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ON THE IRREDUCIBLE DISINTEGRATION OF THE REPRESENTATIONS OF C*–ALGEBRAS

by

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ON THE IRREDUCIBLE DISINTEGRATION OF THE REPRESENTATIONS OF C^* - ALGEBRAS

by Silviu Teleman

In a previous paper we have proved an irreducible disintegration theorem for the representations of C^*- algebras (see [13] , theorem 3.1). We recall that in [13] we associated to any cyclic representation $\pi : \mathcal{C} \to \mathcal{L}(H)$ of the C^*- algebra $\mathcal{C}$ in the complex Hilbert space H, a measure space (P, A, β), where β is a positive, σ -additive measure, such that $\beta(P) = 1$, defined on a σ - algebra A of subsets of the set P. Also, we constructed an integrable field $\{(H_p)_{p\in P}; \Gamma\}$ of Hilbert spaces and a field $(\pi_p)_{p\in P}$ of irreducible representations $\pi_p : \mathcal{C} \to \mathcal{L}(H_p)$, $p \in P$, such that there exists an isometric isomorphism

$$V : H \longrightarrow \int_P^{\oplus} H_p \, d\beta(p)$$

of H on the direct Γ - integral of the field of Hilbert spaces, such that if $x \in H$ and $(\xi_p)_{p\in P} \in V(x)$, then, for any $c \in \mathcal{C}$, we have $(\pi_p(c)\xi_p)_{p\in P} \in V(\pi(c)x)$ and

$$\|\pi(c)x\|^2 = \int_P \|\pi_p(c)\xi_p\|_p^2 \, d\beta(p).$$

The space (P, A, β) and the fields $(H_p)_{p\in P}$, $(\pi_p)_{p\in P}$, as we have constructed them in [13] , depend on the representation π ; also, some of the representations π_p can be degenerated. By analogy with the case of the representations of commutative C^*- algebras (see the Gelfand - Naimark theorem, [10] , ch. IV, § 17. 4) it is desirable to obtain a decomposition theory as canonical as possible.

In what follows, for any C^*- algebra $\mathcal{C}$, we shall construct a measurable space (P, A), a field $(H_p)_{p\in P}$ of Hilbert spaces, a field $(\pi_p)_{p\in P}$ of non-degenerated irreducible representations $\pi_p : \mathcal{C} \to \mathcal{L}(H_p)$, of $\mathcal{C}$, and a vector subspace $\Gamma_0 \subset \prod_{p\in P} H_p$, such that for any cyclic representation $\pi : \mathcal{C} \to \mathcal{L}(H)$ there exists a measure β , defined on A, positive, finite, and such that the L^2 - completion Γ of the space Γ_0 has the properties stated in the above mentioned theorem. As an application we shall give a new generalization to the continuity theorem of P. Lévy, as well as a generalization to the general case of the theorem of S. Bochner (see theorem 2 below).

1. Let $\pi : \mathcal{C} \to \mathcal{L}(H)$ be a cyclic representation of the C^*- algebra $\mathcal{C}$, and let $x_0 \in H$, $\|x_0\| = 1$, be a cylcic vector . Let $\mathcal{Z} \subset (\pi(\mathcal{C}))'$ be a maximal Abelian von

Neumann subalgebra and $\mathcal{B}$ the C^* - algebra generated by $\mathcal{Z}$ and $\pi(\mathcal{C})$. Then we have $\mathcal{Z} \subset \mathcal{B} \subset \mathcal{Z}'$ and $\mathcal{B}' = \mathcal{Z}$.

Let $E(\mathcal{B})$ be the convex, $\sigma(\mathcal{B}^*; \mathcal{B})$ - compact set of the states of $\mathcal{B}$ and $E_0(\mathcal{C}) = \{ f \in \mathcal{C}^*; f \geq 0, \|f\| \leq 1 \}$.

Obviously, $E_0(\mathcal{C})$ is a convex, $\sigma(\mathcal{C}^*; \mathcal{C})$ -compact set and we have

$$\mathrm{ex}\; E(\mathcal{B}) = P(\mathcal{B}), \qquad \mathrm{ex}\; E_0(\mathcal{C}) = P(\mathcal{C}) \cup \{0\},$$

where $P(\mathcal{B})$, respectively $P(\mathcal{C})$ are the sets of the pure states of the C^* - algebra $\mathcal{B}$, respectively $\mathcal{C}$ (see [3] , § 2.5.5.). We define the state $f_0 \in E(\mathcal{B})$ by $f_0(b) = (bx_0 | x_0)$, $b \in \mathcal{B}$. The corestriction $\pi: \mathcal{C} \to \mathcal{B}$ induces an affine mapping $\pi^*: E(\mathcal{B}) \to E_0(\mathcal{C})$ given by $\pi^*(f) = f \circ \pi$, $f \in E(\mathcal{B})$. Obviously, π^* is $(\sigma(\mathcal{B}^*;\mathcal{B}); \sigma(\mathcal{C}^*;\mathcal{C}))$ - continuous. Let α be the central measure associated to $f_0 \in E(\mathcal{B})$ (see [15], théorème 2 ; [11], § 5).

<u>Proposition 1</u>. The direct image $(\pi^*)_*(\alpha)$ of the measure α is an orthogonal measure on $E_0(\mathcal{C})$, which represents $\pi^*(f_0)$.

<u>Proof.</u> a) The measure $(\pi^*)_*(\alpha)$ represents $\pi^*(f_0)$. Indeed, for any $c \in \mathcal{C}$ let us denote by $\lambda_{\mathcal{C}}(c)$ the continuous, affine function, defined on $E_0(\mathcal{C})$ by

$$\lambda_{\mathcal{C}}(c)(f) = f(c), \quad f \in E_0(\mathcal{C}),$$

and by $\lambda_{\mathcal{B}}(b)$ let us denote the analogous continuous, affine function, defined on $E(\mathcal{B})$, for any $b \in \mathcal{B}$. We have

$$\lambda_{\mathcal{C}}(c) \circ \pi^* = \lambda_{\mathcal{B}}(\pi(c)), \quad c \in \mathcal{C},$$

and this implies that

$$(\pi^*)_*(\alpha)(\lambda_{\mathcal{C}}(c)) = \alpha(\lambda_{\mathcal{C}}(c) \circ \pi^*) = \alpha(\lambda_{\mathcal{B}}(\pi(c))) =$$
$$= f_0(\pi(c)) = (\pi^*(f_0))(c), \quad c \in \mathcal{C};$$

the assertion is proved.

b) Let us now remark that the representation π may be identified with the Gelfand-Naimark - Segal representation associated to the state $g_0 = \pi^*(f_0)$ of the C^* - algebra $\mathcal{C}$. Let $K_{\mathcal{C}}: L^\infty((\pi^*)_*(\alpha)) \to (\pi(\mathcal{C}))'$ be the associated mapping (see [11], lemma 3). We have to prove that $K_{\mathcal{C}}$ is a homomorphism of $*$ - algebras (see [11], theorem 7).

Indeed, for any $\varphi \in L^\infty((\pi^*)_*(\alpha))$, and any $c_1, c_2 \in \mathcal{C}$ we have

$$(K_{\mathcal{C}}(\varphi)\pi(c_1)x_0 | \pi(c_2)x_0) = \int_{E_0(\mathcal{C})} \varphi \lambda_{\mathcal{C}}(c_2^* c_1)\, d[(\pi^*)_*(\alpha)] =$$

$$= (\pi^*)_*(\alpha)(\varphi\lambda_{\mathcal{C}}(c_2^* c_1)) = \alpha((\varphi\lambda_{\mathcal{C}}(c_2^* c_1))\circ\pi^*) =$$

$$= \alpha((\varphi\circ\pi^*)(\lambda_{\mathcal{C}}(c_2^* c_1)\circ\pi^*)) = \alpha((\varphi\circ\pi^*)(\lambda_{\mathcal{B}}(\pi(c_2^* c_1)))) =$$

$$= (K_{\mathcal{B}}(\varphi\circ\pi^*)\pi(c_1)x_0 \mid \pi(c_2)x_0),$$

and this proves that we have

$$(*) \qquad K_{\mathcal{C}}(\varphi) = K_{\mathcal{B}}(\varphi\circ\pi^*), \quad \varphi\in L^\infty((\pi^*)_*(\alpha)).$$

From $(*)$ we infer that the operator $K_{\mathcal{C}}$ is a homomorphism of $*$ - algebras; consequently, $(\pi^*)_*(\alpha)$ is an orthogonal measure (see [11] , theorem 7). The proposition is proved.

Remark. From formula $(*)$ and from the fact that im $K_{\mathcal{B}} = \mathcal{B}' = \mathcal{Z}$ we infer that im $K_{\mathcal{C}} \subset \mathcal{Z}$.

Proposition 2. $\pi^*(P(\mathcal{B})) \subset P(\mathcal{C}) \cup \{0\}$.

Proof. a) Let $p\in P(\mathcal{B})$ and $\xi_p^0 \in H_p$ be the cyclic vector associated to the pure state p. Let f_p be the pure state defined on $\pi_p(\mathcal{B})$ by the formula

$$f_p(\pi_p(b)) = (\pi_p(b)\xi_p^0 \mid \xi_p^0) = p(b), \quad b\in\mathcal{B}.$$

For any $z\in\mathcal{Z}$ we have $\pi_p(z) = p(z)1_{H_p}$, and, consequently, for any element $b\in\mathcal{B}$ of the form

$$b = \sum_{i=1}^{n} z_i\pi(c_i) + z_0,$$

where $z_i\in\mathcal{Z}$ and $c_i\in\mathcal{C}$, i = 0, 1, 2, –, n, we have

$$\pi_p(b) = \sum_{i=1}^{n}\pi_p(z_i)(\pi_p\circ\pi)(c_i) + \pi_p(z_0) =$$

$$= \sum_{i=1}^{n} p(z_i)(\pi_p\circ\pi)(c_i) + p(z_0)1_{H_p} \in (\pi_p\circ\pi)(\mathcal{C}) + \mathbb{C}1_{H_p}.$$

It follows that we have the inclusion

$$\pi_p(\mathcal{B}) \subset (\pi_p\circ\pi)(\mathcal{C}) + \mathbb{C}1_{H_p},$$

because the sum from the right-hand member is closed in the norm topology (see [3], § 1.8.).

On the other hand, since the opposed inclusion is obviously true, we have the equality

(1) $$\pi_p(\mathcal{B}) = (\pi_p \circ \pi)(\mathcal{C}) + \mathbb{C}\, 1_{H_p}.$$

b) It follows that $\pi_p(\pi(\mathcal{C}))$ is a two-sided ideal of $\pi_p(\mathcal{B})$, closed for the norm topology; from proposition 2.11.7, from [3], we infer that there exists a decomposition

(2) $$f_p = f'_p + f''_p ,$$

where f'_p and f''_p are positive linear forms on $\pi_p(\mathcal{B})$, such that

$$\| f'_p \| = \| f'_p \mid \pi_p(\pi(\mathcal{C})) \|$$

and

$$f''_p \mid \pi_p(\pi(\mathcal{C})) = 0.$$

Since f_p is pure, from (2) we infer that there exists a number $\lambda \in [0,1]$, such that

$$f'_p = \lambda f_p , \quad f''_p = (1-\lambda) f_p .$$

We infer that we have

$$\lambda = \lambda \| f_p \| = \lambda \| f_p \mid \pi_p(\pi(\mathcal{C})) \|$$

and

$$(1-\lambda) \| f_p \mid \pi_p(\pi(\mathcal{C})) \| = 0.$$

Consequently, $\lambda \neq 0 \Rightarrow \lambda = 1$; it follows that

$$\lambda = 0 \Rightarrow p \circ \pi = 0$$

and

$$\lambda \neq 0 \Rightarrow \| p \circ \pi \| = 1 .$$

With formula (1), the proposition is now an immediate consequence.

Let $E_1(\mathcal{B}) = \{ f \in E(\mathcal{B}) ; \| f \mid \pi(\mathcal{C}) \| = 1 \}$. Obviously, $E_1(\mathcal{B})$ is a convex subset of $E(\mathcal{B})$. Let us denote $P_1(\mathcal{B}) = P(\mathcal{B}) \cap E_1(\mathcal{B})$ and $P_0(\mathcal{B}) = \{ p \in P(\mathcal{B}) ; p \circ \pi = 0 \}$. From proposition 2 we infer that we have

$$P_0(\mathcal{B}) \cap P_1(\mathcal{B}) = \emptyset , \quad P_0(\mathcal{B}) \cup P_1(\mathcal{B}) = P(\mathcal{B}).$$

We obviously have that $f_0 \in E_1(\mathcal{B})$.

Proposition 3. There exists a convex set $Q_1 \subset E_1(\mathcal{B})$, which is Baire measurable in $E(\mathcal{B})$ and has the properties that $f_0 \in Q_1$ and $\alpha(Q_1) = 1$.

Proof. Let $\{u_i\}_{i\in I}$ be an approximative unit in $\mathcal{C}$. Then (see [3], proposition 2.1.5.) we have

$$\lim_i f_0(\pi(u_i)) = 1;$$

it follows that for any $n \in N^*$ there exists an $i_n \in I$, such that

$$1 - \frac{1}{n} < f_0(\pi(u_{i_n})). \tag{1}$$

By induction, we can find a sequence $(j_n)_{n\in N^*}$ of indices from I, such that $j_1 = i_1$ and

$$i_n \leq j_n \leq j_{n+1}, \quad n \in N^*.$$

It follows that $\pi(u_{j_n}) \leq \pi(u_{j_{n+1}})$ and, consequently, we have

$$0 \leq \lambda_{\mathcal{B}}(\pi(u_{i_n})) \leq \lambda_{\mathcal{B}}(\pi(u_{j_n})) \leq \lambda_{\mathcal{B}}(\pi(u_{j_{n+1}})) \leq 1.$$

Let $\varphi = \lim_{n\to\infty} \lambda_{\mathcal{B}}(\pi(u_{j_n}))$. Then $\varphi : E(\mathcal{B}) \to [0,1]$ is a Baire measurable, affine function, and we have

$$\int_{E(\mathcal{B})} \varphi \, d\alpha \doteq 1. \tag{2}$$

Let $Q_1 = \{ f \in E(\mathcal{B}) ;\ \varphi(f) = 1 \}$. From (2) it follows that $\alpha(Q_1) = 1$. Also, for any $f \in Q_1$ we have

$$\lim_i f(\pi(u_i)) = 1,$$

and this shows that $\|f \circ \pi\| = 1$; consequently, we have that $f \in E_1(\mathcal{B})$ and, therefore, the inclusion $Q_1 \subset E_1(\mathcal{B})$ is established. On the other hand, the set Q_1 is obviously Baire measurable, convex and $f_0 \in Q_1$, as a consequence of (1). The proposition is proved.

Corollary 1. The set $E_1(\mathcal{B})$ is α-measurable and $\alpha(E_1(\mathcal{B})) = 1$.

Proof. It is an immediate consequence of proposition 3.

Corollary 2. Any bounded, continuous function $t : E_1(\mathcal{B}) \to \mathbb{C}$ is α-measurable.

Proof. Since the measure α is regular, there exists an increasing sequence $(K_n)_{n\in N}$ of compact sets $K_n \subset E_1(\mathcal{B})$, such that $\alpha(K_n) \uparrow 1$.

Let $K = \bigcup_{n\geq 0} K_n$ and $t_n : E_1(\mathcal{B}) \to \mathbb{C}$ be defined by

$$t_n(f) = \begin{cases} t(f), & f \in K_n \\ 0, & f \in E_1(\mathcal{B}) \setminus K_n . \end{cases}$$

The functions t_n are α - measurable and we have

$$\lim_{n\to\infty} t_n = t\chi_K .$$

It follows that t is α - measurable, because

$$\alpha(E_1(\mathcal{B})\setminus K) = 0.$$

and the corollary is proved.

It is known that the measure α is pseudoconcentrated on $P(\mathcal{B})$ (see [15], theorem 2; [13]). More precisely, if $U\subset E(\mathcal{B})$ is a Baire measurable set, such that $U\cap P(\mathcal{B}) = \emptyset$, then $\alpha(U) = 0$.

It follows that by the formula

$$\beta(U\cap P(\mathcal{B})) = \alpha(U),$$

where $U\subset E(\mathcal{B})$ is Baire measurable, we correctly define a probabilistic measure β on the σ-algebra $A_0(P(\mathcal{B}))$ of all traces on $P(\mathcal{B})$ of the Baire measurable subsets of $E(\mathcal{B})$:

$$A_0(P(\mathcal{B})) = \{ U\cap P(\mathcal{B});\ U\subset E(\mathcal{B}) \text{ is Baire measurable} \}.$$

Proposition 4. $\beta^*(P_0(\mathcal{B})) = 0.$

Proof. With the preceding notations, we have

(1) $$1 = \alpha(Q_1) = \beta(Q_1\cap P(\mathcal{B})),$$

and

(2) $$Q_1\cap P(\mathcal{B}) \subset P_1(\mathcal{B});$$

consequently, we have that $P_0(\mathcal{B})\subset P(\mathcal{B})\cap \complement Q_1$, and the proposition is proved.

Corollary 1. For any bounded, continuous function $t: E_1(\mathcal{B}) \to \mathbb{C}$, the function $t\,|P_1(\mathcal{B})$ is β - measurable and the following equality holds

$$\int_{E_1(\mathcal{B})} t\, d\alpha = \int_{P_1(\mathcal{B})} t\, d\beta.$$

Proof. From corollary 2 of proposition 3, we infer that the function t is α-measurable, where as the set $E_1(\mathcal{B})$ is α -measurable in virtue of corollary 1 of the same proposition. Consequently, the integral in the left-hand member of the preceding equality makes sense, and we have

$$\int_{E_1(\mathcal{B})} t\, d\alpha = \int_{Q_1} t\, d\alpha,$$

by taking into account proposition 3.

Since the set Q_1 is Baire measurable, it follows that there exists an increasing sequence of compact sets $L_n \subset Q_1, n \in \mathbb{N}$, which are Baire measurable and such that $\alpha(L_n) \uparrow 1$. Since the functions $t \mid L_n$, $n \in \mathbb{N}$, are continuous, the functions

$$s_n : E(\mathcal{B}) \to \mathbb{C}, \qquad n \in \mathbb{N},$$

defined by

$$s_n(f) = \begin{cases} t(f), & f \in L_n, \\ 0, & f \in E(\mathcal{B}) \setminus L_n, \end{cases}$$

are Baire measurable and we have

$$\int_{E(\mathcal{B})} s_n \, d\alpha = \int_{P(\mathcal{B})} s_n \, d\beta = \int_{P_1(\mathcal{B})} s_n \, d\beta.$$

(see [13], lemma 1.1.).

If we denote $L = \bigcup_{n \geq 0} L_n$, we have

$$\lim_{n \to \infty} s_n = t \chi_L,$$

and $Q_1 \setminus L$ is a Baire measurable set, for which $\alpha(Q_1 \setminus L) = 0$. It follows that

$$\beta((Q_1 \setminus L) \cap P_1(\mathcal{B})) = 0,$$

and, therefore,

$$\int_{E_1(\mathcal{B})} t \, d\alpha = \lim_{n \to \infty} \int_{E_1(\mathcal{B})} s_n \, d\alpha = \lim_{n \to \infty} \int_{P_1(\mathcal{B})} s_n \, d\beta =$$

$$= \int_{P_1(\mathcal{B})} t \chi_L \, d\beta = \int_{P_1(\mathcal{B})} t \, d\beta,$$

and this concludes the proof.

Proposition 5. For any bounded, continuous, affine function $t : E_1(\mathcal{B}) \to \mathbb{C}$ we have

$$t(f_0) = \int_{E_1(\mathcal{B})} t \, d\alpha.$$

Proof. Since t is continuous at f_0, for any $\varepsilon > 0$ there exists a finite subset $\{b_1, \ldots, b_n\} \subset \mathcal{B}$ such that : $f \in E_1(\mathcal{B})$ and $|f(b_i) - f_0(b_i)| < 1$, $i = 1, 2, -, n$, implies $|t(f) - t(f_0)| < \varepsilon$.

Let $\{A_1, A_2, -, A_m\}$ be a finite partition of $E(\mathcal{B})$, consisting of Borel measurable subsets $A_i \subset E(\mathcal{B})$, such that

$$f', f'' \in A_i \Rightarrow |f'(b_j) - f''(b_j)| < 1, \; j = 1, 2, \ldots, n,$$

for any $i = 1, 2, \ldots, m$. If necessary, we can refine the partition such that, with an arbitrary selection of the points $f_i \in A_i \cap Q_1$, $i = 1, 2, \ldots, m$, we have

$$\left| \int_{E_1(\mathcal{B})} t\, d\alpha - \sum_{i=1}^{m} \alpha(A_i)\, t(f_i) \right| < \varepsilon .$$

Let γ be the Radon measure on $E(\mathcal{B})$, given by

$$\gamma = \sum_{i=1}^{m} \alpha(A_i)\, \varepsilon_{f_i} .$$

It is obvious that the barycenter $b(\gamma)$ of γ is the point $b(\gamma) = \sum_{i=1}^{m} \alpha(A_i) f_i \in E(\mathcal{B})$ and we have $b(\gamma) \in Q_1$. It follows that

$$|b(\gamma)(b_j) - f_0(b_j)| = \left| \sum_{i=1}^{m} \alpha(A_i) f_i(b_j) - \int_{E(\mathcal{B})} f(b_j)\, d\alpha(f) \right| =$$

$$= \left| \sum_{i=1}^{m} \alpha(A_i) f_i(b_j) - \sum_{i=1}^{m} \int_{A_i} f(b_j)\, d\alpha(f) \right| =$$

$$= \left| \sum_{i=1}^{m} \int_{A_i} (f_i(b_j) - f(b_j))\, d\alpha(f) \right| \leq$$

$$\leq \sum_{i=1}^{m} \int_{A_i} |f_i(b_j) - f(b_j)|\, d\alpha(f) < 1,$$

for any j = 1, 2, -, n; consequently, we have

$$|t(b(\gamma)) - t(f_0)| < \varepsilon .$$

Since the function t is affine, we have

$$t(b(\gamma)) = \sum_{i=1}^{m} \alpha(A_i)\, t(f_i),$$

and, consequently,

$$\left| \int_{E_1(\mathcal{B})} t\, d\alpha - t(f_0) \right| < 2\varepsilon .$$

The proposition is proved.

Corollary. For any bounded, affine function $t : E(\mathcal{B}) \longrightarrow \mathbb{C}$, whose restriction to $E_1(\mathcal{B})$ is continuous, we have

$$t(f_0) = \int_{P(\mathcal{B})} t\, d\beta .$$

Proof. It is an immediate consequence of proposition 4, of the corollary to proposition 4 and of proposition 5.

2. Since the function $\pi^* : E(\mathcal{B}) \to E_0(\mathcal{C})$, which was already defined, is continuous, it is Baire measurable; consequently, the direct image $(\pi^*)_*(\alpha)$ of the measure α is defined on the σ-algebra $\mathcal{B}(E_0(\mathcal{C}))$ of all Baire measurable subsets of $E_0(\mathcal{C})$ by the formula

$$[(\pi^*)_*(\alpha)](A) = \alpha((\pi^*)^{-1}(A)), \quad A \in \mathcal{B}(E_0(\mathcal{C})).$$

From the inclusions $(\pi^*)^{-1}(\{0\}) \subset \complement E_1(\mathcal{B}) \subset \complement Q_1$ it follows that

$0=\alpha((\pi^*)^{-1}(\{0\}))$, and therefore, for the Borel measure $(\pi^*)_*(\alpha)$, direct image of α, we have

$$((\pi^*)_*(\alpha))(\{0\}) = 0.$$

Since $\{0\}$ is a closed subset of $E_o(\mathscr{C})$, from the preceding equality if follows that the exterior Baire measure, associated to the measure $(\pi^*)_*(\alpha)$, of the set $\{0\}\subset E_o(\mathscr{C})$ is zero.

Let now $A\subset E_o(\mathscr{C})$ be a Baire measurable subset, such that $A\cap P(\mathscr{C}) = \emptyset$. Then we have

$$((\pi^*)_*(\alpha))(A) = 0.$$

Indeed, from $A\cap P(\mathscr{C}) = \emptyset$ it follows that

$$(\pi^*)^{-1}(A)\cap P(\mathcal{B})\subset(\pi^*)^{-1}(\{0\})\subset \complement Q_1,$$

and this implies that

$$((\pi^*)^{-1}(A)\setminus \complement Q_1)\cap P(\mathcal{B}) = \emptyset;$$

it follows that

$$\alpha((\pi^*)^{-1}(A)\setminus \complement Q_1) = 0,$$

and, therefore,

$$\alpha((\pi^*)^{-1}(A)) = 0,$$

because $\alpha(\complement Q_1)=0$. Consequently, we have

$$((\pi^*)_*(\alpha))(A) = 0.$$

Let $A_o(P(\mathscr{C}))$ be the σ- algebra of the traces on $P(\mathscr{C})$ of all Baire measurable subsets of $E_o(\mathscr{C})$:

$$A_o(P(\mathscr{C})) = \{U\cap P(\mathscr{C});\ U\subset E_o(\mathscr{C}),\text{Baire meas.}\}.$$

From the preceding result it follows that by the formula

$$\gamma(U\cap P(\mathscr{C})) = ((\pi^*)_*(\alpha))(U),\quad U\subset E_o(\mathscr{C})\text{, Baire measurable,}$$ we correctly define a probability measure γ on $A_o(P(\mathscr{C}))$.

In analogy to lemma 1.1. from [13], we can state the following.

Proposition 6. For any bounded, Baire measurable function $t: E_o(\mathscr{C})\to \mathbb{C}$, we have

$$\int_{E_o(\mathscr{C})} t\, d[(\pi^*)_*(\alpha)] = \int_{P(\mathscr{C})} t\, d\gamma.$$

Proof. For any $\varepsilon > 0$ there exists a finite partition $\{E_1, E_2, \ldots, E_n\}$ of $E_o(\mathcal{C})$, consisting of mutually disjoint, Baire measurable subsets of $E_o(\mathcal{C})$, and complex numbers $t_1, t_2, \ldots, t_n \in \mathbb{C}$, such that

$$|t(f) - \sum_{i=1}^{n} t_i \chi_{E_i}(f)| < \varepsilon, \quad f \in E_o(\mathcal{C}).$$

It follows that we have

$$\left| \int_{E_o(\mathcal{C})} t \, d[(\pi^*)_*(\alpha)] - \sum_{i=1}^{n} t_i [(\pi^*)_*(\alpha)](E_i) \right| < \varepsilon,$$

and

$$\left| \int_{P(\mathcal{C})} t \, d\gamma - \sum_{i=1}^{n} t_i \gamma(E_i \cap P(\mathcal{C})) \right| < \varepsilon,$$

whence the proposition is now an immediate consequence, if we take into account the definition of the measure γ.

Let now $A_o(P_1(\mathcal{B}))$ be the σ-algebra of the traces on $P_1(\mathcal{B})$ of all the sets belonging to $A_o(P(\mathcal{B}))$; obviously, $A_o(P_1(\mathcal{B}))$ is the σ-algebra of the traces on $P_1(\mathcal{B})$ of all Baire measurable subsets of $E(\mathcal{B})$.

Since

$$Q_1 \cap P(\mathcal{B}) \subset P_1(\mathcal{B}),$$

from the equality

$$\beta(Q_1 \cap P(\mathcal{B})) = \alpha(Q_1) = 1,$$

it follows that by the formula

$$\beta_1(A \cap P_1(\mathcal{B})) = \alpha(A), \quad A \subset E(\mathcal{B}), \text{ Baire measurable,}$$ we correctly define a probability measure on $A_o(P_1(\mathcal{B}))$.

Let us now consider the mapping $\sigma = \pi^* | P_1(\mathcal{B})$, which is defined on the measurable space $(P_1(\mathcal{B}), A_o(P_1(\mathcal{B})))$, and takes values in the measurable space $(P(\mathcal{C}), A_o(P(\mathcal{C})))$.

The mapping σ is measurable. Indeed, if $A \subset E_o(\mathcal{C})$ is Baire measurable, we have

$$\sigma^{-1}(A \cap P(\mathcal{C})) = ((\pi^*)^{-1}(A)) \cap P_1(\mathcal{B}),$$

and $(\pi^*)^{-1}(A)$ is Baire measurable in $E(\mathcal{B})$.

Proposition 7. $\sigma_*(\beta_1) = \gamma$.

Proof. Let $A \subset E_o(\mathcal{C})$ be an arbitrary Baire measurable set. We have

$$\sigma_*(\beta_1)(A \cap P(\mathcal{C})) = \beta_1(\sigma^{-1}(A \cap P(\mathcal{C}))) = \beta_1((\pi^*)^{-1}(A) \cap P_1(\mathcal{B})) =$$
$$= \alpha((\pi^*)^{-1}(A)) = ((\pi^*)_*(\alpha))(A) = \gamma(A \cap P(\mathcal{C})),$$

and the proposition is proved.

3. For any $p \in P(\mathcal{C})$ let us consider the associated Hilbert space H_p, the associated irreducible representation $\pi_p : \mathcal{C} \to \mathcal{L}(H_p)$, and the associated canonical mapping $\theta_p : \mathcal{C} \to H_p$. We shall define a linear mapping $\theta_{\mathcal{C}} : \mathcal{C} \to \prod_{p \in P(\mathcal{C})} H_p$ by the formula

$$\theta_{\mathcal{C}}(c) = (\theta_p(c))_{p \in P(\mathcal{C})}, \quad c \in \mathcal{C}.$$

Similarly, for any $p \in P(\mathcal{B})$ we shall consider the associated Hilbert space H_p, the associated irreducible representation $\pi_p : \mathcal{B} \to \mathcal{L}(H_p)$, and the associated canonical mapping $\theta_p : \mathcal{B} \to H_p$. We shall consider the linear mappings $\theta_{\mathcal{B}} : \mathcal{B} \to \prod_{p \in P(\mathcal{B})} H_p$ and $\theta'_{\mathcal{B}} : \mathcal{B} \to \prod_{p \in P_1(\mathcal{B})} H_p$, given by

$$\theta_{\mathcal{B}}(b) = (\theta_p(b))_{p \in P(\mathcal{B})}, \quad \theta'_{\mathcal{B}}(b) = (\theta_p(b))_{p \in P_1(\mathcal{B})}, \quad b \in \mathcal{B}.$$

Let $\rho : \prod_{p \in P(\mathcal{B})} H_p \to \prod_{p \in P_1(\mathcal{B})} H_p$ be the canonical mapping.

We shall denote

$$\Gamma(\mathcal{C}) = \operatorname{im} \theta_{\mathcal{C}}, \quad \Gamma(\mathcal{B}) = \operatorname{im} \theta_{\mathcal{B}}, \quad \Gamma_1(\mathcal{B}) = \operatorname{im} \theta'_{\mathcal{B}}.$$

We now define the linear mappings $u : \Gamma(\mathcal{C}) \to \Gamma(\mathcal{B})$ and $u_1 : \Gamma(\mathcal{C}) \to \Gamma_1(\mathcal{B})$ by

$$u(\theta_{\mathcal{C}}(c)) = \theta_{\mathcal{B}}(\pi(c)), \quad u_1(\theta_{\mathcal{C}}(c)) = \theta'_{\mathcal{B}}(\pi(c)).$$

We obviously have $\rho(\Gamma(\mathcal{B})) = \Gamma_1(\mathcal{B})$ and $\rho \circ u = u_1$.

For any $c \in \mathcal{C}$, the function $p \mapsto p(c)$ is measurable and bounded on $(P(\mathcal{C}), A_0(P(\mathcal{C})))$, whereas for any $b \in \mathcal{B}$, the function $p \mapsto p(b)$ is measurable and bounded on $(P(\mathcal{B}), A_0(P(\mathcal{B})))$ respectively on $(P_1(\mathcal{B}), A_0(P_1(\mathcal{B})))$. It follows that we can define the scalar products

$$(\theta_{\mathcal{C}}(c_1) \mid \theta_{\mathcal{C}}(c_2)) = \int_{P(\mathcal{C})} p(c_2^* c_1)\, d\gamma(p), \quad c_1, c_2 \in \mathcal{C},$$

$$(\theta_{\mathcal{B}}(b_1) \mid \theta_{\mathcal{B}}(b_2)) = \int_{P(\mathcal{B})} p(b_2^* b_1)\, d\beta(p), \quad b_1, b_2 \in \mathcal{B},$$

and

$$(\theta'_{\mathcal{B}}(b_1) \mid \theta'_{\mathcal{B}}(b_2)) = \int_{P(\mathcal{B})} p(b_2^* b_1)\, d\beta_1(p), \quad b_1, b_2 \in \mathcal{B},$$

respectively on $\Gamma(\mathcal{C})$, $\Gamma(\mathcal{B})$ and $\Gamma_1(\mathcal{B})$.

We have the following properties

a) u is an isometry of $\Gamma(\mathcal{C})$ into $\Gamma(\mathcal{B})$. Indeed, for any $c_1, c_2 \in \mathcal{C}$ we have

$$(u\theta_{\mathcal{C}}(c_1) \mid u\theta_{\mathcal{C}}(c_2)) = (\theta_{\mathcal{B}}(\pi(c_1)) \mid \theta_{\mathcal{B}}(\pi(c_2))) =$$

$$= \int_{P(\mathcal{B})} p(\pi(c_2^* c_1))\, d\beta(p) = \int_{E(\mathcal{B})} f(\pi(c_2^* c_1))\, d\alpha(f) =$$

$$= \alpha(\lambda_{\mathcal{B}}(\pi(c_2^* c_1))) = \alpha((\lambda_{\mathcal{C}}(c_2^* c_1)) \circ \pi^*) =$$

$$= ((\pi^*)_*(\alpha))(\lambda_{\mathcal{C}}(c_2^* c_1)) = \gamma(\lambda_{\mathcal{C}}(c_2^* c_1)) =$$

$$= \int_{P(\mathcal{C})} p(c_2^* c_1)\, d\gamma(p) = (\theta_{\mathcal{C}}(c_1) \mid \theta_{\mathcal{C}}(c_2)),$$

and the assertion is proved.

b) $\rho \mid \Gamma(\mathcal{B})$ is an isometry of $\Gamma(\mathcal{B})$ on $\Gamma_1(\mathcal{B})$. Indeed, we have

$$(\rho(\theta_{\mathcal{B}}(b_1)) \mid \rho(\theta_{\mathcal{B}}(b_2))) = (\theta^1_{\mathcal{B}}(b_1) \mid \theta^1_{\mathcal{B}}(b_2)) =$$

$$= \int_{P_1(\mathcal{B})} p(b_2^* b_1)\, d\beta_1(p) = \int_{P(\mathcal{B})} p(b_2^* b_1)\, d\beta(p) = (\theta_{\mathcal{B}}(b_1) \mid \theta_{\mathcal{B}}(b_2)),$$

for any $b_1, b_2 \in \mathcal{B}$. The fact that $\rho \mid \Gamma(\mathcal{B})$ is a surjection on $\Gamma_1(\mathcal{B})$ was remarked before.

c) u_1 is an isometry of $\Gamma(\mathcal{C})$ into $\Gamma_1(\mathcal{B})$. This immediately follows from a) and b).

Let us now consider the mappings $U : \Gamma(\mathcal{B}) \to H$ and $U_1 : \Gamma_1(\mathcal{B}) \to H$, given by

$$U(\theta_{\mathcal{B}}(b)) = b x_0, \quad U_1(\theta^1_{\mathcal{B}}(b)) = b x_0, \qquad b \in \mathcal{B}.$$

The first one is, obviously, correctly defined, because $\theta_{\mathcal{B}}(b) = 0 \Rightarrow \theta_p(b) = 0,\ p \in P(\mathcal{B})$; therefore b = 0. Let us now assume that $\theta^1_{\mathcal{B}}(b) = 0$. We then have $\theta_p(b) = 0$, $p \in P_1(\mathcal{B})$, and this implies that

$$0 = \int_{P_1(\mathcal{B})} \|\theta_p(b)\|_p^2\, d\beta_1(p) = \int_{P(\mathcal{B})} \|\theta_p(b)\|_p^2\, d\beta(p) = \int_{P(\mathcal{B})} p(b^* b)\, d\beta(p) =$$

$$= \int_{E(\mathcal{B})} f(b^* b)\, d\alpha(f) = f_0(b^* b) = \| b x_0 \|^2,$$

and this implies that $bx_0 = 0$.

We obviously have $U_1 \circ (\rho \mid \Gamma(\mathcal{B})) = U$, and

$$(U_1 \theta^1_{\mathcal{B}}(b_1) \mid U_1 \theta^1_{\mathcal{B}}(b_2)) = (b_1 x_0 \mid b_2 x_0) = f_0(b_2^* b_1) = \alpha(\lambda_{\mathcal{B}}(b_2^* b_1)) = \int_{E(\mathcal{B})} \lambda_{\mathcal{B}}(b_2^* b_1)\, d\alpha =$$

$$= \int_{P(\mathcal{B})} p(b_2^* b_1)\, d\beta(p) = \int_{P_1(\mathcal{B})} p(b_2^* b_1)\, d\beta_1(p) = (\theta^1_{\mathcal{B}}(b_1) \mid \theta^1_{\mathcal{B}}(b_2)), \quad b_1, b_2 \in \mathcal{B},$$

and this shows that U_1 is an isometry of $\Gamma_1(\mathcal{B})$ into H; it follows that U is an isometry of $\Gamma(\mathcal{B})$ into H.

From the above considerations it follows that we have the commutative diagram

$$\begin{tikzcd}
 & & \Gamma(\mathcal{B}) \arrow[d, "\rho"] \arrow[dr, "U"] & \\
\mathcal{C} \arrow[r, "\theta_{\mathcal{C}}"] & \Gamma(\mathcal{C}) \arrow[ur, "u"] \arrow[dr, "u_1"'] & \Gamma_1(\mathcal{B}) \arrow[r, "U_1"'] & H
\end{tikzcd}$$

and, therefore, the equalities

$$(U \circ u \circ \theta_{\mathcal{C}})(c) = \pi(c)x_0, \quad c \in \mathcal{C},$$

$$U_1 \circ u_1 = U \circ u .$$

It follows that $U \circ u$ is an isometry of $\Gamma(\mathcal{C})$ on a dense subset of H.

4. Let P be an arbitrary set, $(H_p)_{p\in P}$ a field of Hilbert spaces, defined on P, and $\mathcal{A}$ a σ- algebra of subsets of P. Let $\Gamma \subset \prod_{p\in P} H_p$ be a vector subspace, having the following property

$(*)\quad \xi = (\xi_p)_{p\in P} \in \Gamma \Rightarrow$

$\Rightarrow$ the function $p \mapsto \|\xi_p\|_p^2$ is bounded and $\mathcal{A}$-measurable.

From the polarization formula it then follows that for any pair $\xi = (\xi_p)_{p\in P}$, $\eta = (\eta_p)_{p\in P} \in \Gamma$, the function $p \mapsto (\xi_p | \eta_p)_p$, $p \in P$, is bounded and $\mathcal{A}$-measurable.

Let μ be a probability measure, defined on $\mathcal{A}$. We can define a scalar product, on Γ, by the formula $(\xi|\eta)_\mu = \int_P (\xi_p|\eta_p)_p \, d\mu(p)$, $\xi, \eta \in \Gamma$. Let $\Gamma_0 \subset \prod_{p\in P} H_p$ be the set of all vector fields $\xi = (\xi_p)_{p\in P}$, such that

$$\int_P^* \|\xi_p\|_p^2 \, d\mu(p) < +\infty .$$

One can easily show that Γ_0 is a vector subspace of $\prod_{p\in P} H_p$ and, also, that the mapping

$$q : \xi \mapsto \left(\int_P^* \|\xi_p\|_p^2 \, d\mu(p) \right)^{1/2}$$

is a semi-norm on Γ_0 ; we obviously have $\Gamma \subset \Gamma_0$.

Let $\Gamma^2(\mu)$ be the closure of Γ in Γ_0 with respect to the topology determined in Γ_0 by the semi-norm q : it is obvious that $\Gamma^2(\mu)$ is the set of all vector fields $\xi = (\xi_p)_{p\in P} \in \prod_{p\in P} H_p$, which have the property that there exists a sequence $(\xi_n)_{n\geq 0}$ of vector fields $\xi_n = (\xi_{np})_{p\in P} \in \Gamma$, such that

$$\lim_{n\to\infty} \int_P^* \|\xi_{np} - \xi_p\|_p^2 \, d\mu(p) = 0 .$$

Proposition 8. For any $\xi \in \Gamma^2(\mu)$ the function $p \mapsto \|\xi_p\|_p^2$ is μ- integrable.

Proof. Let $\xi = (\xi_p)_{p\in P} \in \Gamma^2(\mu)$ and $\xi_n = (\xi_{np})_{p\in P}$, $n \geq 0$ be a sequence of vector fields from Γ, such that

$$\lim_{n\to\infty} \int_P^* \|\xi_{np} - \xi_p\|_p^2 \, d\mu(p) = 0 .$$

It follows that there exists a subsequence of the sequence $(\xi_n)_{n\geq 0}$, for which we shall maintain the same notation, such that

$$\lim_{n\to\infty} \| \xi_{np} - \xi_p \|_p = 0, \quad \mu\text{-a.e.}$$

It follows that we have

$$\lim_{n\to\infty} \xi_{np} = \xi_p, \quad \mu\text{-a.e.},$$

and we consequently have the following equalities

$$\lim_{n\to\infty} \| \xi_{np} \|_p = \| \xi_p \|_p, \quad \mu\text{-a.e.},$$

$$\| \xi_{mp} - \xi_p \|_p = \lim_{n\to\infty} \| \xi_{mp} - \xi_{np} \|_p, \quad \mu\text{-a.e.}$$

It follows that the functions $p \longmapsto \| \xi_p \|_p$ and $p \longmapsto \| \xi_{mp} - \xi_p \|_p$ are $\mathcal{A}$ - measurable, for any $m \in \mathbb{N}$, after having modified them on a μ - null set from $\mathcal{A}$. We can therefore write the equality

$$q(\xi_m - \xi) = \left(\int_P \| \xi_{mp} - \xi_p \|_p^2 \, d\mu(p) \right)^{1/2},$$

whereas from the inequality

$$| \, \| \xi_{mp} \|_p - \| \xi_p \|_p \, | \leq \| \xi_{mp} - \xi_p \|_p$$

it immediately follows that

$$\int_P | \, \| \xi_{mp} \|_p - \| \xi_p \|_p \, |^2 \, d\mu(p) \leq \int_P \| \xi_{mp} - \xi_p \|_p^2 \, d\mu(p).$$

Hence we immediately infer that

$$\int_P \| \xi_p \|_p^2 \, d\mu(p) < +\infty,$$

and the proposition is proved.

By the formula

$$(\xi, \eta) \longmapsto (\xi \mid \eta) \overset{def}{=} \int_P (\xi_p \mid \eta_p)_p \, d\mu(p), \quad \xi, \eta \in \Gamma^2(\mu),$$

we correctly define a scalar product on $\Gamma^2(\mu)$, which is thus endowed with a structure of a pre-hilbertian space; let $\tilde{\Gamma}^2(\mu)$ be the associated separated pre-hilbertian space : it can be identified with set of all classes of vector fields, which belong to $\tilde{\Gamma}^2(\mu)$, modulo coincidence μ-a.e. .

Proposition 9. The spaces $\Gamma^2(\mu)$ and $\tilde{\Gamma}^2(\mu)$ are complete.

Proof. It will be sufficient to prove that $\Gamma^2(\mu)$ is complete. Let $(\xi_n)_{n\geq 0}$ be a fundamental sequence of vector fields $\xi_n = (\xi_{np})_{p\in P} \in \Gamma^2(\mu)$. By choosing a subsequence we can assume that

$$\int_P \| \xi_{n+1,p} - \xi_{n,p} \|_p^2 \, d\mu(p) < \frac{1}{2^{2n}}, \quad n \in \mathbb{N},$$

and, therefore, from the Schwarz inequality, we get

$$\int_P \|\xi_{n+1,p} - \xi_{n,p}\|_p \, d\mu(p) < \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

The Beppo Levi theorem now implies that the sequence $(\xi_{n,p})_{n\geq 0}$ is a fundamental sequence in H_p, for almost any $p \in P$ (with respect to the measure μ). Let us define $\xi_p = \lim_{n\to\infty} \xi_{n,p}$, for any $p \in P$ at which the sequence $(\xi_{n,p})_{n\geq 0}$ converges in the complete space H_p, and arbitrarily otherwise.
We then have

$$\lim_{n\to\infty} \int_P \|\xi_{n,p} - \xi_p\|_p^2 \, d\mu(p) = 0,$$

and, therefore, $\xi = (\xi_p)_{p\in P} \in \Gamma^2(\mu)$ and $\lim_{n\to\infty} \xi_n = \xi$, in the space $\Gamma^2(\mu)$.
The proposition is proved.

We shall say that the vector fields from $\Gamma^2(\mu)$ are <u>strongly square integrable vector fields.</u> We shall say that the space $\Gamma^2(\mu)$ is the $L^2(\mu)$ - completion of the space Γ whereas the Hilbert space $\tilde{\Gamma}^2(\mu)$ will be called the <u>direct Hilbert integral</u> of the field $(H_p)_{p\in P}$ with respect to the space Γ and the measure μ; we shall also write $\int_P^{\oplus} H_p \, d\mu(p)$, instead of $\tilde{\Gamma}^2(\mu)$, if the omission of the symbol Γ is not creating the danger of confusion.

We shall also say that a vector field $\xi = (\xi_p)_{p\in P}$ is a <u>weakly square integrable vector field</u> if the following conditions are satisfied.

a) there exists a μ-summable function $\varphi : P \to \mathbb{R}$, such that $\|\xi_p\|_p^2 \leq \varphi(p)$, μ-a.e. on P.

b) for any $\eta \in \Gamma^2(\mu)$ the function $p \mapsto (\xi_p | \eta_p)_{p\in P}, p \in P$, is μ-measurable.

From the Schwarz inequality it immediately follows that we have

$$|(\xi_p | \eta_p)_p| \leq \varphi(p)^{1/2} \|\eta_p\|_p, \quad \mu\text{-a.e.},$$

for any weakly square integrable vector field ξ, which satisfies the preceding conditions, and for any strongly square integrable vector field $\eta = (\eta_p)_{p\in P} \in \Gamma^2(\mu)$. We, therefore, infer that the function $p \mapsto (\xi_p | \eta_p)_p$, $p \in P$, is μ-integrable.

We shall say that weakly square integrable vector fields ξ', ξ'' are <u>weakly equivalent</u> if

$$\int_P (\xi'_p | \eta_p)_p \, d\mu(p) = \int_P (\xi''_p | \eta_p)_p \, d\mu(p),$$

for any $\eta = (\eta_p)_{p\in P} \in \Gamma^2(\mu)$.

Proposition 10. Any weakly square integrable vector field is weakly equivalent to a strongly square integrable vector field, which is unique up to a strong equivalence.

Proof. The uniqueness is an immediate consequence of the fact that if $\xi \in \Gamma^2(\mu)$, and if

$$\int_P (\xi_p | \eta_p)_p \, d\mu(p) = 0,$$

for any $\eta \in \Gamma^2(\mu)$, then $\xi_p = 0$, μ-a.e., on P.

Let us now consider a weakly square integrable vector field $\xi = (\xi_p)_{p\in P} \in \prod_{p\in P} H_p$.

Let $\varphi : P \to \mathbb{R}$ be a μ-integrable function, such that

$$\|\xi_p\|_p^2 \leq \varphi(p), \quad \mu\text{-a.e. on } P.$$

We shall define a linear mapping $\ell : \Gamma^2(\mu) \to \mathbb{C}$ by the formula

$$\ell(\eta) = \int_P (\eta_p | \xi_p)_p \, d\mu(p), \quad \eta = (\eta_p)_{p\in P} \in \Gamma^2(\mu).$$

From the Schwarz inequality we infer that

$$|\ell(\eta)| \leq \int_P |(\eta_p|\xi_p)_p| \, d\mu(p) \leq \left(\int_P \varphi \, d\mu\right)^{1/2} \left(\int_P \|\eta_p\|_p^2 \, d\mu(p)\right)^{1/2},$$

and, therefore, since $\Gamma^2(\mu)$ is complete, with the theorem of F. Riesz it follows that there exists a vector field $\xi' = (\xi'_p)_{p\in P} \in \Gamma^2(\mu)$, which is strongly square integrable and such that

$$\ell(\eta) = (\eta | \xi'), \quad \eta \in \Gamma^2(\mu).$$

It follows that ξ' is weakly equivalent to ξ, and the proposition is proved.

Remark. Examples show that, in general, $\Gamma^2(\mu)$ is not a $\mathcal{L}^\infty(\mu)$ - module. If Γ is a $\mathcal{L}^\infty(\mu)$ - module, then $\Gamma^2(\mu)$ has the same property, but the converse is not necessarily true.

Proposition 11. The vector space $\Gamma^2(\mu)$ is a $\mathcal{L}^\infty(\mu)$ - module if, and only if, the following property holds

(*) For any weakly square integrable vector field ξ there exists a strongly square integrable vector field ξ', such that, for any $\eta \in \Gamma^2(\mu)$, the equality

$$(\xi_p - \xi'_p | \eta_p)_p = 0, \quad \mu\text{-a.e. on } P,$$

holds.

Proof. Indeed, let us assume that $\Gamma^2(\mu)$ is a $\mathcal{L}^\infty(\mu)$-module. Let $\xi = (\xi_p)_{p\in P} \in \prod_{p\in P} H_p$ be a weakly square integrable vector field; let $\xi' = (\xi'_p)_{p\in P} \in \Gamma^2(\mu)$ be the strongly square integrable vector field which corresponds to ξ in virtue of proposition 10. We then have

$$(1) \qquad \int_P (\xi_p - \xi'_p | \eta_p)_p \, d\mu(p) = 0,$$

for any $\eta = (\eta_p)_p \in \Gamma^2(\mu)$. By fixing now $\eta \in \Gamma^2(\mu)$, we have $z\eta = (z(p)\eta_p)_{p\in P} \in$

$\in \Gamma^2(\mu)$, for any $z \in \mathcal{L}^\infty(\mu)$; hence, from (1), we get

$$(2) \qquad \int_P \overline{z(p)} \, (\xi_p - \xi'_p \mid \eta_p)_p \, d\mu(p) = 0,$$

for any $z \in \mathcal{L}^\infty(\mu)$; it follows that we have

$$(\xi_p - \xi'_p \mid \eta_p)_p = 0, \quad \mu\text{-a.e. on } P,$$

for any $\eta \in \Gamma^2(\mu)$.

Conversely, let us assume that property $(*)$ holds. It will be sufficient to prove that for any $A \in \mathcal{A}$ and any $\xi = (\xi_p)_{p \in P} \in \Gamma^2(\mu)$, we have $\chi_A \xi = (\chi_A(p)\xi_p)_{p\in P} \in \Gamma^2(\mu)$, where χ_A is the characteristic function of the set A. Obviously, $\chi_A \xi$ is a weakly square integrable vector field.

In virtue of proposition 10, there exists a strongly square integrable vector field $\xi' = (\xi'_p)_{p \in P}$, such that

$$(\chi_A(p)\xi_p \mid \eta_p)_p = (\xi'_p \mid \eta_p)_p, \ \mu\text{-a.e. on } P, \quad \text{for any } \eta \in \Gamma^2(\mu).$$

By making successively $\eta = \xi$ and $\eta = \xi'$, we infer that

$$\|\chi_A(p)\xi_p - \xi'_p\|_p^2 = \chi_A(p)(\chi_A(p)\xi_p - \xi'_p \mid \xi_p)_p - (\chi_A(p)\xi_p - \xi'_p \mid \xi'_p)_p = 0, \quad \mu\text{-a.e. on } P;$$

it follows that $\chi_A \xi \in \Gamma^2(\mu)$, and the proposition is proved.

5. By applying the construction we have just made to the spaces $(P(\mathscr{C}), A_0(P(\mathscr{C})), \gamma, \Gamma(\mathscr{C}))$, $(P(\mathcal{B}), A_0(P(\mathcal{B})), \beta, \Gamma(\mathcal{B}))$ and $(P_1(\mathcal{B}), A_0(P_1(\mathcal{B})), \beta_1, \Gamma_1(\mathcal{B}))$, we respectively obtain the complete pre-hilbertian spaces $\Gamma^2(\gamma)$, $\Gamma^2(\beta)$ and $\Gamma^2(\beta_1)$, and the Hilbert spaces $\tilde{\Gamma}^2(\gamma)$, $\tilde{\Gamma}^2(\beta)$ and $\tilde{\Gamma}^2(\beta_1)$. From what we have proved at § 3, we immediately infer that we have the following commutative diagram

$$\begin{tikzcd}
 & & \tilde{\Gamma}^2(\beta) \arrow[d, "\tilde{\rho}"] \arrow[dr, "\tilde{U}"] & \\
\mathscr{C} \arrow[r, "\tilde{\theta}_{\mathscr{C}}"] & \tilde{\Gamma}^2(\mathscr{C}) \arrow[ur, "\tilde{u}"] \arrow[dr, "\tilde{u}_1"'] & & H \\
 & & \tilde{\Gamma}^2(\beta_1) \arrow[ur, "\tilde{U}_1"'] &
\end{tikzcd}$$

in which $\tilde{u}$, $\tilde{u}_1$, $\tilde{U}$, $\tilde{U}_1$ and $\tilde{\rho}$ are isomorphisms of Hilbert spaces, and we have the following inequalities

$$(*) \qquad \begin{gathered} (\tilde{U} \circ \tilde{u} \circ \tilde{\theta}_{\mathscr{C}})(c) = \pi(c)x_0 , \qquad c \in \mathscr{C} \\ \tilde{U}_1 \circ \tilde{u}_1 = \tilde{U} \circ \tilde{u} . \end{gathered}$$

We obviously have $\tilde{U} = V^{-1}$, where V is the isomorphism from ([13] , theorem 3.1).

Proposition 12. $\Gamma^2(\gamma)$ is a $\mathcal{L}^\infty(\gamma)$-module.

Proof. It will be sufficient to prove that for any function $t \in \mathcal{L}^\infty(\gamma)$ and any $c \in \mathcal{C}$, we have

$$(t(p)\theta_p(c))_{p \in P(\mathcal{C})} \in \Gamma^2(\gamma).$$

Indeed, since the function

$$\varphi : p \longmapsto \|\theta_p(c_0) - t(p)\theta_p(c)\|_p^2 , \quad p \in P(\mathcal{C}),$$

where $c_0 \in \mathcal{C}$ is an arbitrary element, is γ-measurable, by taking into account the fact that we have

$$(\varphi \circ \pi^*)(q) = \|\theta_q(\pi(c_0)) - t(\pi^*(q))\theta_q(\pi(c))\|_q^2 ,$$

for any $q \in P_1(\mathcal{B})$, we infer that

$$\int_{P(\mathcal{C})} \|\theta_p(c_0) - t(p)\theta_p(c)\|_p^2 \, d\gamma(p) = \int_{P_1(\mathcal{B})} \|\theta_p(\pi(c_0)) - t(\pi^*(p))\theta_p(\pi(c))\|_p^2 \, d\beta_1(p) =$$

$$= \int_{P(\mathcal{B})} \|\theta_p(\pi(c_0)) - (t \circ \pi^*)(p)\theta_p(\pi(c))\|_p^2 \, d\beta(p),$$

where the function $t \circ \pi^*$ has been extended arbitrarily on $P_0(\mathcal{B})$. Since $\Gamma^2(\beta)$ is a $\mathcal{L}^\infty(\beta)$-module (see [16] , theorem 1.1 and proposition 11 above), we infer that

$$(t \circ \pi^*)\,\theta_{\mathcal{B}}(\pi(c)) \in \Gamma^2(\beta).$$

From the fact that the representation π is cyclic, we infer that there exists a sequence $(c_n)_{n \in \mathbb{N}}$ of elements of $\mathcal{C}$, such that

$$(2) \quad \lim_{n \to \infty} \int_{P(\mathcal{B})} \|\theta_p(\pi(c_n)) - (t \circ \pi^*)(p)\theta_p(\pi(c))\|_p^2 \, d\beta(p) = 0.$$

From formulas (1) and (2) it follows that

$$\lim_{n \to \infty} \int_{P(\mathcal{C})} \|\theta_p(c_n) - t(p)\theta_p(c)\|_p^2 \, d\gamma(p) = 0,$$

and, therefore, $t\,\theta_{\mathcal{C}}(c) \in \Gamma^2(\gamma)$. The proposition is proved.

Corollary. The system $\{P(\mathcal{C}), A_0(P(\mathcal{C})), \gamma, (H_p)_{p \in P(\mathcal{C})}, \Gamma^2(\gamma)\}$ is an integrable field of Hilbert spaces (in the sense of W. Wils).

Proof. It is an immediate consequence of the preceding results and of ([16] , theorem 1.2).

We resume the results already obtained in the following theorem, which is a generalization of theorem 3.1. from [13] .

Theorem 1. Let $\mathcal{C}$ be an arbitrary C^* - algebra. Then there exist

1^o. A measurable space (P, A_o), where P is a set and A_o is a σ-algebra of subsets of P;

2^o. A field $(H_p)_{p \in P}$ of Hilbert spaces;

3^o. A vector subspace $\Gamma \subset \prod_{p\in P} H_p$, consisting of vector fields $\xi = (\xi_p)_{p\in P}$, which have the property that the function $p \mapsto \|\xi_p\|_p$, $p \in P$, is bounded and A_o-measurable, for any $\xi \in \Gamma$;

4^o. A field $(\pi_p)_{p\in P}$ of non-degenerate irreducible representations $\pi_p : \mathcal{C} \to \mathcal{L}(H_p)$, $p \in P$, such that : for any cyclic representation $\pi : \mathcal{C} \to \mathcal{L}(H)$ there exists a probability measure γ, defined on A_o, and having the following properties :

a) the $L^2(\gamma)$ - completion $\Gamma^2(\gamma)$ of Γ determines an integrable field of Hilbert spaces (in the sense of W. Wils);

b) there exists an isometric isomorphism $V : H \to \int_P H_p \, d\gamma(p)$, such that if $x \in H$ and $V(x)$ is represented by the strongly square integrable vector field $(\xi_p)_{p\in P}$, then, for any $c \in \mathcal{C}$, the vector $V(\pi(c)x)$ is represented by the strongly square integrable vector field $(\pi_p(c)\xi_p)_{p\in P}$, and the equality

$$\|\pi(c)x\|^2 = \int_P \|\pi_p(c)\xi_p\|_p^2 \, d\gamma(p)$$

holds.

Proof. We shall take $P = P(\mathcal{C})$, $A_o = A_o(P(\mathcal{C}))$, the Hilbert spaces H_p, $p \in P$, being those we have considered above; $\Gamma = \Gamma(\mathcal{C})$, whereas the measure γ is that given by proposition 7. The property a) is a consequence of the corollary to proposition 12, whereas property b) is obtained by taking $V = (\tilde{U} \circ \tilde{u})^{-1}$ and by taking into account formulas (*) from § 5. The theorem is proved.

6. Let $g_o \in E(\mathcal{C})$ and μ be a probability Radon measure on $E_o(\mathcal{C})$, which represents g_o, i.e., $g_o = b(\mu)$. The following property is a variant to proposition 3.

Proposition 13. There exists a Baire measurable convex set $Q \subset E_o(\mathcal{C})$, which is contained in $E(\mathcal{C})$ and such that $g_o \in Q$ and $\mu(Q) = 1$.

Proof. Let $(u_j)_{j\in J}$ be an approximate unit of the C^* - algebra $\mathcal{C}$. We have

$$1 = \|g_o\| = \lim_j g_o(u_j),$$

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and, therefore, there exists a sequence $(u_{j_n})_{n\geq 1}$, such that $u_{j_n} \leq u_{j_{n+1}}$ and $g_0(u_{j_n}) \uparrow 1$. It follows that we have $\lambda_{\mathcal{C}}(u_{j_n}) \leq \lambda_{\mathcal{C}}(u_{j_{n+1}})$; let $\psi = \lim_{n\to\infty} \lambda_{\mathcal{C}}(u_{j_n})$. Then $0 \leq \psi \leq 1$ and ψ is a Baire measurable, affine function, defined on $E_0(\mathcal{C})$. We have $\psi(g_0) = 1$ and

$$\int_{E_0(\mathcal{C})} \psi\, d\mu = \lim_{n\to\infty} \int_{E_0(\mathcal{C})} \lambda_{\mathcal{C}}(u_{j_n})\, d\mu = \lim_{n\to\infty} g_0(u_{j_n}) = 1.$$

Let $Q = \{ f \in E_0(\mathcal{C}) \,;\, \psi(f) = 1 \}$. We obviously have that $g_0 \in Q$, Q is a Baire measurable, convex set and $\mu(Q) = 1$. If $f \in Q$, then

$$1 \geq \|f\| = \lim_j f(u_j) \geq \lim_{n\to\infty} f(u_{j_n}) = \psi(f) = 1,$$

and this shows that $\|f\| = 1$. The proposition is proved.

Corollary. The set $E(\mathcal{C})$ is μ-measurable and $\mu(E(\mathcal{C})) = 1$.

In particular, let us consider the Gelfand-Naimark-Segal representation $\pi_{g_0} : \mathcal{C} \to \mathcal{L}(H_{g_0})$, associated to $g_0 \in E(\mathcal{C})$, in the Hilbert space H_{g_0}. Let $\xi_0 \in H_{g_0}$ be the corresponding cyclic vector and denote by $\mathcal{B}$ the C^*- algebra generated in $\mathcal{L}(H_{g_0})$ by $\pi_{g_0}(\mathcal{C})$ and by a maximal Abelian von Neumann algebra $\mathfrak{Z} \subset (\pi_{g_0}(\mathcal{C}))'$. Let $f_0 \in E(\mathcal{B})$ be the state given by

$$f_0(b) = (b\xi_0 | \xi_0), \quad b \in \mathcal{B}.$$

We obviously have $g_0 = f_0 \circ \pi_{g_0} = \pi^*_{g_0}(f_0)$. Let α be the central measure on $E(\mathcal{B})$, associated to f_0.

In virtue of proposition 1, the direct image $(\pi^*_{g_0})_*(\alpha)$ is an orthogonal probability Radon measure on $E_0(\mathcal{C})$, which represents g_0.

If the approximate unit $\{u_i\}_{i\in I}$ of the C^*- algebra $\mathcal{C}$, used in the proof of proposition 3, coincides with the approximate unit used in the proof of proposition 13, and if the subsequences $\{u_{i_n}\}_{n\in\mathbb{N}}$ and $\{u_{j_n}\}_{n\in\mathbb{N}}$ coincide too, then, from the equality

$$\lambda_{\mathcal{C}}(c) \circ \pi^*_{g_0} = \lambda_{\mathcal{B}}(\pi_{g_0}(c)), \quad c \in \mathcal{C},$$

we immediately infer that we have $\psi \circ \pi^* = \varphi$, and, therefore, $Q_1 = (\pi^*)^{-1}(Q)$. Obviously, $E(\mathcal{C})$ is a $(\pi^*_{g_0})_*(\alpha)$ -measurable set, and we have

$$[(\pi^*_{g_0})_*(\alpha)](E(\mathcal{C})) = 1.$$

Proposition 14. a) Any bounded, continuous function $t : E(\mathcal{C}) \to \mathbb{C}$ is $(\pi^*_{g_0})_*(\alpha)$ -measurable;

b) For any bounded, continuous function $t : E(\mathcal{C}) \to \mathbb{C}$ the function $t \,|\, P(\mathcal{C})$ is γ-measurable and we have

$$\int_{E(\mathcal{C})} t\, d((\pi^*_{g_0})_*(\alpha)) = \int_{P(\mathcal{C})} t\, d\gamma.$$

Proof. a) Since the measure α is regular, there exists an increasing sequence $(K_n)_{n\geq 0}$ of compact sets $K_n \subset E(\mathscr{C})$, such that $[(\pi^*_{g_0})_*(\alpha)](K_n)\uparrow 1$. Let $K = \bigcup_{n\geq 0} K_n$ and $t : E_0(\mathscr{C}) \to \mathbb{C}$ be the function defined by

$$t_n(f) = \begin{cases} t(f), & f\in K_n \\ 0, & f\in E_0(\mathscr{C})\setminus K_n. \end{cases}$$

The functions t_n are Borel measurable and we have

$$\lim_{n\to\infty} t_n = t\chi_K.$$

It follows that t is $(\pi^*_{g_0})_*(\alpha)$ -measurable, because $[(\pi^*_{g_0})_*(\alpha)](E(\mathscr{C})\setminus K)=0$.

b) Since the function t is $(\pi^*_{g_0})_*(\alpha)$ -measurable, whereas the set $E(\mathscr{C})$ is $(\pi^*_{g_0})_*(\alpha)$- measurable, it follows that the left hand integral the equality makes sense, and we have

$$\int_{E(\mathscr{C})} t\, d[(\pi^*_{g_0})_*(\alpha)] = \int_{Q} t\, d[(\pi^*_{g_0})_*(\alpha)],$$

as a consequence of proposition 13. Since the set $Q\subset E_0(\mathscr{C})$ is Baire measurable, it follows that there exists an increasing sequence of compact sets $L_n \subset Q,\ n\in\mathbb{N}$, which are Baire measurable, and such that $[(\pi^*_{g_0})_*(\alpha)](L_n)\uparrow 1$. Since the functions $t\,|\,L_n$, $n\in\mathbb{N}$, are continuous, the functions $s_n : E_0(\mathscr{C})\to\mathbb{C}, n\in\mathbb{N}$, given by

$$s_n(f) = \begin{cases} t(f), & f\in L_n \\ 0, & f\in E_0(\mathscr{C})\setminus L_n \end{cases}$$

are Baire measurable and we have

$$\int_{P(\mathscr{C})} s_n\, d\gamma = \int_{E_0(\mathscr{C})} s_n\, d[(\pi^*_{g_0})_*(\alpha)] = \int_{E(\mathscr{C})} s_n\, d[(\pi^*_{g_0})_*(\alpha)],$$

where we have taken into account proposition 6 and the corollary to proposition 13.

By denoting $L = \bigcup_{n\geq 0} L_n$, we have

$$\lim_{n\to\infty} s_n = t\chi_L,$$

and $Q\setminus L$ is a Baire measurable set, for which we have $[(\pi^*_{g_0})_*(\alpha)](Q\setminus L)=0$. It follows that

$$\gamma((Q\setminus L)\cap P(\mathscr{C}))=0,$$

and, therefore, we have

$$\int_{E(\mathscr{C})} t\, d[(\pi^*_{g_0})_*(\alpha)] = \lim_{n\to\infty}\int_{E_0(\mathscr{C})} s_n\, d[(\pi^*_{g_0})_*(\alpha)] =$$

$$= \lim_{n\to\infty}\int_{P(\mathscr{C})} s_n\, d\gamma = \int_{P(\mathscr{C})} t\chi_L\, d\gamma = \int_{P(\mathscr{C})} t\, d\gamma,$$

and the proposition is proved.

Proposition 15. For any bounded, continuous, affine function $t : E(\mathscr{C}) \to \mathbb{C}$ we have

$$t(g_0) = \int_{E(\mathscr{C})} t \, d\left((\pi^*_{g_0})_*(\alpha)\right).$$

Proof. From the fact that $g_0 \in E(\mathscr{C})$, and this implies that t is defined and continuous at g_0, it follows that for any $\varepsilon > 0$ there exists a finite subset $\{c_1, c_2, -, c_n\} \subset \mathscr{C}$, such that $f \in E(\mathscr{C})$, and $|f(c_i) - g_0(c_i)| < 1$, $i = 1, 2, \ldots, n$, imply

$$|t(f) - t(g_0)| < \varepsilon.$$

Let $\{A_1, A_2, -, A_m\}$ be a finite partition of $E_0(\mathscr{C})$, consisting of Borel measurable subsets $A_i \subset E_0(\mathscr{C})$, such that

$$f', f'' \in A_i \implies |f'(c_j) - f''(c_j)| < 1, \quad j = 1, 2, \ldots, n,$$

for any $i = 1, 2, \ldots, m$. If necessary, we can refine the partition, such that, with an arbitrary selection of the points $f_i \in A_i \cap Q$, $i = 1, 2, \ldots, m$, we have

$$\left| \int_{E(\mathscr{C})} t \, d[(\pi^*_{g_0})_*(\alpha)] - \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) t(f_i) \right| < \varepsilon$$

Let ν be the Radon measure on $E_0(\mathscr{C})$, given by

$$\nu = \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) \varepsilon_{f_i} ;$$

it is obvious that the bary center $b(\nu)$ of ν is the point $b(\nu) = \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) f_i \in$ $\in Q$, and, therefore,

$$|b(\nu)(c_j) - g_0(c_j)| = \left| \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) f_i(c_j) - \int_{E_0(\mathscr{C})} \lambda_{\mathscr{C}}(c_j) \, d[(\pi^*_{g_0})_*(\alpha)] \right| =$$

$$= \left| \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) f_i(c_j) - \sum_{i=1}^{m} \int_{A_i} \lambda_{\mathscr{C}}(c_i) \, d[(\pi^*_{g_0})_*(\alpha)] \right| =$$

$$= \left| \sum_{i=1}^{m} \int_{A_i} (f_i(c_j) - f(c_j)) \, d[(\pi^*_{g_0})_*(\alpha)](f) \right| \leq$$

$$\leq \sum_{i=1}^{m} \int_{A_i} |f_i(c_j) - f(c_j)| \, d[(\pi^*_{g_0})_*(\alpha)](f) < 1,$$

for any $j = 1, 2, -, n$; consequently, we have

$$|t(b(\nu)) - t(g_0)| < \varepsilon.$$

Since the function t is affine, we have

$$t(b(\nu)) = \sum_{i=1}^{m} [(\pi^*_{g_0})_*(\alpha)](A_i) t(f_i)$$

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and, therefore,

$$\left| \int_{E(\mathscr{C})} t \, d[(\pi^*_{g_0})_*(\alpha)] - t(g_0) \right| < 2\varepsilon .$$

The proposition is proved.

Corollary. For any bounded, affine function $t : E_0(\mathscr{C}) \to \mathbb{C}$, whose restriction to $E(\mathscr{C})$ is continuous, we have

$$t(g_0) = \int_{P(\mathscr{C})} t \, d\gamma .$$

Proof. It is a immediate consequence of propositions 14 and 15.

7. Let us now consider, for any $f \in E_0(\mathscr{C})$, the associated Hilbert space H_f, the corresponding representation $\pi_f : \mathscr{C} \to \mathscr{L}(H_f)$, the associated cyclic vector $\xi_f \in H_f$, such that $\|\xi_f\|^2 = \|f\|$, and the canonical mapping $\theta_f : \mathscr{C} \to H_f$, which are all obtained by the Gelfand-Naimark-Segal construction. Let $\theta_0 : \mathscr{C} \to \prod_{f \in E_0(\mathscr{C})} H_f$ be the linear mapping given by

$$\theta_0(c) = (\theta_f(c))_{f \in E_0(\mathscr{C})}, \quad c \in \mathscr{C},$$

and $\Gamma_0(\mathscr{C}) \subset \prod_{f \in E_0(\mathscr{C})} H_f$, the vector subspace im θ_0. If we consider in $E_0(\mathscr{C})$ the σ-algebra A of all Baire measurable subsets of $E_0(\mathscr{C})$, it is obvious that condition (*) from §4 is satisfied. Let $\Gamma^2_\mu(\mathscr{C})$ be the vector space of all strongly square integrable vector fields, with respect to a probability regular Borel measure μ, which represents $f_0 \in E(\mathscr{C})$. We have the following generalization of a theorem of E. G. Effros (see [4], th. 4 ; [11], th. 8).

Proposition 16. The system $\{(H_f)_{f \in E_0(\mathscr{C})}; \Gamma^2_\mu(\mathscr{C})\}$ is an integrable field of Hilbert spaces (in the sense of W. Wils) if, and only if, the measure μ is orthogonal.

Proof. We shall define a mapping

$$u : \Gamma_0(\mathscr{C}) \to H_{f_0}$$

by the formula $u(\theta_0(c)) = \theta_{f_0}(c), \; c \in \mathscr{C}$. We have

$$\|\theta_{f_0}(c)\|^2_{f_0} = f_0(c^*c) = \lambda_{\mathscr{C}}(c^*c)(f_0) = \int_{E_0(\mathscr{C})} \lambda_{\mathscr{C}}(c^*c)(f) \, d\mu(f) =$$

$$= \int_{E_0(\mathscr{C})} f(c^*c) \, d\mu(f) = \int_{E_0(\mathscr{C})} \|\theta_f(c)\|^2_f \, d\mu(f) = \|\theta_0(c)\|^2,$$

for any $c \in \mathscr{C}$ (the norm in $\Gamma_0(\mathscr{C})$ is calculated as in §4). It follows that u induces an isometric isomorphism

$$\tilde{u} : \tilde{\Gamma}^2_\mu(\mathscr{C}) \to H_{f_0}$$

of Hilbert spaces.

Let us assume that $\{(H_f)_{f\in E_0(\mathcal{C})}; \Gamma^2_\mu(\mathcal{C})\}$ is an integrable field of Hilbert spaces, in the sense of W. Wils (see [16], def. 1.1, and th. 1.2.). Then, for any Baire measurable function $z: E_0(\mathcal{C}) \to \mathbb{C}$ the multiplication operator $T_z: \Gamma^2_\mu(\mathcal{C}) \to \Gamma^2_\mu(\mathcal{C})$ is defined. It induces a continuous, linear operator $\tilde{T}_z \in \mathcal{L}(\tilde{\Gamma}^2_\mu(\mathcal{C}))$. Let $S_z \in \mathcal{L}(H_{f_0})$ be the operator defined by

$$S_z = \tilde{u}\tilde{T}_z\tilde{u}^{-1}.$$

For any $c_0, c \in \mathcal{C}$ we have

$$S_z \pi_{f_0}(c)\theta_{f_0}(c_0) = (\tilde{u}\tilde{T}_z\tilde{u}^{-1})\pi_{f_0}(c)\theta_{f_0}(c_0) = (\tilde{u}\tilde{T}_z\tilde{u}^{-1})\theta_{f_0}(cc_0) =$$

$$= \tilde{u}[(z(f)\theta_f(cc_0))_{f\in E_0(\mathcal{C})}]^{\sim} = \tilde{u}[(\pi_f(c)z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})}]^{\sim} =$$

$$= \pi_{f_0}(c)(\tilde{u}\tilde{T}_z\tilde{u}^{-1})(\theta_{f_0}(c)) = \pi_{f_0}(c)S_z\theta_{f_0}(c),$$

and this implies that

$$S_z\pi_{f_0}(c) = \pi_{f_0}(c)S_z, \quad c \in \mathcal{C};$$

consequently, we have $S_z \in (\pi_{f_0}(\mathcal{C}))'$. On the other hand, we have

$$(S_z\theta_{f_0}(c_1) \mid \theta_{f_0}(c_2)) = \int_{E_0(\mathcal{C})} z(f)(\theta_f(c_1) \mid \theta_f(c_2))_f \, d\mu(f) =$$

$$= \int_{E_0(\mathcal{C})} z(f) f(c_2^* c_1)\, d\mu(f) = (K_\mu(z)\theta_{f_0}(c_1) \mid \theta_{f_0}(c_2)),$$

for any $c_1, c_2 \in \mathcal{C}$; this shows that we have $S_z = K_\mu(z), z \in \mathcal{L}^\infty(\mu)$. Since the mapping $z \mapsto S_z$ obviously is a $*$ - homomorphism of $\mathcal{L}^\infty(\mu)$ into $\mathcal{L}(H_{f_0})$, from Tomita's theorem we infer that μ is orthogonal (see [11], th. 7). Conversely, if μ is orthogonal, then the system $\{(H_f)_{f\in E_0(\mathcal{C})}; \Gamma^2_\mu(\mathcal{C})\}$ is an integrable field of Hilbert spaces (in the sense of W. Wils). Indeed, it will be sufficient to prove that for any $\theta_0(c_0) = (\theta_f(c_0))_{f\in E_0(\mathcal{C})} \in \Gamma_0(\mathcal{C})$, where $c_0 \in \mathcal{C}$, any $z \in \mathcal{L}^\infty(\mu)$ and any $\varepsilon > 0$, there exists a $c \in \mathcal{C}$, such that we have

$$\| \theta_0(c) - (z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})} \| < \varepsilon.$$

We have

$$\|\theta_0(c) - (z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})}\|^2 = \int_{E_0(\mathcal{C})} \|\theta_f(c) - z(f)\theta_f(c_0)\|_f^2 \, d\mu(f) =$$

$$= \int_{E_0(\mathcal{C})} \|\theta_f(c)\|_f^2 \, d\mu(f) - \int_{E_0(\mathcal{C})} z(f)(\theta_f(c_0) \mid \theta_f(c))_f \, d\mu(f) -$$

$$- \int_{E_0(\mathcal{C})} \overline{z(f)}(\theta_f(c) \mid \theta_f(c_0))_f \, d\mu(f) + \int_{E_0(\mathcal{C})} |z(f)|^2 \|\theta_f(c_0)\|_f^2 \, d\mu(f) +$$

$$+ (K^\mu_{|z|^2}\theta_{f_0}(c_0) \mid \theta_{f_0}(c)).$$

By taking into account the fact that the measure μ is orthogonal, with Tomita's theorem we infer that

$$K^{\mu}_{|z|^2} = K^{\mu}_{z}(K^{\mu}_{z})^* = (K^{\mu}_{z})^* K^{\mu}_{z},$$

and, therefore, from the preceding equality, it follows that

$$\| \theta_0(c) - (z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})} \| = \| \theta_{f_0}(c) - K^{\mu}_{z}\theta_{f_0}(c_0) \| ;$$

since $K^{\mu}_{z}\theta_{f_0}(c_0) \in H_{f_0}$, from the fact that $\theta_{f_0}(\mathcal{C})$ is a dense subset of H_{f_0}, it follows that there exists $c \in \mathcal{C}$, such that

$$\| \theta_{f_0}(c) - (z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})} \| < \varepsilon ,$$

and, therefore, $(z(f)\theta_f(c_0))_{f\in E_0(\mathcal{C})} \in \Gamma^2_{\mu}(\mathcal{C})$. The proposition is proved.

Corollary. If μ is an orthogonal, regular, Borel probability measure on $E_0(\mathcal{C})$, then $A_0(E_0(\mathcal{C}))$ is a dense vector subspace of $L^2(\mu)$.

Proof. Here $A_0(E_0(\mathcal{C}))$ denotes the vector subspace of $C(E_0(\mathcal{C}))$, consisting of all affine, continuous, complex functions, defined on $E_0(\mathcal{C})$ and vanishing at 0. It is obvious that for any $c \in \mathcal{C}$ and $\varphi \in \mathcal{L}^{\infty}(\mu)$ we have

$$\int_{E_0(\mathcal{C})} |\lambda_{\mathcal{C}}(c) - \varphi|^2 d\mu = \int_{E_0(\mathcal{C})} |\lambda_{\mathcal{C}}(c)|^2 d\mu - \int_{E_0(\mathcal{C})} \lambda_{\mathcal{C}}(c)\overline{\varphi}\, d\mu -$$

$$- \int_{E_0(\mathcal{C})} \varphi \overline{\lambda_{\mathcal{C}}(c)}\, d\mu + \int_{E_0(\mathcal{C})} |\varphi|^2 d\mu \leq \int_{E_0(\mathcal{C})} \lambda_{\mathcal{C}}(c^*c)\, d\mu -$$

$$- \int_{E_0(\mathcal{C})} \lambda_{\mathcal{C}}(c)\overline{\varphi}\, d\mu - \int_{E_0(\mathcal{C})} \varphi \overline{\lambda_{\mathcal{C}}(c)}\, d\mu + \int_{E_0(\mathcal{C})} |\varphi|^2 d\mu =$$

$$= f_0(c^*c) - ((K^{\mu}_{\varphi})^* \theta_{f_0}(c) \mid \xi_{f_0}) - (K^{\mu}_{\varphi}\xi_{f_0} \mid \theta_{f_0}(c)) +$$

$$+ (K^{\mu}_{|\varphi|^2}\xi_{f_0} \mid \xi_{f_0}) = \| \theta_{f_0}(c)\|^2 - (\theta_{f_0}(c) \mid K^{\mu}_{\varphi}\xi_{f_0}) -$$

$$- (K^{\mu}_{\varphi}\xi_{f_0} \mid \theta_{f_0}(c)) + \| K^{\mu}_{\varphi}\xi_{f_0}\|^2 = \| \theta_{f_0}(c) - K^{\mu}_{\varphi}\xi_{f_0}\|^2 ,$$

where we have used the Schwarz inequality under the form

$$|\lambda_{\mathscr{C}}(c)|^2(f) = |f(c)|^2 \leq f(c^*c)\,\|f\| \leq f(c^*c) = \lambda_{\mathscr{C}}(c^*c)(f),$$

for any $f \in E_0(\mathscr{C})$, and the fact that the measure μ is orthogonal. The corollary is an immediate consequence of the preceding inequality.

Remark. The preceding corollary is a generalization, to the possibly nonseparable case, of the corollary to theorem 8 from [11].

8. In this section, as an application of theorem 1, we shall give a generalization of the following well-known theorem of P. Lévy.

Let $(F_n)_{n\geq 0}$ be a sequence of distribution functions and $(\varphi_n)_{n\geq 0}$ the corresponding sequence of characteristic functions. If $\lim_{n\to\infty} \varphi_n(s) = \varphi(s)$, for any $s \in \mathbb{R}$, where φ is the characteristic function of the distribution function F, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} f\,dF_n = \int_{\mathbb{R}} f\,dF,$$

for any bounded, continuous function $f : \mathbb{R} \to \mathbb{C}$ (see [7], § 12.2).

The preceding theorem belongs to Harmonic Analysis, since the characteristic functions φ_n are the Stieltjes-Fourier transforms of the Radon measures μ_n, which correspond to the distribution functions F_n, $n \in \mathbb{N}$. Several mathematicians have successively extended this theorem into the frame of Abstract Harmonic Analysis. Thus, U. Grenander, in [5], extended this theorem to the case of locally compact groups satisfying the second countability axiom; R. M. Loynes, in [8], to the case of locally compact groups which satisfy the first countability axiom; P. Martin - Löf, in [9], for arbitrary locally compact groups (see also H. Heyer [6]). In [6] and [9] the presented proofs make use of the Tomita disintegration theory, as it was exposed in [10], but, since this theory was shown not to be correct (see J. L. Taylor [12]), the extensions given in [6] and [9] remained under doubt.

The first correct proof, in which the irreducible disintegration theory apparently is not used, was given by Ch. A. Akemann and M. E. Walter (see [1], proposition 6). In this article P. Lévy's theorem is extended and framed into the theory of the W^*-algebra associated to an arbitrary locally compact group.

In what follows we shall further extend P. Lévy's theorem by framing it into a compacity theorem and by making an explicit use of the Choquet-Bishop-de Leeuw-Wils decomposition theory.

I gratefully acknowledge the useful discusions I had with H. Heyer, who introduced me into the problem and informed me about the pertinent bibliography.

a) Let G be an arbitrary, topological, locally compact group, and $U : G \to \mathscr{L}(H)$ a unitary, continuous representation of G, where H is an arbitrary Hilbert space.

For any $\xi, \eta \in H$, the complex function

$$g \mapsto (U_g \xi | \eta), \quad g \in G,$$

is continous and bounded on G.

Let $M^1(G)$ be the complex vector space of all finite, complex, Radon measures, defined on G. For any $\mu \in M^1(G)$, the integral

$$\int_G (U_g \xi | \eta)\, d\mu(g), \quad \xi, \eta \in H,$$

exists, and there exists a unique, continuous, linear operator $\hat{\mu}(U) \in \mathcal{L}(H)$, such that

$$(\hat{\mu}(U)\xi | \eta) = \int_G (U_g \xi | \eta)\, d\mu(g), \quad \xi, \eta \in H.$$

The operator $\hat{\mu}(U)$ is called the Fourier - Stieltjes transform of the measure μ, corresponding to the representation U (see [3], § 13.3; [6], p. 147), whereas the mapping $\mu \mapsto \hat{\mu}(U)$ is a $*$ - representation of the involutive algebra $M^1(G)$ into $\mathcal{L}(H)$ (see [3], proposition 13.3.1.).

It is well known that in $M^1(G)$ the following topologies can be considered : α) the vague topology : since $M^1(G)$ is the dual of the normed space K (G) of all the continuous, complex functions, defined on G and having compact supports, in which the norm is given by

$$f \mapsto \|f\|_\infty = \sup_{g \in G} |f(g)|, \quad f \in K(G),$$

the vague topology is the topology $\sigma(M^1(G); K(G))$.

β) the $*$-weak topology : $M^1(G)$ is, at the same time, the dual of the Banach space $C_0(G)$ of all continuous, complex functions, defined on G and vanishing at infinity, endowed with the same norm as above. The $*$-weak topology is the topology $\sigma(M^1(G); C_0(G))$ and it is obviously stronger than the vague topology.

The two topologies coincide on the norm bounded subsets of $M^1(G)$, but a vaguely compact set can be unbounded for the norm.

γ) the narrow topology : any measure from $M^1(G)$ can be extended, with the help of regularity, as a Radon measure on the space $C^b(G) = C(\beta G)$ of all bounded, continuous complex functions, defined on G, space which can be canonically identified with the space of all continuous complex functions, defined on the Stone-Čech compactification βG of G. The mentioned extension yields an isometric imbedding of $M^1(G)$ into $C(\beta G)^*$, and the narrow topology is the topology induced by the topology $\sigma(C(\beta G)^*; C(\beta G))$. Obviously, the narrow topology is stronger than the $*$-weak topology (see [2], ch. IX, § 5.3).

δ) The Fourier topology. Let $\mathcal{F}(G) \subset C^b(G)$ be the complex vector space of all functions of the form

$$g \mapsto \sum_{i=1}^{n} (U_g^{(i)} \xi_i | \eta_i),$$

where $U^{(i)} : G \longrightarrow \mathcal{L}(H_i)$ are irreducible, continuous, unitary representations, and $\xi_i, \eta_i \in H_i, 1 \leq i \leq n, n \in \mathbb{N}$. The Fourier topology on $M^1(G)$ is the topology $\sigma(M^1(G); \mathcal{F}(G))$. It is obvious that the Fourier topology is weaker than the narrow topology but, nevertheless, it is a Hausdorff topology.

b) Let P (G) be the set of all continuous functions of positive type, defined on G; let $P_1(G) = \{\varphi \in P(G);\ \varphi(e) = 1\}$. It is known that there exists an affine bijection $\tau : P(G) \to (C^*(G))_+^*$ between P(G) and the set $(C^*(G))_+^*$ of all positive, continuous linear forms, defined on the C^*- algebra C^* (G) of the group G; the bijection τ is given by the formula

$$\tau(\varphi)(\chi(x)) = \int_G \varphi x \, d\mu,$$

where μ is a left-invariant Haar measure on G, $x \in L^1(G)$, $\varphi \in P(G)$ and $\chi : L^1(G) \to C^*(G)$ is the canonical injection (see [3], §§ 2.7, 13.4 and 13.9). Moreover, we have $\|\tau(\varphi)\| = \varphi(e) = \|\varphi\|_\infty$, for any $\varphi \in P(G)$ (see [3], §§ 2.7.5 and 13.4.3).

c) Let $\pi : G \to \mathcal{L}(H)$ be a cyclic, continuous, unitary representation of G and $\rho : C^*(G) \to \mathcal{L}(H)$ the corresponding cyclic representation of the C^*-algebra $C^*(G)$ (see [3], § 1 3.9). Let $\xi \in H$, $\|\xi\| = 1$, be a ρ-cyclic vector and $g_0 \in E(C^*(G))$ the corresponding state, given by

$$g_0(c) = (\rho(c)\xi | \xi), \quad c \in C^*(G).$$

For any maximal, commutative von Neumann algebra $\mathfrak{Z} \subset (\rho(C^*(G)))'$ we shall consider the C^*-algebra $\mathcal{B}$, generated by $\rho(C^*(G))$ and $\mathfrak{Z}$, and the corresponding measures α, β, δ, constructed as in the preceding sections.

The bijection τ induces the bijection $\tau | P_0(G)$, from the set $P_0(G)$ of all pure continuous functions of positive type, defined on G and equal to 1 at $e \in G$, to the set $P(C^*(G))$ of all pure states of the C^*-algebra C^* (G). Let A(G) be the σ-algebra of subsets of $P_0(G)$, which is the reciprocal image through $\tau | P_0(G)$ of the σ-algebra $A_0(P(C^*(G)))$, constructed as in § 2 for the C^*-algebra $\mathcal{C} = C^*(G)$.

<u>Theorem 2.</u> For any $\varphi_0 \in P_1(G)$ there exists a probability measure ν, defined on the σ-algebra A(G) of subsets of $P_0(G)$, such that

$$\hat{\nu}(\varphi_0) = \int_{P_0(G)} \hat{\nu}(\varphi) \, d\nu(\varphi),$$

for any $\nu \in M^1(G)$. In particular, we have

$$\varphi_0(g) = \int_{P_0(G)} \varphi(g) \, d\nu(\varphi),$$

for any $g \in G$.

Proof. From the Raikov theorem (see [3], theorem 13.5.2) it follows that $\tau | P_1(G)$ is a homeomorphism from the space $P_1(G)$, endowed with the compact convergence topology, onto the space $E(C^*(G))$, endowed with the topology induced by $\sigma((C^*(G))^* ; C^*(G))$. For any $\nu \in M^1(G)$ let us consider the function $\tilde{\nu} : (C^*(G))^*_+ \to \mathbb{C}$, given by

$$\tilde{\nu}(f) = \nu(\tau^{-1}(f)), \quad f \in (C^*(G))^*_+ .$$

Obviously, $\tilde{\nu} | E_0(C^*(G))$ is an affine, bounded function, whose restriction to $E(C^*(G))$ is continuous, in virtue of the above mentioned theorem of D. A. Raĭkov and of the finiteness and the regularity of the measure ν. From the corollary to proposition 15 we infer that

$$(*) \qquad \tilde{\nu}(g_0) = \int_{P(C^*(G))} \tilde{\nu}(g)\, d\gamma(g),$$

where $g_0 = \tau(\varphi_0) \in E(C^*(G))$. Let $\mathcal{J}$ be the direct image through $\tau^{-1} | P(C^*(G))$ of the measure γ. It is obvious that for the measure $\mathcal{J}$, thus defined, the first equality from the statement of the theorem holds. The second equality can be obtained by taking $\nu = \varepsilon_g$, the Dirac measure at $g \in G$. The theorem is proved.

The preceding theorem is a generalization to the possibly non-separable case, of the proposition 13.6.8 from [3].

d) Let us now remark that to any measure $\nu \in M^1(G)$ we can injectively associate a function $\hat{\nu} : P_0(G) \to \mathbb{C}$ by the formula

$$\hat{\nu}(\varphi) = \nu(\varphi), \qquad \varphi \in P_0(G);$$

we use here the finiteness of the measure ν. It is obvious that the function $\hat{\nu}$ which we thus associate to the measure ν, is bounded and $\mathcal{J}$-measurable, as a consequence of proposition 14, b) and of the definition of the measure $\mathcal{J}$.

On the other hand, it is easy to see that the mapping $M^1(G) \ni \nu \mapsto \hat{\nu} \in \mathbb{C}^{P_0(G)}$ is a homeomorphism from $M^1(G)$, endowed with the Fourier topology, into $\mathbb{C}^{P_0(G)}$, endowed with the simple convergence topology (i.e., the Tikhonov topology).

Theorem 3. The subsets of $M^1_+(G)$, which are metrizable and compact for the Fourier topology, are compact for the narrow topology.

Proof. Let $M \subset M^1_+(G)$ be a compact, metrizable subset (with respect to the Fourier topology). Here $M^1_+(G)$ is the set of all positive, finite Radon measures on G.

a) M is norm-bounded. Indeed, let $1 \in P_0(G)$ be the constant function, equal to 1 on G. We have $\hat{\nu}(1) = \|\nu\|$, for any $\nu \in M^1_+(G)$; since the mapping

$\nu \mapsto \hat{\nu}(1)$ is continuous, the set $\{\|\nu\|;\ \nu\in M\}$ is compact in $\mathbb{R}$.

b) Any ultrafilter $\mathfrak{U}$ on M converges for the topology induced on M by the weak topology $\sigma(M^1(G);\ P(G))$. Indeed, let $\mathfrak{U}$ be an ultrafilter on M. Since M is compact for the Fourier topology, there exists

(1) $$\lim \mathfrak{U} = \nu_0 \in M.$$

Let us now show that for any $\varphi \in P(G)$, we have

$$\lim_{\nu\in\mathfrak{U}} \hat{\nu}(\varphi) = \hat{\nu}_0(\varphi).$$

(here we have denoted by $\hat{\nu}(\varphi)$ the integral of the function φ with respect to the finite Radon measure ν). Indeed, to prove this, we must show that for any $\varepsilon > 0$ there exists a set $U_\varepsilon \in \mathfrak{U}$, such that

$$|\hat{\nu}(\varphi) - \hat{\nu}_0(\varphi)| < \varepsilon,\quad \nu \in U_\varepsilon.$$

If this be not true, there would exist a $\varphi_0 \in P(G)$, and an $\varepsilon_0 > 0$, such that, for any $U\in\mathfrak{U}$ there exists a $\nu_U \in U$, for which

$$|\hat{\nu}_U(\varphi_0) - \hat{\nu}_0(\varphi_0)| \geq \varepsilon_0.$$

Let $\{V_n\}_{n\geq 1}$ be a countable basis of neighbourhoods of ν_0 for the Fourier topology ($V_n \subset M$). From (1) we infer that $V_n \in \mathfrak{U}$, $n \geq 1$. Consequently, for any $n \in \mathbb{N}^*$, there exists a $\nu_n \in V_n$, such that

(2) $$|\hat{\nu}_n(\varphi_0) - \hat{\nu}_0(\varphi_0)| \geq \varepsilon_0,$$

and, obviously, we have $\lim_{n\to\infty} \nu_n = \nu_0$, for the Fourier topology.

Let $g_0 = \tau(\varphi_0) \in (C^*(G))^*_+$ and ϑ be the positive measure, defined on the σ-algebra A(G) of subsets of $P_0(G)$, given by the theorem 2. Let $\tilde{A}(G)$ be the complete σ-algebra of subsets of $P_0(G)$, generated by A(G) with respect to ϑ. Then all the functions $\hat{\nu}: P_0(G)\to\mathbb{C}$, $\nu\in M^1(G)$, are $\tilde{A}(G)$-measurable, and we have

(3) $$\hat{\nu}_n(\varphi_0) = \int_{P_0(G)} \hat{\nu}_n(\varphi)\,d\vartheta(\varphi),\qquad n\in\mathbb{N}.$$

By taking into account the fact that

$$\sup_{n\in\mathbb{N}}\ \sup_{\varphi\in P_0(G)} |\hat{\nu}_n(\varphi)| < +\infty,$$

from the Lebesgue dominated convergence theorem and from theorem 2 we infer that

$$\lim_{n\to\infty} \hat{\nu}_n(\varphi_0) = \int_{P_0(G)} \hat{\nu}_0(\varphi)\,d\vartheta(\varphi) = \hat{\nu}_0(\varphi_0),$$

and this contradicts inequality (2).

c) If $\varphi, \psi \in K(G)$, then $\varphi * \psi$ is a linear combination of four continuous functions of positive type (see [3], corollary 13.6.5). From we have just proved in b), it follows that for any ultrafilter $\mathfrak{U}$ on M, which converges for the Fourier topology to $\nu_0 \in M$, we have

$$\lim_{\nu \in \mathfrak{U}} \hat{\nu}(\varphi * \psi) = \hat{\nu}_0(\varphi * \psi).$$

d) Let now $\varphi \in K(G)$ and $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit in K(G) (with respect to the convolution). We have

$$\lim_{\lambda \in \Lambda} (u_\lambda * \varphi) = \varphi,$$

uniformly on G, and, therefore,

$$\lim_{\nu \in \mathfrak{U}} \hat{\nu}(\varphi) = \hat{\nu}_0(\varphi),$$

for any $\varphi \in$ K(G). Indeed, this follows from the inequalities

$$|\hat{\nu}(\varphi) - \hat{\nu}_0(\varphi)| \leq |\hat{\nu}(\varphi - u_\lambda * \varphi)| + |\hat{\nu}(u_\lambda * \varphi) - \hat{\nu}_0(u_\lambda * \varphi)| +$$

$$+ |\hat{\nu}_0(u_\lambda * \varphi) - \hat{\nu}_0(\varphi)| \leq \|\hat{\nu}\| \, \|\varphi - u_\lambda * \varphi\|_\infty + |\hat{\nu}(u_\lambda * \varphi) - \hat{\nu}_0(u_\lambda * \varphi)| +$$

$$+ \|\hat{\nu}_0\|_\infty \, \|u_\lambda * \varphi - \varphi\|_\infty .$$

e) Let now $\varepsilon > 0$ be an arbitrary positive number, $\mathfrak{U}$ an ultrafilter on M and $\varphi \in C^b(G)$. Let ν_0 be the limit of the ultrafilter $\mathfrak{U}$ for the Fourier topology. There exists a compact set $K \subset G$, such that

$$\nu_0(K) \geq \|\nu_0\| - \varepsilon .$$

Let now $\varphi_0 \in K_+(G)$ be such that

$$\chi_K \leq \varphi_0 \leq 1 .$$

We then have

$$\|\nu_0\| - \varepsilon < \hat{\nu}_0(\varphi_0) \leq \|\nu_0\| ,$$

and, therefore,

$$\nu_0(1 - \varphi_0) < \varepsilon .$$

Since $\lim_{\nu \in \mathfrak{U}} \|\nu\| = \|\nu_0\|$, from what we have just proved it follows that there exists a $U \in \mathfrak{U}$, such that

$$\|\nu_0\| - \varepsilon < \|\nu\| < \|\nu_0\| + \varepsilon ,$$

and

$$\hat{\nu}_0(\varphi_0) - \varepsilon < \hat{\nu}(\varphi_0) < \hat{\nu}_0(\varphi_0) + \varepsilon,$$

for any $\nu \in \mathcal{U}$. We infer that we have

$$|\hat{\nu}(\varphi) - \hat{\nu}_0(\varphi)| \leq 3\varepsilon \|\varphi\|_\infty + |\hat{\nu}(\varphi\varphi_0) - \hat{\nu}_0(\varphi\varphi_0)|,$$

for any $\nu \in \mathcal{U}$. By taking into account the fact that $\varphi\varphi_0 \in K(G)$, from what we have just proved in d), it follows that we have

$$\lim_{\nu \in \mathcal{U}} \hat{\nu}(\varphi) = \hat{\nu}_0(\varphi),$$

for any $\varphi \in C^b(G)$. Consequently, any ultrafilter on M converges for the narrow topology. Consequently, M is compact for this topology.

Corollary 1. On the ~~norm-bounded~~ subsets of $M^1_+(G)$, which are compact and metrizable for the Fourier topology, the narrow topology coincides with the Fourier topology.

Corollary 2. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of finite, positive Radon measures on the locally compact group G. If there exists a finite positive Radon measure ν, such that

$$\lim_{n\to\infty} \hat{\nu}_n(\varphi) = \hat{\nu}(\varphi),$$

for any pure, continuous function φ, of positive type, defined on G, then the same equality holds for any bounded, continuous, complex function φ, defined on G.

This corollary extends P. Lévy's theorem to arbitrary locally compact groups.

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