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BOUNDARY CONTROL PROBLEMS WITH CONVEX COST CRITERION

by

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BOUNDARY CONTROL
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Abstract. A class of boundary-distributed linear control systems in Banach spaces is studied. A maximum principle for a convex control problem associated with such system is obtained.

1. INTRODUCTION. In [1], [3] (see also [4], Chapter IV) we have studied control problems with convex cost criterion associated with linear evolution equations of the form

$$x' = Ax + Bu, \quad (' = \frac{d}{dt}); \quad 0 < t < \infty \tag{1.1}$$

in a Hilbert space E, where A is the infinitesimal generator of a strongly continuous semigroup on E and B is a linear continuous operator from a control space U to E. This setting is very suited for distributed control systems but is quite inadequate for boundary control systems. In fact to represent in the form (1.1) a such system we are led to consider unbounded operators B. In [9] Fattorini has developed a general theory of boundary control systems. Starting from his model we introduce here a general class of boundary control systems for which a "mild" solution exists for each summable controller. For such systems we consider the control problem with convex cost criterion and prove a maximum principle in the subdifferential form (see Theorem 1 below). It turns out that the class of boundary control systems studied here is large enough to include the principal boundary control systems of parabolic type (see Section 3).

An other general approach to the boundary control problems with quadratic cost has developed by Lions [10] (see also [6] for a semigroup approach to this problem).

Before conclude this section we set forth some notations which will be used in sequel.

1° Given a Banach space X and a real interval $[0,T]$ we denote by $C(0,T;X)$ the Banach space of all continuous functions $x: [0,T] \longrightarrow X$ endowed with standard norm.

2° For each $1 \leq p < \infty$ denote by $L^p(0,T;X)$ the space of all p-summable functions on $[0,T]$ with values in X. The usual modification in case $p = \infty$.

3° Given another Banach space Y we denote by $L(X,Y)$ the algebra of linear continuous operators from X to Y endowed with the usual norm $\|\cdot\|_{L(X,Y)}$.

4° Given a closed, densely defined linear operator A on X we denote by $D(A)$ its domain endowed with the graph norm.

5° Let $\varphi : X \longrightarrow \overline{R} =]-\infty, +\infty]$ be a lower semicontinuous convex function. The subdifferential $\partial\varphi : X \longrightarrow X^*$ is defined by

$$(1.2) \quad \partial\varphi(x_0) = \{ x_0^* \in X^*;\ \varphi(x_0) - \varphi(x) \leq (x_0^*, x_0 - x) \quad \text{for all } x \in X \}.$$

Here X^* is the dual space of X (which is assumed real) and $(\cdot,\cdot)$ the pairing between X and X. For other properties of $\partial\varphi$ which will be used in the sequel we refer the reader to [4], [8], [13] and [14].

2. BOUNDARY CONTROL SYSTEMS To begin with let us briefly describe Fattorini's theory of boundary-distributed control system (see [9]).

Let E be a (real or complex) Banach space and let σ be a closed, linear densely defined operator in E. Let τ be a linear operator (the boundary operator) with domain in E and range in some Banach space X. Finally, let U_1 and U_2 be two Banach spaces which in sequel will be reffered to as the control spaces of the system.

The control system we shall consider is

$$(2.1)\quad y(t) = \sigma y(t)+B_1u_1(t)+f(t), \quad \tau y(t)=B_2u_2(t) \quad \text{over } [0,T]$$

with initial value condition

$$(2.2)\quad y(o) = y^o$$

where $B_1:U_1 \longrightarrow E$ and $B_2:U_2 \longrightarrow X$ are linear continuous operators and $[0,T]$ is a fixed interval. The controllers $u_1(\cdot)$ and $u_2(\cdot)$ are summable functions on $[0,T]$ with values in U_1 and U_2, respectively. We shall call u_1, u_2 the distributed and boundary control, respectively. Here f is a given E-valued summable function.

In applications the state space E is a space of functions on some domain Ω of the Euclidean space R^n, σ is a partial differential operators on Ω and τ a partial differential operator acting on the boundary Γ of Ω.

<u>Assumption</u> I $D(\sigma) \subset D(\tau)$ and the restriction of τ to $D(\sigma)$ is continuous relative to graph norm of $D(\sigma)$.

Let $A:E \longrightarrow E$ be the linear operator defined by

$$(2.3)\quad D(A)=\left\{y \in D(\sigma);\ \tau y=o\right\},\ Ay=\sigma y \quad \text{for } y\in D(A).$$

Assumption II The operator A is the infinitesimal generator of a strongly continuous semigroup $\{S(t); t \geqslant o\}$ on E.

Assumption III There exists a linear continuous operator $B: U_2 \longrightarrow E$ such that

(2.4) $\sigma B \in L(U_2, E), \quad \tau(Bu) = B_2 u$ for all $u \in U_2$

(2.5) $\|Bu\|_E \leqslant C \|B_2 u\|_X$ for all $u \in U_2$

where C is some positive constant.

In terms of A and B system (2.1) can be written as

(2.6) $$y' = Az + B_1 u_1 + \sigma B u_2 + f, \quad o \leq t \leq T$$
$$y = z + Bu_2 .$$

If $u_2(\circ)$ is continuously differentiable on $[0,T]$ then z can be defined as a "mild" solution to the Cauchy problem

$$z' = Az + B_1 u_1 + \sigma B u_2 - Bu_2' + f$$
$$z(o) = y^o - Bu_2(o).$$

Thus, in this we may define the solution y to the system (2.1), (2.2) by the variation of constant formula

(2.7) $$y(t) = S(t)(y^o - Bu_2(o)) + Bu_2(t) + \int_o^t S(t-s)(B_1 u_1(s) + \sigma B u_2(s) - Bu_2'(s) + f(s))ds .$$

Since the differentiability of controller u_2 represents an unrealistic and severe requirement, we are led to extend the concept of solution to (2.1), (2.2) for general $u_2 \in L^1(0,T;U_2)$.

Integrating (formally) by parts in (2.7) we get

(2.8) $$y(t) = S(t)y^o - \int_o^t A\, S(t-s)Bu_2(s)ds + \int_o^t S(t-s)(B_1 u_1(s) + \sigma B u_2(s) + f(s))ds .$$

In general, unless we impose further assumptions on S(t) and B, the right hand side of (2.8) is not well defined.

Assumption IV For each $t \in [0,T]$ and $u \in U_2$,

$S(t)Bu \in D(A)$. There exists a positive function $\gamma \in L^1(0,T)$ such that

(2.9) $\quad \|AS(t)B\|_{L(U_2,E)} \leq \gamma(t) \quad$ a.e. $t \in]0,T[$.

Since $S(t)Bu \in D(A)$ for all $u \in U_2$, by the closed graph theorem we deduce that the operator $AS(t)B$ is continuous from U_2 to E so that (2.9) makes sense.

Assumption IV implies that for every $u_2 \in L^1(0,T;U_2)$, function $t \longrightarrow \int_0^t AS(t-s)\ Bu_2(s)ds$ is well defined as an element of $L^1(0,T;E)$. By definition, for each $y_o \in E$, $f \in L^1(0,T;E)$, $u_1 \in L^1(0,T;U_1)$ and $u_2 \in L^1(0,T;U_2)$, the function $y \in L^1(0,T;E)$ defined by (2.8) is the solution of distributed-boundary control system (2.1), (2.2).

Since the function $t \longrightarrow \int_0^t S(t-s)Bu_2(s)ds$ belongs to $L^1(0,T;D(A))$, $y(\cdot)$ may be expressed in the following equivalent form

$$y(t)=S(t)y_o-A\int_0^t S(t-s)Bu_2(s)ds+\int_0^t S(t-s)(B_1u_1(s)+\sigma Bu_2(s)+ + f(s))ds \quad \text{a.e. } t\in]0,T[.$$

Let χ_o be a fixed number in $\varrho(A)$ (the resolvent of A) and let $\Pi = A - \chi_o I$ (I is the identity operator). Thus y may be regarded as solution to distributed control system

(2.10)
$$w' = Aw + D_1u_1 + Du_2 + \Pi^{-1} f$$
$$y = \Pi w$$

where

(2.11) $\quad D_1=(A-\chi_oI)^{-1}B_1,\ D=(A-\chi_oI)^{-1}(\sigma B-\chi_oB)-B.$

Denote by U the product space $U_1 \times U_2$ and by $\Lambda : U \longrightarrow E$ the linear continuous operator given by

(2.12) $\quad \Lambda(u_1,u_2)=D_1u_1+D_2u_2,\quad$ for $u_1 \in U_1$, $u_2 \in U_2$

Then we may rewrite system (2.10) as

$$(2.13)\quad \begin{aligned} w' &= Aw + \Lambda u + \Pi^{-1}f, \quad 0 \leq t \leq T \\ y &= \Pi w. \end{aligned}$$

Thus, we are led to interpret the solution y to (2.1) as the observed value of a control system of the form (2.13) with unbounded observation operator Π (we refer the reader to [7] for definition and theory of observation for infinite dimensional systems).

Remark 1 If $u_2 \in L^p(0,T;U_2)$ and $\gamma \in L^{p'}(0,T)$; $1/p+1/p'=1$ then we see by (2.8) and (2.9) that $y \in C(0,T;E)$.

3 EXAMPLES It should be observed that Assumptions IV has some severe implications on system (2.1). In particular if the range R(B) of B is, say, all of E then the semigroup S(.) must be analytic (see e.g. [17], p.254). However, this condition is less restrictive than it might at first appear to be. We shall see here that it is satisfied by the main boundary control systems governed by parabolic equations.

Mixed Dirichlet problem Let Ω be a bounded and open subset of R^n with a sufficiently smooth boundary Γ.

Consider the boundary control system

$$(3.1)\quad \begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f && \text{in } Q = \Omega \times [0,T] \\ &y|_{\Gamma} = u && \text{for } t \in [0,T] \\ &y(x,0) = y_0(x) && \text{for } x \in \Omega, \end{aligned}$$

where $u \in L^2(\Sigma)(\Sigma = \Gamma \times [0,T])$, $y_0 \in L^2(\Omega)$ and $f \in L^2(Q)$.

To formulate this as a boundary control system of the form (2.1) we define $E = U_1 = L^2(\Omega)$, $X = H^{-1/2}(\Gamma)$, $U_2 = L^2(\Gamma)$, $B_1 \equiv 0$, $B_2 \equiv I$ and ($H^k(\Omega)$, $H^\alpha(\Gamma)$ are usual Sobolev spaces on Ω and Γ)

$$(3.2)\quad D(\sigma) = \left\{ y \in L^2(\Omega);\ \Delta y \in L^2(\Omega) \right\}, \qquad \sigma = \Delta .$$

The operator τ is the "trace" operator $\tau y = y|_\Gamma$ which is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $y \in D(\sigma)$ (see Lions-Magenes [12] Vol.1). The operator A is given by

$$(3.3) \qquad A = \Delta \ , \ D(A) = H_0^1(\Omega) \cap H^2(\Omega) \ .$$

Clearly Assumptions I and II are satisfied. To verify III and IV we define the linear operator $B: U_2 = L^2(\Omega) \longrightarrow L^2(\Gamma)$ by $Bu = w_u$ where $w_u \in L^2(\Omega)$ is the unique (generalized) solution to the Dirichlet boundary-value problem

$$(3.4) \qquad \begin{aligned} \Delta w_u &= o \quad \text{in } \Omega \\ w_u|_\Gamma &= u. \end{aligned}$$

In other words,

$$(3.5) \qquad \int_\Omega w_u \Delta \psi \, dx = \int_\Gamma u \frac{\partial \psi}{\partial n} d\sigma \quad \text{for all } \psi \in H_0^1(\Omega) \cap H^2(\Omega) \ .$$

Here $\frac{\partial \psi}{\partial n}$ denotes the outward normal derivative of ψ which is well defined as an element of $H^{1/2}(\Gamma)$. We need the following lemma

LEMMA 1 <u>For every</u> $u \in H^{-1/2}(\Gamma)$ <u>problem</u> (3.4) <u>has a unique solution</u> $w_u \in L^2(\Omega)$ <u>satisfying</u>

$$(3.6) \qquad \| w_u \|_{L^2(\Omega)} \leq C_1 \| u \|_{H^{-1/2}(\Gamma)} \ .$$

<u>If</u> $u \in H^{1/2}(\Gamma)$ <u>then</u> $w_u \in H^1(\Omega)$ <u>and</u>

$$(3.7) \qquad \| w_u \|_{H^1(\Omega)} \leq C_2 \| u \|_{H^{1/2}(\Gamma)} .$$

<u>Here</u> Ci, i = 1,2 <u>are positive constants independent of</u> u.

<u>Proof</u>. Let $u \in H^{-1/2}(\Gamma)$. The existence of w_u satisfying (3.5) is well-known (see e.g. [10], p.72). It follows from the fact that the operator Δ with domain $H_0^1(\Omega) \cap H^2(\Omega)$ is onto on $L^2(\Omega)$ and the functional $\psi \longrightarrow \int_\Gamma u \frac{\partial \psi}{\partial n} d\sigma$ is continuous on $H_0^1(\Omega) \cap H^2(\Omega)$. Also the uniqueness of such

w_u is immediate. On the other hand by "trace" inequality

(3.8) $$\left\| \frac{\partial \psi}{\partial n} \right\|_{H^{1/2}(\Gamma)} \leq C \|\psi\|_{H_0^1(\Omega) \cap H^2(\Omega)}$$

and by (3.5) we see that

$$|w_u(\varphi)| \leq C\|u\|_{H^{-1/2}(\Gamma)} \|A^{-1}\varphi\|_{H^2(\Omega)} \quad \text{for all } \varphi \in L^2(\Omega)$$

thereby proving (3.6).

Suppose now that $u \in H^{1/2}(\Gamma)$. Then again by "trace" theorem there is $y_u \in H^1(\Omega)$ such that $\tau y_u = u$ and

(3.9) $$\|y_u\|_{H^1(\Omega)} \leq C\|u\|_{H^{1/2}(\Gamma)}$$

where C is independent of u. On the other hand, it follows from (3.5) and Green's Formula that the function $z = y_u - w_u$ satisfies the equation

$$\int_\Omega z\Delta\psi \, dx = \int_\Omega \text{grad } y_u \text{ grad } \psi \, dx$$

for all $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$. Hence

(3.10) $$|z(\varphi)| \leq C\|y_u\|_{H^1(\Omega)} \|\varphi\|_{H^{-1}(\Omega)} \quad \text{for all } \varphi \in L^2(\Omega).$$

This shows that $z \in H_0^1(\Omega)$. Hence $w_u \in H^1(\Omega)$. Furthermore, by (3.9) and (3.10) we get (3.7) as claimed. This completes the proof of the lemma.

In particular, Lemma 1 shows that Assumption III is satisfied. As regards Assumption IV, we observe first that by (3.6) and (3.7) it follows that $Bu \in (L^2(\Omega), H^1(\Omega))_{1/2}$ for all $u \in L^2(\Gamma)$ and

(3.11) $$\|Bu\|_{H^{1/2}(\Omega)} \leq C\|u\|_{L^2(\Gamma)} \quad \text{for all } u \in L^2(\Gamma).$$

Here $(L^2(\Omega), H^1(\Omega))_{1/2}$ denotes the interpolation space $\left\{ y(x,0);\ y \in L^2(R^+; H^1(\Omega)), \frac{\partial y}{\partial t} \in L^2(R^+; L^2(\Omega)) \right\}$ (see e.g. [11]).

Inasmuch as the semigroup S(.) generated by A is analytic (see e.g. [12] Vol.II) we have

(3.12) $$\| AS(t)y_o\|_{L^2(\Omega)} \leq C\ t^{-1} \| y_o\|_{L^2(\Omega)}$$

for all $y_o \in L^2(\Omega)$ and $t > o$.

On the other hand according to an interpolation result due to Lions [11], we have for each $y_o \in H^1(\Omega)$,

(3.13) $$\| AS(t)y_o\|_{L^2(\Omega)} \leq C\ t^{-3/4} \| y_o\|_{H^1(\Omega)} \quad \text{for } t > o .$$

Thus interpolating between spaces $H^1(\Omega)$ and $L^2(\Omega)$ we see by (3.11), (3.12) and (3.13) that

(3.14) $$\| AS(t)Bu\|_{L^2(\Omega)} \leq C\ t^{-7/8} \| u\|_{L^2(\Omega)}$$

for all $u \in L^2(\Omega)$ and $t > o$. Here C is a positive constant independent of u and t. In other words, inequality (2.9) holds with $\gamma(t) = C\ t^{-7/8}$.

Thus we have shown that <u>system</u> (3.3) <u>satisfies Assumptions</u> I <u>up to</u> IV <u>with</u> A <u>and</u> B <u>defined by</u> (3.3) <u>and</u> (3.4) <u>respectively</u>.

<u>Mixed Neumann problem</u>. Consider the boundary control system

(3.15)
$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y = f && \text{in } Q = \Omega \times [0,T] \\ &\frac{\partial y}{\partial n} + \alpha y = u && \text{in } \Sigma = \Gamma \times [0,T] \\ &y(o,x) = y_o(x)\ , && x \in \Omega \end{aligned}$$

where $y_o \in L^2(\Omega)$, $f \in L^2(Q)$ and u (the boundary control) is in $L^2(\Gamma)$. Here α is a nonnegative constant.

Define $E = L^2(\Omega)$, $U_2 = X = L^2(\Gamma)$, $B_1 \equiv 0$, $B_2 \equiv I$, $\sigma y = \Delta y$, $D(\sigma) = H^2(\Omega)$ and $\tau y = \alpha y + \frac{\partial y}{\partial n}$. The operator A is given by

(3.16) $$Ay = \Delta y \quad \text{on } D(A) = \left\{ y \in H^2(\Omega);\ \alpha y + \frac{\partial y}{\partial n} = o \right\}.$$

Define the operator $B: L^2(\Gamma) \longrightarrow L^2(\Omega)$ by $Bu = z_u$ where $z_u \in H^1(\Omega)$ is the solution to boundary-value problem

(3.17)
$$z_u - \Delta z_u = 0 \quad \text{in} \quad \Omega$$
$$\alpha z_u + \frac{\partial z_u}{\partial n} = u \quad \text{in} \quad \Gamma .$$

Consider ou the product space $H^1(\Omega) \times H^1(\Omega)$ the bilinear functional

(3.18) $$a(y,\varphi) = \int_\Omega (y\varphi + \text{grad } y \text{ grad } \varphi)dx - \int_\Gamma (u-\alpha y)\varphi \, d\sigma$$

where $u \in H^{-1/2}(\Gamma)$ (the integral $\int_\Gamma u\varphi \, d\sigma$ must be regarded as the value of u at $\tau\varphi \in H^{1/2}(\Gamma)$). Since a is coercive, there is $z_u \in H^1(\Omega)$ satisfying $a(z_u,\varphi) = 0$ for all $\varphi \in H^1(\Omega)$. In other words, $z_u = Bu$ is the solution to (3.17). From (3.18) we see also that

(3.19) $$\|z_u\|_{H^1(\Omega)} \leq C \|u\|_{H^{-1/2}(\Gamma)} .$$

In particular, we have shown that Assumption III holds. To verify Assumption IV we notice that since the operator $-A$ is self-adjoint and positive, we have

(3.20) $$\|AS(t)y_0\|_{L^2(\Omega)} \leq C \, t^{-1/2} \|y_0\|_{D((-A)^{1/2})}$$

for all $t > 0$ and $y_0 \in D((-A)^{1/2})$. Since $D((-A)^{1/2}) = H^1(\Omega)$ we may rewrite (3.20) as

$$\|AS(t)y_0\|_{L^2(\Omega)} \leq C \, t^{-1/2} \|y_0\|_{H^1(\Omega)}$$

for all $t > 0$ and $y_0 \in H^1(\Omega)$. The latter combined with (3.19) yields

(3.21) $$\|AS(t)Bu\|_{L^2(\Omega)} \leq C \, t^{-1/2} \|u\|_{L^2(\Gamma)}$$

for all $t > 0$ and $u \in L^2(\Gamma)$. Hence Assumption IV holds with $\gamma(t) = C \, t^{-1/2}$.

Remark 2. It should be emphasized that estimate (3.21) is not optimal. Observing that the operator B maps $L^2(\Gamma)$ into $H^{3/2}(\Omega)$ a sharper estimate can be obtained.

Remark 3. In the preceding examples the Laplacian can be replaced by any second order symmetric and elliptic differential operator on Ω.

3. INTEGRAL CONVEX COST CRITERIA. In this section we consider the following unconstrained boundary-distributed control problem: minimize

$$\text{(P)} \qquad \int_0^T L_o(t,y,u_1,u_2)dt + \ell(y(o),y(T))$$

in $y \in C(0,T;E)$, $u_i \in L^p(0,T;U_i)$; $i = 1,2,$ subject to state equation (2.1).

Here $1 < p < \infty$ and $L_o : [0,T] \times E \times U_1 \times U_2 \longrightarrow \bar{R} =]-\infty,+\infty]$, $\ell : E \times E \longrightarrow \bar{R}$ are given functions which will be precised later.

From now on we shall assume that the spaces E, U_1 and U_2 are reflexive and strictly convex together with their duals E^*, U_1^* and U_2^*.

We denote by U the product space $U_1 \times U_2$ and denote by $|\cdot|$ (resp. $\|\cdot\|$) the norm in E (resp. U). The pairing between E, E^* and U, U^* will be denoted by $(\cdot,\cdot)$ and $\langle \cdot,\cdot \rangle$, respectively.

Finally we denote by $F : E \longrightarrow E^*$ and $\Xi : U \longrightarrow U^*$ the duality mapping of E and U, respectively. It should be recalled (see e.g. [2], p.13) that under our assumptions F and Ξ are single valued, injective and demicontinuous (i.e. strongly-weakly continuous).

We shall assume also that Assumptions I ~ IV are satisfied and that function γ in condition (2.9) belongs to $L^r(0,T)$ where

$$p(p-1)^{-1} < r \leq \infty .$$

As remarked earlier, the solution y to system (2.1) is continuous on $[0,T]$ and hence (P) makes sense.

As seen in Section 2, the state system (2.1) can be brought into the form (2.13) where $u = (u_1,u_2) \in U_1 \times U_2 = U$, $\Pi = A - \chi_o I(\chi_o \in \varrho(A))$ and Λ is given by (2.12). Let $L: [0,T] \times E \times U \longrightarrow \bar{R}$ be defined by

$$L(t,y,u) = L_o(t,y,u_1,u_2) \quad \text{for} \quad u = (u_1,u_2).$$

Then problem (P) can be equivalently expressed as: minimize

$$\text{(P)} \qquad \int_o^T L(t,y,u)dt + \ell(y(o),y(T))$$

in $y \in C(0,T;E)$ and $u \in L^p(0,T;U)$ subject to

$$\text{(4.1)} \qquad \begin{aligned} w' &= Aw + \Lambda u + \Pi^{-1} f \\ y &= \Pi w. \end{aligned}$$

Here $f \in L^1(0,T;E)$ is a given function. The solution of (4.1) must be understood in the sense of (2.8), i.e.,

$$(4.1)' \quad y(t)=S(t)y(o)+\int_o^t \Pi S(t-s)\Lambda u(s)ds+\int_o^t S(t-s)f(s)ds, \quad o \leq t \leq T.$$

We notice that Assumption IV and the condition imposed on p imply that

$$\text{(4.2)} \qquad \| \Pi S(t)\Lambda \|_{L(U,E)} \leq \zeta(t) \quad \text{for } t \in [0,T]$$

where $\zeta \in L^{p'}(0,T)$; $1/p + 1/p' = 1$.

Beside the above assumptions on E, U, A and B further hypotheses on L_o and ℓ must be imposed.

(A). The functions L(t) and ℓ are lower semicontinuous and convex on $E \times U$ (for each t) and $E \times E$, respectively. Furthermore, the following conditions hold.

(a) For each $(y,u) \in E \times U$ the functions $L(t,y,v): [0,T] \longrightarrow \bar{R}$ and $J^L(t,y,v): [0,T] \longrightarrow E \times U$ are Lebesgue measurable.

(b) There exist $r_o \in L^2(0,T;E)$, $s_o \in L^\infty(0,T;U)$ and

$g_o \in L^1(0,T)$ such that for all $(y,u) \in E \times U$,

$$L(t,y,u) \geqslant (y, r_o(t)) + \langle v, s_o(t) \rangle + g_o(t), \quad \text{a.e. } t \in]0,T[.$$

(c) For each $y_o \in E$ there is a neighbourhood $\mathcal{O}$ of y_o, functions $\alpha \in L^{p'}(0,T)$, $\beta \in L^p(0,T)$ and a mapping $\Sigma : [0,T] \times \mathcal{O} \longrightarrow U$ such that

(4.3) $\quad L(t,y,\ \Sigma(t,y)) \leq \alpha(t) \qquad$ a.e. $t \in]0,T[$

(4.4) $\quad \| \Sigma(t,y) \| \leq \beta(t) \qquad$ a.e. $t \in]0,T[$

for all $y \in \mathcal{O}$.

Here $J^L_\mu(t,y,u) = (y_\mu, u_\mu)$ denotes the solution to equation (see e.g. [2] , p.41)

(4.5) $\quad \{F(y_\mu - y),\ \Xi(u_\mu - u)\} + \mu \partial L(t,y\ ,u\ ) \ni o$

where $\partial L(t): E \times U \longrightarrow E^* \times U^*$ is the subdifferential of $L(t)$.

We notice that condition (a) implies that $L(t,y(t),u(t))$ is a Lebesgue measurable function of t whenever $y(\cdot)$ and $u(\cdot)$ are Lebesgue measurable functions (this may be seen from formula (5.4) below). It can be shown that if spaces E and U are separable then (a) is satisfied iff L is a convex normal integrand in the sense of Rockafellar (see [15]). If L is independent of t then conditions (a) and (b) automatically hold. As regards (c) it is satisfied in particular if the Hamiltonian associated with L is finite on $E \times U$ (other situations in which (c) holds are discussed in [4], p.217).

An end point pair $(y_1, y_2) \in E \times E$ is called attainable for problem (P) if there exist functions $y \in C(0,T;E)$ and $u \in L^p(0,T;U)$ satisfying system (4.1) and such that

(4.6) $\quad L(t,y,u) \in L^1(0,T);\ y(o) = y_1,\ y(T) = y_2.$

The set of all attainable pairs will be denoted by C_L.

Denote also by $D(\ell) = \{(y_1,y_2) \in E \times E;\ \ell(y_1,y_2) < +\infty\}$

the effective domain of ℓ. Our next assumption is

(B) There is $(y_1, y_2) \in C_L \cap D(\ell)$ such that one of the following two conditions hold

(4.7) $$y_2 \in \text{int}\{x \in E;\ (y_1, x) \in D(\ell)\}$$

(4.8) $$y_2 \in \text{int}\{x \in E;\ (y_1, x) \in C_L\}.$$

It might be noticed that in general (4.8) fails for infinite dimensional systems because it requires the complete controllability.

The main result of this paper, Theorem 1 below may be regarded as a maximum principle for our boundary-distributed control problem.

THEOREM 1. <u>Suppose that all above hypotheses on system</u> (2.1) <u>and functions</u> L, ℓ <u>are satisfied. Then a given pair</u> (y_o, u_o) <u>is optimal in problem</u> (P) <u>if and only if there exist the functions</u> $p_o \in C(0,T;E^*) \cap L^{p'}(0,T;D(\Lambda^* \Pi^*))$ <u>and</u> $q_o \in L^1(0,T;E^*)$ <u>which satisfy along with</u> y_o <u>and</u> u_o <u>system</u>

(4.9) $$\begin{aligned} w_o' &= A w_o + \Lambda u_o + \Pi^{-1} f, \quad \text{on } [0,T] \\ y_o &= \Pi w_o \end{aligned}$$

(4.10) $$p_o' = -A^* p_o + q_o \quad \text{on } [0,T]$$

(4.11) $$(q_o(t), \Lambda^* \Pi^* p_o(t)) \in \partial L(t, y_o(t), u_o(t)) \quad \text{a.e. } t \in]0,T[$$

<u>and transversality conditions</u>

(4.12) $$(p_o(o), -p_o(T)) \in \partial \ell(y_o(o), y_o(T)).$$

Here $\partial L(t) : E \times U \longrightarrow E^* \times U^*$ and $\partial \ell : E \times E \longrightarrow E^* \times E^*$ stand for subdifferentials of $L(t)$ and ℓ, respectively.

Of course equation (4.9) must be considered in the sense of (4.1)', i.e.,

$$y_o(t) = S(t) y_o(o) + \int_o^t \Pi S(t-s) \Lambda u_o(s) ds + \int_o^t S(t-s) f(s) ds,$$

while (4.10) is taken in the "mild" sense, i.e.,

(4.13) $$p_o(t) = S^*(T-t)p_o(T) - \int_t^T S^*(s-t)q_o(s)ds, \quad o \leq t \leq T$$

where $S^*(.)$ is the semigroup generated by the adjoint A^* of A. By Λ^*, Π^* we have denoted the adjoints of Λ and Π respectively.

Some insight into the problem and Theorem can be gained from the following simple example. Minimize

(4.14) $$\int_Q g(x,y(x,t))dxdt + \int_\Sigma h(u(\sigma,t))d\sigma$$

in $y \in C(0,T;L^2(\Omega))$ and $u \in L^p(0,T;L^2(\Omega))$ subject to

(4.15) $$\frac{\partial y}{\partial t} - \Delta y = o \qquad \text{in } Q = \Omega \times [0,T]$$

(4.16) $$y = u \qquad \text{in } \Sigma = \Gamma \times [0,T]$$

(4.17) $$y(x,o) = y^o(x) \qquad x \in \Omega$$

where $y^o \in L^2(\Omega)$. The function $g: \Omega \times R \longrightarrow R$ is continuous and convex in y, measurable in x and satisfies

$$|g(x,y)| \leq C|y|^2 + \zeta(x) \quad \text{a.e. } x \in \Omega, \quad y \in R$$

where $\zeta \in L^2(\Omega)$. As regards the function $h: R \longrightarrow \bar{R}$ it will be assumed convex, lower semicontinuous and <u>cofinite</u> i.e.,

(4.18) $$\lim_{|u| \to \infty} h(u)/|u| = +\infty .$$

In particular we may take function h as

$$h(u) = \begin{cases} h_o(u) & u \in U_o \\ +\infty & \text{otherwise} \end{cases}$$

Where h_o is a continuous convex function on real axis and U_o is a bounded and closed interval.

Clearly Assumptions (A) and (B) are satisfied where $E = L^2(\Omega)$, $U = U_2 = L^2(\Gamma)$

$$L(t,y,u) = \int_\Omega g(x,y(x))dx + \int_\Gamma h(u(\sigma))d\sigma$$

and ℓ is defined by

$$\ell(y_1, y_2) = o \quad \text{if} \quad y_1 = y_1^o \quad \text{and} = +\infty \quad \text{if} \quad y_1 \neq y_o.$$

In this case we have also $A = \Pi = \Delta$, $D(A) = H_o^1(\Omega) \cap H^2(\Omega)$ and $D = -B$ where $B: L^2(\Gamma) \longrightarrow L^2(\Omega)$ is defined by equation (3.4). Then as easily seen by (3.5) the adjoint B^* in given by

$$B^* y = \frac{\partial}{\partial n} (\Delta)^{-1} y \quad \text{for all } y \in L^2(\Omega)$$

so that system (4.10), (4.11) becomes

(4.19) $$\frac{\partial p_o}{\partial t} + \Delta p_o = q_o \quad \text{in } Q$$

(4.20) $$q_o \in \partial_y g(x, y_o) \quad \text{in } Q$$

(4.21) $$\frac{\partial p_o}{\partial n} \in -\partial h(u_o) \quad \text{in } \Sigma$$

while transversality conditions (4.12) reduce to

(4.22) $$y_o(x,o) = y^o(x), \; p_o(x,T) = o \quad \text{a.e. } x \in \Omega.$$

Since condition (2.9) holds with a function $\gamma \in L^r(0,T)$ where $1 \leq r < 8/7$, we must choose p in the control space $L^p(0,T; L^2(\Gamma))$ such that $p > 8$.

Thus by Theorem 1, a given pair (y_o, u_o) is optimal in problem (4.14) iff there exist $p_o \in C(0,T;L^2(\Omega)) \cap L^{p'}(0,T; H_o^1(\Omega) \cap H^2(\Omega))$ and $q_o \in L^1(0,T;L^2(\Omega))$ satisfying (4.19) $\sim$ (4.22).

4. PROOF OF THEOREM 1. Since the sufficiency is straightforward we confine ourselves to prove the necessity of conditions (4.9) $\sim$ (4.12) for optimality. The idea of the proof comes from the author work [3] (see also [4], Chapter IV).

Let L_μ and ℓ_μ, $\mu > o$, denote the functions

(5.1) $$L_\mu(t,y,u) = \inf\left\{ \frac{1}{2\mu} (|y-\tilde{y}|^2 + \|u-\tilde{u}\|^2) + L(t,\tilde{y},\tilde{u}) ; (\tilde{y},\tilde{u}) \in E \times U \right\} = L(t, J_\mu^L(y,u)) + \frac{\mu}{2} \|\partial L_\mu(t,y,u)\|^2$$

(5.2) $\ell_\mu(y_1,y_2) = \inf\left\{ \frac{1}{2\mu}(|y_1-\tilde{y}_1|^2+|y_2-\tilde{y}_2|^2)+\ell(\tilde{y}_1,\tilde{y}_2);\right.$

$$\left.(\tilde{y}_1,\tilde{y}_2)\in E\times E\right\} = \ell(J_\mu^\ell(y_1,y_2))+\frac{\mu}{2}\|\partial\ell_\mu(y_1,y_2)\|^2.$$

We recall (see [4], p.107) that $L_\mu(t)$ and ℓ_μ are Gâteaux differentiable on $E\times U$ and $E\times E$, respectively. Their differential $\partial L_\mu(t)$ and $\partial\ell_\mu$ are given by

(5.3) $$\partial L_\mu(t,y,u) = \mu^{-1}(G_1(y,u)-J_\mu^L(t,y,u))$$

(5.4) $$\partial\ell_\mu(y_1,y_2) = \mu^{-1}(G_2(y_1,y_2)-J_\mu^\ell(y_1,y_2))$$

where $G_1 = (F,\Xi)$ and $G_2 = (F,F)$ are the duality mappings of $E\times U$ and $E\times E$, respectively. It should be observed that in virtue of Assumption (A) and equality (5.1), $L_\mu(t,y(t),u(t))$ is a Lebesgue measurable function of t whenever $y(\cdot)$ and $u(\cdot)$ are Lebesgue measurable.

Let (y_o,u_o) be any optimal pair of problem (P) and let $w_o = \Pi^{-1}y_o$. Consider the approximating problem

(5.5) $$\inf\left\{\int_o^T (L_\mu(t,y,u)+p^{-1}\|u-u_o\|^p)dt+\ell_\mu(y(o),y(T)) + \frac{1}{2}|y(o)-y_o(o)|^2\right\}$$

where the infimum is taken over all $u\in L^p(0,T;U)$ and $y\in C(0,T;E)$ satisfying system (4.1)'. By condition (b) in Assumption (A) and by (5.1), (5.3) we see that for all $y\in E$ and $u\in U$, we have

(5.6) $$L_\mu(t,y,u)\geq (r_o,y)+\langle s_o,u\rangle+g_o \quad \text{a.e. } t\in\,]0,T[$$

where $g_o\in L^1(0,T)$ is independent of μ.

In particular it follows by (5.6) that problem (5.5) has for each $\mu>0$ a solution (y_μ,u_μ) (unique because U is strictly convex). Using the fact that $L_\mu(t)$, ℓ_μ and the norms in U and E are Gâteaux differentiable, we find that (y_μ,u_μ) satisfy

(5.7) $\int_0^T ((\partial_y L_\mu(t,y_\mu,u_\mu),y)+\langle \partial_u L_\mu(t,y_\mu,u_\mu)+\|u_\mu-u_o\|^{p-2}\,\Xi\,(u_\mu-$

$-u_o),\ u\rangle\,)dt+(\partial_1 \ell_\mu(y_\mu(o),y_\mu(T))+F(y_\mu(o)-y_o(o)),y(o))+$

$+(\partial_2 \ell_\mu(y_\mu(o),y_\mu(T)),y(T)) = o$

for all $u\in L^p(0,T;U)$ and y satisfying (4.1)'.

We see by (5.3) and (5.6) that $\partial_y L_\mu(t,y_\mu,u_\mu)\in L^p(0,T;E^*)$ and $\partial_u L_\mu(t,y_\mu,u_\mu)\in L^p(0,T;U^*)$. Let $p_\mu\in C(0,T;E^*)$ be defined by

(5.8) $\quad p_\mu(t)=S^*(T-t)p_\mu(T)-\int_t^T S^*(s-t)\,\partial_y L_\mu(s,y_\mu(s),u_\mu(s)ds$

where

(5.9) $\qquad -p_\mu(T)=\partial_2 \ell_\mu(y_\mu(o),y_\mu(T))$

($\partial_2 \ell$ denotes the differential relative to second argument).

We observe that by (4.2) the operator $\Lambda^* S^*(t)\Pi^*$ is continuous from E^* to U^* for each $t\in[0,T]$ and

$$\|\Lambda^* S^*(t)\Pi^*\|_{L(E^*,U^*)} \le \zeta(t) \quad \text{for } t\in[0,T].$$

Inasmuch as $S^*(t)\Pi^* = \Pi^* S^*(t)$ on $D(A^*)$ we may infer that

(5.10) $\quad \|\Lambda^*\Pi^* S^*(t)\|_{L(E^*,U^*)} \le \zeta(t),\quad t\in[0,T].$

In particular, it follows that $\Lambda^*\Pi^* p_\mu \in L\ (0,T;U^*)$. Thus substituting y by (4.1)' in (5.7) we get after some calculations involving Fubini's theorem that

(5.11) $\Lambda^*\Pi^* p_\mu+\|u_o-u_\mu\|^{p-2}\,\Xi\,(u_o-u_\mu)=\partial_u L_\mu(t,y_\mu,u_\mu)$, a.e. $t\in\,]0,T[$

and using (5.9) we find the transversality equations

(5.12) $\{p_\mu(o)+F(y_o(o)-y_\mu(o)),-p_\mu(T)\} = \partial \ell_\mu(y_\mu(o),y_\mu(T)).$

By (5.5) we have

(5.13) $\int_0^T (L_\mu(t,y_\mu,u_\mu)+p^{-1}\|u_\mu-u_o\|^p)dt+\ell_\mu(y_\mu(o),y_\mu(T))+$

$+\frac{1}{2}|y_\mu(o)-y_o(o)|^2 \le \int_0^T L(t,y_o,u_o)dt+\ell(y_o(o),y_o(T)).$

because $L_\mu \leq L$ and $\ell_\mu \leq \ell$ for all $\mu > 0$. In particular, it follows that $\{u_\mu\}$ is bounded in $L^p(0,T;U)$ and by (4.1)' this implies that $\{y_\mu\}$ is bounded in $C(0,T;E)$. Thus extracting, a subsequence if necessary, we may assume that

$$(5.14)\quad \begin{aligned} &u_\mu \longrightarrow \tilde{u} && \text{weakly in } L^p(0,T;U) \\ &y_\mu \longrightarrow \tilde{y} && \text{weakly-star in } L^\infty(0,T;E) \\ &y_\mu(t) \longrightarrow \tilde{y}(t) && \text{weakly in } E \text{ for each } t \in [0,T] \end{aligned}$$

Clearly $(\tilde{y},\tilde{u}) \in C(0,T;E) \times L^p(0,T;U)$ satisfy equation (4.1). On the other hand, we have

$$\int_0^T L(t,\tilde{y},\tilde{u})dt + \ell(\tilde{y}(0),\tilde{y}(T)) \geqslant \int_0^T L(t,y_0,u_0)dt + \ell(y_0(0),y_0(T))$$

and

$$(5.15)\quad \liminf_{\mu \to 0} \int_0^T L_\mu(t,y_\mu,u_\mu)dt \geqslant \int_0^T L(t,\tilde{y},\tilde{u})dt$$

$$(5.16)\quad \liminf_{\mu \to 0} \ell_\mu(y_\mu(0),y_\mu(T)) \geqslant \ell(\tilde{y}(0),\tilde{y}(T))$$

which in conjuction with (5.13) and (5.14) imply

$$(5.17)\quad u_\mu \longrightarrow u_0 \quad \text{strongly in } L^p(0,T;U)$$

$$(5.18)\quad y_\mu \longrightarrow y_0 \quad \text{strongly in } C(0,T;E) \; .$$

The justification of inequalities (5.15), (5.16) is seen by invoking relations (5.1) $\sim$ (5.4) and the weak lower semicontinuity of ℓ on $E \times E$ and of convex integrand $\int_0^T L(t,y,u)dt$ on $L^p(0,T;E) \times L^p(0,T;U)$.

We have in mind to pass to limit in equations (5.8), (5.11) and (5.12). To this purpose some a priori estimates on p_μ are necessary. The first is given by

LEMMA 1 $\{p_\mu(T);\ 0 < \mu \leq 1\}$ <u>is a bounded subset of</u> E .

<u>Proof</u>. Since the proof is essentially the same as that of Lemma 2 in [3] (see also [4] p.230) it will be only outlined.

First we assume that condition (4.7) holds i.e., there exist $y \in C(0,T;E)$ and $u \in L^p(0,T;U)$ satisfying (4.1)' and such that

$$L(t,y,u) \in L^1(0,T), \; y(T) \in \operatorname{int}\{x \in E;\ (y(o),x)) \in D(\ell)\}.$$

Therefore, there is $\varrho > o$ and $C > o$ such that

$$\ell(y(o,y(T)+\varrho h) \leq C \text{ for all } h \in E, |h| \leq 1.$$

Next by (5.12), we have

$$(5.19) \quad (p_\mu(o),y_\mu(o)-y(o))-(p_\mu(T),y_\mu(T)-y(T)-\varrho h) + +(F(y\ (o)-y_\mu(o)),y_\mu(o)-y(o)) \geqslant \ell_\mu(y_\mu(o),y_\mu(T)) - -\ell_\mu(y(o),y(T)+\varrho h)$$

while by (4.1)', (5.8) and (5.11) we see that

$$(p_\mu(o),y_\mu(o)-y(o))-(p_\mu(T),y_\mu(T)-y(T)) \geqslant \int_o^T (L_\mu(t,y_\mu,u_\mu) + + p^{-1}\|u_\mu-u_o\|^p)dt - \int_o^T (L(t,y,u) + p^{-1}\|u-u_o\|^p)dt\ ;$$

Combining the latter with (5.19) we get

$$(5.20) \qquad |p_\mu(T)| \leq C \quad \text{for all} \quad \mu > o$$

as claimed (In the sequel we shall denote by C several positive constants independent of μ)

Let $\varphi : E \times E \longrightarrow \bar{R}$ be the convex function defined by

$$\varphi(h_1,h_2)=\inf\left\{\int_o^T (L(t,y,u)+p^{-1}\|u\|^p)dt; y(o)=h_1,\ y(T) = h_2; \ (y,u) \text{ satisfies } (4.1)'\right\}.$$

Clearly φ is lower semicontinuous and its effective domain is the very set C_L. If condition (4.8) holds, then there exist $y \in C(0,T;E)$ and $u \in L^p(0,T;U)$ satisfying (4.1)' and such that

$$\varphi(y(o),\ y(T)+\ \varrho h) \leq C \quad \text{for all } |h| \leq 1 \ ,$$

Then proceeding as in [1] we find that $\{|p_\mu(T)|\}$ is bounded.

We continue the proof of theorem by noticing that in virtue of Assumption (A) there exist $\alpha \in L^{p'}(0,T)$, $\beta \in L^p(0,T)$, $\varrho > o$ and $v_h : [0,T] \longrightarrow U$ such that $\| v_h(t)\| \leq \beta(t)$ a.e. $t \in]0,T[$

and for all $b \in E$, $|b| \leq 1$,

(5.21) $\qquad L(t, y_o(t) + \varrho h, v_h(t)) \leq \alpha(t)$ a.e. $t \in]0,T[$.

Next by (5.11) and definition of ∂L_μ we have ;

$(\partial_y L_\mu(t, y_\mu, u_\mu), y_\mu - y_o - \varrho h) + \langle \Lambda^* \Pi^* p_\mu + \Xi(u_o - u_\mu) \| u_o - u_\mu \|^{p-2},$
$u_\mu - v_h \rangle \geqslant L_\mu(t, y_\mu, u_\mu) - \alpha(t)$, a.e. $t \in]0,T[$.

Invoking (5.6) and (5.18) the latter yields for a sufficiently small μ,

(5.21) $\quad |\partial_y L_\mu(t, y_\mu, u_\mu)| \leq C(\| u_\mu \| + \beta)(\| u_o - u_\mu \|^{p-1} +$
$\qquad + \| \Lambda^* \Pi^* p_\mu \|) + \delta(t)$, a.e. $t \in]0,T[$

where $\delta \in L^{p'}(0,T)$. We set $q_\mu = \partial_y L_\mu(t, y_\mu, u_\mu)$. Now taking into account Lemma 1 and (5.8),(5.10),(5.21) we obtain

(5.22) $\quad \| \Lambda^* \Pi^* p_\mu(t) \| \leq C(\zeta(T-t) + \int_t^T \zeta(s-t)(1 + \| u_\mu(s) \|)$
$\qquad (\| \Lambda^* \Pi^* p_\mu(s) \| + \| u_o(s) - u_\mu(s) \|^{p-1}) ds + 1$ for all $t \in [0,T]$.

Next by Young's inequality we have

$$\left(\int_\vartheta^T \left(\int_t^T \zeta(s-t) \| u_\mu(s) \| \; \| \Lambda^* \Pi^* p_\mu(s) \| \, ds \right)^{p'} dt \right)^{1/p'} \leq$$

$$\leq \left(\int_0^{T-\vartheta} | \zeta(t) |^{p'} dt \right)^{1/p'} \int_\vartheta^T \| u_\mu(s) \| \; \| \Lambda^* \Pi^* p_\mu(s) \| \, ds \leq$$

$$\varkappa(T-\vartheta) \left(\int_\vartheta^T \| \Lambda^* \Pi^* p_\mu(t) \|^{p'} dt \right)^{1/p'} \quad \text{for} \quad 0 \leq \vartheta \leq T$$

where $\lim_{t \to 0} \varkappa(t) = 0$. We may there fore conclude from (5.22) that $\left\{ \int_{T-\nu}^T \| \Lambda^* \Pi^* p_\mu \|^{p'} dt \right\}$ is bounded where ν is some positive constant. By (5.8) we see that $\{ | p_\mu(t) | \}$ are uniformly bounded on $[T-\nu, T]$. Now reasoning as above with T replaced by $T-\nu$ we get after some steps that $\{ \Lambda^* \Pi^* p_\mu \}$ is bounded in $L^{p'}(0,T;U^*)$ and

(5.23) $\qquad |p_\mu(t)| \leq C \quad$ for $\quad t \in [0,T]$.

It should be observed that (5.21) also implies that

$\{q_\mu\} \subset L^1(0,T;E^*)$ is equibounded and the measures $\{\vartheta(\theta) = \int_\theta q_\mu(t)dt;\ \theta$ measurable subset of $[0,T]\}$ are uniformly absolutely continuous. Then according to Dunford-Pettis criterion in Banach spaces (see [5]), the set $\{q_\mu\}$ is weakly compact in $L^1(0,T;E^*)$. Hence there exists a subsequence (again denoted q_μ) such that for $\mu \longrightarrow 0$,

(5.24) $\quad q_\mu \longrightarrow q_0 \quad$ weakly in $L^1(0,T;E^*)$.

Extracting further subsequences, we may also assume that

(5.25) $\quad p_\mu(T) \longrightarrow p_T \quad$ weakly in E^*

(5.26) $\quad p_\mu \longrightarrow p_0 \quad$ weakly-star in $L^\infty(0,T;E^*)$

(5.27) $\quad \Lambda^*\Pi^*p_\mu \longrightarrow \Lambda^*\Pi^*p_0 \quad$ weakly in $L^{p'}(0,T;U)$.

It follows from (5.25) and (5.8) that for each $t \in [0,T]$,

$$p_\mu(t) \longrightarrow p_0(t) = S^*(T-t)p_T - \int_t^T S^*(s-t)q_0(s)ds$$

in the weak topology of E^*.

Since $y_\mu(t) \longrightarrow y_0(t)$ uniformly on $[0,T]$ and F is demicontinuous on E we may pass to limit in (5.12) to obtain

(5.28) $\quad (p_0(0),-p_0(T)) \in \partial\ell(y_0(0),y_0(T))$.

The justification of the final assertion is seen by recalling that $\partial\ell_\mu(y_\mu(0),-y_\mu(T)) \in \partial\ell(J_\mu^\ell(y_\mu(0),y_\mu(T)))$ and the fact that $\partial\ell$ is demiclosed in $E \times E$ (see e.g. [2], Chapter II). To conclude the proof it remains to verify equation (4.11).

By (5.11) and definition of ∂L_μ we have

$$L_\mu(t,y_\mu,u_\mu) \leq L_\mu(t,y,u)+(q_\mu,y_\mu-y)+\langle\Lambda^*\Pi^*p_\mu+\|u-u_\mu\|^{p-2}.$$
$$\Xi(u-u_\mu),u_\mu-u\rangle$$

for all $u \in L^p(0,T;U)$ and all $y \in C(0,T;E)$.

Integrating over $[0,T]$ and letting μ tend to zero we obtain

(5.29) $\quad \int_0^T L(t,y_0,u_0)dt \leq \int_0^T L(t,y,u)dt + \int_0^T ((q_1,y_0-y) +$

$$+\langle \Lambda^* \Pi^* p_o \quad , u_o - u\rangle \quad)dt$$

(We have used here relations (5.15) $\sim$ (5.18), (5.27) and the demicontinuity of the duality mapping Ξ .) Since in (5.29) the pair $(y,u) \in C(0,T;E) \times L^p(0,T;U)$ is arbitrary we may conclude by a standard argument that

$$(q_o(t), \Lambda^* \Pi^* p_o(t)) \in \partial L(t, y_o(t), u_o(t)) \quad \text{a.e. } t \in]0,T[$$

thereby completing the proof.

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