F(\omega_1) F(\omega_2)$ then we say that the semi-spectral measure is spectral. We shall denote usualy by F a semi-spectral measure and by E a spectral measure.

If $\mathcal{K}$ is a Hilbert space, E an $\mathcal{L}(\mathcal{E})$-valued spectral measure on $\mathbb{T}$, and V a bounded operator from $\mathcal{E}$ into $\mathcal{K}$, then if we put for any $\omega \in B(\mathbb{T})$, $F(\omega) = V^* E(\omega) V$, then clearly we obtain an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$. Conversely, using the celebrated Naimark dilation theorem [7], if F is an $\mathcal{L}(\mathcal{E})$-valued spectral measure on $\mathbb{T}$, then there exist a Hilbert space $\mathcal{K}$, a bounded operator V from $\mathcal{E}$ into $\mathcal{K}$ and $\mathcal{L}(K)$-valued spectral measure on $\mathcal{K}$ such that

(4.1) $$F(\omega) = V^* E(\omega) V \qquad (\omega \in B(\mathbb{T})).$$

Let us remark that any semi-spectral measure is completely-positive in the following sense : for any finite system $\varphi_1, \ldots, \varphi_n$ in $C(\mathbb{T})$ and any $a_1, \ldots, a_n$ in $\mathcal{E}$ we have

(4.2) $$\sum_{i,j} \int_{\mathbb{T}} \varphi_j \overline{\varphi_i} \, d(F(t) a_j, a_i) \geq 0 .$$

Indeed we have :

$$\sum_{i,j} \int_{\mathbb{T}} \varphi_j \overline{\varphi_i} \, d(F(t)a_j, a_i)_{\mathcal{E}} = \sum_{i,j} \int \varphi_j \overline{\varphi_i} \, d(E(t)Va_j, Va_i)_{\mathcal{K}} =$$

$$= \| \sum_i \int \varphi_i \, dE(t) a_i \|^2 \geq 0 .$$

The triplet $[\mathcal{K}, V, E]$ is called spectral dilation of F. In the supplementary condition of minimality $\mathcal{K} = \bigvee_{\omega \in B(\mathbb{T})} E(\omega) V \mathcal{E}$, the spectral dilation of F is unique up to a unitarity which conserves the operator V.

Let now $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an L^2-bounded analytic function and V_Θ the bounded operator from $\mathcal{E}$ into $H^2(\mathcal{F})$, associated to $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ as in section 3. We shall consider V_Θ as an operator from $\mathcal{E}$ into $L^2(\mathcal{F})$ (via the geometric isomorphism between $H^2(\mathcal{F})$ and $L^2_+(\mathcal{F})$. Let $E^\times_{\mathcal{F}}$ be the spectral measure attached to the shift operator (multiplication by e^{it}) in $L^2(\mathcal{F})$.

We shall denote by F_Θ the $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$ defined as

(4.3) $$F_\Theta(\omega) = V_\Theta^* E_{\mathcal{F}}^{\times}(\omega) V_\Theta \qquad (\omega \in B(\mathbb{T})).$$

We shall call F_Θ the semi-spectral measure attached to $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$.

If F is an $\mathcal{L}(\mathcal{E})$ - valued semi-spectral measure which admits as a spectral dilation a triplet $[L^2(\mathcal{F}), V, E_{\mathcal{F}}^{\times}]$ such that $V\mathcal{E} \subset L^2_+(\mathcal{F})$ then, if we construct as in section 3 the L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which verifies $V_\Theta = V$, then clearly $F_\Theta = F$.

THEOREM 1. Let $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$, $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ be two L^2-bounded analytic functions, the second one being outer, F_Θ, F_{Θ_1} be theirs semi-spectral measures. Suppose

(4.4) $$F_\Theta \leq F_{\Theta_1}.$$

Then there exists a contractive analytic function $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ such that

(4.5) $$\Theta(\lambda) = \Theta_2(\lambda)\, \Theta_1(\lambda) \qquad (\lambda \in \mathbb{D}).$$

If in (4.4) the equality holds, then $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ is inner. If moreover $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is outer then $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ is unitary constant.

Proof. For any function $h \in H^2(\mathcal{F}_1)$ of the form $h = \sum_0^n e^{ikt} V_{\Theta_1} a_k$, let us put

(4.6) $$Qh = \sum e^{ikt} V_\Theta a_k$$

we have

$$\| Qh \|^2_{H^2(\mathcal{F})} = \| \sum_0^n e^{ikt} V_\Theta a_k \|^2 = \sum_{k,j} \int_0^{2\pi} e^{i(k-j)t} d(F_\Theta h_k, h_j) \leq$$

$$\leq \sum_{k,j} \int_0^{2\pi} e^{i(k-j)t} d(F_{\Theta_1} h_k, h_j) = \| h \|^2_{H^2(\mathcal{F}_1)}.$$

We have used here (4.4) in the completely-positivity form (4.2). Since $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$, is outer it results that (4.6) gives rise to a contraction Q from $H^2(\mathcal{F}_1)$ into $H^2(\mathcal{F})$. Clearly $Qe^{it} = e^{it}Q$. From Lemma Q it results that there exists a contractive analytic function $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ such that

$$(Qh)(\lambda) = \Theta_2(\lambda)\, h(\lambda) \qquad (h \in H^2(\mathcal{F})).$$

If we take $h(\lambda) = \Theta_1(\lambda)a$ we obtain

$$\Theta(\lambda)\, a = (V_\Theta a)(\lambda) = (Qa)(\lambda) = \Theta_2(\lambda)\, \Theta_1(\lambda)\, a.$$

If in (4.4) equality holds then clearly $Q = \Theta_{2+}$ is an isometry, thus $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ is inner. If moreover $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is outer, then clearly $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ is outer, thus it is unitary constant.

THEOREM 2. Let F be an $\mathcal{L}(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and $[\mathcal{K}, V, E]$ its minimal spectral dilation. There exists a unique L^2-bounded outer function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ with the properties :

(1) $$F_{\Theta_1} \leq F.$$

(2) <u>For any L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $F_\Theta \leq F$, we have</u> also $F_\Theta \leq F_{\Theta_1}$.

<u>The equality holds in (1) if and only if</u>

(4.7) $$\bigcap_{n\geq 0} U^n \mathcal{K}_+ = \{0\}$$

<u>where U is the unitary operator which corresponds to the spectral measure E and</u> $\mathcal{K}_+ = \bigvee_0^\infty U^n V\mathcal{E}$.

<u>Proof.</u> Aplying Wold decomposition (cf [12]) to the isometry $U_+ = U|\mathcal{K}_+$ we obtain

$$\mathcal{K}_+ = \bigoplus_0^\infty U^n \mathcal{F}_1 \oplus \mathcal{R},$$

where $\mathcal{F}_1 = \mathcal{K}_+ \ominus U\mathcal{K}_+$ and $\mathcal{R} = \bigcap_{n\geq 0} U^n \mathcal{K}_+$.

Clearly

$$\mathcal{K} = \bigoplus_{-\infty}^{+\infty} U^n \mathcal{F}_1 \oplus \mathcal{R}.$$

Let P be the orthogonal projection of $\mathcal{K}$ onto the subspace $\bigoplus_{-\infty}^{+\infty} U^n \mathcal{F}_1$. Then we have

(4.8) $$PU = UP.$$

Denote by $X_{\mathcal{F}}$ the canonical isomorphism between $\bigoplus_{-\infty}^{+\infty} U^n \mathcal{F}_1$ and $L^2(\mathcal{F}_1)$ (Fourier representation), and define $V_{\Theta_1} : \mathcal{E} \to L^2(\mathcal{F}_1)$ by

$$V_{\Theta_1} a = X_{\mathcal{F}_1} PVa \qquad (a \in \mathcal{E}).$$

Clearly then $V_{\Theta_1}\mathcal{E} \subset L^2_+(\mathcal{F}_1)$ and

$$\bigvee_0^\infty e^{int} V_{\Theta_1}\mathcal{E} = \bigvee_0^\infty e^{int} X_{\mathcal{F}_1} PV\mathcal{E} = X_{\mathcal{F}_1} \bigvee_0^\infty U^n PV\mathcal{E} = X_{\mathcal{F}_1} P \bigvee_0^\infty U^n V\mathcal{E} =$$
$$= X_{\mathcal{F}_1} P\mathcal{K}_+ = X_{\mathcal{F}_1} \bigoplus_0^\infty U^n \mathcal{F}_1 = L^2_+(\mathcal{F}_1).$$

We obtain that the L^2-bounded analytic function attached to $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ corresponding to V_{Θ_1} as in section 3, is outer. For any analytic polynomial p we have

$$\int_0^{2\pi} |p|^2 d(F_{\Theta_1} a, a)_{\mathcal{E}} = \|pV_{\Theta_1} a\|^2_{L^2(\mathcal{F})} = \|pX_{\mathcal{F}} PVa\|^2_{L^2(\mathcal{F})} = \|p(U)PVa\|^2_{\mathcal{K}} =$$
$$= \|Pp(U)Va\|^2_{\mathcal{K}} \leq \|p(U)Va\|^2 = \int_0^{2\pi} |p|^2 d(Fa, a)_{\mathcal{E}}.$$

Thus

$$F_{\Theta_1} \leq F.$$

We have equality iff $X_{\mathcal{F}} PV = X_{\mathcal{F}} V$, i.e. iff $PV = V$, i.e. iff $\mathcal{R} = \{0\}$, i.e. iff $\bigcap_{n\geq 0} U^n \mathcal{K}_+ = \{0\}$.

Let now $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be another L^2-bounded analytic function such that $F_\Theta \leq F$. Let us put for an element $k \in \mathcal{K}_+$ of the form $k = \sum_0^m U^k V a_k$

(4.9) $$Xk = \sum e^{ikt} V_\Theta a_k.$$

We have

$$\|Xk\|^2 = \sum_{l,j} \int_0^{2\pi} e^{i(l-j)t} d(F_\Theta(t) a_l, a_j) \leq$$

$$\leq \sum_{\ell,j} \int e^{i(\ell-j)t} d(F(t)a_\ell, a_j) = \| \sum_j U^j V a_j \|^2 = \|k\|^2 .$$

Thus (4.9) gives rise to a contraction X from $\mathcal{K}_+$ into $L^2_+(\mathcal{F})$ such that

$$XU = e^{it}X.$$

We have

$$X\mathcal{R} = X \bigcap_n U^n \mathcal{K}_+ \subset \bigcap_n X U^n \mathcal{K}_+ = \bigcap_n e^{int} X \mathcal{K}_+ \subseteq \bigcap_n e^{int} L^2_+(\mathcal{F}) = \{0\}.$$

Thus XP = X. We have then for any analytic polynomial $\underline{p}$

$$\int |p|^2 d(F_\Theta a, a) = \|pV_\Theta a\|^2 = \|X\, p(U)\, V a\|^2 = \|XPp(U)\, Va\|^2 \leq \|Pp(U)\, Va\|^2 =$$

$$= \|X_{\mathcal{F}_1} Pp(U)\, Va\|^2 = \|p(e^{it}) X_{\mathcal{F}_1} P V a\|^2 = \|p V_{\Theta_1} a\|^2 = \int |p(e^{it})|^2 d(F_{\Theta_1} a, a)$$

i.e. $\qquad F_\Theta \leq F_{\Theta_1}$.

Clearly any L^2-bounded outer functions $\{\mathcal{E}, \mathcal{F}_1', \Theta_1'(\lambda)\}$ which verifies (1) and (2), verifies also $F_{\Theta'} = F_{\Theta_1}$ and from Theorem 1 it results that they differ by a unitary constant factor.

COROLLARY. <u>Any L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ has a unique factorization of the form</u>

$$\Theta(\lambda) = \Theta_i(\lambda)\Theta_e(\lambda) \qquad (\lambda \in \mathbb{D})$$

<u>in the inner and the outer parts.</u>

An L^2-bounded outer function $\{\mathcal{E}, \mathcal{F}, \Delta(\lambda)\}$ will be called <u>evaluation function</u> of $\mathcal{E}$ in $\mathcal{F}$ if V_Θ is an isometry from $\mathcal{E}$ into $H^2(\mathcal{F})$.

PROPOSITION 1. <u>An L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Delta(\lambda)\}$ is an evaluation function if and only if $\mathcal{E}$ can be isometricaly embeded in $H^2(\mathcal{F})$ as a cyclic subspace for the shift operator in $H^2(\mathcal{F})$ such that</u>

$$\Theta(\lambda)a = a(\lambda) \qquad (\lambda \in \mathbb{D}) .$$

In this case we have necessary $\dim \mathcal{F} \leq \dim \mathcal{E}$.

<u>Proof.</u> If $\mathcal{E}$ is a cyclis subspace of $H^2(\mathcal{F})$ then clearly (4.8) defines an evaluation function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$. Conversely, if $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an evaluation function then V_Θ is an isometricaly embeding of $\mathcal{E}$ in $H^2(\mathcal{F})$. Since $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is by definition outer, then $V_\Theta\mathcal{E}$ is a cyclic subspace for the shift operator on $H^2(\mathcal{F})$.

Let $\mathcal{E}$ be a cyclic subspace in $H^2(\mathcal{F})$ and denote by P the orthogonal projection of $H^2(\mathcal{F})$ on $\mathcal{F}$. If $f \in \mathcal{F}$ and for any $a \in \mathcal{E}$ we have

$$(f, Pa) = 0$$

then clearly $(f, \lambda^n a) = 0$ for any $n \geq 0$ and $a \in \mathcal{E}$. From the cyclicity of $\mathcal{E}$ it results f=0. It results that $\overline{P\mathcal{E}} = \mathcal{F}$, i.e. $\dim \mathcal{F} \leq \dim \mathcal{E}$.

THEOREM 3. Any L^2-contractive analytic function can be factorized in the form

(4.10) $$\Theta(\lambda) = M(\lambda)\,\Delta(\lambda) \qquad \lambda \in \mathbb{D},$$

where $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ is an evaluation function and $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ is a contractive analytic function.

Proof. Denote by $D_\Theta = [I - V_\Theta^* V_\Theta]^{1/2}$ and put

$$dF = dF_\Theta + D_\Theta^2\, dt$$

Since $F_\Theta(\mathbb{T}) = V_\Theta^* V_\Theta$ we have $F(\mathbb{T}) = I$. Moreover $F = F_\Omega$ where $\{\mathcal{E}, \mathcal{F}_1, \Omega(\lambda)\}$ is the L^2-bounded analytic function defined as

$$\Omega(\lambda) = \Theta(\lambda) \oplus D_\Theta$$

and $\mathcal{F}_1 = \mathcal{F} \oplus \overline{D_\Theta \mathcal{E}}$. Let $\{\mathcal{E}, \mathcal{E}_1, \Delta(\lambda)\}$ be the outer part of $\{\mathcal{E}, \mathcal{F}_1, \Omega(\lambda)\}$. Then clearly $F_\Omega = F_\Delta$ and consequently $V_\Delta^* V_\Delta = F_\Delta(\mathbb{T}) = I$ and $F_\Theta \le F_\Delta$. Applying Theorem 1, we obtain the desired factorization.

5. Boundary limits

In section 4, we attached to any L^2-bounded analytic function the bounded operator V_Θ from $\mathcal{E}$ into $L^2_+(\mathcal{F})$ and the semi-spectral measure F_Θ. If F is an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure, we attached to F its maximal outer function $\{\mathcal{E}, \mathcal{F}, \Theta_1(\lambda)\}$ such that $F_{\Theta_1} \le F$. In this way we obtained elements for both theorems Fatou and Szegö. We need "only" a desintegration for the operator V_Θ of the following type : there exists a strongly (or in norm) measurable function $t \longrightarrow \Theta(e^{it})$ defined on $\mathbb{T}$ with values in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ such that for any $a \in \mathcal{E}$ we have :

(5.1) $$(V_\Theta a)(e^{it}) = \Theta(e^{it})a \qquad \text{a.e.}$$

In this case, clearly we have $\Theta(\lambda) \longrightarrow \Theta(e^{it})$ strongly a.e. when $\lambda \longrightarrow e^{it}$ nontangentially,

(5.2) $$dF_\Theta = \Theta(e^{it})^* \Theta(e^{it})\, dt$$

and

(5.3) $$\Theta(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(t-s)\, \Theta(e^{is})\, ds$$

in strong sense.

If in Theorem 2 we have $F_{\Theta_1} = F$ then

$$dF = \Theta_1(e^{it})^* \Theta_1(e^{it})\, dt$$

which corresponds to Szegö factorization theorem.

In case $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is a bounded function, we have a such desintegration for V_Θ in strong sens (cf. [12]). Indeed, let $\{a_n\}$ be a dense set in $\mathcal{E}$ and ω a total set in $\mathbb{T}$, such that

$\Theta(\lambda)\, a_n \longrightarrow (V_\Theta a_n)(e^{it})$ for any $t \in \omega$ and a_n, as λ tends to e^{it} non-tangentially. Since $\|\Theta(\lambda)\| \le M$ then $\Theta(\lambda)$ tends strongly to a bounded operator $\Theta(e^{it})$ when λ tends nontangentially to e^{it}. Then the function $t \longrightarrow \Theta(e^{it})$ is clearly strongly measurable and $(V_\Theta a)(t) = \Theta(e^{it})a$ a.e. Even in this case $t \longrightarrow \Theta(e^{it})$ is not necessary norm-measurable, thus $\Theta(e^{it})$ is not a boundary function in norm sense.

For a general L^2-bounded analytic function $\Theta(\lambda)$ does not exist a desintegration for the operator V_Θ. If, for example, $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is the evaluation function, $\mathcal{E} = H^2$, $\mathcal{F} = \mathbb{C}$ and $\Theta(\lambda) = a(\lambda)$, and $t \longrightarrow \Theta(e^{it})$ is the boundary function (in strong sense) for $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$, then it results that there exists a total set ω in $\mathbb{T}$ such that any function a in H^2 has a radial limit when $\lambda \longrightarrow e^{it}$, $t \in \omega$, which is clearly impossible.

Theorem 3 permits us to reduce the difficulties in construction of boundary limit to such a type of obstructions.

THEOREM 4. Let $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an L^2-contractive analytic function. Then there exist a Hilbert space $\mathcal{E}_1$ and a bounded analytic function $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$ such that $\mathcal{E}$ can be isometrically embedded in $H^2(\mathcal{E}_1)$,

$$\Theta(\lambda)a = M(\lambda)\, a(\lambda) \qquad a \in \mathcal{E} \subset H^2(\mathcal{E}_1)$$

and

$$(V_\Theta a)(e^{it}) = M(e^{it})\, a(e^{it}) \qquad \text{a.e.}$$

where $t \longrightarrow M(e^{it})$ is the boundary function of $\{\mathcal{E}_1, \mathcal{F}, M(\lambda)\}$.

We can interpret $t \longrightarrow M(e^{it})$ as a boundary function of $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ (modulo the evaluation of $\mathcal{E}$ into $H^2(\mathcal{E}_1)$). If $\mathcal{E}_1 \subset \mathcal{E}$, then $M(\lambda) = \Theta(\lambda)|\mathcal{E}_1$, thus $\Theta(\lambda)$ has strong boundary limit on the subspace of constant functions in $\mathcal{E}$. In general we obtained a desintegration for V_Θ by composing the simultan desintegration of the elements of $\mathcal{E}$ as analytic functions on $\mathcal{E}_1$ and of V_M as a multiplication by $\mathcal{L}(\mathcal{E}_1, \mathcal{F})$-valued strongly measurable and bounded function $t \longrightarrow M(e^{it})$ on $L^2_+(\ )$.

If an L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ has boundary limit in the Fatou (strong) sense, then clearly the operator $\Theta : D(\Theta) \subset L^2(\mathcal{E}) \longmapsto L^2(\mathcal{F})$ constructed as in the last part of section 3 is closable. The converse assertion seems to be an interesting probleme whose solution we don't know up to now.

As we already remarked, Szegö Theorem for an $\mathcal{L}(\mathcal{E})$- valued semi-spectral measure F consists- modulo the above discussions-actually characterises in terms of F the fact that in the factorization theorem (Theorem 2) one has $F_{\Theta_1} = F$ (or at least $\Theta_1 \neq 0$).

Let F be an $\mathcal{L}(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and $[\mathcal{K}, V, U]$ its unitary dilation. Let us put

(5.4) $$(\Delta[F]a,a) = \inf \sum_{k,j=0}^{n} \int_0^{2\pi} e^{i(j-k)t} d(F(t)a_j, a_k)$$.

where the infimum is taken over all finite system $a_o, a_1, \dots, a_n$ in $\mathcal{E}$ such that $a_o = a$.

We have

$$(\Delta[F]a,a) = \inf_{a_o=a, a_1, \dots a_n \in \mathcal{E}} \sum_{k,j} \int_0^{2\pi} e^{i(k-j)t} d(F(t)a_k, a_j)_{\mathcal{E}} =$$

$$= \inf_{a_o=a, a_1, \dots, a_n \in \mathcal{E}} \sum_{k,j} \int_0^{2\pi} e^{i(k-j)t} d\left(E(t)Va_k, Va_j\right)_{\mathcal{K}} = \inf_{a_o=a, a_1, \dots, a_n \in \mathcal{E}} \sum (U^{k-j}Va_k, Va_j)_{\mathcal{K}} =$$

$$= \inf_{a_1, \dots, a_n \in \mathcal{E}} \| Va - \sum_{k=1}^{n} U^k Va_k \|^2 = \| (I-P_1)Va \|^2 = (V^*(I-P_1)Va, a)$$

where P_1 is the orthogonal projection of $\mathcal{K}$ on $\bigvee_1^{\infty} U^k V\mathcal{E}$.

It results that $\Delta[F]$ is a positive operator on $\mathcal{E}$, and we call it the Szegö (or prediction - error) operator of F. The name is justified because if $F = \mu$ is scalar valued, then

$$\Delta[\mu] = \inf_{p_o} \int_0^{2\pi} |1 - p_o|^2 d\mu$$

where the infimum is taken over all analytic polynomial p_o which vanish in origin.

Now we can state the following generalization of Szegö-Komogorov-Krein theorem:

THEOREM 5. Let F be an $\mathcal{L}(\mathcal{E})$ - valued semi-spectral measure on $\mathbb{T}$ and $\Delta[F]$ its prediction-error operator. Then we have :

(i) $\Delta[F] = 0$ if and only if does not exist an L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ $\Theta(\lambda) \neq 0$ such that $F_{\Theta} \leq F$.

(ii) If $\Delta = 0$ there exists an unique maximal (in the sense of Theorem 2) outer L^2-bounded analytic function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ such that $F_{\Theta_1} \leq F$, $\dim \mathcal{F}_1 = \dim (\Delta\mathcal{E})$, and

$$\Delta[F] = \Delta[F_{\Theta}] = \Theta(0)^* \Theta(0) .$$

Proof. If $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an L^2-bounded analytic function and $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ its outer part then $F_{\Theta} = F_{\Theta_1}$ and

$$(\Delta[F_{\Theta}]a,a) = (\Delta[F_{\Theta_1}]a,a) = \inf_{a_1, \dots, a_n \in \mathcal{E}} \| V_{\Theta_1} a - \sum_{k=1}^{n} e^{ikt} V_{\Theta_1} a_k \|^2 =$$

$$= \inf_{h \in H^2(\mathcal{F}_1),\, h(0)=0} \| V_{\Theta_1} a - h \|^2 = \| (V_{\Theta_1} a)(0) \|^2 = \| \Theta_1(0) a \|^2$$

Hence if $F_{\Theta} \leq F$ we have

$$\Theta_1(0)^* \Theta_1(0) = \Delta[F_{\Theta}] \leq \Delta[F] .$$

Let now $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ be the maximal outer function of F given by Theorem 2. Then if P is the orthogonal projection of $\mathcal{K}$ onto $\bigoplus_{-\infty}^{+\infty} U^n \mathcal{F}_1$ we have

$$(I - P_1)P = P(I - P_1) = I - P_1 .$$

Therefore

$$(\Delta[F]a, a) = \|(I-P_1)Va\|^2 = \|(I-P_1)PVa\|^2 =$$

$$= \|(P - P_1P)PVa\|^2 = (\Delta[F_{\Phi_1}]a, a)$$

i.e.

$$\Delta[F] = \Delta[F_{\Phi_1}] = \Phi_1(0)^*\Phi_1(0).$$

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