A SURVEY ON
REPRESENTATIONS OF THE UNITARY GROUP $U(\infty)$

by

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This paper surveys some results concerning the representation problem for the unitary group $U(\infty)$. It is based on the results of the papers :[7],[15],[16],[17],[18],[19],[21],[22].

For such groups, which are not locally compact, the representation theory, as for the canonical commutation and anticommutation relations of mathematical physics, deals with special classes of representations and with a global study based on the use of some associated C^*-algebras.

The developed methods are as well applicable for the related groups $O(\infty)$, $Sp(\infty)$, $SO(\infty)$, $SU(\infty)$ and, moreover, some of the results presented below are sufficiently general to include also these other groups.

A survey on infinite-dimensional spin groups was given by R.J.Plymen [12].

Contents

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Notations

Let H be a complex separable infinite-dimensional Hilbert space, $L(H)$ the algebra of all bounded linear operators on H, $K(H)$ the ideal of compact operators, $C_1(H)$ the Banach space of nuclear operators endowed with the norm $\|X\|_1 = \mathrm{Tr}(|X|)$ and $C_2(H)$ the Hilbert space of Hilbert-Schmidt operators endowed with the norm $\|X\|_2 = \mathrm{Tr}(X^*X)^{1/2}$.

Let $U(H)$ be the group of unitary operators endowed with the strong-operator topology, $U_0(H) = \{V \in U(H) \; ; \; V - I \in K(H)\}$ endowed with the norm topology, $U_1(H) = \{V \in U(H) \; ; \; V - I \in C_1(H)\}$ endowed with the metric $d_1(V', V'') = \|V' - V''\|_1$ and $U_2(H) = \{V \in U(H) \; ; \; V - I \in C_2(H)\}$ endowed with the metric $d_2(V', V'') = \|V' - V''\|_2$.

Consider also the topological group $U(\infty)$ which is the direct limit of the classical unitary groups $U(n)$ with respect to the inclusions

$$U(n) \ni V \longmapsto \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} \in U(n+1) .$$

There are different realizations of $U(\infty)$ as a subgroup of $U(H)$. If $\{e_n\}_{n\in\mathbb{N}}$ is any orthonormal basis of H and H_n denotes the linear span of $\{e_1, \ldots, e_n\}$, then we can identify $U(\infty)$ with

$$\{V \in U(H) \; ; \; V \,|\, H \ominus H_n = I \,|\, H \ominus H_n \text{ for some } n \in \mathbb{N}\}.$$

Then $U(\infty) \subset U_1(H) \subset U_0(H) \subset U(H)$ and $U(\infty)$ is dense in all these groups with respect to their respective topologies.

Similarly, starting with the classical groups $O(n)$, $Sp(n)$, $SO(n)$ or $SU(n)$, one can define the direct limit groups $O(\infty)$, $Sp(\infty)$, $SO(\infty)$ or $SU(\infty)$ respectively. Also, the group $S(\infty)$ of finite permutations of $\mathbb{N}$ can be viewed as the direct limit of the symmetric groups $S(n)$ of all permutations of $\{1, \ldots, n\}$.

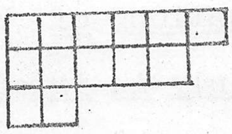
By a representation of a topological group we shall always mean continuous unitary representation on a Hilbert space.

§ 1. Segal-Kirillov representations.

The classical theorem of H. Weyl [23] shows that all irreducible representations of $U(n)$ are realized in spaces of tensors of determined symmetry types classified by decreasing n-tuples of integers $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n$ called "signatures".

In the infinite-dimensional case a similar result holds for the group $U_0(H)$.

1.1. For a positive signature $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n \geqslant 0$ consider the "Young diagram"



the rows of which have lengths $m_1, \ldots, m_n$ respectively and insert in squares the numbers $1, 2, \ldots, m$, where

$$m = m_1 + \ldots + m_n ,$$

filling first the first column, then the second one and so on. Let P and Q be the subgroups of the symmetric group $S(m)$ consisting of those permutations which conserve the rows of the Young diagram, and respectively its columns (horizontal and vertical permutations). Let $\varepsilon(\sigma)$ denote the sign of $\sigma \in S(m)$.

Consider the representation $\tilde{\rho}$ of $U(H)$ on $H^m = H \otimes \ldots \otimes H$, ($m$ times), given by

$$\tilde{\rho}(V)\left(\bigotimes_{j=1}^{m} \xi_j\right) = \bigotimes_{j=1}^{m} V\xi_j \quad ; \quad V \in U(H) ,$$

and the representation π of $S(m)$ on H^m given by

$$\pi(\sigma)\left(\bigotimes_{j=1}^{m} \xi_j\right) = \bigotimes_{j=1}^{m} \xi_{\sigma^{-1}(j)} \quad ; \quad \sigma \in S(m) ,$$

and define the linear map $R : H^m \longrightarrow H^m$ by

$$R = \sum_{(p,q) \in P \times Q} \varepsilon(q)\, \pi(qp) .$$

Then $R(H^m)$ is an invariant subspace for the $\tilde{\rho}(V)$, $V \in U(H)$, and the restriction ρ of $\tilde{\rho}$ to $R(H^m)$ is an irreducible representation of $U(H)$ ([23],[7],[15]).

1.2. The same construction applied to H_k instead of H with respect to the signature $m_1 \geqslant \ldots \geqslant m_k \geqslant 0$, ($m_j = 0$ for $j > n$), yields an irreducible representation ρ_k of $U(H_k)$ and it is apparent that $\rho | U(\infty)$ is the natural direct limit of the ρ_k's.

1.3. The representations ρ of $U(H)$ described in 1.1, were first considered by I.E.Segal([15]) who proved that these are the only irreducible representations of $U(H)$ which, when restricted to any $U(H_n)$, decompose only in irreducible representations of $U(H_n)$ corresponding to positive signatures. In Segal's terminology these are called "physical representations".

1.4. A slight modification of the construction in 1.1 is possible in order to associate an irreducible representation of $U_o(H)$ with an arbitrary (not necessarily positive) signature and A.A.Kirillov ([7]) showed that any irreducible representation of $U_o(H)$ is obtained in this way. However, the general construction involves mixed tensors (tensor products like $H \otimes \ldots \otimes H \otimes \bar{H} \otimes \ldots \otimes \bar{H}$) and the argument is no longer a straightforward extension of the finite-dimensional case (see[7],Lemma 1).

1.5. A.A.Kirillov ([7]) also shows that every representation of $U_o(H)$ is a discrete direct sum of irreducible representations. A similar statement is proved by I.E.Segal ([15]) for physical representations of $U(H)$.

1.6. For $U(\infty)$ however, there are many other irreducible representations, for instance arbitrary direct limits of irreducible representations of the $U(H_n)$'s.

Let, for each $n \in \mathbb{N}$, ρ_n be an irreducible representation of $U(H_n)$ and assume that $\rho_n \prec \rho_{n+1}$, i.e., $\rho_{n+1} | U(H_n)$ contains ρ_n. If ρ_n corresponds to the signature $m_1^{(n)} \geqslant \ldots \geqslant m_n^{(n)}$, then

$$\rho_n \prec \rho_{n+1} \iff m_{j-1}^{(n+1)} \geqslant m_{j-1}^{(n)} \geqslant m_j^{(n+1)}, \quad (1 \leq j \leq n+1),$$

and in this case the multiplicity $[\rho_{n+1} : \rho_n]$ of ρ_n in ρ_{n+1} is exactly one ([23]).

Since $\rho_n \prec \rho_{n+1}$, there are isometric imbeddings

$$i_n : H_{\rho_n} \longrightarrow H_{\rho_{n+1}}$$

such that $(\rho_{n+1} | U(H_n)) \circ i_n = i_n \circ \rho_n$ and, moreover, since $[\rho_{n+1} : \rho_n] = 1$, the i_n's are unique up to a scalar factor of module 1. On the completion H_ρ of the direct limit of the H_{ρ_n}'s along the i_n's there is a natural representation ρ of $U(\infty)$ which is independent of the choice of the i_n's.

<u>The representation</u> $\rho \sim (\rho_1 \prec \rho_2 \prec \ldots)$ <u>is irreducible and two such representations</u> $\rho \sim (\rho_1 \prec \rho_2 \prec \ldots)$ <u>and</u> $\rho' \sim (\rho_1' \prec \rho_2' \prec \ldots)$ <u>are equivalent if and only if</u> ρ_n <u>is equivalent to</u> ρ_n' <u>for all sufficiently large</u> n's (Ş.Strătilă, D.Voiculescu, [17], III.2).

1.7. Later (§ 5) we shall associate a C^*-algebra $A(U(\infty))$ with the factor representations of $U(\infty)$ and (§ 6) we shall characterize its primitive ideal space. Since $A(U(\infty))$ is not of type I, there appear all the pathologies known for this case. Let us mention that A.A.Kirillov ([7]) already pointed out that $U(\infty)$ is not of type I.

1.8. Results similar to those above presented for $U(\infty)$ are valid for the groups $O(\infty)$, $Sp(\infty)$, $SO(\infty)$, $SU(\infty)$ ([7],[17]).

§ 2. Infinite tensor product representations.

In the situation considered in 1.1, a theorem of H.Weyl ([23]; for infinite-dimensional H see A.A.Kirillov [7]) asserts that the commutant of $\tilde{\rho}(U(H))$ is the linear span of $\pi(S(m))$. A similar result holds for infinite tensor products. However the representations of $U_1(H)$ arising in this way are factorial of type II_∞, as we shall see.

2.1. Consider an arbitrary orthonormal system α in H,

$$\alpha = (a_1, a_2, \ldots, a_n, \ldots)$$

and define the Hilbert space $\mathcal{H}^\alpha$ as the von Neumann infinite tensor product of a sequence of copies of H along the sequence α ([11]). There are natural representations ρ^α of $U_1(H)$ and π^α of $S(\infty)$ on $\mathcal{H}^\alpha$ such that

$$\rho^\alpha(V)\Big(\bigotimes_{j=1}^{\infty} \xi_j\Big) = \bigotimes_{j=1}^{\infty} V\xi_j \quad ; \quad V \in U_1(H),$$

$$\pi^\alpha(\sigma)\Big(\bigotimes_{j=1}^{\infty} \xi_j\Big) = \bigotimes_{j=1}^{\infty} \xi_{\sigma^{-1}(j)} \quad ; \quad \sigma \in S(\infty),$$

for all decomposable vectors $\bigotimes_{j=1}^{\infty} \xi_j \in \mathcal{H}^\alpha$.

Note that ρ^α is the representation of $U_1(H)$ associated with the function of positive type

$$\varphi^\alpha(V) = \prod_{j=1}^{\infty} (Va_j | a_j) \quad ; \quad V \in U_1(H).$$

2.2. Theorem. (Ş.Strătilă, D.Voiculescu [17], V). *The representation ρ^α of $U_1(H)$ on $\mathcal{H}^\alpha$ is a factor representation of type II_∞ and the commutant of $\rho^\alpha(U_1(H))$ is the von Neumann algebra generated by $\pi^\alpha(S(\infty))$.*

2.3. The proof of this Theorem goes as follows(cf.[17]).

First one considers the case $\alpha \subset \{e_n\}_{n\in\mathbb{N}}$ where $\{e_n\}_{n\in\mathbb{N}}$ is the orthonormal basis by means of which one realizes $U(\infty) \subset U(H)$ and one defines $\rho^\alpha(V)$ only for $V \in U(\infty)$, which is clearly legitimate. In this case one proves, by an approximation argument using the commutation theorem of H.Weyl, that the commutant of $\rho^\alpha(U(\infty))$ is generated by $\pi^\alpha(S(\infty))$. It is well known that the left regular representation of $S(\infty)$ is factorial of type II_1 and from this one infers that $\pi^\alpha(S(\infty))$ generates a type II_1 factor , hence ρ^α is a type II factor representation of $U(\infty)$. By constructing an infinite family of mutually orthogonal and equivalent projections in $\rho^\alpha(U(\infty))''$, one shows that ρ^α is actually of type II_∞ . Moreover, by a direct computation one shows that the function of positive type φ^α is uniformly continuous with respect to the metric of $U_1(H)$ and hence ρ^α extends to a representation of $U_1(H)$.

The general case now follows due to the fact that every unitary $W \in U(H)$ defines an automorphism $V \longmapsto W^*VW$ of $U_1(H)$.

2.4. Now, given two orthonormal systems $\alpha = \{a_n\}$; $\beta = \{b_n\}$ in H , it is natural to ask for necessary and sufficient conditions in order that the representations ρ^α and ρ^β be equivalent

A reasonable conjecture might be that this is the case if and only if there exist a permutation σ of $\mathbb{N}$, an operator $U \in U_1(H)$ (or, may be, $U \in U_2(H)$?) and $\theta_n \in \mathbb{C}$, $|\theta_n| = 1$ such that $Ub_n = \theta_n a_{\sigma(n)}$, $(n\in\mathbb{N})$. We could prove only the following fact :

<u>Proposition</u>. (Ş.Strătilă,D.Voiculescu,[17],V). <u>If</u> ρ^α <u>and</u> ρ^β <u>are equivalent then there are finite sets</u> $F_\alpha \subset \mathbb{N}$, $F_\beta \subset \mathbb{N}$ <u>and a bijective map</u> $\sigma : \mathbb{N} \setminus F_\beta \longrightarrow \mathbb{N} \setminus F_\alpha$ <u>such that</u>

$$\lim_n \| b_n - \theta_n a_{\sigma(n)} \| = 0 \quad \underline{\text{for suitable}} \quad \theta_n \in \mathbb{C} , \ |\theta_n| = 1$$

Let us also mention that if $a_n = e_{k_n}$, $b_n = e_{j_n}$ where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H and $\{k_n\}$, $\{j_n\}$ are strictly increasing sequences of positive integers, then ([17], V.1.7)

$$\rho^{\alpha} \simeq \rho^{\beta} \iff (\exists)\ n_o \in \mathbb{N} \ \underline{\text{such that}}\ k_n = j_n \ (\forall)\ n \geqslant n_o \ .$$

§ 3. The characters.

For U(n) the determination of all irreducible representations is equivalent to the determination of all its characters, that is of all indecomposable central functions χ of positive type on U(n) with $\chi(1) = 1$. The explicit formula of characters of U(n) is due to H. Weyl ([23]) .

For U(oo), the determination of all its characters means the classification not of irreducible but of finite factor representations (types I_n and II_1). As we shall see, in the infinite-dimensional case there are similarities with the case of commutative groups due to the fact that U(oo) is stable under the direct sum operation.

3.1. More generally, let G be a group and Γ be the set of classes of conjugate elements in G . For $g \in G$ let $\hat{g} \in \Gamma$ denote its conjugacy class and for a central function χ on G let $\tilde{\chi}$ denote the corresponding function on Γ . Any homomorphism $\varphi : G \times G \longrightarrow G$ defines an operation "$\oplus$" on :

$$\hat{g}_1 \oplus \hat{g}_2 = \varphi(g_1, g_2)^{\wedge} \ .$$

In what follows <u>we shall assume the existence of a homomorphism</u> φ <u>such that the operation "$\oplus$" on Γ be commutative, associative and with neutral element</u> $\hat{e}$ (where $e \in G$ is the neutral element of G). Then there are homomorphisms $\varphi_n : G^n \longrightarrow G$ such that $\varphi_n(g_1, \ldots, g_n)^{\wedge} = \hat{g}_1 \oplus \ldots \oplus \hat{g}_n$, and $\hat{g} \longmapsto (\hat{g})^* = (g^{-1})^{\wedge}$ is an homomorphism of Γ .

3.2. Theorem. (D.Voiculescu[21]). Let χ be a central function of positive type on G with $\chi(e) = 1$. Then χ is the character of a finite factor representation of G if and only if

$$\tilde{\chi}(\gamma_1 \oplus \gamma_2) = \tilde{\chi}(\gamma_1)\tilde{\chi}(\gamma_2) \quad ; \quad \gamma_1, \gamma_2 \in \Gamma .$$

The simple proof of this theorem will be given in 3.8. In some particular cases this result was found, with rather complicated proofs, by E.Thoma ([19],[20]) and D.Voiculescu ([22]).

The following corollaries are obvious :

3.3. Corollary (D.Voiculescu,[21]). The tensor product of two finite factor representations of G is still a finite factor representation.

3.4. Corollary (D.Voiculescu,[21]). Let G, G' be two groups satisfying the hypotheses in 3.1, and $\omega : G' \longrightarrow G$ a homomorphism such that $\varphi \circ (\omega \times \omega) = \omega \circ \varphi'$. If ρ is a finite factor representation of G, then $\rho \circ \omega$ is a finite factor representation of G'.

Below we consider some examples.

3.5. If G is commutative, then $\Gamma = G$ and we can take φ as the group operation in G. Thus, Theorem 3.2 implies the well known characterization of characters of commutative groups.

3.6. If $G = U(\infty)$, then the conjugacy class $\hat{V}$ of $V \in U(n)$ is determined by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of V together with their multiplicities. We shall therefore write :

$$\hat{V} = (\lambda_1, \ldots, \lambda_n, 1, 1, \ldots) .$$

The group $U(\infty)$ can be as well realized on $H\oplus H$ with respect to the orthonormal basis $\{e_1, e_2, \dots\} \oplus \{e_1, e_2, \dots\}$. If $V_1, V_2 \in U(\infty) \subset U(H)$, then $V_1 \oplus V_2 \in U(\infty) \subset U(H\oplus H)$. The map

$$\varphi : U(\infty) \times U(\infty) \ni (V_1, V_2) \longmapsto V_1 \oplus V_2 \in U(\infty)$$

is a homomorphism which induces on Γ the operation

$$(\lambda_1, \dots, \lambda_n, 1, \dots) \oplus (\mu_1, \dots, \mu_m, 1, \dots) = (\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m, 1, \dots)$$

satisfying all the requirements in 3.1.

Thus, by Theorem 3.2, a continuous central function χ of positive type on $U(\infty)$ with $\chi(1) = 1$, is a continuous character of $U(\infty)$ if and only if

$$\tilde{\chi}(\lambda_1, \dots, \lambda_n, 1, \dots) = \chi(\lambda_1) \dots \chi(\lambda_n)$$

for all $(\lambda_1, \dots, \lambda_n, 1, 1, \dots) \in \Gamma$. Thus $\tilde{\chi}$, and hence χ, is determined by $p = \chi | U(1)$:

(1) $$\chi(V) = \det p(V) \quad ; \quad V \in U(\infty) .$$

Since p is a continuous function on $U(1)$ we have a Fourier expansion

(2) $$p(z) = \sum_{n \in \mathbb{Z}} c_n z^n ,$$

where $c_n \geq 0$ and $\sum_{n\in\mathbb{Z}} c_n = 1$ because p is of positive type and $p(1) = 1$. Moreover $\chi | U(n)$ is of positive type, which is equivalent to the fact that the coefficients in its development with respect to the characters of $U(n)$ are all positive. These coefficients can be computed and we are led to the following condition (cf[22]):

(3) $$\det((c_{m_i + (j-i)})_{1 \leq i,j \leq n}) \geq 0$$

for all integers $m_1 \geq \dots \geq m_n$.

The series (2) with $c_n = 0$ for $n < 0$, $c_0 = 1$ and satisfying the conditions (3) were studied by E.Thoma ([19]) in connection with the characters of the group $S(\infty)$. In this case, the result of E.Thoma shows that p(z) is of the form

$$(4)\quad p(z) = z^m p_o(z) \quad ; \quad p_o(z) = \prod_{i=1}^{\infty} \frac{1-a_i}{1-a_i z} \cdot \prod_{j=1}^{\infty} \frac{1+b_j z}{1+b_j} \cdot e^{\lambda(z-1)} \quad ,$$

with $m \in \mathbb{Z}$, $m \geqslant 0$; $0 \leqslant a_i < 1$, $\sum_i a_i < +\infty$; $b_j \geqslant 0$, $\sum_j b_j < +\infty$; $\lambda \geqslant 0$.

The formulas (1) and (4) determine the characters of all finite factor representations of $U(\infty)$ which, when restricted to the $U(n)$'s, decompose only in irreducible representations of $U(n)$ with positive signatures.

3.7. Similar characterizations hold for the groups $O(\infty)$, $Sp(\infty)$, $SO(\infty)$, $SU(\infty)$ or for the group $GL(\infty,k)$ over a finite field k considered by E.Thoma ([20]). Owing to Corollary 3.4, one obtains characters of $O(\infty)$, $Sp(\infty)$, etc, by restricting the above determined characters of $U(\infty)$.

In case $G = S(\infty)$, the conjugacy class $\hat{\sigma}$ of $\sigma \in S(\infty)$ is determined by the decomposition of σ into cycles. If σ_1, σ_2 are two finite permutations of $\mathbb{N}$, then they determine a finite permutation $\sigma_1 \sqcup \sigma_2$ of the disjoint union $\mathbb{N} \sqcup \mathbb{N}$ identified with $\mathbb{N}$, and this procedure yields the direct sum operation on the corresponding Γ. Actually, all the characters of $S(\infty)$ were determined by E.Thoma ([19]).

3.8. Proof of Theorem 3.2.

Necessity of the multiplicativity condition. Let ρ be a finite factor representation of G and assume $\chi(g) = \mathrm{Tr}\,\rho(g)$, where Tr is the normalized trace, $\mathrm{Tr}\, I = 1$. Consider

$$\rho_1(g) = \rho(\varphi(g,e)) \quad , \quad \rho_2(g) = \rho(\varphi(e,g))$$

and F , A , B the von Neumann algebras generated by $\rho(G)$, $\rho_1(G)$, $\rho_2(G)$ respectively. The restriction of the trace of F to A and

B yields faithful traces on A and B. Since $\hat{g} \oplus \hat{e} = \hat{e} \oplus \hat{g} = \hat{g}$, we have $\mathrm{Tr}(\rho_1(g)) = \mathrm{Tr}(\rho_2(g)) = \mathrm{Tr}(\rho(g))$. We infer from this that F, A, B are isomorphic, in particular A and B are also factors. Since A and B are commuting subfactors of F, it follows that

$$\begin{aligned}\tilde{\chi}(\hat{g}_1 \oplus \hat{g}_2) &= \chi(\varphi(g_1, g_2)) = \mathrm{Tr}(\rho_1(g_1)\rho_2(g_2)) = \\ &= \mathrm{Tr}(\rho_1(g_1))\mathrm{Tr}(\rho_2(g_2)) = \tilde{\chi}(\hat{g}_1)\,\tilde{\chi}(\hat{g}_2).\end{aligned}$$

<u>Sufficiency of the multiplicativity condition.</u> Consider K the set of central functions $\chi : G \longrightarrow \mathbb{C}$ of positive type such that $\chi(e) = 1$. We show that for $\chi \in K$, $\gamma_i \in \Gamma$ and $a_i \in \mathbb{C}$, $(1 \leq i \leq n)$, we have

$$(5) \qquad \sum_{1 \leq i, j \leq n} a_i \bar{a}_j \,\tilde{\chi}(\gamma_i \oplus \gamma_j^*) \geq 0 .$$

Indeed, for $m \in \mathbb{N}$ let

$$c_{km+p} = a_{k+1}/m \quad ; \quad (0 \leq k \leq n-1 \,,\, 1 \leq p \leq m)$$

and $g_s \in G$, $(1 \leq s \leq mn)$, be such that

$$\hat{g}_{km+p} = \gamma_{k+1} \quad ; \quad (0 \leq k \leq n-1 \,,\, 1 \leq p \leq m).$$

Then defining

$$g'_{km+p} = \varphi_{mn}(e, \dots, e, g_{km+p}, e, \dots, e)$$

(with g_{km+p} on the $(km+p)$-place), we have

$$(g'_{k_1 m+p_1}(g'_{k_2 m+p_2})^{-1})^{\wedge} = \begin{cases} \gamma_{k_1} \oplus \gamma_{k_2}^* & \text{if } (k_1, p_1) \neq (k_2, p_2) \\ \hat{e} & \text{if } (k_1, p_1) = (k_2, p_2) \end{cases}$$

Since χ is of positive type, this gives

$$0 \leq \sum_{1 \leq i, j \leq mn} c_i \bar{c}_j \,\chi(g'_i (g'_j)^{-1}) =$$

$$= \sum_{1 \leq p, q \leq n} a_p \bar{a}_q \,\tilde{\chi}(\gamma_p \oplus \gamma_q^*) + \frac{1}{m} \sum_{k=1}^{n} |a_i|^2 \,(1 - \tilde{\chi}(\gamma_i \oplus \gamma_i^*)) .$$

Letting $m \longrightarrow +\infty$, this gives (5).

Since Γ is a semigroup with an involutive automorphism $\gamma \longmapsto \gamma^*$, $\ell^1(\Gamma)$ has the structure of an involutive Banach algebra and the set P of functions $f \in \ell^\infty(\Gamma)$ such that $f(\hat{e}) = 1$ and

$$\sum_{1 \le i,j \le n} a_i \bar{a}_j \, f(\gamma_i \oplus \gamma_j^*) \geq 0$$

for all $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{C}$, $\gamma_1, \ldots, \gamma_n \in \Gamma$, is the set of states of $\ell^1(\Gamma)$. Thus, relation (5) together with the boundedness of functions of positive type on G, shows that $\chi \longmapsto \tilde{\chi}$ is an injective affine map $K \longrightarrow P$.

Now, if $\tilde{\chi}$ is multiplicative on Γ, then the corresponding representation of $\ell^1(\Gamma)$ is one-dimensional and hence irreducible, Using (2.5.4, 2.5.5 in [3]) it follows that $\tilde{\chi}$ is an extreme point of K, and hence χ is then à fortiori an extreme point of K, that is the character of a finite factor representation.

3.9. In the above proof of necessity we saw that every finite representation of G generates a factor which contains two commuting subfactors isomorphic to itself. This implies that for G as above, a finite factor representation is either of type II_1 or of type I_1 (cf.[22]).

3.10. Using a desintegration procedure, D.Voiculescu ([22]) showed that <u>every central function of positive type on $U(\infty)$ can be extended by continuity to $U_1(H)$</u>.

§ 4. KMS-functions of positive type.

In this section we introduce KMS-functions of positive type on topological groups endowed with a one-parameter automorphism group, by a straightforward analogy with KMS-states on C^*-dynamical systems. This will be applied to a certain class of functions of positive type on $U_1(H)$ derived from the character formula 3.6.(4) by replacing a scalar by a positive operator. This notion proves useful since in general we don't dispose of any corresponding C^*-dynamical system, and moreover, it leads to easy computations.

4.1. Let G be a topological group and $\mathbb{R} \ni t \longmapsto \alpha_t \in \mathrm{Aut}(G)$ a one-parameter automorphism group such that for every $g \in G$ the map $\mathbb{R} \ni t \longmapsto \alpha_t(g) \in G$ is continuous.

A continuous function $\theta : G \longrightarrow \mathbb{C}$ of positive type is called KMS with respect to $(\alpha_t)_{t\in\mathbb{R}}$ if for every $g,h \in G$ there is a bounded continuous function $F_{g,h}$ defined on the strip $S = \{z \in \mathbb{C}\ ;\ 0 \leq \mathrm{Re}\, z \leq 1\}$ with complex values which is analytic in the interior of S and such that, for all $t \in \mathbb{R}$,

$$F_{g,h}(it) = \theta(g\alpha_t(h)) \quad , \quad F_{g,h}(1+it) = \theta(\alpha_t(h)g) \quad .$$

Using the Kaplansky density theorem and the Phragmen-Lindelöf principle, it is easy to prove the following result:

Proposition. *Let θ be a continuous function of positive type on G, KMS with respect to the continuous one-parameter automorphism group $(\alpha_t)_{t\in\mathbb{R}}$ of G. Let ρ be the cyclic representation of G, with cyclic vector η, associated to θ. Then the representation ρ is in standard form, that is η is also a separating vector for the von Neumann algebra $\rho(G)''$ and, denoting by $(\sigma_t^{\omega_\eta})_{t\in\mathbb{R}}$ the modular automorphism group of $\rho(G)''$ associated to the vector state ω_η, we have*

$$\sigma_t^{\omega_\eta}(\rho(g)) = \rho(\alpha_t(g)) \quad ; \quad t \in \mathbb{R}\, ,\ g \in G\, .$$

4.2. Recall that the characters of $U(\infty)$ corresponding to positive signatures are given by the formulas 3.6.(1) and (4). For a fixed p as in 3.6.(4), let

$$R = (\sup_i a_i)^{-1} > 1 .$$

Then $p(z)$ is an analytic function in the open disk $\{z \in \mathbb{C} \; ; |z| < R\}$ and $p_0(z) \neq 0$ on some neighborhood of the segment $[0, +\infty)$.

For every $B \in L(H)$, $\|B\| < R$, $B \geq 0$, one defines a function $\theta = \theta_{p,B}$ on $U_1(H)$ by

$$\theta_{p,B}(V) = \det(V^m)\det(p_0(BV)p_0(B)^{-1}) \quad ; \quad V \in U_1(H) .$$

Moreover, if $\operatorname{Ker} B = 0$, we have a continuous one-parameter group $\{\alpha_t\}_{t \in \mathbb{R}}$ of automorphisms of $U_1(H)$ defined by

$$\alpha_t(V) = B^{it} V B^{-it} \quad ; \quad V \in U_1(H).$$

The main result presented in this section is the following theorem :

4.3. Theorem (Ş.Strătilă, D.Voiculescu, [18], §4). *For every* p *as in* 3.6.(4) *and every* $B \in L(H)$, $\|B\| < R$, $B \geq 0$, $\theta_{p,B}$ *is a continuous function of positive type on* $U_1(H)$. *If, moreover,* $\operatorname{Ker} B = 0$, *then* $\theta_{p,B}$ *is* KMS *with respect to the automorphism group* $\{\alpha_t\}_{t \in \mathbb{R}}$.

4.4. The proof of the fact that $\theta = \theta_{p,B}$ is of positive type consists of approximating the operator B by diagonable operators and thus reducing the problem to the finite dimensional case.

If $H = H_n$ is finite-dimensional, then

$$\tau_n : U(H_n) \ni V \longmapsto \det p(V)$$

is a central function of positive type on $U(H_n)$ and

$$\tau_n(V) = \sum_\mu c_\mu \chi_\mu(V) \quad ; \quad V \in U(H_n) ,$$

where μ runs over all the positive signatures $m_1 \geq \ldots \geq m_n \geq 0$

and $c_\mu \geq 0$. Then, for $V \in U(H_n)$ we have

$$\det(p(B))\Theta(V) = \tau_n(BV)$$
$$= \sum_\mu c_\mu \chi_\mu(BV)$$
$$= \sum_\mu c_\mu \operatorname{Tr}(\rho_\mu(B)\rho_\mu(V))$$

which shows that Θ is of positive type on $U(H_n)$.

4.5. To see that $\Theta = \Theta_{p,B}$ is KMS with respect to $\{\alpha_t\}_{t\in\mathbb{R}}$, consider $V,W \in U_1(H)$ such that $V - I$, $W - I$ are finite rank and define

$$F_{V,W}(z) = \det(V^m W^m)\det(p_o(VB^z WB^{1-z})p_o(B)^{-1}) .$$

Then $F_{V,W}$ is a well-defined, bounded continuous function on the strip $S = \{z \in \mathbb{C} \ ; \ 0 \leq \operatorname{Re} z \leq 1\}$, analytic in the interior of S and

$$F_{V,W}(it) = \Theta(V\alpha_t(W)) \quad , \quad F_{V,W}(1+it) = \Theta(\alpha_t(W)V) .$$

§ 5. The associated C^*-algebra and applications.

In this section we associate a C^*-algebra to a direct limit of compact groups which reflects the factor representations of the direct limit group. This C^*-algebra turns out to be approximately finite dimensional (AF-algebra) and for such algebras we give a diagonalization that reduces their study to dynamical systems of a particular nature. The ideals of the AF-algebra and some classes of representations can be easily handled using the associated dynamical system.

5.1. Let G be the direct limit of the compact separable groups

$$\{e\} = G_o \subset G_1 \subset \ldots \subset G_n \subset G_{n+1} \subset \ldots$$

such that G_n be Haar negligible in G_{n+1}. We denote by $A(G)$ the envelopping C^*-algebra of the involutive normed algebra defined as the direct limit of $\sum_{j=1}^{n} L^1(G_j) \subset M(G_n)$ along the isometric imbeddings induced by $M(G_n) \subset M(G_{n+1})$. Then :

Theorem (Ş.Strătilă, D.Voiculescu, [17],II.1). $A(G)$ is a C^*-algebra whose factor representations are in bijection with the factor representations of G and those of the G_n's. This bijection conserves the type and the equivalence of representations.

5.2. The C^*-algebra $A = A(G)$ is an AF-algebra, that is, there is an ascending sequence $\{A_n\}_{n\geq 0}$ of finite dimensional C^*-subalgebras in A with

(1) $$A = \overline{\bigcup_n A_n} .$$

For an arbitrary AF-algebra (1) we have a diagonalization method in the sense of the following theorem.

Theorem. (Ş.Strătilă, D.Voiculescu,[17],I.1). For an AF-algebra A there exists

a) a maximal abelian $*$-subalgebra C in A ;

b) a conditional expectation $P : A \longrightarrow C$;

c) a subgroup U of the unitary group of A ;

such that

(i) $u^*Cu = C$ for all $u \in U$;

(ii) $P(u^*xu) = u^*P(x)u$ for all $u \in U$, $x \in A$;

(iii) A = c.l.m.(UC) = c.l.m.(CU) .

Moreover, let Ω be the Gelfand spectrum of C and Γ be the group of homeomorphisms of Ω induced by U .

Consider the Hilbert space $\ell^2(\Omega)$ with orthonormal basis $\{t \; ; \; t \in \Omega\}$ and denote by $(\cdot|\cdot)$ the scalar product. Each $f \in C(\Omega)$ defines a "multiplication operator" T_f on $\ell^2(\Omega)$ by

$$T_f(h) = fh \quad ; \quad h \in \ell^2(\Omega)$$

and each element $\gamma \in \Gamma$ defines a "permutation operator" V_γ on $\ell^2(\Omega)$ by

$$V_\gamma(h)(t) = h(\gamma^{-1}(t)) \quad ; \quad t \in \Omega \ , \ h \in \ell^2(\Omega) \ .$$

Let

$$A(\Omega, \Gamma)$$

be the C^*-algebra generated in $L(\ell^2(\Omega))$ by the operators T_f and V_γ $(f \in C(\Omega) \ , \ \gamma \in \Gamma)$.

Then there exists a $*$-isomorphism

$$A \simeq A(\Omega, \Gamma)$$

such that

$$P(x)(t) = (xt|t) \quad ; \quad t \in \Omega \ , \ x \in A \ .$$

5.3. The diagonalization of A presented in 5.2 is not at all canonical ; among other things it depends on the expression (1) of A as an AF-algebra.

However, as shown subsequently by W.Krieger (9), any two dynamical systems (Ω, Γ) , (Ω', Γ') arising from the same A by such a diagonalization are isomorphic.

5.4. In the case $A = A(G)$, the dynamical system ((G), (G)) can be described as follows :

The points of Ω are the symbols

$$t = (\rho_0(t) \to \ldots \to \rho_{n-1}(t) \xrightarrow{k_n(t)} \rho_n(t) \to \ldots) \ ,$$

where $1 \leq n < n_0(t) \leq +\infty$, $\rho_n(t) \in \widehat{G}_n$ and $1 \leq k_n(t) \leq [\rho_n(t) : \rho_{n-1}(t)] \neq 0$. If $t \in \Omega$ and $\omega \subset \Omega$, then $t \in \overline{\omega}$ if and only if either

(a) $t \in \omega$

or

(b) $n_0(t) = +\infty$ and for every $m \in \mathbb{N}$ there is $s \in \omega$ with

$$\rho_n(s) = \rho_n(t)\ ,\ k_n(s) = k_n(t)\quad ;\quad (\forall)\quad n \le m$$

or

(c) $n_o(t) < +\infty$ and the set

$$\left\{\rho_{n_o(t)+1}(s)\ ;\ s \in \omega\ ,\ \rho_n(s) = \rho_n(t)\ ,\ k_n(s) = k_n(t)\ ,\ (\forall)\ n \le n_o(t)\right\}$$

is an infinite set .

For a permutation σ of the set

$$\left\{t \in \Omega\ ;\ n_o(t) = m\ ,\ \rho_m(t) = \rho_m\right\}\ ,$$

($m \in \mathbb{N}$, $\rho_m \in \hat{G}_m$) , let $\gamma = \gamma(m,\rho_m,\sigma)$ be the transformation of Ω such that

$$\gamma(t) = t \quad \text{if either}\quad n_o(t) < m \quad \text{or}\quad \rho_m(t) \ne \rho_m$$

and

$$\gamma(t) = (\sigma(\rho_o(t) \xrightarrow{k_1(t)} \dots \xrightarrow{k_m(t)} \rho_m(t)) \xrightarrow{k_{m+1}(t)} \rho_{m+1}(t) \longrightarrow \dots)$$

in the contrary case. The transformation group $\Gamma(G)$ is generated by these $\gamma(m,\rho_m,\sigma)$.

For details see ([17], II.2).

5.5. Concerning the ideals of an AF-algebra A we record the following result

Theorem. (Ş.Strătilă, D.Voiculescu, [17], I.2). The primitive ideals of A are in bijection with the closures of the orbits of Γ in Ω .

5.6. Let μ be a Γ-quasi-invariant probability measure on Ω Then μ can be regarded as a state of the commutative C^*-algebra $C \cong C(\Omega)$ and therefore

$$\varphi_\mu = \mu \circ P$$

is a state of A . Let π_μ be the cyclic representation of A associated to φ_μ . Then π_μ is a standard representation and, in fact, π_μ coincides with the representation given by the Krieger construction ([8]) applied to the dynamical system (Ω, Γ, μ).

π_μ is a factor representation if and only if μ is Γ-ergodic.

π_μ is a finite representation if and only if μ is equivalent to some Γ-invariant probability measure on Ω. Moreover, every finite representation of A is quasiequivalent to some π_μ.

π_μ is semifinite if and only if the transformation group Γ is μ-measurable, i.e., there exists a Γ-invariant sigma-finite positive measure on Ω, equivalent to μ.

For details and more precise characterizations of the type of π_μ see ([17], I.3).

5.7. For μ as in 5.6, there is a unique $*$-representation ρ_μ of $A(\Omega, \Gamma)$ on $L^2(\Omega, \mu)$ such that

$$\rho_\mu(T_f)h = fh \quad ; \quad h \in L^2(\Omega, \mu) \quad , \quad f \in C(\Omega)$$

and

$$(\rho_\mu(V_\gamma)h)(t) = (d\mu^\gamma/d\mu)^{1/2}(t)\ h(\gamma^{-1}(t)) \quad ; \quad t \in \Omega \ , \ h \in L^2(\Omega, \mu) \ , \ \gamma \in \Gamma .$$

Then

$$\rho_\mu \text{ is irreducible} \iff \mu \text{ is ergodic}$$

and

$$\rho_{\mu_1} \text{ is equivalent to } \rho_{\mu_2} \iff \mu_1 \text{ is equivalent to } \mu_2 .$$

(see [17], I.3).

§ 6. The primitive ideal space of $A(U(\infty))$.

For the group $U(n)$, the determination of characters, the determination of irreducible representations or the determination of primitive ideals of $C^*(U(n))$ amount to the same thing. Of course this is no longer true for a non-type I group, in particular for $U(\infty)$.

Here we give a complete description of the primitive ideal space of the C^*-algebra $A(U(\infty))$ associated to $U(\infty)$ as in § 5.

6.1. Let Ω and Γ be the compact space and its group of transformations provided by the diagonalization of the AF-algebra $A(U(\infty))$. By 5.4, the points of Ω are of the form

$$t = (\rho_1 \prec \rho_2 \prec \dots \prec \rho_n \prec \dots)$$

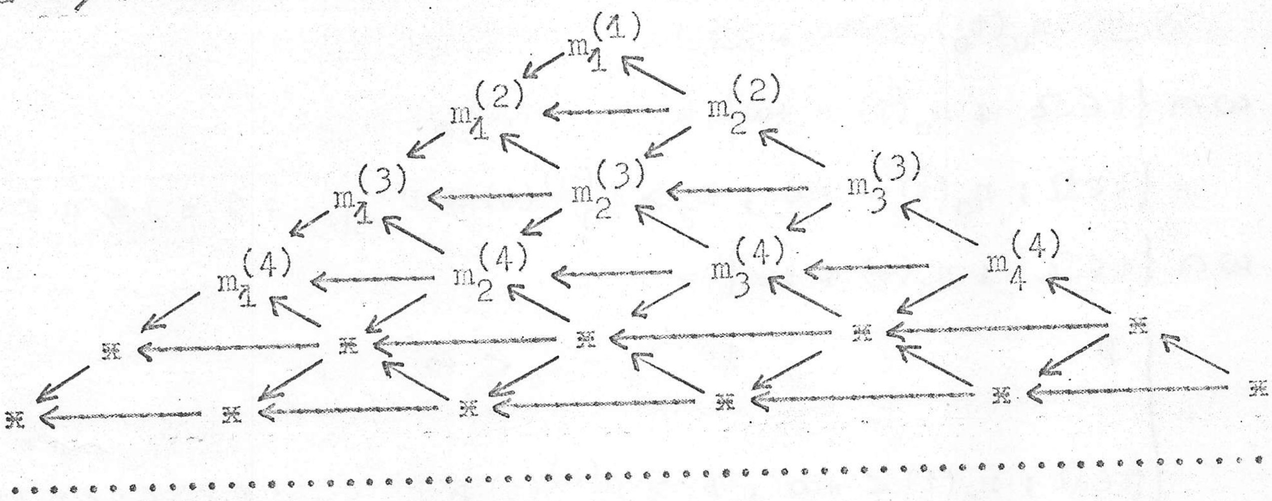
where ρ_n is an irreducible representation of $U(n)$ (note that if $\rho_n \prec \rho_{n+1}$, then $[\rho_{n+1} : \rho_n] = 1$). According to the description with signatures of the ρ_n's and of the relations $\rho_n \prec \rho_{n+1}$ (see 1.6), it follows that the points of Ω are the symbols

$$t = \{(m_j^{(n)}(t))_{1 \leq j \leq n}\}_{1 \leq n < n_o(t)}$$

where

$$1 \leq n_o(t) \leq +\infty \quad , \quad m_j^{(n)}(t) \in \mathbb{Z} \text{ and } m_{j-1}^{(n)}(t) \geqslant m_{j-1}^{(n-1)}(t) \geqslant m_j^{(n)}(t) .$$

Therefore, a point of Ω looks like the picture below, where $a \longrightarrow b$ means $a \leq b$.



The description of the topology on Ω and the description of the transformation group Γ follow obviously from section 5.4. Roughly speaking, a point $t \in \Omega$ with $n_o(t) = +\infty$ is adherent to a set $\omega \subset \Omega$ if for every "horizontal line" in the picture of t one can find a point in ω with the same "beginning" until that line. Also, the generators of Γ change these beginnings among themselves, leaving fixed the rest of the picture.

6.2. We recall from 5.5 that the primitive ideals of $A(U(\infty))$ correspond in a canonical way to the closures of the orbits of Γ. In the next Lemma we determine these sets. First, some notations. For $t \in \Omega$ and $1 \leq j < n_o(t)$ we define

$$L_j(t) = \sup \{ m_j^{(n)}(t) \ ; \ j \leq n < n_o(t) \} \in \mathbb{Z} \cup \{+\infty\}$$

$$M_j(t) = \inf \{ m_{n-j+1}^{(n)}(t) \ ; \ j \leq n < n_o(t) \in \mathbb{Z} \cup \{-\infty\} \ .$$

These definitions can be easily visualised on the picture of t.

It is clear that $\{L_j(t)\}_j$ is decreasing, $\{M_k(t)\}_k$ is increasing and $L_j(t) \geq M_k(t)$ for all j,k.

<u>Lemma</u>. <u>Consider</u> $t_o \in \Omega$, $L_j = L_j(t_o)$, $M_j = M_j(t_o)$ <u>and denote by</u> $\omega = \overline{\Gamma(t_o)}$ <u>the closure of the</u> t_o-<u>orbit of</u> Γ <u>in</u> Ω. <u>Then</u> :

1) <u>If</u> $n_o(t_o) < +\infty$, <u>then</u>

$$\omega = \{ t \in \Omega \ ; \ n_o(t) = n_o(t_o) \ , \ m_j^{(n_o(t)-1)}(t) = L_j \ ; \ 1 \leq j < n_o(t) \} \ .$$

2) <u>If</u> $n_o(t_o) = +\infty$, <u>then</u>

$$\omega \cap \{ t \in \Omega \ ; \ n_o(t) = +\infty \} =$$

$$= \{ t \in \Omega \ ; \ n_o(t) = +\infty \ , \ L_j \geq m_j^{(n)}(t) \geq M_{n-j+1} \ ; \ 1 \leq j \leq n < +\infty \}$$

$$\omega \cap \{ t \in \Omega \ ; \ n_o(t) < +\infty \} =$$

$$= \begin{cases} \emptyset & \text{if } L_1 - M_1 < +\infty \\ \{ t \in \Omega \ ; \ n_o(t) < +\infty \ , \ L_j \geq m_j^{(n)}(t) \geq M_{n-j+1} \ ; \ 1 \leq j \leq n < n_o(t) \} & \text{if } L_1 - M_1 = +\infty \end{cases}$$

6.3. The next Lemma answers a natural converse question.

Lemma. For any given $L_j \in \mathbb{Z} \cup \{+\infty\}$, $M_j \in \mathbb{Z} \cup \{-\infty\}$, $(j \in \mathbb{N})$, *such that*

$$+\infty \geq L_1 \geq L_2 \geq \dots \geq L_j \geq \dots \geq M_j \geq \dots \geq M_2 \geq M_1 \geq -\infty$$

there exists a point $t_o \in \Omega$ *with* $n_o(t_o) = +\infty$ *such that*

$$L_j = L_j(t_o) \quad , \quad M_j = M_j(t_o) \quad ; \quad j \in \mathbb{N} .$$

6.4. Using the above remarks and lemmas, one obtains the following result :

Theorem (Ş.Strătilă, D.Voiculescu, [17], III.1). *The primitive spectrum of the* C^**-algebra* $A(U(\infty))$ *can be identified with the set of all the symbols*

$$\xi = \{L_j(\xi), M_j(\xi)\}_{1 \leq j < n_o(\xi)}$$

where either $n_o(\xi) = +\infty$ *and, for all* $1 \leq j < +\infty$, *we have*

$$\mathbb{Z} \cup \{+\infty\} \ni L_j(\xi) \geq L_{j+1}(\xi) \geq M_{j+1}(\xi) \geq M_j(\xi) \in \mathbb{Z} \cup \{-\infty\} ,$$

or $n_o(\xi) \in \mathbb{N}$ *and, for all* $1 \leq j < n_o(\xi)$, *we have*

$$\mathbb{Z} \ni M_{n_o(\xi)-j}(\xi) = L_j(\xi) \geq L_{j+1}(\xi) = M_{n_o(\xi)-j-1}(\xi) .$$

Namely, if ρ is a factor representation of $U(\infty)$ (or of some $U(k)$), then the kernel of ρ corresponds to the symbol

$$L_j = \sup\{\sup\{m_j^{(n)} \ ; \ n \geq j\}\}$$
$$M_j = \inf\{\inf\{m_{n-j+1}^{(n)} \ ; \ n \geq j\}\}$$

where the first sup and the first inf are taken over all signatures $(m_1^{(n)}, \dots, m_n^{(n)}) \in \widehat{U(n)}$ which appear in $\rho|U(n)$.

The points $\xi \in \mathrm{Prim}(A(U(\infty))$ with $n_o(\xi) = +\infty$ correspond to factor representations of $U(\infty)$, while the points ξ with $n_o(\xi) = n_o \in \mathbb{N}$ correspond to factor representations of $U(n_o-1)$.

The topology on the space $\mathrm{Prim}(A(U(\infty)))$ can also be described (see [17], III.1.5). For instance, the one point set $\{\xi_\infty\} \subset \mathrm{Prim}(A(U(\infty)))$, where $L_j(\xi_\infty) = \infty$, $M_j(\xi_\infty) = -\infty$ for all j, is everywhere dense.

For $\xi \in \mathrm{Prim}(A(U(\infty)))$, $\{L_j(\xi)\}_j$ will be called the <u>upper signature</u> of ξ and $\{M_j(\xi)\}_j$ will be called the <u>lower signature</u> of ξ.

6.5. In 1.6 we have associated with every point $t = (\rho_1 \prec \ldots \ldots \prec \rho_n \prec \ldots)$ of Ω an irreducible representation ρ_t of $U(\infty)$ which is the direct limit of the ρ_n's.

On the other hand, let μ be a completely atomic Γ-quasi-invariant probability measure concentrated on the Γ-orbit $\Gamma(t)$. Then, for all $\gamma \in \Gamma$ we have

$$\mu(\{\gamma(t)\}) > 0$$

and, for all Borel sets $B \subset \Omega$ we have

$$\mu(B) = \sum_{s \in \Gamma(t) \cap B} \mu(\{s\}) .$$

Clearly, μ is Γ-ergodic and therefore the representation ρ_μ is irreducible (see 5.7).

Moreover, $\mathrm{Ker}\, \rho_\mu$ corresponds to $\overline{\Gamma(t)}$ and the representations ρ_μ and ρ_t are equivalent (Ş.Strătilă, D.Voiculescu, [17], III.2).

Thus, <u>every primitive ideal of</u> $A(U(\infty))$ <u>which corresponds to a factor representation of</u> $U(\infty)$ <u>is the kernel of an irreducible representation of</u> $U(\infty)$ <u>which is a direct limit of irreducible representations of the</u> $U(n)$<u>'s.</u>

6.6. The following result shows that the representations of $U(\infty)$ corresponding to bounded signatures can be extended to <u>norm-continuous</u> representations of $U_1(H)$ (while, usually, only the strong-continuity is required).

Proposition (Ş.Strătilă, D.Voiculescu, [18], 2.8). Let σ be a continuous representation of $U(\infty)$ such that, for any $n \in \mathbb{N}$, $\sigma | U(H_n)$ contains only irreducible representations of signatures $(m_1 \geqslant \ldots \geqslant m_n)$ with $|m_j| \leq M < +\infty$. Then:

$$\|\sigma(V') - \sigma(V'')\| \leq M \|V' - V''\|_1 \quad ; \quad V', V'' \in U(\infty) .$$

6.7. Consider a primitive ideal J of $A(U(\infty))$ corresponding to a bounded upper and lower signature, and let $A/J = A(U(\infty))/J$. Using the above proposition one obtains the following

Corollary (Ş.Strătilă, D.Voiculescu, [18], 2.9). There is a canonical norm-continuous representation

$$\rho_J : U_1(H) \longrightarrow A/J$$

and, for every unitary $W \in U(H)$ there exists a unique $*$-automorphism α_W of A/J such that

$$\alpha_W(\rho_J(V)) = \rho_J(WVW^*) \quad ; \quad V \in U_1(H) .$$

The mapping

$$U(H) \ni W \longmapsto \alpha_W \in \mathrm{Aut}(A/J)$$

is a representation, continuous with respect to the strong-operator topology on $U(H)$ and the point-norm topology on $\mathrm{Aut}(A/J)$.

Since for arbitrary signatures a similar result does not hold, we had to replace in § 4 the C^*-algebra KMS condition by a group-theoretic KMS condition.

6.8. For the finite-dimensional group $U(n)$ it is known that the signatures classify the symmetry types of tensors over H_n. For instance, the signatures of the form $(1,1,\ldots,1,0,\ldots,0)$ correspond to antisymmetric tensors while the signatures of the form $(m,0,\ldots,0)$ correspond to symmetric tensors.

For the infinite-dimensional case, we can say that the upper and lower signatures correspond to symmetry types of tensors over H, or that the primitive ideal spectrum of $A(U(\infty))$ is described in terms of symmetry types of tensors over H. In fact we shall see later that the representations of $U(\infty)$ corresponding to upper signature $L_j = 1$, $(j \in \mathbb{N})$, and lower signature $M_j = 0$, $(j \in \mathbb{N})$, are realized in spaces of antisymmetric tensors over H.

§ 7. Representations in antisymmetric tensors.

In this section we consider a particular class of KMS-functions of positive type on $U_1(H)$, among those presented in § 4, for which we give a complete classification according to type and quasi-equivalence, and also we show how the corresponding representations can be realized in spaces of antisymmetric tensors over H. All these representations, when restricted to $U(\infty)$, correspond to the upper signature $L_j = 1$ and to the lower signature $M_j = 0$, $(j \in \mathbb{N})$.

Also, these representations are related to the restrictions of the gauge-invariant generalited free states ([13]) of the CAR-algebra to the subalgebra of gauge-invariant elements. In fact, the main result presented in this section can be viewed as the classification according to type and quasi-equivalence of these restrictions.

7.1. Let $A \in L(H)$, $0 \leq A \leq I$, and define

$$\Psi_A(V) = \det((I-A) + AV) \quad ; \quad V \in U_1(H).$$

Then ψ_A is a continuous function of positive type on $U_1(H)$. Actually, ψ_A corresponds, as in § 4, to the character given by $p(z) = (1 + z)/2$ and to $B = A(I-A)^{-1}$. Let ρ_A be the associated cyclic representation of $U(\infty)$. We shall also consider

$$W_A = A^{1/2} + i(I-A)^{1/2} \in U(H) .$$

For $T \in L(H)$ let $\sigma(T)$ and $\sigma_{ess}(T)$ be the spectrum and the essential spectrum of T, respectively.

For two projections $P, Q \in L(H)$, such that $P - Q$ is a compact operator, we denote by $cd(P,Q)$ the relative codimension of Q in P, i.e., the index of the Fredholm operator $QP : PH \longrightarrow QH$, or, equivalently,

$$cd(P,Q) = \dim(P - s(PQP)) - \dim(Q - s(QPQ)) ,$$

where $s(T)$ denotes the support projection of T.

The main result is the following theorem.

7.2. Theorem (Ş. Strătilă, D. Voiculescu, [18], 3.1). Let $A, B \in L(H)$, $0 \leq A \leq I$, $0 \leq B \leq I$. Then :

1°. ψ_A is of type I $\Longleftrightarrow$ $A(I-A) \in C_1(H)$. In this case ρ_A is a direct sum of irreducible representations.

2°. ψ_A is factorial and of type I $\Longleftrightarrow$ A is a projection. In this case ρ_A is irreducible.

3°. ψ_A is factorial but not of type I $\Longleftrightarrow$ $A(I-A) \notin C_1(H)$. In this case :

a) ψ_A is of type II_1 $\Longleftrightarrow$ $A - pI \in C_2(H)$ for some $p \in (0,1)$;

b) ψ_A is of type II_∞ $\Longleftrightarrow$ $A(I-A)(A-pI)^2 \in C_1(H)$ for some $p \in (0,1)$ and $\{0,1\} \cap \sigma_{ess}(A) \neq \emptyset$;

c) ψ_A is of type III $\Longleftrightarrow$ $A(I-A)(A-pI)^2 \notin C_1(H)$ for all $p \in (0,1)$.

4°. If A, B are projections, then $\psi_A \sim \psi_B \Longleftrightarrow A - B \in C_2(H)$ and $cd(A,B) = 0 \Longleftrightarrow$ there exists $V \in U_2(H)$ such that $VAV^* = B$.

5^o. *If* $A(I-A) \notin C_1(H)$, $B(I-B) \notin C_1(H)$, *then* $\psi_A \sim \psi_B \Longleftrightarrow$ $W_A - W_B \in C_2(H)$.

In the above statement the sign "$\sim$" stands for the quasi-equivalence of the associated cyclic representations.

In what follows we sketch the main ideas of the proof.

7.3. Let $\Omega = \Omega(U(\infty))$, $\Gamma = \Gamma(U(\infty))$ and $\omega \subset \Omega$ the Γ-invariant subset of Ω corresponding to the primitive ideal J of $A(U(\infty))$ with upper signature $L_j = 1$ and lower signature $M_j = 0$, $(j \in \mathbb{N})$. Let G denote the restriction of Γ to ω . Then the C^*-algebra $A(U(\infty))/J$ is $*$-isomorphic to the AF-algebra $A(\omega, G)$ constructed as in 5.2 from the couple (ω, G). By 6.7, we may consider $U_1(H) \subset A(\omega, G)$ and then the function of positive type ψ_A extends to a state φ_A of $A(\omega, G)$, the type problem and the quasi-equivalence problem being thus transfered for φ_A .

7.4. After identifications, we get $\omega = \{0,1\}^{\aleph_0}$ and G consists of transformations $\gamma_{n,\sigma}$,

$$\gamma_{n,\sigma}(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \ldots) = (\sigma(\alpha_1, \ldots, \alpha_n), \alpha_{n+1}, \ldots) ,$$

where $n \in \mathbb{N}$, and σ is a bijection of the set $\{0,1\}^n$ which preserves the sum of the components of the elements :

$$\alpha, \beta \in \{0,1\}^n , \sigma(\alpha) = \beta \Longrightarrow \alpha_1 + \ldots + \alpha_n = \beta_1 + \ldots + \beta_n .$$

7.5. If A is diagonable, with eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$, and $A(I-A)$ is injective, then we can define the measure $\mu = \mu_A$ on ω as the product of the measures μ_n on $\{0,1\}$ defined by $\mu_n(\{0\}) = p_n = 1 - \lambda_n$, $\mu_n(\{1\}) = q_n = \lambda_n$. Note that in this case μ is a Γ-quasi-invariant probability measure .

Moreover, in this case the state φ_A on $A(\omega,G)$ is of the form

$$\varphi_A = \mu_A \circ P$$

where $P : A(\omega,G) \longrightarrow C(\omega)$ is the conditional expectation. Therefore (see 5.6), the factor and type problems for φ_A are reduced to the G-ergodicity of μ and to the μ-measurability of G, respectively. The corresponding results are :

Theorem. (Ş.Strătilă, D.Voiculescu, [17], IV.4). μ *is G-ergodic if and only if* $\sum_{n=1}^{\infty} p_n(1-p_n) = +\infty$.

Theorem. (Ş.Strătilă, D.Voiculescu, [18], 1.2). G *is* μ*-measurable if and only if* $\sum_{n=1}^{\infty} p_n(1-p_n)(p_n-p)^2 < +\infty$ *for some* $p \in (0,1)$.

7.6. On the other hand, let G' be the group of transformations on ω consisting of all finite permutations of the coordinates of a point $\alpha \in \{0,1\}^{\aleph_0}$, i.e., the direct sum of the permutation groups on each $\{0,1\}$. The AF-algebra $A(\omega,G')$ is $*$-isomorphic to the so-called CAR-algebra, associated to the canonical anticommutation relations.

Since $G \subset G'$, we have $A(\omega,G) \subset A(\omega,G')$, namely $A(\omega,G)$ identifies with the subalgebra of "gauge-invariant" elements of the CAR-algebra. Moreover, the states φ_A are exactly the restrictions of the "gauge-invariant generalized free states" of the CAR-algebra (see [13]) .

Using the methods and the results of R.T.Powers and E.Størmer ([13]) for the classification of gauge-invariant generalized free states of the CAR-algebra, the problem for φ_A is also reducible to the case A diagonable (see [18] § 3).

Let us mention that every φ_A is quasi-equivalent to a similar state with A diagonable.

7.7. Remark that Theorem 7.2 solves also the classification problem, according to type and quasi-equivalence, for the restrictions of gauge-invariant generalized free states of the CAR-algebra, to the gauge-invariant subalgebra. This is different from the classification of the non-restricted states given in ([13]) .

7.8. For A diagonable, say $Ae_n = \lambda_n e_n$, $0 < \lambda_n < 1$, $\sum_n \lambda_n(1-\lambda_n) < +\infty$, the representation ρ_A can be realized as follows on a space of antisymmetric tensors.

Consider $X_n = \bigoplus_{k=0}^{n} (\wedge^k H_n \otimes \wedge^k H_n)$, then, with the convention $(a\otimes b)\wedge(c\otimes d) = (a\wedge c)\otimes(b\wedge d)$,

$$J_n : X_{n-1} \ni \xi \longmapsto \xi \wedge ((1-\lambda_n)^{1/2}\, 1\otimes 1 + \lambda_n^{1/2}\, e_n \otimes e_n) \in X_n,$$

and the representation

$$\rho_n = (\text{natural representation}) \otimes 1$$

of $U(n)$ on X_n .

Then ρ_A is unitarily equivalent to the direct limit of the ρ_n's along the J_n's (see [17], IV).

7.9. For an arbitrary $A \in L(H)$, $0 \leq A \leq I$, with $\operatorname{Ker} A = \operatorname{Ker}(I-A) = 0$, let $B = A(I-A)^{-1}$. By 6.7, the automorphism group $\{B^{it} \cdot B^{-it}\}_t$ of $U_1(H)$ extends to a $*$-automorphism group $\{\alpha_t\}_t$ of $A(\omega,G)$. For the states φ_A one can prove something more than was asserted in 4.3, namely φ_A <u>satisfies the</u> C^*-KMS-<u>conditions with respect to</u> $\{\alpha_t\}_{t\in\mathbb{R}}$ (see [18], 4.7).

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