# LINEAR PREDICTOR FOR STATIONARY PROCESSES IN COMPLETE CORRELATED ACTIONS

by

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November 1977

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## 1. Introduction

In this paper we shall continue the study of prediction theory of a stationary process, considered as time evolution in a correlated action which was began in [4] . As in the precedent paper, we shall follow the line of Wiener and Masani prediction schema for (finite) multivariate stationary process [7] , [8] .

The notion of completion of a correlated action, which we shall introduce in section 2, will permite us to give a precise meaning to the predictible part of the process and consequently to formulate more precisely the prediction problems (section 3). Since some results from [4 ] are used here in a slightly different context, we prefere to outline their proof. In section 4, under the suplimentary condition of boundedness on the spectral distribution of the process, similar to Wiener-Masani boundedness condition [8] , we shall determine the predictible part of the process by means of a linear (infinite) Wiener filter. The solution of prediction problems are given in terms of Taylor coefficients of maximal outer function which factorizes the spectral distribution of the process (see[3] ) .

The reader will notice that we permanently used the ideas from the Sz.-Nagy and C.Foiaş model for contraction [6] to give an operator or functional model for prediction based on an operator valued positive definite map (on the integers) which corresponds to an infinite variate (discrete) stationary process.

## 2. Complete correlated actions

The notion of correlated action was introduced in [4] as the triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$, where $\mathcal{E}$ is a Hilbert space (the space of the parameters), $\mathcal{H}$ a right $\mathcal{L}(\mathcal{E})$ - module (the state space), and $\Gamma : \mathcal{H} \times \mathcal{H} \to \mathcal{L}(\mathcal{E})$ is an $\mathcal{L}(\mathcal{E})$-valued map (the correlation) with the properties:

(i) $\Gamma[h,h] \geq 0$, $\Gamma[h,h] = 0 \Rightarrow h = 0$.

(ii) $\Gamma[h_1, h_2] = \Gamma[h_2, h_1]^*$.

(iii) $\Gamma[\sum_i A_i h_i, \sum_j B_j g_j] = \sum_{i,j} A_i^* \Gamma[h_i, g_j] B_j$.

Let now $\mathcal{E}$, $\mathcal{K}$ be two Hilbert spaces and $\mathcal{H} = \mathcal{L}(\mathcal{E}, \mathcal{K})$. Putting for $A \in \mathcal{L}(\mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{K})$

$$AV = VA$$

where VA is the usual composition of operators, then $\mathcal{H}$ becames a right $\mathcal{L}(\mathcal{E})$-module. If we consider $\Gamma$ defined by

$$\Gamma[V_1, V_2] = V_1^* V_2 \tag{2.1}$$

then obvious $\Gamma$ satisfies the properties (i) and (ii). For (iii) we have

$$\Gamma[\sum_i A_i V_i, \sum_j B_j W_j] = (\sum_i V_i A_i)^* (\sum_j W_j B_j) =$$

$$= \sum_{i,j} A_i^* V_i^* W_j B_j = \sum_{i,j} A_i^* \Gamma[V_i, W_j] B_j.$$

Hence $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is a correlated action. In fact, as the following Proposition shows, any correlated action can be embeedded into one of this type.

PROPOSITION 1. Let $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ be a correlated action. There exist a Hilbert space $\mathcal{K}$ and an algebraic imbedding $h \to V_h$ of the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{H}$ into the right $\mathcal{L}(\mathcal{E})$-module $\mathcal{L}(\mathcal{E}, \mathcal{K})$ with the properties

$$\Gamma[h_1, h_2] = V_{h_1}^* V_{h_2} \qquad h_1, h_2 \in \mathcal{H}. \tag{2.2}$$

(2.3) The elements of the form $\gamma_{(a,h)} = V_h a$, when $a \in \mathcal{E}$ and $h \in \mathcal{H}$ span a dense subspace in $\mathcal{K}$.

This imbedding is unique up to ~~an~~ unitary equivalence.

Proof. The proof follows the construction of the Aronszajn reproducing kernel Hilbert space [1], [2]. Let $\Lambda = \mathcal{E} \times \mathcal{H}$ and $\gamma_{(a,h)}$ be the complex valued function defined on $\Lambda$ by

$$\gamma_{(a,h)}(b,g) = (\Gamma[g,h]a, b)_{\mathcal{E}}. \tag{2.4}$$

On the linear span of these functions we define the form

$$\left\langle \sum_j \gamma_{(a_j,h_j)}, \sum_k \gamma_{(a_k,g_k)} \right\rangle = \sum_{j,k} (\Gamma[g_k, h_j]a_j, b_k)_{\mathcal{E}}.$$

For $a_1, \ldots, a_n \in \mathcal{E}$, choose $a \in \mathcal{E}$ and $A_j \in \mathcal{L}(\mathcal{E})$ such that $A_j a = a_j$. We have

$$\left\langle \sum_j \gamma_{(a_j,h_j)}, \sum_k \gamma_{(a_k,h_k)} \right\rangle = \sum_{j,k} (\Gamma[h_k, h_j]a_j, a_k)_{\mathcal{E}} =$$

$$= \sum_{j,k} (\Gamma[h_k, h_j]A_j a, A_k a)_{\mathcal{E}} = \sum_{j,k} (A_k^* \Gamma[h_k, h_j]A_j a, a)_{\mathcal{E}} =$$

$$= \left(\Gamma\left[\sum_k A_k h_k, \sum_j A_j h_j\right]a, a\right) \geq 0.$$

Thus $\langle \cdot, \cdot \rangle$ is a sesquilinear semi-positive definite form. The

Hilbert space $\mathcal{K}$ is obtained in the usual way from this form.

For any $h \in \mathcal{H}$ we define

$$(2.5) \qquad V_h a = \gamma_{(a,h)} \qquad a \in \mathcal{E}.$$

Using (2.5) and (2.4) we have

$$\|V_h a\|^2_{\mathcal{K}} = \|\gamma_{(a,h)}\|^2_{\mathcal{K}} = (\Gamma[h,h]a,a)_{\mathcal{E}} \leq \|\Gamma[h,h]\| \cdot \|a\|^2$$

therefore $V_h \in \mathcal{L}(\mathcal{E},\mathcal{K})$.

For any $h_1, h_2 \in \mathcal{H}$ we have

$$(\Gamma[h_1,h_2]a,b)_{\mathcal{E}} = \langle \gamma_{(a,h_2)}, \gamma_{(b,h_1)} \rangle =$$
$$= \langle V_{h_2} a, V_{h_1} b \rangle = (V^*_{h_1} V_{h_2} a, b)_{\mathcal{E}}$$

Hence

$$\Gamma[h_1,h_2] = V^*_{h_1} V_{h_2}$$

and the property (2.2) is verified. The property (2.3) it results from the construction of the Hilbert space $\mathcal{K}$.

If $h \longrightarrow V'_h$ is an other imbedding of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E},\mathcal{K}')$ which verifies (2.2) and (2.3), then setting

$$XV'_h a = V_h a$$

we obtain an unitary operator $X:\mathcal{K}' \longrightarrow \mathcal{K}$ such that

$$XV'_h = V_h .$$

The proof of the Proposition is finished.

The Hilbert space $\mathcal{K}$, uniquely attached to the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ as in Proposition 1, is called the <u>measuring space</u> of the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$.

We say that the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ is a <u>complete</u>

correlated action, if the map $h \longrightarrow V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E},\mathcal{K})$ is onto.

Recall that a $\Gamma$-stationary (discret) process in the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ is a sequence $\{f_n\}_{-\infty}^{+\infty}$ of elements in $\mathcal{H}$ such that $\Gamma[f_n, f_m]$ depends only of the difference m-n and not on m and n separately.

For a $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$ we use the following notation

$$\mathcal{H}_n^f = \{h \in \mathcal{H} \mid h = \sum_{i \leq n} A_i f_i \ , \ A_i \in \mathcal{L}(\mathcal{E})\}$$

$$\mathcal{K}_n^f = \bigvee_{k=-\infty}^{n} V_{f_k}\mathcal{E}$$

$$\mathcal{K}_\infty^f = \bigvee_{-\infty}^{+\infty} V_{f_k}\mathcal{E} \ .$$

Remark that we also have

$$\mathcal{K}_n^f = \bigvee_{h \in \mathcal{H}_n^f} V_h \mathcal{E} \ .$$

We say that the $\Gamma$-stationary processes $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ are stationary cross-correlated if $\Gamma[f_n, g_m]$ depends only of the difference m-n.

PROPOSITION 2. For any $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$ there exists an unitary operator $U_f$ on $\mathcal{K}_\infty^f$ such that

$$V_{f_n} = U_f^n V_{f_0} \ . \qquad (2.6)$$

The $\Gamma$-stationary process $\{g_n\}_{-\infty}^{+\infty}$ is stationary cross-correlated with $\{f_n\}_{-\infty}^{+\infty}$ iff there exists an unitary operator $U_{fg}$ on

$$\mathcal{K}_\infty^{fg} = \mathcal{K}_\infty^f \vee \mathcal{K}_\infty^g$$

<u>such that</u>

$$U_f = U_{fg} \mid \mathcal{K}^f_\infty \quad \underline{\text{and}} \quad U_g = U_{fg} \mid \mathcal{K}^f_\infty .$$

<u>Proof</u>. Setting on the generators of $\mathcal{K}^f_\infty$

$$U_f V_{f_n} a = V_{f_{n+1}} a \quad ,$$

clearly we obtain an unitary operator on $\mathcal{K}^f_\infty$ as in Proposition.

Let $\{f_n\}^{+\infty}_{-\infty}$ and $\{g_n\}^{+\infty}_{-\infty}$ be stationary cross-correlated processes, and $U_f$ , $U_g$ be as above. Then if we put

$$(2.7) \qquad U_{fg}(V_{f_n} a + V_{g_m} b) = V_{f_{n+1}} a + V_{g_{m+1}} b$$

then we have

$$\| U_{fg}(V_{f_n} a + V_{g_m} b)\|^2 = \| V_{f_{n+1}} a + V_{g_{m+1}} b \|^2 = \| \gamma_{(a,f_{n+1})} + \gamma_{(b,g_{m+1})} \|^2 =$$

$$= \langle \gamma_{(a,f_{n+1})} + \gamma_{(b,g_{m+1})} , \gamma_{(a,f_{n+1})} + \gamma_{(b,g_{m+1})} \rangle =$$

$$= (\Gamma[f_{n+1}, f_{n+1}]a, a) + (\Gamma[g_{m+1}, g_{m+1}]b, b) + 2\,\mathrm{Re}\,(\Gamma[f_{n+1}, g_{m+1}]b, a) =$$

$$= (\Gamma[f_n, f_n]a, a) + (\Gamma[g_m, g_m]b, b) + 2\,\mathrm{Re}\,(\Gamma[f_n, g_m]b, a) =$$

$$= - - - = \| V_{f_n} a + V_{g_m} b \|^2 .$$

It results that (2.7) defines an unitary operator $U_{fg}$ on $\mathcal{K}^{fg}_\infty$ which extends both $U_f$ and $U_g$ .

The unitary operator $U_f$ is called the <u>shift operator</u> attached to the $\Gamma$-stationary process $\{f_n\}^{+\infty}_{-\infty}$ and $U_{fg}$ the <u>extended shift</u> of the stationary cross-correlated processes $\{f_n\}^{+\infty}_{-\infty}$ and $\{g_n\}^{+\infty}_{-\infty}$ .

Let us remark that from (2.6) it results:

$$\mathcal{K}^f_\infty = \bigvee_{-\infty}^{+\infty} U_f^n V_f \mathcal{E}$$

where $V_f = V_{f_0}$.

In what follows, we use the following notation:

$$\mathcal{K}_+ = \bigvee_0^\infty U_f^{*n} V_f \mathcal{E} = \mathcal{K}_0 \tag{2.8}$$

and

$$U_+ = U_f^* \mid \mathcal{K}_+ . \tag{2.9}$$

The Γ-stationary process $\{g_n\}_{-\infty}^{+\infty}$ is called <u>white noise</u> process, provided $\Gamma[g_n, g_m] = 0$ for $n \neq m$.

We say that the process $\{f_n\}_{-\infty}^{+\infty}$ <u>containes</u> the white noise process $\{g_n\}_{-\infty}^{+\infty}$ if:

(i) $\{g_n\}_{-\infty}^{+\infty}$ is stationary cross-correlated with $\{f_n\}_{-\infty}^{+\infty}$ and

$$\Gamma[f_n, g_m] = 0 , \quad m > n .$$

(ii) $V_g \mathcal{E} \subset \mathcal{K}^f_+$

(iii) $\operatorname{Re} \Gamma[f_n - g_n, g_n] \geq 0$.

The $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called <u>deterministic</u> if it contains no non-zero white noise process.

We say that the $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$ is called a <u>moving average</u> of a white noise $\{g_m\}_{-\infty}^{+\infty}$ if $\{f_n\}_{-\infty}^{+\infty}$ contains $\{g_n\}_{-\infty}^{+\infty}$ and $\mathcal{K}^g_\infty = \mathcal{K}^f_\infty$.

THEOREM 1. (Wold decomposition in time domain). <u>The $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$ admits an unique decomposition of the form</u>

$$f_n = u_n + v_n \tag{2.10}$$

<u>where</u> $\{u_n\}_{-\infty}^{+\infty}$ <u>is a moving average of a white noise</u> $\{g_n\}_{-\infty}^{+\infty}$

contained in $\{f_n\}_{-\infty}^{+\infty}$, $\{v_n\}_{-\infty}^{+\infty}$ is a deterministic process, and $\Gamma[u_n, v_m] = 0$ for any n, m. The white noise $\{g_n\}_{-\infty}^{+\infty}$ is the maximal white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$

Proof. Using the embedding $h \to V_h$ of $\mathcal{H}$ into $\mathcal{L}(\mathcal{E}, \mathcal{K})$ and (2.6) we can consider

$$f_n = U_f^n V_f .$$

By the Wold decomposition of the isometric operator $U_+$ on $\mathcal{K}_+$ we have

$$\mathcal{K}_+ = M_+(\mathcal{F}) \oplus \mathcal{R} \tag{2.11}$$

where

$$\mathcal{F} = \mathcal{K}_+ \ominus U_+\mathcal{K}_+ , \quad M_+(\mathcal{F}) = \bigoplus_0^\infty U_+^n \mathcal{F} \quad \text{and} \quad \mathcal{R} = \bigcap_{n\geq 0} U_+^n \mathcal{K}_+ .$$

Let P be the orthogonal projection of $\mathcal{K}_+$ onto $M_+(\mathcal{F})$ and $P_\mathcal{F}$ be the orthogonal projection of $\mathcal{K}_+$ on the wandering subspace $\mathcal{F}$. If we put $u_n = U_f^n P V_f$, $v_n = U_f^n (I-P) V_f$ and $g_n = U_f^n P_\mathcal{F} V_f$, then (2.10) is obvious and we have

$$\Gamma[u_n, v_n] = V_f^* P U_f^{m-n} (I-P) V_f = V_f^* U_f^{m-n} P(I-P) V_f = 0 .$$

Because

$$\Gamma[g_n, g_m] = V_f^* P_\mathcal{F} U_f^{m-n} P_\mathcal{F} V_f = 0 \quad , \qquad n \neq m$$

it results that $\{g_n\}_{-\infty}^{+\infty}$ is a white noise process. The $\Gamma$-stationary white noise process $\{g_n\}_{-\infty}^{+\infty}$ is contained in $\{u_n\}_{-\infty}^{+\infty}$. Indeed we have :

(i) $\{g_n\}_{-\infty}^{+\infty}$ is stationary cross-correlated with $\{u_n\}_{-\infty}^{+\infty}$ and

$$\Gamma[u_n, g_m] = V_f^* P U_f^{m-n} P_\mathcal{F} V_f = 0$$

for $m > n$ .

(ii) $V_g \mathcal{E} = P_\mathcal{F} V_f \mathcal{E} \subset P V_f \mathcal{E} \subset \mathcal{K}_+^u$.

(iii) $\Gamma[u_n - g_n, g_n] = \Gamma[u_n, g_n] - \Gamma[g_n, g_n] =$

$$= V_f^* P P_{\mathcal{F}} V_f - V_f^* P_{\mathcal{F}} V_f = 0 .$$

Since we clearly have

(2.12) $$\mathcal{K}_\infty^g = \mathcal{K}_\infty^u = \mathcal{M}(\mathcal{F})$$

it follows that the process $\{u_n\}_{-\infty}^{+\infty}$ is a moving average of the white noise process $\{g_n\}_{-\infty}^{+\infty}$.

Let us see that the white noise $\{g_n\}_{-\infty}^{+\infty}$ is also contained in the $\Gamma$-stationary process $\{f_n\}_{-\infty}^{+\infty}$.

(1) For any $a \in \mathcal{E}$ and $m > n$ we have

$$(\Gamma[f_n, g_m]a, a)_{\mathcal{E}} = (V_f^* U_f^{m-n} P_{\mathcal{F}} V_f a, a)_{\mathcal{E}} = (P_{\mathcal{F}} V_{\cancel{f}} a, U_+^{m-n} V_f a)_{\mathcal{E}} = 0$$

We have also:

(2) $V_g \mathcal{E} = P_{\mathcal{F}} V_f \mathcal{E} \subset \mathcal{K}_+^f$

(3) $\Gamma[f_n - g_n, g_n] = \Gamma[f_n, g_n] - \Gamma[g_n, g_n] = V_f^* P_{\mathcal{F}} V_f - V_f^* P_{\mathcal{F}}^2 V_f = 0$

Hence the white noise $\{g_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$.

Let $\{g'_n\}_{-\infty}^{+\infty}$ be an other white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$. We shall see that $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$, too. Firstly we see that

(2.14) $$V_{g'} \mathcal{E} \subset \mathcal{F}.$$

Indeed, remarking that the extended shift $U_{fg'} = U_f$, for any $a, a_n \in \mathcal{E}$ we have

$$(V_{g'} a, U_f^{*n+1} V_f a_n)_{\mathcal{K}} = (V_f^* U_f^{n+1} V_{g'} a, a_n)_{\mathcal{E}} =$$

$$= (\Gamma[f_0, g'_{n+1}]a, a_n) = 0$$

because $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$. Therefore

$$V_{g'} \mathcal{E} \subset \mathcal{F} \subset \mathcal{M}_+(\mathcal{F}) = \mathcal{K}_+^g .$$

From (2.14) it results that for $m > n$ we have

$$\Gamma[g_n, g'_n] = V_f^* P_{\mathcal{F}} U_f^{m-n} V_{g'} = 0.$$

Because

$$\Gamma[g_n - g'_n, g'_n] = \Gamma[g_n, g'_n] - \Gamma[g'_n, g'_n] = V_f^* P_{\mathcal{F}} V_{g'} - \Gamma[g'_n, g'_n] =$$

$$= V_f^* V_{g'} - \Gamma[g'_n, g'_n] = \Gamma[f_n, g'_n] - \Gamma[g'_n, g'_n] = \Gamma[f_n - g'_n, g'_n]$$

it results (by the fact that $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$) that $\mathrm{Re}\,\Gamma[g_n - g'_n, g'_n] \geq 0$ . Hence $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$, i.e. $\{g_n\}_{-\infty}^{+\infty}$ is the maximal white noise contained in $\{f_n\}_{-\infty}^{+\infty}$ .

Let $\{l_n\}_{-\infty}^{+\infty}$ be a white noise contained in $\{v_n\}_{-\infty}^{+\infty}$ . Then we have

$$\Gamma[u_n, l_m] = V_f^* P U_f^{m-n} V_l = V_f^* P U_f^{m-n} (I-P) V_l = 0.$$

It follows that $\{l_n\}_{-\infty}^{+\infty}$ and $\{f_m\}_{-\infty}^{+\infty}$ are cross-correlated and $\Gamma[f_n, l_m] = 0$.

The fact that $V_l \mathcal{E} \subset \mathcal{K}_+^f$ is obvious, and

$$\mathrm{Re}\,\Gamma[f_n - l_n, l_n] = \mathrm{Re}\,\Gamma[u_n, l_n] + \mathrm{Re}\,\Gamma[v_n - l_n, l_n] \geq 0.$$

Therefore the white noise $\{l_n\}_{-\infty}^{+\infty}$ is contained in $\{f_n\}_{-\infty}^{+\infty}$, and by the maximality of $\{g_n\}_{-\infty}^{+\infty}$ in $\{f_n\}_{-\infty}^{+\infty}$ it results that $\{l_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$ . We have then

$$\Gamma[g_n, l_n] = V_f^* P_{\mathcal{F}} V_l = V_f^* P_{\mathcal{F}} (I-P) V_l = 0.$$

Hence

$$\Gamma[l_n, l_n] = \mathrm{Re}\,\Gamma[g_n, l_n] - \mathrm{Re}\,\Gamma[g_n - l_n, l_n] \leq 0$$

which implies $l_n = 0$.

If we consider

(2.13) $\qquad f_n = u'_n + v'_n$

an other decomposition of the form (2.10) and $\{u'_n\}_{-\infty}^{+\infty}$ is a moving average of the white noise $\{g'_n\}_{-\infty}^{+\infty}$ contained in $\{f_n\}_{-\infty}^{+\infty}$, then, by the maximality of $\{g_n\}_{-\infty}^{+\infty}$ it follows that $\{g'_n\}_{-\infty}^{+\infty}$ is contained in $\{g_n\}_{-\infty}^{+\infty}$. Hence $V_{g'}\mathcal{E} \subset \mathcal{K}_+^g = \mathcal{M}_+(\mathcal{F})$. Moreover $V_{g'}\mathcal{E} \subset \mathcal{F}$. Indeed,

$$(V_{g'}a, U_f^{*\,n+1} V_f a_n)_{\mathcal{K}} = (V_f^* U_f^{n+1} V_{g'} a, a_n)_{\mathcal{E}} = (\Gamma[f_0, g'_{n+1}]a, a_n)_{\mathcal{E}} = 0$$

and $V_{g'}\mathcal{E}$ is orthogonal on $U_+\mathcal{K}_+^f$, i.e. $V_{g'}\mathcal{E} \subset \mathcal{F}$.

From (2.13) we have

$$V_f = V_{u'} + V_{v'} \tag{2.15}$$

and

$$\mathcal{K}_\infty^f = \mathcal{K}_\infty^{u'} \oplus \mathcal{K}_\infty^{v'} . \tag{2.16}$$

Let us denote by $\mathcal{F}_1 = \mathcal{F} \ominus \overline{V_{g'}\mathcal{E}}$ and $q_n = U_f^n P_{\mathcal{F}_1} V_f$. Then it is obvious that $\{q_n\}_{-\infty}^{+\infty}$ is a white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$ and because:

(i) $(\Gamma[v'_n, q_m]a, a) = (V_{v'}^* U_f^{m-n} P_{\mathcal{F}_1} V_f a, a) = (P_{\mathcal{F}_1} V_f a, U_f^{*\,m-n} V_{v'} a) = 0$,

(ii) $V_q\mathcal{E} \subset \mathcal{K}_+^{v'}$ (by the fact that $V_q\mathcal{E} \perp \mathcal{K}_+^u$ and (2.16)),

(iii) $\mathrm{Re}\,\Gamma[v'_n - q_n, q_n] = \mathrm{Re}\,\Gamma[v'_n, q_n] - \Gamma[q_n, q_n] =$

$$= V_{v'}^* P_{\mathcal{F}_1} V_f - V_f^* P_{\mathcal{F}_1} V_f = V_f^* P_{\mathcal{F}_1} V_f - V_f^* P_{\mathcal{F}_1} V_f = 0,$$

it results that the white noise process $\{q_n\}_{-\infty}^{+\infty}$ is contained in the deterministic process $\{v'_n\}_{-\infty}^{+\infty}$, i.e. $q_n = 0$. Therefore $\mathcal{F}_1 = \{0\}$ and consequently $\overline{V_{g'}\mathcal{E}} = \mathcal{F}$. Hence we obtain that $\mathcal{K}_\infty^{u'} = \mathcal{K}_\infty^{g'} = \mathcal{M}(\mathcal{F})$, $\mathcal{K}_\infty^{v'} = \mathcal{R}$, and by (2.15), (2.16) it follows that $V_{u'} = PV_f$. So we have $u' = u$ and $v' = v$.

The proof of the theorem is finished.

The process $\{g_n\}_{-\infty}^{+\infty}$ is the innovation part of the process $\{f_n\}_{-\infty}^{+\infty}$ and it is called the <u>innovation-process</u> associated with $\{f_n\}_{-\infty}^{+\infty}$.

## 3. <u>Prediction problems</u>

Let $\{f_n\}_{-\infty}^{+\infty}$ be a $\Gamma$-stationary process in the complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$. Denote by

$$(3.1) \qquad \mathcal{H}_0 = \{h \in \mathcal{H} \mid h = \sum_{k \leq 0} A_k f_k \,, A_k \in \mathcal{L}(\mathcal{E})\}$$

where $A_k$ are finitely non-zero operators. Following Wiener and Masani [7], we call $\mathcal{H}_0$ the <u>present and past</u> of the process $\{f_n\}_{-\infty}^{+\infty}$ and interpret it as the total information obtained acting on the process up to the present moment (t = 0).

To predict the process at the next moment (t = 1) means to obtain the best information about $f_1$ in terms of the elements in $\mathcal{H}_0$. The following proposition will precisize this .

<u>PROPOSITION 3. Let $\{f_n\}_{-\infty}^{+\infty}$ be a $\Gamma$-stationary process and $\{g_n\}_{-\infty}^{+\infty}$ be the maximal white noise contained in it. Setting</u>

$$(3.2) \qquad \hat{f}_1 = f_1 - g_1$$

<u>we have</u> $\Gamma[\hat{f}_1 \,, g_1] = 0$ <u>and</u>

$$(3.3) \qquad \Gamma[f_1 - \hat{f}_1 \,, f_1 - \hat{f}_1] = \inf_{h \in \mathcal{H}_0} \Gamma[f_1 - h \,, f_1 - h]$$

<u>where the infimum is taken in the set of the positive operators in</u> $\mathcal{L}(\mathcal{E})$.

<u>For any</u> $a \in \mathcal{E}$ <u>we have</u>

$$(3.4)\qquad (\Gamma[f_1-\hat{f}_1, f_1-\hat{f}_1]a,a)=\inf\sum_{j,k=0}^{m}(\Gamma[f_j,f_k]a_j,a_k)$$

where the infimum is taken over all finite sistems $a_1, \ldots, a_m$ in $\mathcal{E}$ and $a_0 = a$.

Proof. (See [4]). For any $a\in\mathcal{E}$ we have

$$(\Gamma[f_1-\hat{f}_1, f_1-\hat{f}_1]a,a)_{\mathcal{E}}=(\Gamma[g_1,g_1]a,a)=(V_f^*P_{\mathcal{F}}V_f a,a)=\|P_{\mathcal{F}}V_f a\|^2=$$

$$=\inf_{k\in U_+\mathcal{K}_+}\|V_f a-k\|^2=\inf_{a_1,\ldots,a_m\in\mathcal{E}}\|V_f a+\sum_{k=1}^{m}U_f^{*k}V_f a_k\|^2=$$

$$=\inf\|\sum_{k=0}^{m}U_f^{*k}V_f a_k\|^2=\inf\sum_{k,j=0}^{m}(V_f^*U_f^{k-j}V_f a_j,a_k)_{\mathcal{E}}=$$

$$=\inf_{\substack{a_1,\ldots,a_m\in\mathcal{E}\\ a_0=a}}\sum_{k,j=0}^{m}(\Gamma[f_j,f_k]a_j,a_k)_{\mathcal{E}}$$

thus (3.4) is proved.

Let now $h=\sum_{k=0}^{m}A_k f_{-k}$ be an arbitrary element in $\mathcal{H}_0$. For any $a\in\mathcal{E}$, setting $a_k=-A_k a$ we obtain

$$(\Gamma[f_1-h,f_1-h]a,a)_{\mathcal{E}}=(\Gamma[f_1-\sum_{k=0}^{m}A_k f_{-k}, f_1-\sum_{j=0}^{m}A_j f_{-j}]a,a)_{\mathcal{E}}=$$

$$=\sum_{j,k=-1}^{m}(\Gamma[f_{-k},f_{-j}]a_j,a_k)_{\mathcal{E}}=\sum_{j,k=-1}^{m}(\Gamma[f_j,f_k]a_j,a_k)_{\mathcal{E}}=\sum_{j,k=0}^{m+1}(\Gamma[f_j,f_k]a_j,a_k)_{\mathcal{E}}.$$

From (3.4) it results that

$$\Gamma[f_1-\hat{f}_1, f_1-\hat{f}_1]\le\Gamma[f_1-h, f_1-h]$$

Let $A$ be a positive operator in $\mathcal{L}(\mathcal{E})$ such that for any $h\in\mathcal{H}_0$

$$A\le\Gamma[f_1-h, f_1-h].$$

For any $a\in\mathcal{E}$ and $a_1, \ldots, a_n\in\mathcal{E}$ we choose $A_k\in\mathcal{L}(\mathcal{E})$

such that $A_k a = a_k$ . Then we obtain

$$(Aa,a) \leq \left(\Gamma\left[f_1-\sum_{k=1}^{m} A_k f_{-k}, f_1-\sum_{j=1}^{m} A_j f_{-j}\right]a,a\right)=\sum_{k,j=0}^{m}(\Gamma[f_j,f_k]a_j,a_k).$$

Using again (3.4) it results that $A \leq \Gamma[f_1-\hat{f}_1 , f_1-\hat{f}_1]$.

By this proposition we see that if in some way we can determine $\hat{f}_1$ , then it contains the best information we can extract acting on the process up to the moment $t=0$ , about $f_1$ . This justifies us to call $\hat{f}_1$ the <u>predictible part</u> of $f_1$ and $\Delta[f] = \Gamma[f_1-\hat{f}_1 , f_1-\hat{f}_1]$ the <u>prediction-error operator</u>.

Now we can formulate more precisely the prediction problems in the following manner:

(1) To determine a sequence of finite operators $(A_1 ,\ldots,A_m)_{(N)}$ in $\mathcal{L}(\mathcal{E})$ such that $(\sum_k A_k f_{-k})_{(N)}$ tends strongly in $\mathcal{L}(\mathcal{E},\mathcal{K})$ to $\hat{f}_1$.

(2) To compute the prediction-error operator $\Delta[f]$.

As in the Wiener-Kolmogorov theory of prediction what is supposed to be known is the <u>correlation function</u>

$$\Gamma(n) = \Gamma[f_{m+n} , f_n] .$$

It is clear that $\Gamma(n)$ is an $\mathcal{L}(\mathcal{E})$ -valued positive definite function on the group of integers. Using Naimark dilation theorem, we can represent $\Gamma(n)$ on the form

$$\Gamma(n) = \int_0^{2\pi} e^{-int} dF(t)$$

where F is an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on the unidimensional torus, so called <u>spectral ~~density~~ distribution</u> of the process $\{f_n\}_{-\infty}^{+\infty}$. It is easy to verify that $[\mathcal{K}_\infty^f, V_f, E]$ where E is the spectral measure of the unitary operator $U_f^*$ , is the spectral dilation of F. When no confusion arrise we denote it by $[\mathcal{K}, V, E]$. In [3] we attached to any $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure F an outer $L^2$-bounded analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which is maximal with the property that its semi-spectral measure $F_\Theta$ verifies $F_\Theta \le F$. (See for details [3]). In [3] and [4] we also proved that

$$(\Delta[f]a,a) = \inf_{a_0=a, a_1, \ldots, a_m \in \mathcal{E}} \sum_{k,j=0}^{m} \int_0^{2\pi} e^{i(k-j)t} d(F(t)a_k, a_j) =$$

$$= \inf \sum_{k,j=0}^{m} \int_0^{2\pi} e^{i(k-j)t} d(F_\Theta(t)a_k, a_j) =$$

$$= (\Theta(0)^* \Theta(0)a, a).$$

In fact $F_\Theta$ is the spectral distribution of the moving average part $\{u_n\}_{-\infty}^{+\infty}$ of $\{f_n\}_{-\infty}^{+\infty}$. We also have $0 \le \Delta[f] \le \Gamma(0)$, $\Delta[f] = 0$ iff $\{f_n\}_{-\infty}^{+\infty}$ is deterministic, $\Delta[f] = \Gamma(0)$ iff $\{f_n\}_{-\infty}^{+\infty}$ is white noise process, $\Delta[f] \geq \Delta[p]$ for any white noise process $\{p_n\}_{-\infty}^{+\infty}$ contained in $\{f_n\}_{-\infty}^{+\infty}$ and $\Delta[f] = \Delta[g]$ if $\{g_n\}_{-\infty}^{+\infty}$ is the maximal white noise process contained in $\{f_n\}_{-\infty}^{+\infty}$.

Concerning the first part of prediction problems, to determine the predictible part $\hat{f}_1$ of $f_1$, it is rather difficult one. From the formulas $\Gamma[\hat{f}_1, g_1] = 0$, $\Gamma[h, g_1] = 0$ for any $h \in \mathcal{H}_0$ and

$$\Gamma[g_1, g_1] = \inf_{h \in \mathcal{H}_0} \Gamma[f_1 - h, f_1 - h]$$

we can interpret $f_1 = \hat{f}_1 + g_1$ like on orthogonal (in $\Gamma$) decomposition of $f_1$ with respect to $\mathcal{H}_0$. From this it results a kind of closness of $\hat{f}_1$ to $\mathcal{H}_0$, but the problem to describe this closness by an aproximation procedure seems to be very complicate. However, under some supplementary boundedness condition on the spectral ~~density~~ distribution F, similar to that imposed by Wiener and Masani in the matrix valued case [8], we shall determine, in the next section, $\hat{f}_1$ as a sum (in strong sense) of an infinite series of elements from $\mathcal{H}_0$.

## 4. Linear predictor

The supplementary boundedness condition on F is the following: there exists a constant $c > 0$ such that

$$(4.1) \qquad \frac{1}{2\pi} c\, dt \leq F \leq \frac{1}{2\pi} c^{-1} dt$$

We shall begin with the following

PROPOSITION 4. *Let F be an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\mathbb{T}$, $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be its maximal outer function, and $G = \Theta(0)^* \Theta(0)$. Then F verifies the condition (4.1) if and only if $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is a bounded analytic function which has a bounded analytic inverse, $F_\Theta = F$, $\dim \mathcal{E} = \dim \mathcal{F}$ and there exists an identification of $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ with an invertible bounded analytic function $\{\mathcal{E}, \mathcal{E}, \Phi(\lambda)\}$ such that*

$$(4.2) \qquad \Phi(0) = G^{1/2}$$

*Proof.* Let $\{\mathcal{E}, \mathcal{E}, \Phi(\lambda)\}$ be an identification for $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ as in proposition and $\{\mathcal{E}, \mathcal{E}, \Psi(\lambda)\}$ be its inverse. Then there exist the Fatou limits $\Phi(e^{it})$ and $\Psi(e^{it})$ and

$$(4.3) \qquad dF = dF_\Phi = \frac{1}{2\pi} \Phi(e^{it})^* \Phi(e^{it})\, dt \ .$$

For any trigonometric polynomial p and $a \in \mathcal{E}$ we have

$$\int_0^{2\pi} |p(e^{it})|^2 d(F(t)a, a) = \frac{1}{2\pi} \int_0^{2\pi} \| \Phi(e^{it}) p(e^{it}) a \|^2 \leq$$

$$\leq \| \Phi \|^2 \frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^2 \|a\|^2 dt$$

and

$$\int_0^{2\pi} |p(e^{it})|^2 d(F(t)a, a) = \frac{1}{2\pi} \int_0^{2\pi} \| \Phi(e^{it}) p(e^{it}) a \|^2 dt \geq$$

$$\geq \frac{1}{2\pi} \| \Psi \|^{-2} \int_0^{2\pi} \| \Psi(e^{it}) \Phi(e^{it}) p(e^{it}) a \|^2 dt =$$

$$= \frac{1}{2\pi} \|\Psi\|^{-2} \int_0^{2\pi} |p(e^{it})|^2 \|a\|^2 dt$$

where $\Phi$ and $\Psi$ are the multiplication operators in $\mathcal{L}(\mathcal{E})$ generated by $\phi(e^{it})$ respectively $\Psi(e^{it})$. It results that for any positive continous function $\varphi$ on $\mathbb{T}$ we have

$$\frac{1}{2\pi} \|\Psi\|^{-2} \int_0^{2\pi} \varphi\, dt \leq \int_0^{2\pi} \varphi\, dF \leq \frac{1}{2\pi} \|\Phi\|^2 \int_0^{2\pi} \varphi\, dt$$

i.e. F verifies (4.1).

Conversely, suppose that F verifies (4.1). If $[\mathcal{K}, V, E]$ is the spectral dilation of F and U is the unitary operator corresponding to $E$, then

$$X\Big(\sum_n U^n V a_n\Big) = \sum_n e^{int} a_n$$

defines an invertible operator from $\mathcal{K}_+$ to $H^2(\mathcal{E})$ which intertwines U with the shift operator on $H^2(\mathcal{E})$. Then clearly

$$X\Big(\bigcap_{n\geq 0} U^n \mathcal{K}_+\Big) = \bigcap_{n\geq 0} U^n X \mathcal{K}_+ = \{0\}.$$

Thus $\bigcap_{n\geq 0} U^n \mathcal{K}_+ = \{0\}$ which implies (by the factorization theorem [3]) that we have $F_\Theta = F$. Obvious (4.1) implies that $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded and the corresponding $\Theta_+$ is a bounded operator with bounded inverse $\Theta_+^{-1}$. The operator $\Theta_+^{-1}$ intertwines the shifts, thus it becomes from a bounded analytic function $\{\mathcal{F}, \mathcal{E}, \Omega(\lambda)\}$ which is the inverse of $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$.

Let us consider the operator $X : \mathcal{F} \to \mathcal{E}$ defined by $X = G^{1/2}\Omega(0)$ where $G = \Theta(0)^* \Theta(0)$.

We have

$$\|Xa\|^2 = \|G^{1/2}\Omega(0)a\|^2 = (G\Omega(0)a, \Omega(0)a) =$$

$$= (\Theta(0)^*\Theta(0)\Omega(0)a, \Omega(0)a) = \|\Theta(0)\Omega(0)a\|^2 = \|a\|^2.$$

Hence X is an unitary operator from $\mathcal{F}$ onto $\mathcal{E}$.

If we put

$$\Phi(\lambda) = X\,\Theta(\lambda) \qquad \lambda \in D$$

then we have

$$\Phi(0) = X\,\Theta(0) = G^{1/2}\,\Omega(0)\,\Theta(0) = G^{1/2}.$$

Clearly that $\{\mathcal{E}, \mathcal{E}, \Phi(\lambda)\}$ is an other identification of the same function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ . The proof is finished.

Let now $\{f_n\}_{-\infty}^{+\infty}$ be a $\Gamma$-stationary process whose spectral density F verifies (4.1). Its prediction-error operator $\Delta[f] = G$ is then an invertible operator on $\mathcal{E}$. Let $\{g_n\}_{-\infty}^{+\infty}$ be the maximal white noise contained in $\{f_n\}_{-\infty}^{+\infty}$. Denote

(4.4) $$h_n = G^{-1/2} g_n .$$

Then $\{h_n\}_{-\infty}^{+\infty}$ is a white noise process such that

$$\Gamma[h_n, h_n] = I_{\mathcal{E}}$$

The process $\{h_n\}_{-\infty}^{+\infty}$ is called the normalised innovation process of $\{f_n\}_{-\infty}^{+\infty}$.

Let $\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ be the maximal outer function of F identified as in Proposition 4. Then the geometric model for prediction can be drown as follows:

$$\mathcal{K} = L^2(\mathcal{E}), \qquad \mathcal{K}'_+ = L^2_+(\mathcal{E})$$

$$V = \Theta|\mathcal{E}, \qquad (Va)(t) = \Theta(e^{it})\,a$$

U = the multiplication with $e^{it}$ on $L^2(\mathcal{E})$.

We have also the following identification for our processes as operators from $\mathcal{E}$ into $L^2(\mathcal{E})$:

$$f_n \; : \; a \longrightarrow e^{-int} \Theta(e^{it}) a$$

$$g_n \; : \; a \longrightarrow e^{-int} \Theta(0) a = e^{-int} G^{1/2} a$$

$$h_n \; : \; a \longrightarrow e^{-int} a \, .$$

Let us write also the Taylor expansions of the function $\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ and its inverse $\{\mathcal{E}, \mathcal{E}, \Omega(\lambda)\}$ as follows:

$$\Theta(\lambda) = G^{1/2} + \sum_{k=1}^{\infty} \Theta_k \lambda^k \tag{4.5}$$

$$\Omega(\lambda) = G^{-1/2} + \sum_{k=1}^{\infty} \Omega_k \lambda^k \, . \tag{4.6}$$

PROPOSITION 5. <u>Let $\{f_n\}_{-\infty}^{+\infty}$ be a $\Gamma$-stationary process whose spectral distribution F verifies the boundedness condition (4.1). Then we have</u>

$$f_n = \sum_{k=0}^{\infty} \Theta_k h_{n-k} \tag{4.7}$$

and

$$h_n = \sum_{k=0}^{\infty} \Omega_k f_{n-k} \tag{4.8}$$

<u>where the series are supposed to be convergent in the strong topology on $\mathcal{L}(\mathcal{E}, \mathcal{K})$.</u>

<u>Proof</u>. Working with the above identifications, for any $a \in \mathcal{E}$ we have

$$\sum_{k=0}^{\infty} \Theta_k h_{n-k} a = \sum_{k=0}^{\infty} e^{-i(n-k)t} \Theta_k a = e^{-int} \sum_{k=0}^{\infty} e^{ikt} \Theta_k a =$$

$$= e^{-int} \Theta(e^{it}) a = f_n a$$

and

$$\sum_{k=0}^{\infty} \Omega_k f_{n-k} a = \sum_{k=0}^{\infty} e^{-i(n-k)t} \Theta(e^{it}) \Omega_k a =$$

$$= e^{-int} \Theta(e^{it}) \sum_{k=0}^{\infty} e^{ikt} \Omega_k a = e^{-int} \Theta(e^{it}) \Omega(e^{it}) a =$$

$$= e^{-int} a = h_n a .$$

The convergence of the series and the commutation of the operators with the summations which appeared above is verifiable in an obvious manner.

THEOREM 2. Let $\{f_n\}_{-\infty}^{+\infty}$ be a $\Gamma$-stationary process whose spectral distribution $F$ verifies the boundedness condition (4.1), $\{\mathcal{E}, \mathcal{E}, \Theta(\lambda)\}$ be the attached maximal outer function and $\{\mathcal{E}, \mathcal{E}, \Omega(\lambda)\}$ be its inverse. Then the predictible part $\hat{f}_n$ of $f_n$ is given by

$$(4.9) \qquad \hat{f}_n = \sum_{j=0}^{\infty} E_j f_{(n-1)-j}$$

where $E_j$ is

$$(4.10) \qquad E_j = \sum_{p=0}^{j} \Theta_{p+1} \Omega_{j-p} .$$

The prediction-error operator $\Delta[f]$ is

$$\Delta[f] = \Theta(0)^* \Theta(0) .$$

Proof. From (4.7) and (4.8) we obtain

$$\hat{f}_n = f_n - g_n = \sum_{k=0}^{\infty} \Theta_k h_{n-k} - G^{1/2} h_n = \sum_{k=1}^{\infty} \Theta_k h_{n-k} =$$

$$= \sum_{k=1}^{\infty} \Theta_k \sum_{s=0}^{\infty} \Omega_s f_{n-k-s} = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \Theta_{p+1} \Omega_s f_{n-p-s} =$$

$$= \sum_{j=0}^{\infty} \Big( \sum_{p+s=j} \Theta_{p+1} \Omega_s \Big) f_{(n-1)-j} = \sum_{j=0}^{\infty} \Big( \sum_{p=0}^{j} \Theta_{p+1} \Omega_{j-p} \Big) f_{(n-1)-j} \,.$$

The convergence of the series and the commutations involved are easily verifiable.

In such a way we can obtain the predictible part $\hat{f}_n$ of $f_n$ using the linear (infinite) filter $E_1$ , $E_2$ ,...., so called the <u>linear predictor</u> or <u>Wiener filter</u> for prediction.

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