ON INTERTWINING DILATIONS.V

by

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I n t r o d u c t i o n : The interest of a functional labelling of all intertwining dilations ([1]) of a given contraction A, intertwining two contractions T' and T (i.e. $T'A = AT$) was stressed in [18] , where such a labelling, involving analytic and non-analytic operator-valued functions, was used in the study of some pure operator theory questions. More recently, in [11] , a functional labelling, by means of contractive analytic operator-valued functions, was shown to play a central role in an electrical engineering problem, in the case when $T' = T$ are contractions of class $C_o(N)$ (in the sense of [16] , Ch.IX, Sec.3). However, in the cases $T' = S^*$, $T = S$, where S is a Jordan operator (on a finite dimensional space) or a unilateral shift, this kind of labelling was already obtained by Schur (implicitely, for the numerical case, in his classical research on extrapolation [14]) and by Adamjan - Arov - Krein (explicitely, for the operatorial case, in their basic research on Hankel operators [1] , [2] , [3] , [4]).

The general case (considered for instance in [17] , [16] , [10] [8] , [5] etc), namely arbitrary contractions A, T', T and arbitrary contractive (but not necessarily strictly contractive) intertwining dilations, seems to have not been considered. The first aim of this paper is to feel this gap by showing that in this most general case there exists also a labelling by contractive analytic operator valued functions. This labelling was suggested by the previous papers [6] , [9] , [7] .

In establishing this labelling (in Sec.4 below) we shall establish another new one. Namely we shall show that the contractive intertwining dilations can be labelled by sequences $\{\Gamma_n\}_{n=1}^{\infty}$ of contractions such that Γ_1 acts between two suitable spaces while, for $n \geq 2$, Γ_n acts between the closed ranges of $I - \Gamma^*_{n-1}\Gamma_{n-1}$ and $I - \Gamma_{n-1}\Gamma^*_{n-1}$ (see Sec.3, below). This labelling was imposed to us by a problem in geophysics (where the Γ_n's have a concrete physical meaning) and by its numerical treatment . These connections will be discussed elsewhere. However in Sec. 5 we give an application of our results to the classification of all Ando's isometric dilations of a pair of commuting contractions [5] .

Finally, let us remark that at this stage of our research the explicite connection of this paper to [18] is still an open (and seemingly, basic) question.

Also, we take this opportunity to thank our colleague Gr.Arsene for the useful discussions on the subject of this Note.

1. We shall start by giving the main notations and by recalling some basic facts concerning contractive intertwining dilations.

Let H and H' be some Hilbert spaces ([2]) and let $L(H, H')$ denote the algebra of all operators from H to H'; in case $H=H'$, $L(H, H)$ will be denoted simply by $L(H)$. For two contractions $T \in L(H)$, $T' \in L(H')$ we denote by $I(T', T)$ the set of the $A \in L(H, H')$ intertwining T' and T, i.e. such that $T'A = AT$. Let $U \in L(K)$, $U' \in L(K')$ be the minimal isometric dilations of T, T' respectively ; for $n = 0,1,2,\dots$, let P_n, P'_n denote the orthogonal projections of K, K' onto

$$H_n = \begin{cases} H & (n=0) \\ H + L + UL + \dots + U^{n-1}L & (n\geq 1), \end{cases}$$

respectively

$$\overset{\circ}{H}{}'_n = \begin{cases} H' & (n=0) \\ H' + L' + U'L' + \dots + U'^{\,n-1}L' & (n\geq 1), \end{cases}$$

where

$$L = ((U-T)H)^{-}, \quad L' = ((U'-T'H')^{-}.$$

Also we set $P = P_o$, $P' = P'_o$ and

$$T_n = P_n U | H_n, \quad T'_n = P'_n U' | H_n \qquad (n = 0,1,2,\ldots,);$$

obviously $T_o = T$, $T'_o = T'$ and U, U′ are also minimal isometric dilations of T_N, T'_N, respectively (N = 0,1,2,...,). In the sequel A will be a contraction $\in I(T',T)$.

By a <u>contractive intertwining dilation</u>, respectively <u>N^{th}-partial intertwining dilation</u> (shortly denoted in the sequel by CID, resp. N - PCID) of A we mean an operator $A_\infty \in L(K, K')$, resp. $A_N \in L(H_N, H'_N)$ such that

$$(1.1) \quad \| A_\infty \| \leq 1, \; A_\infty \in I(U', U), \quad P'A_\infty = AP,$$

respectively

$$(1.1)_N \quad \| A_N \| \leq 1, \; A_N \in I(T'_N, T_N), \; P'A_N = A(P | H_N).$$

Thus, the operator

$$(1.2)_n \quad A_n = P'_n A_\upsilon | H_n$$

where

$$n = 0, 1, 2, \ldots, \quad \text{if } \upsilon = \infty$$
$$n = 0, 1, 2, \ldots, N \text{ if } \upsilon = N,$$

is an n - PCID of A, and

$$(1.3)_n \quad P'_n A_{n+1} = A_n (P_n | H_{n+1})$$

for $0 \leq n < \upsilon$; moreover in the first case we have

$$(1.4) \qquad A_\infty = \text{strong} \lim_{n \to \infty} A_n P_n .$$

It is also easy to verify that, conversely, if a sequence of n - PCID's A_n (n=0,1,2,...,) satisfies the conditions $(1.3)_n$ (n= 0,1,2,...,), then the strong limit in (1.4) exists and defines a CID of A. Therefore we can state the following :

R e m a r k 1.1. There exists an one-to-one correspondance (given by $(1.2)_n$,n = 0,1,2,..., and (1.4)) between the CID's of A and the sequences $\{ A_n \}_{n=0}^{\infty}$ of n-PCID's of A, A_n satisfying $(1.3)_n$ (n = 0,1,2,...,).

In order to facilitate the exposition, we shall give now several useful facts, which actually resume the original construction of a CID (see [17] , [10] , [16] , [7]). To this aim, let T, T' and A be as above. We set (³)

$$(1.5) \quad \begin{cases} F_A = \{D_A Th + (U-T)h : h \in H\}^- \\ R_A = (\mathcal{D}_A + L) \ominus F_A \end{cases}$$

and

$$(1.5)' \quad \begin{cases} F'_A = \{D_A h \oplus (U'-T')Ah : h \in H\}^- \\ R'_A = (\mathcal{D}_A \oplus L') \ominus F'_A . \end{cases}$$

L e m m a 1.1. <u>Let</u> T, T' <u>and</u> A <u>be as above</u>. <u>Then</u>

$$(1.6)\quad C(D_A Th + (U-T)h) = (U' - T')Ah \qquad (h \in H)$$

defines a contraction $C = C_A \in L(F_A, L')$. Moreover the formula

$$(1.7)\quad W_A D_{C^*} l' = R'_A (0_{\mathcal{D}_A} \oplus l') \qquad (l' \in L'),$$

where R'_A denotes the orthogonal projection of $\mathcal{D}_A \oplus L'$ onto R'_A, defines a unitary operator from $\mathcal{D}_{C_A^*}$ onto R'_A.

P r o o f. Let $i_{L'}$ and ω be the operators defined by

$$i_{L'}(l') = 0_{\mathcal{D}_A} \oplus l' \in \mathcal{D}_A \oplus L' \qquad (l' \in L'),$$

$$\omega(D_A Th + (U-T)h) = D_A h \oplus (U'-T')Ah \quad (h \in H).$$

Obviously $i_{L'}$ is unitary from L' to $\{0\} \oplus L' \subset \mathcal{D}_A \oplus L'$; also, ω is unitary from F_A to F'_A since, by virtue of [16], Sec.II.1, we have

$$\begin{aligned}
||D_A Th + (U-T)h||^2 &= ||D_A Th||^2 + ||(U-T)h||^2 = \\
&= ||Th||^2 - ||ATh||^2 + ||D_T h||^2 = ||h||^2 - ||ATh||^2 = \\
&= ||D_A h||^2 + ||Ah||^2 - ||T'Ah||^2 = ||D_A h||^2 + ||D_{T'} Ah||^2 = \\
&= ||D_A h||^2 + ||(U'-T')Ah||^2 = ||D_A h \oplus (U'-T')Ah||^2
\end{aligned}$$

for all $h \in H$. We shall consider $i_{L'}$ as operator from L' to $\mathcal{D}_A \oplus L'$ and we shall extend ω on the whole of $\mathcal{D}_A + L$, by setting $\omega r = 0_{\mathcal{D}_A} \oplus L$ for all $r \in R_A$. Then $C_A = i^*_{L'} \omega | F_A$, hence C_A is a contraction and

$$C^*_A = \omega^* i_{L'}, \quad C_A C^*_A = i^*_{L'} \omega\omega^* i_{L'}, \quad D^2_{C^*_A} = i^*_{L'} R'_A i_{L'}.$$

It follows that

$$||D_{C^*_A} l'||^2 = (D^2_{C^*_A} l', l') = (i^*_{L'} R'_A i_{L'} l', l') =$$

$$= \|R'_A i_{L'} l'\|^2 = \|R'_A(0_{\mathcal{D}_A} \oplus l')\|^2 \qquad (l' \in L')$$

and consequently that W_A is an isometric operator from $\mathcal{D}_{C_A^*}$ to R'_A. If $d_o \oplus l'_o \in R'_A$ is orthogonal to the range of W_A then

$$(l'_o, l') = (d_o \oplus l'_o, 0 \oplus l') = (d_o \oplus l'_o, W_A D_{C_A^*} l') = 0 \qquad (l' \in L')$$

whence $l'_o = 0$. But, by $(1.5)'$,

$$(d_o, D_A h) = (d_o \oplus l'_o, D_A h \oplus (U'-T')Ah) = 0 \qquad (h \in H),$$

whence $d_o = 0$, since $d_o \in \mathcal{D}_A$. Thus $d_o \oplus l'_o = 0$ and consequently we conclude that W_A is unitary.

L e m m a 1.2. Let T, T' and A be as in Lemma 1.1. Then the formula

$$(1.8) \quad C(A_1)\ (D_A P + I - P) | H_1 = (I-P)A_1$$

establishes an one-to-one correspondence between the 1-PCID A_1 of A and all contractions

$$(1.9) \quad C : \mathcal{D}_A + L \longrightarrow L',\ C \mid F_A = C_A.$$

Moreover the formula

$$(1.10) \quad X(A_1) D_{C(A_1)} (D_A P + I - P) \mid H_1 = D_{A_1}$$

defines a unitary operator from $\mathcal{D}_{C(A_1)}$ to $\mathcal{D}_{A_1}$.

P r o o f. Let A_1 be a 1-PCID of A.

Then, since by $(1.1)_1$,

$$(1.11)\quad \|(I-P')A_1h_1\|^2=\|A_1h_1\|^2-\|P'A_1h_1\|^2 =$$
$$= \|A_1h_1\|^2-\|APh_1\|^2\leq\|h_1\|^2-\|APh_1\|^2=$$
$$= \|h_1\|^2-\|Ph_1\|^2+\|D_APh_1\|^2=\|(I-P)h_1\|^2+$$
$$+ \|D_APh_1\|^2 = \|(D_AP + I - P)h_1\|^2 \qquad (h_1 \in H_1),$$

we infer that indeed the formula (1.8) defines a contraction $C=C(A_1)$ from $\mathcal{D}_A + L$ to $L' = (I-P)H_1$. Moreover, since

$$(D_AP + I - P)T_1h = D_ATh + (U-T)h \qquad (h\in H),$$

we have also

$$C(D_ATh + (U-T)h)= (I-P')A_1T_1h=(I-P')T_1'A_1h=$$
$$=(I-P')T_1'PA_1h=(I-P')T_1'Ah=(U-T')Ah=C_A(D_ATh+(U-T)h) \quad (h\in H),$$

i.e. $C|F_A = C_A$. Also, from

$$A_1 = P'A_1 + (I-P')A_1 = AP|H_1 + (I-P')A_1$$

we obtain

$$(1.12)\quad A_1=(AP + C(D_AP + I - P))|H_1.$$

This formula shows that A_1 is uniquely determined by $C=C(A_1)$. Let now C be any contraction enjoying the properties (1.9) and let $A_1 \in L(H_1)$ be defined by (1.12). Then, the relation $P'A_1=AP|H_1$ is plain and therefore

$$T_1'A_1h_1=T_1'P'A_1h_1 = T_1'APh_1=T'APh_1+(U'-T')APh_1= =T'APh_1+C_A(D_AT + U -T)Ph_1= ATPh_1+C_A(D_AT + U- - T)Ph_1=ATPh_1+C(D_AT+U-T)Ph_1=APT_1Ph_1+C(D_APT_1Ph_1+ + (1-P)T_1)h_1=(AP+C(D_AP+I-P))T_1h_1=A_1T_1h_1 \qquad (h_1 \in H_1),$$

i.e. $A_1 \in I(T_1', T_1)$. Moreover, from (1.12) we infer

$$(1.13)\quad ||h_1||^2-||A_1h_1||^2= ||h_1||^2-||APh_1||^2- -||C(D_AP + I-P)h_1||^2=||D_C(D_AP+I-P)h_1||^2 \qquad (h_1 \in H_1).$$

This shows that A_1 is a contraction. We have thus verified that A_1 enjoys the properties $(1.1)_1$. The last statement of the lemma follows now readily from (1.13).

R e m a r k 1.2. The basic existence theorem [17], [16] for a CID A_∞ of a contraction $A \in I(T', T)$, where T', T are as above, follows from the preceding lemmas by the following simple recurrent construction.

Set $A_o = A$ and set $C_1 = C_AQ_1$ where Q_1 denote the orthogonal projection of $\mathcal{D}_{A_o}+L$ onto F_{A_o}. Define A_1 as the 1 - PCID such that $C(A_1) = C_1$. Repeat the some precedure with A_1, UL and $U'L'$ in the role of A_o, L and L' and obtain A_2, and so on. Finally one obtains a sequence $\{A_n\}_{n=0}^{\infty}$ of n - PCID's A_n of A satisfying the conditions $(1.3)_n$ (n=0,1,2,...) and consequently a CID A_∞ of A, by virtue of Remark 1.1.

2. Let T, T' and A, $||A||\le 1$, be as in Sec.1. Let moreover A_N be an N-PCID of A and $\overset{\circ}{A}_n$ be the operator defined by $(1.2)_n$ (n=1,2,...,N-1). From Lemma 1.2 it follows readily that A_N is uniquely determined, and also uniquely determines, a string $\{C_n\}_{n=1}^{N}$ of $U'^{\,n-1}L'$ - valued contractions C_n (n=1,2,..N-1), namely the string

$\{C(A_n)\}_{n=1}^{N}$. However, the definition of the string $\{C(A_n)\}_{n=1}^{N}$ explicitely involves, besides the operators U, U' (i.e. the minimal isometric dilations of T, T', respectively) and A, also the operators $A_1,\dots,A_{N-1}$. In order to get rid of the explicite reference to $A_1,\dots,A_{N-1}$ in the characterization of A_N by a string of $U'^{\,n-1}\mathcal{L}'$ - valued contraction C_n $(n=1,2,\dots,N)$, we firstly introduce the following :

D e f i n i t i o n 2.1. A string or a sequence $\{C_n\}_{1\le n<\upsilon}$ (where $\upsilon = 1,2,\dots,\infty$; in this last casewe shall set $\upsilon - 1 = \infty$) of operators

$$(2.1)\qquad C_n:\mathcal{D}_{n-1} + U^{n-1}\mathcal{L} \mapsto U'^{\,n-1}\mathcal{L}' \qquad (1\le n<\upsilon)$$

is called <u>A-cascade</u> if each C_n $(1\le n <\upsilon)$ is a contraction,

$$(2.2)\qquad \mathcal{D}_o=\mathcal{D}_A,\quad \mathcal{D}_n=\mathcal{D}_{C_n} \qquad (1\le n<\upsilon-1),$$

$$(2.3)_1\qquad C_1|F_A=C_A \quad ,$$

$$(2.3)_2\qquad C_2(D_{C_1}(D_ATh+(U-T)h)+Ul_1)=U'C_1(D_Ah+l_1)\quad (h\in\mathcal{H},\ l_1\in\mathcal{L})$$

(in case $\upsilon>2$), and

$$(2.3)_n\qquad C_n(D_{C_{n-1}}(D_{C_{n-2}}(\dots(D_{C_1}(D_ATh + (U-T)h)+Ul_1)\dots)+U^{n-2}l_{n-2}) + U^{n-1}l_{n-1}) =$$
$$=U'C_{n-1}(D_{C_{n-2}}(\dots(D_{C_1}(D_Ah+l_1) + Ul_2)\dots)+ U^{n-2}l_{n-1}) \qquad (h\in\mathcal{H},l_1,\dots,l_{n-1}\in\mathcal{L})$$

for all $3\le n<\upsilon$ (in case $\upsilon>3$).

In the next two lemmas, $\{C_n\}_{1\le n<\upsilon}$ will be any fixed A-cascade string or sequence; also the spaces $\mathcal{D}_n$ $(0\le n<\upsilon)$ will have the same meaning as in the preceding definition.

L e m m a 2.1. There exists a unique string (or sequence, respectively) $\{Y_n\}_{1\le n<\upsilon}$ of isometric operators

$$(2.4)\qquad Y_n : \mathcal{D}_{n-1} \mapsto \mathcal{D}_n \qquad (1\le n<\upsilon)$$

such that

$$(2.5)_1\qquad Y_1 D_A h = D_{C_1}(D_A Th+(U-T)h) \qquad (h\in H),$$

and (if $\upsilon>2$)

$$(2.5)_n\qquad Y_n D_{C_{n-1}}(d+U^{n-2}l)=D_{C_n}(Y_{n-1}d+U^{n-1}l) \qquad (d\in\mathcal{D}_{n-2}, l\in L)$$

for all n, $2\le n<\upsilon$.

P r o o f. We have, by $(2.3)_1$,

$$\begin{aligned}||D_A h||^2 &= ||Th||^2-||ATh||^2+||D_T h||^2-\\ &- ||D_{T'}Ah||^2=||D_A Th+(U-T)h||^2-\\ &- ||U'-T')Ah||^2=||D_A Th + (U-T)h||^2-\\ &- ||C_A(D_A Th+(U-T)h||^2=||D_{C_1}(D_A Th+(U-T)h||^2 \quad (h\in H),\end{aligned}$$

thus, indeed, $(2.5)_1$ defines an isometric operator from $\mathcal{D}_o=\mathcal{D}_A$ to $\mathcal{D}_1=\mathcal{D}_{C_1}$. We assume now that $(2.5)_n$ (for n=m-1, $2\le m<\upsilon$) defines an isometric operator Y_{m-1}, obviously in a unique manner. Then the definition $(2.3)_m$ can be written under the form

$$C_m(Y_{m-1}d+U^{m-1}l)=U'C_{m-1}(d+U^{m-2}l) \qquad (d\in\mathcal{D}_{m-2}, l\in L);$$

consequently we have

$$||D_{C_{m-1}}(d+U^{m-2}l)||^2=||d+U^{m-2}l||^2-$$

$$-||C_{m-1}(d+U^{m-2}l||^2=||d||^2+||U^{m-2}l||^2-$$

$$-||U'C_{m-1}(d+U^{m-2}l)||^2=||Y_{m-1}d||^2+||U^{m-1}l||^2-$$

$$-||C_m(Y_{m-1}d+U^{m-1}l)||^2=||Y_{m-1}d+U^{m-1}l||^2-$$

$$-||C_m(Y_{m-1}d+U^{m-1}l)||^2=||D_{C_m}(Y_{m-1}d+U^{m-1}l||^2$$

$$(d\in\mathcal{D}_{m-2},\ l\in L).$$

These relations show that, indeed, $(2.5)_m$ defines the searched isometric operator Y_m. Thus the lemma is proved by recurrence.

R e m a r k 2.1. By virtue of Lemma 2.1, it is easy to infer that the definitions $(2.3)_n$ (for $2\leq n<\upsilon$) can be written under the compact form

$$(2.6)_n \quad C_n(Y_{n-1}d+U^{n-1}l)=U'C_{n-1}(d+U^{n-2}l) \qquad (d\in\mathcal{D}_{n-2},\ l\in L).$$

Also let us notice that

$$(2.7)_n \quad \mathcal{D}_n\subset\mathcal{D}_{n-1}+U^{n-1}L,\quad \mathcal{D}_n\subset H_n$$

$(1\leq n<\upsilon-1)$ and $\mathcal{D}_o\subset H$.

Now let us consider any isometric operator $X:\mathcal{D}_1\mapsto H_1$. By $(2.7)_n$ $(2\leq n<\upsilon)$, we can attach to X the string (or sequence) $\{X_n\}_{1\leq n<\upsilon}$ of unitary operators

$$(2.8) \quad X_n:\mathcal{D}_n+U^nL\mapsto R_n = \text{Range}\ (X_n) \qquad (1\leq n<\upsilon)$$

by following recurrent manner :

$$(2.9) \quad R_1=\text{Range}\ (X)+UL$$

$$(2.10)_n \quad X_n|U^n\overset{\circ}{L} = I_{U^nL}$$

$(1\leq n<\upsilon)$,

$$(2.11)_1 \quad X_1|\mathcal{D}_1=X$$

and, in case $\upsilon>2$,

$$(2.11)_n \quad X_n|\mathcal{D}_n=X_{n-1}|\mathcal{D}_n$$

$(2\leq n<\upsilon)$. This sequence will be improperly called the $\{C_n\}_{1\leq n<\upsilon}$-extension of X.

Let moreover A_1 be the 1-PCID of A such that $C(A_1)=C_1$ (see Lemma 1.2) and let $\{X_n\}_{1\leq n<\upsilon}$ be the $\{C_n\}_{1\leq n<\upsilon}$ - extension of $X(A_1)$; then by virtue of (2.1) and (2.8) $C_n'=C_{n+1}X_n\in L(\mathcal{R}_n,U'^nL')$ $(1\leq n<\upsilon-1)$. For convenience, the string (or sequence) $\{C_n'\}_{1\leq n<\upsilon-1}$ will be called the reduced string (or sequence) of $\{C_n\}_{1\leq n<\upsilon}$ (here plainly one assumes that $\upsilon>2$).

L e m m a 2.2. The reduced string (or sequence) is A_1-cascade.

P r o o f. It is plain that C_n' $(1\leq n<\upsilon)$ are contractions from $\mathcal{D}'_{n-1}+U^{n-1}(UL)$ to $U'^{n-1}(U'L')$, where $\mathcal{D}_0'=\mathcal{D}_{A_1}$ and $\mathcal{D}'_{n-1}=X_n\mathcal{D}_n$ $(1\leq n<\upsilon)$. Moreover, since $C_n'X_n=C_{n+1}$ $(1\leq n<\upsilon-1)$, we have

$$X_n^*C_n'^*C_n'X_n=C_{n+1}^*C_{n+1}, \quad C_n'^*C_n'X_n=X_nC_{n+1}^*C_{n+1}$$

whence

$$(2.12)_n \quad D_{C_n'}X_n=X_nD_{C_{n+1}}$$

$(1\leq n<\upsilon-1)$. Since, by $(2.11)_2$, (2.1), (2.2), $(2.10)_1$, $(2.11)_1$ and (1.10), we have

$$\mathcal{D}_1'=X_2\mathcal{D}_2= \quad X_1\mathcal{D}_2\subset X_1(\mathcal{D}_1+UL)=X(A_1)\mathcal{D}_{C_1}+UL=$$
$$=\mathcal{D}_{A_1}+UL=\mathcal{D}_o+UL$$

and, since (if $\upsilon>3$ and $2\leq n<\upsilon-1$) we have, by $(2.11)_{n+1}$, (2.2), $(2.12)_n$ and (2.8)

$$\mathcal{D}_n'=X_{n+1}\mathcal{D}_{n+1}=X_n\mathcal{D}_{n+1}=X_n\mathcal{D}_{C_{n+1}}=X_n(D_{C_{n+1}}(\mathcal{D}_n+U^nL))^-=$$
$$=(D_{C_n'}R_n)^-=\mathcal{D}_{C_n'},$$

the relations (2.1) and (2.2) are satisfied by $\{C_n'\}_{1\leq n<\upsilon}$ (of course with υ, $\mathcal{D}_n$, L and L' replaced by $\upsilon-1$, $\mathcal{D}_n'$, UL and $U'L'$). Also, by virtue of $(2.10)_1$ and (1.10), we have

$$(2.13)_1\quad C_1'(D_{A_1}(h+l_1)+Ul_2)=C_2X_1^{-1}(D_{A_1}(h+l_1)+Ul_2)=$$
$$=C_2(X_1^{-1}D_{A_1}(h+l_1)+Ul_2)=C_2(D_{C_1}(D_Ah+l_1)+Ul_2)$$
$$(h\in H,\ l_1,\ l_2\in L)$$

$$(2.13)_n\quad C_n'(D_{C_{n-1}'}(\ldots D_{C_1'}(D_{A_1}(h+l_1)+Ul_2)+U^2l_3)\ldots)+U^nl_{n+1})=$$
$$=C_{n+1}X_n^{-1}(D_{C_{n-1}'}(\ldots)+U^nl_{n+1})=C_{n+1}(X_n^{-1}D_{C_{n-1}'}(\ldots)+$$
$$+U^nl_{n+1})=C_{n+1}\ (X_n^{-1}\cdot X_{n-1}D_{C_n}X_{n-1}^{-1}(\ldots)+U^nl_{n+1})=$$
$$=C_{n+1}(D_{C_n}X_{n-1}^{-1}(\ldots)+U^nl_{n+1})=\ldots=$$
$$=C_{n+1}(D_{C_n}(\ldots X_2^{-1}(D_{C_1'}(D_{A_1}(h+l_1)+Ul_2)+U^2l_3)\ldots)+$$
$$+U^nl_{n+1})=C_{n+1}(D_{C_n}(\ldots(X_2^{-1}X_1D_{C_2}X_1^{-1}(D_{A_1}(h+l_1)+Ul_2)+$$
$$+U^2l_3)\ldots)+U^nl_{n+1})=C_{n+1}(D_{C_n}(\ldots(D_{C_2}X_1^{-1}(D_{A_1}(h+l_1)+Ul_2)+$$
$$+U^2l_3)\ldots)+U^nl_{n+1})=C_{n+1}(D_{C_n}(\ldots(D_{C_2}(X_1^{-1}D_{A_1}(h+l_1)+Ul_2)+$$
$$+U^2l_3)\ldots)+U^nl_{n+1})=C_{n+1}(D_{C_n}(\ldots(D_{C_2}(D_{C_1}(D_Ah+l_1)+$$
$$+Ul_2)+U^2l_3)\ldots)+U^nl_{n+1})\qquad(h\in H,\ l_1,\ldots,l_{n+1}\in L),$$

$$(D_{C_1}(D_A h+l_1)+Ul_2)+U^2 l_3)\ldots)+U^n l_{n+1}) \quad (h\in H, l_1\ldots l_{n+1}\in L),$$

where we used in order the relations $(2.10)_n$, $(2.12)_{n-1}$, $(2.11)_n$, $(2.10)_{n-1}$, ..., $(2.10)_2$, $(2.12)_1$, $(2.11)_2$, $(2.10)_1$, $(2.11)_1$ and (1.10). Now, from $(2.13)_1$, $(2.3)_2$ and (1.8), we infer

$$C_1'(D_{A_1}T_1(h+l)+(U-T_1)(h+l))=C_1'(D_{A_1}(Th+(U-T)h)+Ul)=$$
$$=C_2(D_{C_1}(D_A Th+(U-T)h)+Ul)=U'C_1(D_A h+l)=U'(P_1'-P')A_1(h+l)$$
$$=(U'-T_1')A_1(h+l) \qquad (h\in H,\ l\in L) \quad ,$$

thus C_1' satisfies $(2.3)_1$ (of course with F_A and C_A replaced by F_{A_1} and C_{A_1}). Also, (in case $\upsilon>3$) from $(2.13)_2$, $(2.3)_3$ and $(2.13)_1$ we infer

$$C_2'(D_{C_1'}(D_{A_1}T_1(h+l)+(U-T_1)(h+l))+U^2 l_1)=$$
$$=C_2'(D_{C_1'}(D_{A_1}(Th+(U-T)h)+Ul)+U^2 l_1)=$$
$$=C_3(D_{C_2}(D_{C_1}(D_A Th+(U-T)h)+Ul)+U^2 l_1)=$$
$$=U'C_2(D_{C_1}(D_A h+l)+Ul_1)=U'C_1'(D_{A_1}(h+l)+Ul_1) \quad (h\in H, l, l_1\in L),$$

thus C_1', C_2' satisfy $(2.3)_2$ (of course with A and L, L' replaced by A_1 and UL, $U'L'$ respectively). Finally, in a similar way, one verifies that (in case $\upsilon>4$), by virtue of $(2.13)_n$, $(2.3)_{n+1}$ and $(2.13)_{n-1}$, the string $\{C_n'\}_{1\le n<\upsilon-1}$ satisfies the relations $(2.3)_n$ for all n, $3\le n<\upsilon-1$ (of course, again with A, L, L' replaced by A_1, UL, $U'L'$). This finishes the proof of the lemma.

L e m m a 2.3. Let A_1 be an 1-PCID of A. Any A_1-cascade string (or sequence) $\{C_n'\}_{1\le n<\upsilon-1}$ is the reduced string of a uniquely determined A-cascade string (or sequence) of contractions $\{C_n\}_{1\le n<\upsilon}$.

P r o o f. If the string (or sequence) $\{C'_n\}_{1\leq n<\nu-1}$ is the reduced string (or sequence) of $\{C_n\}_{1\leq n<\nu}$, then this last one must be defined in the following manner : Firstly

$$(2.14)_1 \quad C_1 = C(A_1), X_1|\mathcal{D}_{C_1} = X(A_1), X_1|UL = I_{UL}$$

(thus $\mathcal{D}_{C_1} \subset \mathcal{H}_1$ and X_1 is a unitary operator from $\mathcal{D}_{C_1}+UL$ to $\mathcal{R}_1=\mathcal{D}_{A_1}+UL$), secondly

$$(2.14)_n \begin{cases} C_n = C'_{n-1} X_{n-1} \\ X_n(d_n+U^n l) = X_{n-1}d_n + U^n l \qquad (d_n \in \mathcal{D}_{C_n},\ l \in L) \end{cases}$$

$(2\leq n<\upsilon)$, where X_n is viewed as operator from $\mathcal{D}_{C_n}+U^nL$ to $\mathcal{R}_n=(X_{n-1}\mathcal{D}_{C_n})^- + U^nL$. These definitions are consistent if they imply recurrently

$$X_{n-1}\mathcal{D}_{C_n} \subset \mathcal{D}_{C_{n-1}} \qquad (2\leq n<\upsilon).$$

However, we shall prove, by induction, even more, namely that

$$(2.15)_n \quad X_{n-1}\mathcal{D}_{C_n} = \mathcal{D}_{C'_{n-1}}$$

$$(2.16)_n \quad \mathcal{D}_{C_n} \subset \mathcal{H}_n$$

and that X_n is unitary (for $2\leq n<\upsilon$) (for n=2, the last two statements are, by virtue of $(2.16)_1$, obviously true). We start by noticing that if for some $n, 2\leq n<\upsilon$, the first relation $(2.14)_n$ makes sense and if X_{n-1} is unitary then, by the same argument as in the proof of Lemma 2.2, we infer the validity of $(2.12)_{n-1}$, whence that of $(2.15)_n$. Thus, by virtue of $(2.14)_1$, $(2.15)_2$ is also valid, so that we have completed the first induction step. In case $\upsilon>3$, we can therefore assume that the statements are always valid for n-1, n being fixed, $2 \le n < \upsilon$. Then, be virtue of $(2.15)_{n-1}$ and the fact that $\{C'_n\}_{1 \le n < \upsilon - 1}$ is A_1-cascade, the first relation $(2.14)_n$ makes sense ; thus, by virtue of the above discussion on $(2.15)_n$, we infer that this relation is valid. Therefore, using once again the fact that $\{C'_n\}_{1 \le n < \upsilon - 1}$ is A_1- cascade, from the second relations $(2.14)_n$ we obtain that X_n is unitary, while from the second relation $(2.14)_{n-1}$ and $(2.16)_{n-1}$ we obtain $(2.16)_n$. Thus the n^{th} inductive step is completed and consequently the string (or sequence) $\{C_n\}_{1 \le n < \upsilon}$ is consistently defined. By this very definition, it is plain that $\{C_n\}_{1 \le n < \upsilon}$ satisfies the conditions (2.1), (2.2) and $(2.3)_1$. Now we can establish, as in the proof of Lemma 2.2, the relations $(2.13)_n$ $(1 \le n < \upsilon - 1)$ and subsequently infer the relations $(2.3)_n$ $(2 \le n < \upsilon)$ for $\{C_n\}_{1 \le n < \upsilon}$ from the fact that $\{C'_n\}_{1 \le n < \upsilon}$, being A_1-cascade, satisfies $(2.3)_n$($1 \le n < \upsilon - 1$; of course with A, L and L' replaced by A_1, $U'L'$, respectively). In this manner we conclude that $\{C_n\}_{1 \le n < \upsilon}$ is A-cascade. Actually, the proof of the lemma is now completed.

We can, and shall, now define a mapping from A-cascade strings to PCID's of A, for any contractions T, T' and $A \in I(T', T)$. Namely, for given T, T' ; A, N=1,2,..., and A-cascade string $\{C_n\}_{n=1}^{N}$ (of length N) we shall define an N-PCID $A_N(A;C_1, \dots, C_N)$ by the following recurrent formula

$$(2.17)_1 \qquad A_1(A;C_1) = A_1$$

where A_1 is the 1-PCID (yielded by Lemma 1.2) such that $C(A_1) = C_1$, and

$$(2.17)_N \qquad A_N(A;\, C_1, \dots, C_N) = A_{N-1}(A_1; C'_1, \dots, C'_{N-1})$$

where $\{C'_n\}_{n=1}^{N-1}$ is the reduced string of $\{C_n\}_{n=1}^{N}$ (N=2,3, ...). (Actually, one should write $A_N(A;\, T', T;\, U', U;\, C_1, \dots, C_N)$ instead of $A_N(A;\, C_1, \dots C_N)$

since this operator depends also on T, T' and the concrete constructions of the isometric dilations U, U' of T, T' respectively; thus $(2.17)_N$ be should written under the form

$$A_N(A;T',T;U',U;C_1,\ldots,C_N)=A_{N-1}(A_1;T'_1,T_1;U',U;C'_1,\ldots,C'_{N-1}).$$

However, when no confusion seems possible, we shall not complicate the notations with these precisenesses).

The consistence of the definitions $(2.17)_N$ (N=2,3,...,) is a direct consequence of Lemma 2.2 and the fact that any (N-1)-PCID of an 1-PCID of A is an N-PCID of A. Also by an obvious inductive argument it follows that

$$(2.18)_N \quad P'_{N-1}A_N(A;C_1,\ldots,C_N)=A_{N-1}(A;C_1,\ldots,C_{N-1})(P_{N-1}|H_N)$$

(N=2,3,...,), i.e. $A_N(A;C_1,\ldots,C_N)$ is an 1-PCID of $A_{N-1}(A;C_1,\ldots C_{N-1})$.

P r o p o s i t i o n 2.1. <u>For</u> N=1,2,..., <u>and</u> T, T', $A\in I(T',T)$ <u>fixed, the mapping</u>

$$(2.19)_N \quad \{C_n\}_{n=1}^{N}\mapsto A_N(A;C_1,\ldots,C_N)$$

<u>establishes an one-to-one correspondence between the A-cascade</u> [illegible] <u>strings (of length N) and the N-PCID's of A.</u>

P r o o f. For N=1, the statement in the proposition reduces to the first statement in Lemma 1.2. Therefore we assume that the statement is also true for $N=m-1\geq 1$. Let moreover A_m be an m-PCID of A and let A_1 be the 1-PCID of A defined by $(1.2)_1$ (with $\nu=m$). Then, A_m is an (m-1)-PCID of A_1, thus by the inductive assumption, there exists a uniquely determined A_1-cascade string $\{C'_n\}_{n=1}^{m-1}$ such that

$(2.20)_m \quad A_m = A_{m-1}(A;\ C'_1,\dots,C'_{m-1}).$

By virtue of Lemma 2.3, there exists a unique A-cascade string $\{C_n\}_{n=1}^{N}$ such that $\{C'_n\}_{n=1}^{m-1}$ is the (A_1-cascade) reduced string of $\{C_n\}_{n=1}^{m}$; moreover (see $(2.14)_1$) $C_1=C(A_1)$. Therefore from $(2.20)_m$ and $(2.17)_m$ we infer that A_m is of the form

$$A_m = A_m(A;C_1,\ C_2,\dots,C_m),$$

where $\{C_n\}_{n=1}^{m}$ is the above (uniquely determined) A-cascade string. This finishes the proof of the proposition.

L e m m a 2.4. <u>Within the frame of Proposition 2.1, we have</u>

$$(2.21)_N \quad \|D_{A_N}h\| = \|D_{C_N}D_{C_{N-1}}\dots D_{C_1}D_A h\| \quad (h\in H),$$

<u>where</u> $A_N=A_N(A;\ C_1,\ C_2,\dots,C_N)$ <u>and</u> $\{C_n\}_{n=1}^{N}$ <u>is an A-cascade string.</u>

P r o o f. The relation $(2.21)_1$ follows directly from Lemma 1.2.

Let the relation $(2.21)_N$ be true for $N=m-1\geq 0$. Then from $(2.17)_m$ we infer

$$(2.22) \quad \|D_{A_m}h\| = \|D_{A'_{m-1}}h\| = \|D_{C'_{m-1}}\dots D_{C'_1}D_{A_1}h\| \quad (h\in H)$$

where $A'_{m-1}=A_{m-1}(A_1;C'_1,\dots,C'_{m-1})$, A_1 is the 1-PCID of A defined by $(1.2)_1$ (with $\upsilon=m$) and $\{C'_n\}_{n=1}^{m-1}$ is the reduced string of $\{C_n\}_{n=1}^{m}$. But, by virtue of Lemma 1.2 we have

$$D_{A_1}h = X(A_1)D_{C_1}D_A h$$

so that, if $\{X_n\}_{n=1}^{N}$ is the $\{C_n\}_{n=1}^{N}$- extension of $X(A_1)$, we obtain

$$D_{C'_{N-1}} \ldots D_{C'_1} D_{A_1} h = D_{C'_{N-1}} \ldots D_{C'_1} X_1 D_{C_1} D_A h =$$
$$= D_{C'_{N-1}} \ldots D_{C'_2} X_1 D_{C_2} D_{C_1} D_A h = D_{C'_{N-1}} \ldots D_{C'_2} X_2 .$$
$$D_{C_2} D_{C_1} D_A h = \ldots = D_{C'_{N-1}} X_{N-2} D_{C_{N-1}} \ldots D_{C_1} D_A h =$$
$$= X_{N-1} D_{C_N} D_{C_{N-1}} \ldots D_{C_1} D_A h \qquad (h \in H),$$

where we used, in order, the relation $(2.11)_1$, $(2.12)_1$, $(2.11)_2, \ldots,$ $(2.11)_{N-1}$, $(2.12)_{N-1}$. Since X_{N-1} is unitary, from (2.22) it follows that $(2.21)_m$ is also valid. This completes the proof.

P r o p o s i t i o n 2.2. <u>The mapping</u>

$$(2.23) \qquad \{C_n\}_{n=1}^{\infty} \longleftrightarrow A_\infty(A;\ C_1,\ C_2, \ldots,) = \text{strong}\lim A_N(A; C_1, \ldots, C_N) P_N$$

<u>establishes an one-to-one correspondence between all the A-cascade sequences and all the CID's of A. Moreover</u> $A_\infty = A_\infty(A; C_1, C_2, \ldots)$ <u>is an isometry if and only if</u>

$$(2.24) \qquad \|D_{C_N} D_{C_{N-1}} \ldots D_{C_1} D_A h\| \underset{N\to\infty}{\to} 0 \qquad (h \in H).$$

P r o o f. The first statement of proposition follows at once from Remark 1.1, Proposition 2.1 and $(2.18)_N$ $(N=2,3,\ldots,)$. Concerning the second, we remark that (2.24), holds if and only if

$$(2.25) \qquad \|D_{A_N} h\| \underset{N\to\infty}{\to} 0 \qquad (h \in H),$$

where

$$A_N = A_N(A;\ C_1, \ldots, C_N) \qquad (N=1,2,\ldots,).$$

From the first statement it follows that

$$||D_{A_\infty}h||^2=||h||^2-||A_\infty h||^2=||h||^2-\lim_{N\to\infty}||A_N h||^2=$$

$$=\lim_{N\to\infty}||D_{A_N}h||^2$$

and consequently (2.25) (or equivalently (2.24)) holds if and only if $D_{A_\infty}|\mathcal{H}=0$, that is if, $A_\infty|\mathcal{H}$ is isometric. Thus it remains only to prove that the last property implies that A_∞ is isometric. Or, since $A_\infty U = U'A_\infty$ it follows at once that $A_\infty|U^n\mathcal{H}$ is isometric for all $n=0,1,2,\ldots,$; in its turn, this implies that

$$D_{A_\infty}|U^n\mathcal{H}=0 \qquad (n=0,1,2,\ldots,).$$

Since the spaces $U^n\mathcal{H}(n=0,1,2,\ldots,)$ span $\mathcal{K}$, we conclude that $D_{A_\infty}=0$, i.e. A_∞ is isometric.

3. Proposition 2.1 and 2.2 reduce the study of all PCID's and CID's of an $A\in I(T',T)$ (where T, T' and A are some given contractions) to that of the A-cascade strings and sequences. However an A-cascade string or sequence is a rather involved concept. Therefore we shall show that the study can be actually confined to more transparent concepts, one of which is defined in the following.

D e f i n i t i o n 3.1. A string (or sequence) $\{\Gamma_n\}_{1\le n<\upsilon}$ of operators will be called an A-<u>choice string</u> (<u>or sequence</u>) if each $\Gamma_n (1\le n<\upsilon)$ is a contraction acting from $\mathcal{R}_A$ to $\mathcal{R}'_A$ (if n=1) and from $\mathcal{D}_{\Gamma_{n-1}}$ to $\mathcal{D}_{\Gamma^*_{n-1}}$ (if $n\ge 2$). (Thus if $\{\Gamma_n\}_{n=1}^N$ is an A-choice string, then for <u>any</u> contraction $\Gamma_{N+1}\in L(\mathcal{D}_{\Gamma_N}, \mathcal{D}_{\Gamma^*_N})$, $\{\Gamma_n\}_{n=1}^{N+1}$ is also an A-choice string; this is the justification of the terminology).

In this section we shall establish a natural connection between the A-cascade strings (or sequences) and the A-choice strings (or sequences). To this aim we need some simple facts, rather known, which, for the sake of completeness, will be collected in the following

L e m m a 3.1. Let G, G' and G_o be some Hilbert spaces, G_o being a subspace of G, and let $C_o : G_o \to G$ be a contraction. Then the formula

$$(3.1)\quad \begin{cases} D_{C_o^*}\Gamma(C_o,C) = C|G \ominus G_o \\ C(C_o,\Gamma) = C_oQ + D_{C_o^*}\Gamma(I-Q) \end{cases}$$

(where Q denotes the orthogonal projection of G onto G_o), establishes an one-to-one correspondence between all the contractions $C: G \to G'$ such that

$$(3.2)\quad C|G_o = C_o ,$$

and all the contractions $\Gamma: G \ominus G_o \to \mathcal{D}_{C_o^*}$.

Moreover the formula

$$(3.3)\quad \begin{cases} ZD_\Gamma = RD_C|G \ominus G_o \\ Z_*D_{\Gamma^*}D_{C_o^*} = D_{C^*} \\ Z'D_{C_o} = D_C|G_o \end{cases}$$

(where R denotes the orthogonal projection of $\mathcal{D}_C$ onto $\mathcal{D}_C \ominus (D_CG_o)^-$ define unitary operators $Z = Z(C_o,C)$ from $\mathcal{D}_\Gamma$ to $\mathcal{D}_C \ominus (D_CG_o)^-$, $Z_* = Z_*(C_o,C)$ from $\mathcal{D}_{\Gamma^*}$ to $\mathcal{D}_{C^*}$ and $Z' = Z'(C_o,C)$ from $\mathcal{D}_{C_o}$ to $(D_CG_o)^-$; also

$$(3.4)\quad Z\mathcal{D}_\Gamma = \mathcal{D}_C \ominus Z'\mathcal{D}_{C_o}.$$

P r o o f. Let $C : G \to G'$ be a contraction enjoying the property (3.2). Then,

$$(3.5)\quad \|QC^*g'\|^2+\|(I-Q)C^*g'\|^2=\|C^*g'\|^2\le\|g'\|^2 \quad (g'\in G')$$

and

$$(QC^*g',g_o)=(C^*g',g_o)=(g',Cg_o)=(g',C_og_o)=(C_o^*g',g_o)$$
$$(g'\in G',\ g_o\in G_o)$$

whence

$$(3.6)\quad QC^* = C_o^*$$

and therefore, by (3.5),

$$(3.7)\quad \|(I-Q)C^*g'\|\le\|D_{C_o^*}g'\| \qquad (g\in G').$$

It follows that there exists a unique contraction

$$\Gamma^*:\mathcal{D}_{C_o^*} \to G\ominus G_o \text{ such that}$$

$$(3.8)\quad \Gamma^*D_{C_o^*} = (I-Q)C^*.$$

Consequently setting $\Gamma(C_o,C) = \Gamma \in L(G\ominus G_o,\ \mathcal{D}_{C_o^*})$ we obtain the first relation (3.1). Conversely, if we are given a contraction $\Gamma:G\ominus G_o\mapsto\mathcal{D}_{C^*}$, and if we define $C = C(C_o,\ \Gamma)$ by the second relation (3.1), then plainly (3.2) and (3.8) are satisfied ; consequently we obtain (3.6) with the same argument as above. It follows

$$(3.9)\quad \|C^*g'\|^2=\|QC^*g'\|^2+\|(I-Q)C^*g'\|^2=\|C_o^*g'\|^2+$$
$$+\ \|\Gamma^*D_{C_o^*}g'\|^2\le\|C_o^*g'\|^2+\|D_{C_o^*}g'\|^2=\|g'\|^2 \quad (g'\in G')$$

hence C is a contraction; finally, (3.8) shows that $\Gamma(C_o,C)=\Gamma$.

This completes the proof of the first statement in the lemma. The statements on Z_* and Z' follow readily from (3.9) and (3.2), respectively. Concerning the statement on Z, we note that

$$\|D_C g\|^2=\|g\|^2-\|Cg\|^2=\|(I-Q)g\|^2+\|Qg\|^2-\|C_o Qg+$$

$$+D_{C_o}{}^*\Gamma(I-Q)g\|^2=\|(I-Q)g\|^2+\|Qg\|^2-\|C_o Qg\|^2-2\mathrm{Re}(C_o Qg,$$

$$D_{C_o}{}^*\Gamma(I-Q)g)-\|D_{C_o}{}^*\Gamma(I-Q)g\|^2=\|(I-Q)g\|^2+\|D_{C_o}Qg\|^2-$$

$$-2\mathrm{Re}(D_{C_o}{}^*C_o Qg,\Gamma(I-Q)g)-\|D_{C_o}{}^*\Gamma(I-Q)g\|^2=\|(I-Q)g\|^2+$$

$$+\|D_{C_o}Qg\|^2-2\mathrm{Re}(C_o D_{C_o}Qg,\Gamma(I-Q)g)-\|D_{C_o}{}^*\Gamma(I-Q)g\|^2=$$

$$=\|D_\Gamma(I-Q)g\|^2+\|C_o^*\Gamma(I-Q)g\|^2+\|D_{C_o}Qg\|^2-2\mathrm{Re}(D_{C_o}Qg,$$

$$C_o^*\Gamma(I-Q)g)=\|D_\Gamma(I-Q)g\|^2+\|D_{C_o}Qg-C_o^*\Gamma(I-Q)g\|^2 \quad ,\ (g\in G)$$

whence

$$(3.10)\qquad \|D_C g+D_C g_o\|^2=\|D_\Gamma g\|^2+\|D_C g_o-C^*\Gamma g\|^2 \quad (g\in G\ominus G_o,\ g_o\in G_o).$$

But since,

$$C^*\Gamma(G\ominus G_o)\subset C^*\mathcal{D}_{C^*}\subset\mathcal{D}_C\ ,$$

from the relation (3.10) it follows

$$\|RD_C g\|^2=\inf_{g_o\in G_o}\|D_C g+D_C g_o\|^2=\|D_\Gamma g\|^2 \quad (g\in G\ominus G_o).$$

This shows that the definition of Z is meaningful and that Z is unitary. Since (3.4) is now obvious, the proof is completed.

We now return to the aim of this section, stated before Lemma 3.1., by considering an A-cascade string $\{C_n\}_{n=1}^N$ (where T, T' and A are as in Sec.2). We set

$$(3.11)\begin{cases} G_{01} = F_A, G_1 = \mathcal{D}_A + L,\ G_1' = L' \\ G_{0n} = Y_{n-1}\mathcal{D}_{n-2} + U^{n-1}L, G_n = \mathcal{D}_{n-1} + U^{n-1}L,\ G_n' = U'^{n-1}L' \quad (n>1) \end{cases}$$

and we define the contractions

$$C_{0n} : G_{0n} \to G_n' \qquad (n=1,2,\dots,N)$$

by

$$(3.12)\begin{cases} C_{01} = C_A \\ C_{0n}|Y_{n-1}\mathcal{D}_{n-2} = U'C_{n-1}|\mathcal{D}_{n-2}, C_{0n}|U^{n-1}L = U'C_{n-1}|U^{n-1}L \quad (n>1). \end{cases}$$

By virtue of $(2.3)_1$ and $(2.6)_n$ (for $n>1$) we have

$$(3.13)\ C_n|G_{0n} = C_{0n} \qquad (n=1,2,\dots,N).$$

Therefore, Lemma 3.1 yields the operators

$$(3.14)_n\ \Gamma_n' = \Gamma(C_{0n}, C_n), Z_n = Z(C_{0n}, C_n), Z_{*n} = Z_*(C_{0n}, C_n) \text{ and } Z_n' = Z'(C_{0n}, C_n)$$

for $n=1,2,\dots,N$. (In the sequel, when a more precise notation will seem necessary, we shall write $\Gamma_n' = \Gamma_n'(C_1,\dots,C_n)$, $Z_n = Z_n(C_1,\dots,C_n),\dots$ instead of Γ_n', $Z_n,\dots$).

L e m m a 3.2. <u>For</u> $2 \le n \le N$, <u>the range of</u> Z_{n-1}' <u>is</u> $Y_{n-1}\mathcal{D}_{n-2}$ <u>and</u> Γ_n' <u>is a contraction from</u> $Z_{n-1}\mathcal{D}_{\Gamma_{n-1}'}$ <u>to</u> $U'\mathcal{D}_{C_{n-1}^*}$.

P r o o f. For proving

$$(3.15)_n \qquad Z'_{n-1}\mathcal{D}_{C_{0,n-1}} = Y_{n-1}\mathcal{D}_{n-2}$$

for $n=2,\ldots,N$, we note firstly that $(3.15)_2$ follows from (3.3) and $(2.5)_1$, by the relations

$$Z'_1 D_{C_A}(D_A Th+(U-T)h)=D_{C_1}(D_A Th+(U-T)h)=Y_1 D_A h \quad (h\in H).$$

For $n>2$, we have, by virtue of (3.3), (3.12) and $(2.5)_{n-1}$

$$Z'_{n-1}D_{C_{0,n-1}}(Y_{n-2}d+U^{n-2}l)=D_{C_{n-1}}(Y_{n-2}d+U^{n-2}l)=$$

$$=Y_{n-1}D_{C_{n-2}}(d+U^{n-3}l) \qquad (d\in\mathcal{D}_{n-3},\ l\in L),$$

from which $(3.15)_n$ follows at once. Concerning the second statement in the lemma, we notice first that Lemma 3.1 yields

$$\Gamma'_n\in L(G_n\ominus G_{0n},\mathcal{D}_{C^*_{0n}}).$$

But, by virtue of (3.11), $(3.15)_n$ and (3.4) we have firstly

$$(3.16)_n \quad G_n\ominus G_{0n}=\mathcal{D}_{n-1}\ominus Y_{n-1}\mathcal{D}_{n-2}=\mathcal{D}_{C_{n-1}}\ominus Z'_{n-1}\mathcal{D}_{C_{0,n-1}}=Z_{n-1}\mathcal{D}_{\Gamma'_{n-1}},$$

while, by virtue of (3.12), we have secondly

$$C_{0n}C^*_{0n}=U'C_{n-1}C^*_{n-1}U'^*|U'^{n-1}L'$$

whence

$$D^2_{C^*_{0n}}=U'D^2_{C^*_{n-1}}U'^*|U'^{n-1}L',\ D_{C^*_{0n}}=U'D_{C^*_{n-1}}U'^*|U'^{n-1}L'$$

and hence

$$(3.17)_n \quad \mathcal{D}_{C^*_{0n}}=U'\mathcal{D}_{C^*_{n-1}}.$$

We can thus conclude that

$$\Gamma_n' \in L(Z_{n-1}\mathcal{D}_{\Gamma_{n-1}'}\,;\, U'\mathcal{D}_{C_{n-1}^*}),$$

completing the proof of the lemma.

We shall associate to our A-cascade string $\{C_n\}_{n=1}^N$ an A-choice string $\{\Gamma_n\}_{n=1}^N$ in the following manner. We set

$$(3.18)_1 \qquad \Gamma_1 = W_A\Gamma_1',$$

and

$$(3.19)_1 \qquad W_1 = I_{\mathcal{D}_{\Gamma_1}}, W_{*1} = W_A^*|\mathcal{D}_{\Gamma_1^*} .$$

Since, $W_A \in L(\mathcal{D}_{C_A^*}, R_A')$ is unitary (see Lemma 1.1), we have obviously

$$D_{\Gamma_1} = D_{\Gamma_1'},\ W_A D_{\Gamma_1'^*} D_{\Gamma_1^*} W_A$$

hence the operators

$$(3.20)_1 \qquad W_1 : \mathcal{D}_{\Gamma_1} \mapsto \mathcal{D}_{\Gamma_1'}, W_{*1} : \mathcal{D}_{\Gamma_1^*} \mapsto \mathcal{D}_{\Gamma_1'^*},$$

defined by the formula $(3.19)_1$, are unitary ; moreover we have also

$$(3.21)_1 \qquad \Gamma_1 \in L(R_A, R_A'),$$

thus $\{\Gamma_1\}$ is an A-choice string (of length 1); this will be associated to our A-cascade string if N=1. If N>1, we appeal to the following

L e m m a 3.3. <u>Let</u> $N>1$. <u>Then the formulas</u> $(3.18)_1$, $(3.19)_1$, <u>and</u>, <u>for</u> $2\le n\le N$,

$$(3.18)_n \qquad \Gamma_n = W^*_{*,n-1} Z^*_{*,n-1} U'^* \Gamma'_n Z_{n-1} W_{n-1}$$

$$(3.19)_n \qquad W_n = Z_{n-1} W_{n-1} | \mathcal{D}_{\Gamma_n}, \quad W_{*n} = U' Z_{*,n-1} W_{*,n-1} | \mathcal{D}_{\Gamma_n^*}$$

<u>define an A-choice string</u> $\{\Gamma_n\}_{n=1}^N$ <u>and unitary operators</u>

$$(3.20)_n \qquad W_n : \mathcal{D}_{\Gamma_n} \mapsto \mathcal{D}_{\Gamma'_n}, \quad W_{*n} : \mathcal{D}_{\Gamma_n^*} \mapsto \mathcal{D}_{\Gamma'^*_n}$$

$(1\le n\le N)$.

P r o o f. Proceeding by recurrence, we notice that the statements concerning Γ_1, W_1 and W_{*1} were already established above. Assuming that those concerning W_{m-1} and $W_{*,m-1}$ (where $m-1\ge 1$, $m\le N$) are also established we infer by virtue of Lemma 3.2 and $(3.20)_{m-1}$ that the relation

$$(3.21)_n \qquad \Gamma_n \in L(\mathcal{D}_{\Gamma_{n-1}}, \mathcal{D}_{\Gamma^*_{n-1}})$$

is valid for $n=m$. From this we obtain

$$Z_{m-1} W_{m-1} \Gamma_m^* \Gamma_m = \Gamma_m'^* \Gamma'_m Z_{m-1} W_{m-1},$$

$$U' Z_{*,m-1} W_{*,m-1} \Gamma_m \Gamma_m^* = \Gamma'_m \Gamma'^*_m U' Z_{*,m-1} W_{*,m-1},$$

whence

$$Z_{m-1} W_{m-1} D_{\Gamma_m} = D_{\Gamma'_m} Z_{m-1} W_{m-1},$$

$$U' Z_{*,m-1} W_{*,m-1} D_{\Gamma_m^*} = D_{\Gamma'^*_m} U' Z_{*,m-1} W_{*,m-1}.$$

From these relations it follows readily that indeed the formula $(3.19)_m$ define the unitary operators $(3.20)_m$. Thus all the operators W_n and W_{*n} $(1\leq n\leq N)$ are unitary and $(3.21)_n$ is true for all n, $1\leq n\leq N$, which means that $\{\Gamma_n\}_{n=1}^N$ is an A-choice sequence. This finishes the proof.

It is plain that the operators Γ_n, W_n and W_{*n} $(1\leq n\leq N)$ yielded by the preceding argument depend only on $C_1, C_2, \ldots, C_n$ (and of course on A, T, T′ and U, U′). Therefore we shall denote them by $\Gamma_n(C_1,\ldots,C_n)$, $W_n(C_1,\ldots,C_n)$ and $W_{*n}(C_1,\ldots,C_n)$. (When a confusion seems possible we shall explicitate also the dependence on A, T, T′, U, U′, for instance $\Gamma_n(A; T, T'; U, U'; C_1,\ldots,C_n)$ for Γ_n etc.

P r o p o s i t i o n 3.1. For $\upsilon=2,3,\ldots,\infty$ and T, T′, $A\in I(T',T)$, fixed, the mapping

$$(3.22)_\upsilon \qquad \{C_n\}_{1\leq n<\upsilon} \mapsto \{\Gamma_n(C_1,\ldots,C_n)\}_{1\leq n<\upsilon}$$

establishes an one-to-one correspondence between all the A-cascade strings (if $\upsilon<\infty$), respectively sequences (if $\upsilon=\infty$) and all the A-choice strings (of length $\upsilon-1$, if $\upsilon<\infty$), respectively sequences (if $\upsilon=\infty$).

P r o o f. The case $\upsilon=\infty$ follows immediately from the case $\upsilon<\infty$. Since the case $\upsilon=2$, is a direct consequence of Lemma 3.1, we shall assume now that the proposition is valid if $\upsilon=m\geq 2$.

Let $\{\Gamma_n\}_{1\leq n<m+1}$ be any A-choice string. Then by our assumption there exists a unique A-cascade string $\{C_n\}_{1\leq n<m}$ such that

$$(3.23)_n \qquad \Gamma_n=\Gamma_n(C_1, C_2,\ldots,C_m)$$

$(1\leq n<m)$. Therefore, by virtue of Lemmas 3.1 and 3.2, the operators

$$\Gamma'=\Gamma'_{m-1}(C_1,\ldots,C_{m-1}) : \mathcal{G}_{m-1}\ominus\mathcal{G}_{0,m-1} \mapsto \mathcal{D}_{C^*_{0,m-2}}$$

and

$$Z=Z_{m-1}= Z(C_1,\dots,C_{m-1}): \mathcal{D}_{\Gamma'_{m-1}} \mapsto G_m \oplus G_{0m} \qquad \text{(see (3.16)}_m\text{)}$$

$$Z_*=Z_{*,m-1}= Z_*(C_1,\dots,C_{m-1}): \mathcal{D}_{\Gamma'^*_{m-1}} \mapsto \mathcal{D}_{C^*_{m-1}}$$

$$W=W_{m-1}= W(C_1,\dots,C_{m-1}): \mathcal{D}_{\Gamma_{m-1}} \mapsto \mathcal{D}_{\Gamma'_{m-1}}$$

$$W_*=W_{*,m-1}= W_*(C_1,\dots,C_{m-1}): \mathcal{D}_{\Gamma^*_{m-1}} \mapsto \mathcal{D}_{\Gamma'^*_{m-1}}$$

are also uniquelly determined, and Γ' is a contraction while Z, Z_*, W, W_* are unitary. Setting

$$(3.24) \quad \Gamma' = U'Z_*W_*\Gamma_m W^*Z^*$$

we obtain a contraction from $G_m \oplus G_{0m}$ to $\mathcal{D}_{C^*_{0m}}$ (see (3.11), (3.12) and $(3.17)_m$). By virtue of Lemma 3.1, there exists a uniquely determined contraction $C_m: G_m \mapsto G'_m$ such that $C_m = C(C_{0m}, \Gamma')$. Comparing (3.11), (3.12) and (3.13) (in the case $n=m$) with $(2.6)_m$ we see that $\{C_n\}_{1\le n<m+1}$ is an A-cascade string.

Comparing (3.24) with $(3.18)_m$ we finally see that $(3.21)_m$ is also valid. Thus we verified that the mapping $(3.22)_{m+1}$ is surjective. Since the last term in $\{C_n\}_{1\le n<m+1}$ is necessarily of the form $C_m=C(C_{0m}, \Gamma')$, where Γ' is given by (3.24), the mapping is also injective.

Now the proposition is concluded by induction.

4. In this section we shall associate the CID's of an $A \in I(T',T)$ with T, T' and A as in the preceding sections, to a more usual concept, namely to contractive analytic functions ([16], Ch.V). As preparation, we shall now discuss the preceding sections in a very particular case, namely that of an arbitrary contraction Γ from $\mathcal{R}$ to $\mathcal{R}'$ (where $\mathcal{R}$ and $\mathcal{R}'$ are two Hilbert spaces), considered as

intertwining the corresponding null operators O_R, $O_{R'}$, i.e. $\Gamma \in I(O_{R'}, O_R)$.

On this purpose, for the operator O_R we shall choose as minimal isometric dilation $V_{R'}$ the canonical multiplication shift

$$V_R f(z) = zf(z) \qquad (|z| < 1)$$

on $H^2(R)$, where R is identified to the space of constant functions in $H^2(R)$ (4); the minimal isometric dilation $V_{R'}$ of $O_{R'}$ will be chosen in the obvious similar way. Since any CID of Γ is a contraction intertwining V_R and $V_{R'}$, it is the multiplication operator by a contractive analytic function $\{R, R', \Gamma(z)\}$ (see [16], Ch.V, Sec.3), which obviously must satisfy the condition $\Gamma(0)=\Gamma$. Since the converse fact is also obvious, we can state the following consequence of our previous reults.

L e m m a 4.1. Let $\Gamma: R \mapsto R'$ be an arbitrary fixed contraction. Then Propositions 2.2 and 3.1 with $T = O_R$, $T' = O_{R'}$, and $\Lambda=\Gamma\in I(T';T)$ yield an one-to-one correspondence between all the contractive analytic functions $\{R, R', \Gamma(z)\}$ such that $\Gamma(0)=\Gamma$ and all the Γ-choice sequences.

R e m a r k 4.1. We recall that within the frame of the preceding discussion, (1.5) and (1.5)' take the form

$$(4.1) \quad \begin{cases} F_\Gamma = V_R R \\ R_\Gamma = (\mathcal{D}_\Gamma + V_R R) \ominus F_\Gamma = \mathcal{D}_\Gamma \end{cases}$$

and

$$(4.1)' \begin{cases} F_\Gamma' \doteq \{D_\Gamma r \oplus V_R, \Gamma r : r \in R\} \\ R_\Gamma' = (\mathcal{D}_\Gamma \oplus V_R, R') \ominus F_\Gamma' = \{r \oplus V_R' r' : D_\Gamma r + \\ \quad + \Gamma^* r' = 0, r \in R, r' \in R'\}. \end{cases}$$

L e m m a 4.2. <u>The formula</u>

$$(4.2) \quad \omega(\Gamma) r' = (-\Gamma^* r') \oplus V_R, D_{\Gamma^*} r' \qquad (r' \in \mathcal{D}_{\Gamma^*})$$

<u>defines a unitary operator from</u> $\mathcal{D}_{\Gamma^*}$ to R_Γ'

P r o o f. It is obvious, by virtue of (4.1)' and of the relation $D_\Gamma \Gamma^* = \Gamma^* D_{\Gamma^*}$, that (4.2) defines an isometric operator $\omega(\Gamma)$ from $\mathcal{D}_{\Gamma^*}$ to R_Γ'.

Moreover, if we are given $r \oplus Vr' \in R_\Gamma'$, then setting

$$r_1' = D_{\Gamma^*} r' - \Gamma r$$

we obtain $r_1' \in \mathcal{D}_{\Gamma^*}$ and

$$\omega(\Gamma) r_1' = (-\Gamma^* D_{\Gamma^*} r' + \Gamma^* \Gamma r) \oplus V_R, (D_{\Gamma^*}^2 r' - D_{\Gamma^*} \Gamma r) =$$
$$= (r - D_\Gamma(\Gamma^* r' + D_\Gamma r)) \oplus V_R, (r' - \Gamma(\Gamma^* r' + D_\Gamma r)) = r \oplus V_R' r'.$$

This finishes the proof of the lemma.

Let now $T', T, A \in I(T', T)$ be some arbitrary contractions (of course, together with some fixed minimal isometric dilations U, U' of T, T'). For an A-choice sequence $\{\Gamma_n\}_{n=1}^{\infty}$ we set

$$(4.3)_n \quad \gamma_n(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) \; (=\gamma_n) = \omega(\Gamma_1) \Gamma_{n+1}$$

$(1 \le n < \infty)$.

Since (see Definition 3.1)

$$\mathcal{D}_{\Gamma_1^*} \supset \mathcal{D}_{\Gamma_2^*} \supset \ldots \supset \mathcal{D}_{\Gamma_n^*} \supset \ldots ,$$

the definition $(4.3)_n$ $(1 \leq n < \infty)$ makes sense.

L e m m a 4.3. <u>The mapping</u>

$$(4.4) \qquad \{\Gamma_n\}_{n=1}^{\infty} \mapsto \{\Gamma_1, \{\gamma_n(\Gamma_1, \ldots, \Gamma_n)\}_{n=1}^{\infty}\}$$

<u>establishes an one-to-one correspondence between all A-choice sequences and all pairs formed by a contraction</u> $\Gamma : R_A \to R'_A$ <u>(considered as belonging to</u> $I(O_R', O_R)$ <u>) and a</u> Γ <u>- choice sequence.</u>

P r o o f. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be an A-choice sequence and let $\{\gamma_n\}_{n=1}^{\infty}$ be the sequence yielded by $(4.3)_n$ $(1 \leq n < \infty)$. It is obvious that, by virtue of Lemma 4.2, we have

$$(4.5)_n \qquad D_{\gamma_n} = D_{\Gamma_{n+1}}$$

$(1 \leq n < \infty)$ and, using also (4.1),

$$(4.6)_1 \qquad \gamma_1 : R_{\Gamma_1} = \mathcal{D}_{\Gamma_1} \mapsto \omega(\Gamma_1)\ \mathcal{D}_{\Gamma_1^*} = R'_{\Gamma_1},$$

where, as already indicated above, Γ_1 is regarded as belonging to to $I(O_R', O_R)$; moreover, we have also

$$(4.7)_n \quad D^2_{\gamma_n^*} = \omega(\Gamma_1)\ D^2_{\Gamma_{n+1}^*} \quad \omega(\Gamma_1)^*, D_{\gamma_n^*}\,\omega(\Gamma_1) = \omega(\Gamma_1)\ D_{\Gamma_{n+1}^*}$$

$(1 \leq n < \infty)$.

From $(4.5)_n$ and $(4.7)_n$, we infer readily that

$$(4.8)_n \qquad \mathcal{D}_{\gamma_n} = \mathcal{D}_{\Gamma_{n+1}}, \ \mathcal{D}_{\gamma_n^*} = \omega(\Gamma_1)\mathcal{D}_{\Gamma_{n+1}^*}$$

$(1 \le n < \infty)$. Consequently γ_{n+1} is a contraction from $\mathcal{D}_{\gamma_n}$ to $\mathcal{D}_{\gamma_n^*}$ $(1 \le n < \infty)$. Together with $(4.6)_1$, this shows that $\{\gamma_n\}_{n=1}^{\infty}$ is a Γ_1-choice sequence. If we are given now a pair $\{\Gamma_1, \{\gamma_n\}_{n=1}^{\infty}\}$ formed by a contraction $\Gamma_1 : R_A \to R_A'$ (regarded as belonging to $I(\mathcal{O}_R', \mathcal{O}_R)$) and a Γ_1 - choice sequence $\{\gamma_n\}_{n=1}^{\infty}$, then there may exists only one A-choice sequence $\{\Gamma_n\}_{n=1}^{\infty}$ which is mapped by (4.4) onto our given pair, namely that given by the formula

$$(4.9)_n \quad \Gamma_{n+1} = \omega(\Gamma_1)^* \gamma_n .$$

It is now easy to infer that if Γ_{n+1} $(1 \le n < \infty)$ are actually defined by $(4.9)_n$, then $(4.5)_n$, $(4.7)_n$ and consequently $(4.8)_n$ are also satisfied for all n=1,2,...,; obviously, it follows that $\{\Gamma_n\}_{n=1}^{\infty}$ is an A-choice sequence. This concludes the proof.

We are now in state to formulate the main result of this section. To this aim let T , T', $A \in I(T', T)$, (as well as U and U') be as above. Let A_∞ be a CID of A and let

$$\Lambda_2 : A_\infty \mapsto \{C_n\}_{n=1}^{\infty}$$

be the inverse mapping of that given in Proposition 2.2, let

$$\Lambda_3 : \{C_n\}_{n=1}^{\infty} \mapsto \{\Gamma_n\}_{n=1}^{\infty}$$

be the mapping given by Proposition 3.1, let

$$\Lambda_4 : \{\Gamma_n\}_{n=1}^{\infty} \mapsto \{\Gamma_1, \{\gamma_n\}_{n=1}^{\infty}\}$$

that given by Lemma 4.3 and finally let

$$\Lambda_5 : \{\Gamma_1, \{\gamma_n\}_{n=1}^{\infty}\} \to \{R_A, R_A', \Gamma(z)\}$$

be the inverse mapping of that given by Lemma 4.1.
Then, the bijectivity property of these mappings yields directly the following

P r o p o s i t i o n 4.1. <u>The mapping</u>

$$\Lambda_1 = \Lambda_5 \circ \Lambda_4 \circ \Lambda_3 \circ \Lambda_2$$

<u>establishes an one-to-one correspondence between all the CID's of A and all the contractive analytic</u> $L(R_A, R_A')$ - <u>valued functions.</u>

R e m a r k 4.2. The uniqueness theorem for CID's given in [6] is a direct corollary of Proposition 4.1. Indeed, by virtue of this proposition, there exists a unique CID of A, if and only if there exists only one contractive analytic function $\{R_A, R_A', \Gamma(z)\}$. Obviously this happens if and only if at least one of the spaces R_A or R_A' reduce to $\{0\}$, i.e. (see [16], Ch.VII) if at least one of the factorization $A \cdot T$ or $T_3' \cdot A$ is regular.

Let us present a particular case which might be instructive. On this purpose, we shall denote by $i_{L'}$, the natural isometric identification of L' with the subspace $\{0\} \oplus L'$ of $\mathcal{D}_A \oplus L'$ and by P_{L_*} the orthogonal projection of K onto

$$L_* = ((I - UT^*) H)^- ,$$

where the notation is, as usual, that of Sec.1. Also let us firstly give the following

L e m m a 4.4. The operators

$$i^*_{L'} | R'_A : R'_A \mapsto L', \quad P_{L_*} | R_A : R_A \mapsto L_*$$

are injective.

P r o o f. Let $P_{L_*} r = 0$; $r \in R_A$ or equivalently $r = Uh_1$ for some $h_1 \in H$ and

$$(T^* D_A P + U^*(I-P))r = 0.$$

It follows

(4.10) $T^* D_A Th_1 + (I-T^*T)h_1 = 0$, $h_1 = T^*(I-D_A)Th_1$.

But $0 \le I-D_A \le I$ implies $||Th_1||^2 \le ||h_1||^2 = (T^*(I-D_A)Th_1, h_1) = ||(I-D_A)^{\frac{1}{2}} Th_1||^2 \le ||Th_1||^2$,
whence

(4.11) $(I-D_A)^{\frac{1}{2}} Th_1 = Th_1$ and $D_A Th_1 = 0$.

From (4.10) it follows

$$||(U-T)h_1||^2 = ((1-T^*T)h_1, h_1) = 0,$$

whence

(4.12) $r = Uh_1 = Th_1 \in \overset{\circ}{H}$.

Since $r \in \mathcal{D}_A$, from (4.11) and (4.12), we infer that $r=0$. This proves the injectivity of $P_{L_*} | R_A$. Concerning the injectivity of

$i^*_{L'}|R'_A$, we notice firstly that if $r' \in R'_A$, $i^*_{L'}r' = 0$, then $r' = d \oplus 0$ with some $d \in \mathcal{D}_A$ and secondly that

$$(r', D_A h \oplus (U'-T')Ah) = 0 \qquad (h \in H)$$

implies $D_A d = 0$, $d=0$, thus $i^*_{L'}|R'_A$ is also injective. Thus the lemma is proved.

By virtue of the preceding lemma and of [16], Ch.II, Sec.I we have

$$\dim R_A \le \delta_{T^*} \overset{\text{def.}}{=} \operatorname{rank} D_{T^*} = \dim L_*$$
$$\dim R'_A \le \delta_{T'} \overset{\text{def.}}{=} \operatorname{rank} D_{T'} = \dim L'.$$

Therefore, from Lemma 4.4 and Proposition 4.1, we can now readily obtain the following

C o r o l l a r y 4.1. Assume that, within the frame of Proposition 4.1, we have $\delta_{T^*} = \delta_{T'} = 1$. Then either the set of all CID's of A is a singleton or it is in an one-to-one correspondence (explicitely given by Λ_1) with the unit ball of H^∞ (i.e. the set of all complex-valued analytic functions $u(z)$ on the unit disk $D=\{z:|z|<1\}$ such that $|u(z)|\le 1$ for all $z \in D$).

It is plain that in this corollary, the first case occurs if

$$\min(\dim R_A, \dim R'_A)$$

(see Remark 4.2), while the second one if

$$\dim R_A = \dim R'_A = 1.$$

5.1. We shall apply now Proposition 4.1 to the labelling of all classes of isomorphic Ando dilations. To be more precise, for a pair $\{T_1, T_2\}$ of some fixed commuting contractions on some Hilbert space H, there always exists (as shown in a celebrate short note by Ando [5]) a pair $\{U_1, U_2\}$ of commuting isometric operators on some Hilbert space K containing H as a (closed linear) subspace and such that

$$PU_1^{n_1} U^{n_2} | H = T_1^{n_1} T_2^{n_2} \qquad (n_1, n_2 = 0,1,2,\ldots,),$$

where P denote the orthogonal projection of K onto H. Obviously we can and shall also suppose that

$$(5.2) \quad K = \bigvee_{n_1, n_2 \geq 0} U_1^{n_1} U_2^{n_2} H.$$

Any such pair $\{U_1, U_2\}$ will be called an <u>Ando dilation</u> of $\{T_1, T_2\}$. Two Ando dilations $\{U_1, U_2\}$, $\{U_1', U_2'\}$ are called isomorphic if there exists a unitary operator W from the space K, on which operate U_1 and U_2, to the space K' on which operate U_1' and U_2' such that

$$(5.3) \quad WU_j = U_j' W \ (j=1,2), \ W|H = I_H.$$

Let now U on $\mathbb{K}$ be a fixed minimal isometric dilation of $T = T_1$. Obviously any Ando dilation $\{U_1'', U_2''\}$ is isomorphic with some Ando dilation $\{U_1, U_2\}$ operating on a space K containing $\mathbb{K}$ (as closed linear subspace), and such that

$$(5.4) \quad U_1 | \mathbb{K} = U.$$

Let $\{U_1', U_2'\}$ be another such Ando dilation, isomorphic "by W" to $\{U_1, U_2\}$. Then by virtue of (5.3) we have

$$WU^n h = WU_1^n h = U_1'^n Wh = U_1'^n h = U^n h$$

for all $h \in H$, $n = 0, 1, 2, \ldots$, therefore

(5.5) $\quad W|\mathcal{K} = I_{\mathcal{K}}$.

By virtue of this discussion, we can and we shall consider from now on only Ando dilations satisfying (5.4). With this convened, we state the following

L e m m a 5.1. <u>For</u> $T = T_1$ <u>and</u> $A = T_2$, <u>the formula</u>

(5.6) $\quad \hat{A} = P_{\mathcal{K}} U_2 | \mathcal{K}$

(<u>where</u> $P_{\mathcal{K}}$ <u>denotes the orthogonal projection of</u> K <u>onto</u> $\mathcal{K}$) <u>establishes an one-to-one correspondence between all classes of isomorphic Ando dilations of</u> $\{T_1, T_2\}$ <u>and all the CID's of A</u>.

P r o o f. First we remark that

(5.7) $\quad U_1^* \mathcal{K} \subset \mathcal{K}$,

i.e. that $\mathcal{K}$ is reducing U_1. This was, for instance, proven in [13]. For the sake of completeness let us sketch the proof. On this purpose we infer easily from (5.1) and (5.2) that

(5.8) $\quad PU_1 = T_1 P = TP$, $PU_2 = T_2 P = AP$;

from the first relation (5.8) it follows that

$$(5.9)\quad U_1^*|H = T_1^* = T^*,$$

whence, for $h \in H$,

$$U_1^* U^n h = \begin{cases} T^* h & \text{if } n = 0 \\ U^{n-1} h & \text{if } n = 1,2,\ldots, \end{cases}$$

so that, since these $U^n h$'s span K, (5.7) is true. We conclude thus that

$$(5.10)\quad P_K U_1 = U_1 P_K .$$

Now the fact that for a given $\{U_1, U_2\}$ the formula (5.6) defines a CID $\hat{A}$ of A can be easily obtained from (5.10) and the second relation (5.8). Moreover if $\{U_1', U_2'\}$ is another Ando dilation of $\{T_1, T_2\}$, isomorphic (by W) to $\{U_1, U_2\}$, then $WP_K = P_K = P_K W$, so that

$$P_K U_2' | K = P_K W U_2 | K = P_K U_2 | K = \hat{A}.$$

Thus we can conclude that (5.6) defines a mapping from the classes of isomorphic Ando dilations of $\{T_1, T_2\}$ to the set of the CID's of A. Let now $\hat{A}$ be a CID of A. Let U_2 on K be a minimal dilation of $\hat{A}$. Since U is an isometric operator commuting to A it has a unique CID (as operator in K commuting with U_2; namely consider in Remark 4.2 the case when A is isometric and observe that in this particular case we have $R_A = \{0\}$), which we shall denote by U_1. The pair $\{U_1, U_2\}$ is an Ando dilation of $\{T_1, T_2\}$ satisfying the property (5.6). Indeed, (5.6) is satisfied by the very definition of U_2, while

$$P U_1^{n_1} U_1^{n_2} = PP_K U_1^{n_1} U_2^{n_2} = PU^{n_1} P_K U_2^{n_2} =$$

$$= PU^{n_1} \hat{A}^{n_2} P_K = T^{n_1} P\hat{A}^{n_2} P_K = T^{n_1} A^{n_2} PP_K = T^{n_1} A^{n_2} P =$$

$$= T_1^{n_1} T_2^{n_2} P$$

for all $n_1, n_2 = 0, 1, 2, \ldots,$.

Moreover, since U is isometric, it follows directly that $U_1 | K = U$, whence

$$\| U_1 \sum_{n=0}^{N} U_2^n k_n \|^2 = \| \sum_{n=0}^{N} U_2^n Uk_n \|^2 = \sum_{N \geq n \geq m \geq 0} (U_2^{n-m} Uk_n, Uk_m) +$$

$$+ \sum_{0 \leq n < m \leq N} (Uk_n, U_2^{m-n} Uk_m) = \sum_{N \geq n \geq m \geq 0} (\hat{A}^{n-m} Uk_n, Uk_m) + \sum_{0 \leq n \leq m \leq N}$$

$$(Uk_n, \hat{A}^{m-n} Uk_m) = \sum_{N \geq n \geq m \geq 0} (U\hat{A}^{n-m} k_n, Uk_m) + \sum_{0 \leq n \leq m \leq N} (Uk_n,$$

$$U\hat{A}^{m-n} k_m) = \sum_{N \geq n \geq m \geq 0} (\hat{A}^{n-m} k_n, k_m) + \sum_{0 \leq n \leq m \leq N} (k_n, \hat{A}^{m-n} k_m)$$

$$= \| \sum_{n=0}^{N} U_2^n k_n \|^2$$

for all $k_1, k_2, \ldots, k_N$, $N = 0, 1, \ldots$ Therefore U_1 is indeed isometric.

Finally , the fact that the relation (5.2) is also satisfied, follows from

$$\bigvee_{n_1, n_2 \geq 0} U_1^{n_1} U_2^{n_2} H = \bigvee_{n_2 \geq 0} U_2^{n_2} \bigvee_{n_1 \geq 0} U_1^{n_1} H =$$

$$= \bigvee_{n_2 \geq 0} U_2^{n_2} \bigvee_{n_1 \geq 0} U^{n_1} H = \bigvee_{n_2 \geq 0} U_2^{n_2} K = K$$

because U and U_2 are minimal isometric dilations of $T(=T_1)$ and A, respectively.

It remains to prove that the mapping yielded by (5.6), is one-to-one. But this follows at once from the preceding construction, since if

$$P_{\mathcal{K}}U_2'|\mathcal{K} = P_{\mathcal{K}}U_2|\mathcal{K} \; (=\hat{A})$$

for two Ando dilations $\{U_1, U_2\}, \{U_1', U_2'\}$, the isometries U_2 and U_2' are actually minimal isometric dilations of $\hat{A}$, thus isomorphic, say by the unitary operator W. But then

$$WU_1U_2^{n_2} k = WU_2^{n_2}Uk = U_2'^{n_2} WUk = U_2'^{n_2}Uk = U_1'U_2'^{n_2}k = U_1'W U_2^{n_2}k$$

for all the elements $U_2^{n_2}k$ ($k \in \mathcal{K}$, $n_2 = 0,1,2,\ldots$). Since these elements span the space on which operate U_1 and U_2, we infer that (5.3) is valid (of course in the special case satisfying (5.5)), thus $\{U_1, U_2\}$ and $\{U_1', U_2'\}$ are isomorphic.

P r o p o s i t i o n 5.1. Let $\{T_1, T_2\}$ be a pair of commuting contractions on $\mathcal{H}$ and let, for $i = 1$, $j=2$ or $i = 2$, $j = 1$,

(5.10) $\mathcal{R}_{ij}=(\mathcal{D}_{T_i}\oplus\mathcal{D}_{T_j})\ominus\{D_{T_i}T_jh \oplus D_{T_j}h : h\in \mathcal{H}\}$.

There exists an one-to-one (explicite) correspondence between all classes of isomorphic Ando dilations of $\{T_1, T_2\}$ and all the contractive analytic $L(\mathcal{R}_{21}, \mathcal{R}_{12})$ - valued functions.

P r o o f. We set $T = T_1$ and $A = T_2$. By virtue of Lemma 5.1 and Proposition 4.1. we have an explicite one-to-one correspondence from the classes of isomorphic Ando dilations of $\{T_1, T_2\}$ and all contractive analytic functions $\{\mathcal{R}_A, \mathcal{R}_A', B(z)\}$. Or by virtue of [16], Ch.II, Sec.1, there exists a unitary (canonical) identification

$\varphi: D_Th\to(U-T)h$, of $\mathcal{D}_T=\mathcal{D}_{T_1}$ with L. Thus $\varphi_1 = \left[I_{\mathcal{D}_{T_2}}, \varphi\right]$ identifies $\mathcal{D}_{T_2}\oplus\mathcal{D}_{T_1}$ to $\mathcal{D}_{T_2}\oplus L$ and takes $\mathcal{R}_{21}$ onto $\mathcal{R}_A$, while $\varphi_1' = I_{\mathcal{D}_{T_2}}\oplus\varphi$ identifies $\mathcal{D}_{T_2}\oplus\mathcal{D}_{T_1}$ to $\mathcal{D}T_2\oplus L$ and takes $\mathcal{R}_{21}' = \mathcal{D}_{T_2}\oplus\mathcal{D}_{T_1}\ominus\{D_{T_2}h + D_{T_1}T_2h : h\in\mathcal{H}\}$ onto $\mathcal{R}_A'$.

Denoting ϕ the unitary operator from $\mathcal{D}_{T_2} \oplus \mathcal{D}_{T_1}$ to $\mathcal{D}_{T_1} \oplus \mathcal{D}_{T_2}$ which intertwines the coordinates, we obtain by

$$A(z) = \phi\varphi'^{*} B(z)\varphi_1 | R_{21} \qquad (|z|<1),$$

the mapping yielding the one-to-one correspondence between the set of all contractive analytic function$\{R_A, R'_A, B(z)\}$ and that of those of the form$\{R_{21}, R_{12}, A(z)\}$. Plainly this concludes the proof.

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FOOTNOTES

(1) For the terminology and partly for notations, which are essentially those of [16] , [8] , [6] , see the next Section 1.

(2) Hilbert spaces will be considered complex and their subspaces, if not specified, will be assumed to be linear and closed. Operators will always be assumed to be linear and bounded; also when confusion might occur the identity operator I and the null operator 0 on a Hilbert space $\mathcal{G}$ will be denote by $I_{\mathcal{G}}$ and $O_{\mathcal{G}}$, respectively.

(3) Recall that for any operator C from a Hilbert space $\mathcal{G}$ to another one $\mathcal{G}'$, D_C denotes the defect operator $((I-C^*C)^2)^{1/4}$ and $\mathcal{D}_C = (D_C\mathcal{G})^-$; if $||C||\leq 1$, then obviously $D_C = (1-C^*C)^{1/2}$.

(4) For the Hardy spaces $H^2(\mathcal{R})$, where $\mathcal{R}$ is a Hilbert space, see [16] , Ch.V.