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A MINIMUM MODULUS THEOREM AND APPLICATIONS TO ULTRADIFFERENTIAL OPERATORS

by

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A minimum modulus theorem and applications to ultradifferential operators

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Ioana Cioranescu and László Zsidó .

In this work we give a minimum modulus theorem which enable us to prove the invertibility of a large class of ultradifferential operators.

It is known that the invertibility of convolution operators defined by ultradistributions S with compact support is equivalent to the existence of a cèrtain lower estimation for the modulus of the Fourier transform of S (see [1] ,[3],[8],[9]).While usual differential operators with constant coefficients are all invertible even in the space of Schwartz's distributions,the following problem is still open: is every ultradifferential operator invertible in the corresponding ultradistributions space or at least in the "union " of all ultradistributions?

In [2] Ch.Ch.Chou positively solved this problem for elliptic ultradifferential operators.For the general case some results are given by the same author in[1];unfortunately,the invertibility is proved under very restictive conditions on the considered ultradistributions space.

The aim of this work is to give a general minimum modulus theorem, improving the well known theorem of L.Ehrenpreis [7] and which yields to an invertibility result in ultradistributions spaces satisfying less restrictive conditions then those of Ch.Ch.Chou.In particular we prove that all ultradifferential operators of class $\left\{ k!\left(\prod_{j=2}^{k} \ln j\right)^{\alpha} \right\}$ with $\alpha > 1$,are invertible,while Chou's result works only for $\alpha > 2$.

§ 1. A minimum modulus theorem.

Let f be an entire function with $f(0) = 1$ and let $a_1, a_2, \dots$ be its zeros indexed such that $|a_1| \leq |a_2| \leq \dots$ .We denote for each $r > 0$

$$M_f(r) = \sup_{|z|=r} |f(z)|$$

and

$$n_f(r) = \text{the numbers of } a_k \text{ with } |a_k| \leq r \quad .$$

Then, by the Jensen formula (see [14]), we have

(1.1) $$n_f(r) \leq \ln M_f(er) \quad , \quad r > 0 \quad .$$

The following result is a refinement of the minimum modulus theorem of L.Ehrenpreis from [7] (for other variants see [8] and [9]); on its proof we used technics both from [7] and from [15], Section 23.

1.1. Theorem. *Let f be an entire function of finite exponential type with $f(0)=1$; then for every $r_0 > 0$ and $0 < \tau < \frac{1}{8e}$, there is an r' with $r_0 \leq r' \leq (1+\tau)r_0$, such that*

$$\inf_{|z|=r'} \ln|f(z)| \geq -6\ln M_f(2er_0)\ln\frac{1}{\tau} - 8\sum_{j=1}^{+\infty} \frac{\ln M_f(2^j er_0)}{4^j} \quad .$$

Proof. Let us define the entire function g by

$$g(z) = f(z)f(-z) \quad , \quad z \in \mathbf{C} \quad .$$

If for $r > 0$, $z_r \in \mathbf{C}$ is such that $|z_r| = r$ and $|f(z_r)| = \inf_{|z|=r} |f(z)|$, then we have

$$\inf_{|z|=r} |g(z)| \leq |f(z_r)| \cdot |f(-z_r)| \leq \inf_{|z|=r} |f(z)| . M_f(r) \quad ,$$

so that

(1.2) $$\inf_{|z|=r} \ln|f(z)| \geq -\ln M_f(r) + \inf_{|z|=r} \ln|g(z)| \quad , \quad r > 0 \; .$$

If $a_1, a_2, \dots$ are the zeros of f, indexed such that $|a_1| \leq |a_2| \leq \dots$, by Hadamard's factorization theorem ([14], 8.22 and 8.24.), we have

$$\sum_{k=1}^{+\infty} \frac{1}{|a_k|^2} < +\infty \quad \text{and} \quad g(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{a_k^2}\right) \quad , \quad z \in \mathbb{C} \ .$$

Let $r_o > 0$ and $0 < \tau < \frac{1}{8e}$ be fixed and denote

$$n' = n_f((1-\tau)r_o) \ , \ n'' = n_f(2r_o) \ .$$

We define the entire functions g_1, g_2, g_3 by

$$g_1(z) = \prod_{k \le n'} \left(1 - \frac{z^2}{a_k^2}\right) \ ,$$

$$g_2(z) = \prod_{n' < k \le n''} \left(1 - \frac{z^2}{a_k^2}\right) \ ,$$

$$g_3(z) = \prod_{k \geq n''} \left(1 - \frac{z^2}{a_k^2}\right) \ .$$

Then

$$g = g_1 g_2 g_3 \ .$$

Let r with $r_o \le r \le (1+\tau)r_o$ and put $n = n_f(r)$; let further $z \in \mathbb{C}$ such that $|z| = r$. We can write

$$g_1(z) = \frac{z^{2n}}{a_1^2 \cdots a_n^2} \cdot \frac{a_{n'+1}^2 \cdots a_n^2}{z^{2(n-n')}} \prod_{k \le n'} \left(\frac{a_k^2}{z^2} - 1\right) \ .$$

As for $1 \le k \le n$, $|a_k| \le |z|$, we get

$$(1.3) \qquad \left| \frac{z^{2n}}{a_1^2 \cdots a_n^2} \right| \geq 1 \ .$$

Next, since for $k > n'$, $|a_k| > (1-\tau)r_o$, we have

$$\left| \frac{a_{n'+1}^2 \cdots a_n^2}{z^{2(n-n')}} \right| \geq \left(\frac{1-\tau}{1+\tau}\right)^{2(n-n')} \ .$$

Using the inequality

$$\frac{1-u}{1+u} \geq u \quad , \quad 0 < u \le \sqrt{2} - 1 \ ,$$

it follows that

(1.4) $$\left|\frac{a_{n'+1}^2\cdots a_n^2}{z^{2(n-n')}}\right| \geqslant \tau^{2(n-n')} .$$

Finally, for $1\le k\le n'$, we have

$$\left|\frac{a_k^2}{z^2}-1\right| \geqslant \frac{(r-|a_k|^2)^2}{r^2} \geqslant \frac{(r-(1-\ \ )r_o)^2}{r^2} \geqslant \frac{(r-(1-\ \ )r)^2}{r^2} = \tau^2 ,$$

hence

(1.5) $$\left|\prod_{k\le n'}\left(\frac{a_k^2}{z^2}-1\right)\right| \geqslant \tau^{2n'} .$$

By (1.3),(1.4) and (1.5)

$$|g_1(z)| \geqslant \tau^{2n'} \geqslant \alpha^{2n_f(2r_o)} ,$$

consequently

(1.6) $$\ln|g_1(z)| \geqslant -2n_f(2r_o)\ln\frac{1}{\tau} .$$

Let us estimate now $|g_3(z)|$, for $|z|=r, r_o\le r\le(1+\tau)r_o$. We have

$$\ln|g_3(z)| \geqslant \sum_{k>n'} \ln\left(1-\left|\frac{z}{a_k}\right|^2\right) =$$

$$= \sum_{j=2} \sum_{n_f(2^{j-1}r_o)<k\le n_f(2^j r_o)} \ln\left(1-\left|\frac{z}{a_k}\right|^2\right) .$$

Using the inequality

$$\ln(1-u) \geqslant -\frac{u}{1-u} , \quad 0<u<1 ,$$

we obtain

$$\ln|g_3(z)| \geqslant -\sum_{j=2} \sum_{n_f(2^{j-1}r_o)<k\le n_f(2^j r_o)} \frac{\left|\frac{z}{a_k}\right|^2}{1-\left|\frac{z}{a_k}\right|^2} .$$

Since for $j\geqslant 2$ and $n_f(2^{j-1}r_o)<k\le n_f(2^j r_o)$ we successively have

$$\left|\frac{z}{a_k}\right| \leq \frac{r_o(1+\tau)}{2^{j-1}r_o} \leq \frac{1+\frac{1}{8}}{2^{j-1}} = \frac{9}{2^{j+2}},$$

$$\frac{\left|\frac{z}{a_k}\right|^2}{1-\left|\frac{z}{a_k}\right|^2} \leq \frac{\frac{81}{4^{j+2}}}{1-\frac{81}{4^{j+2}}} = \frac{81}{4^{j+2}-81} \leq \frac{8}{4^j},$$

then

$$\ln|g_3(z)| \geqslant -8\sum_{j=2}^{+\infty}\frac{n_f(2^j r_o)}{4^j}. \tag{1.7}$$

Finally we observe that

$$g_2(z) = \frac{\prod_{n'<k\leq n''}(a_k^2-z^2)}{a_{n'+1}^2\cdots a_{n''}^2} \tag{1.8}$$

and we apply the Boutroux-Cartan theorem (see [15], Section 2.2.) to the polynomial at the numerator of the fraction from (1.8). Thus for $z\in\mathbb{C}$ outside of $2(n''-n')$ circles with the sum of their diameters less then $8er_o\tau^2$, we have

$$\left|\prod_{n'<k\leq n''}(a_k^2-z^2)\right| \geqslant \left(\frac{2er_o\tau^2}{e}\right)^{2(n''-n')} = (2r_o\tau^2)^{2(n''-n')}$$

so that

$$|g_2(z)| \geqslant \tau^{4(n''-n')} \geqslant \tau^{4n''}. \tag{1.9}$$

Since $8er_o^2 < r_o\tau$, there exists r' with $r_o\leq r'\leq(1+\tau)r_o$ such that every $z\in\mathbb{C}$, $|z| = r'$, is outside of the above $2(n''-n')$ circles.

By (1.9), for $|z| = r'$ holds

$$\ln|g_2(z)| \geqslant -4n_f(2r_o)\ln\frac{1}{\tau}. \tag{1.10}$$

Introducing the estimations (1.6), (1.7) and (1.10) in (1.2), we find

$$\inf_{|z|=r'} \ln|f(z)| \geq -\ln M_f(2r_o) - 6n_f(2r_o)\ln\frac{1}{\tau} - 8\sum_{j=2}^{+\infty}\frac{n_f(2^j r_o)}{4^j} .$$

Using (1.1) in the above inequality, the proof of the theorem is complete.

q.e.d.

If we take in the above theorem, for example $\tau = e^{-4}$, then we reobtain the minimun modulus theorem of L.Ehrenpreis ([4], Th.6). In our result, unlike those of similar kind of O.von Grudzinski from [8] and [9], the dependence of the minimun of the modulus on the parameter τ has a more simple form which is very useful in the applications. This is illustrated by the following

1.2. Corollary. Let $\alpha : (0,+\infty) \longrightarrow (0,+\infty)$ be an increasing function with

$$\int_1^{+\infty} \frac{\alpha(r)}{r^2}\,dr < +\infty \quad \text{and} \quad \int_1^{+\infty} \frac{\alpha(r)}{r^2}\ln\frac{r}{\alpha(r)}\,dr < +\infty .$$

Then there exists an increasing function $\beta : (0,+\infty) \longrightarrow (0,+\infty)$ such that

$$\int_1^{+\infty} \frac{\beta(r)}{r^2}\,dr < +\infty ,$$

and with the property:

if f is an entire function satisfying

$$\ln|f(z)| \leq c\,\alpha(|z|) + c' , \quad z \in \mathbb{C} , \tag{1.11}$$

for some $c,c' > 0$, then there exist $d,d' > 0$ such that for every $r_o > 0$

$$\sup_{r_o \leq r \leq r_o + d\beta(r_o)} \ \inf_{|z|=r} \ln|f(z)| \geq -d\beta(r_o) - d' .$$

Proof. Since $\lim\limits_{r\to+\infty}\frac{\alpha(r)}{r} = 0$, (see [12]), there exists $c_o > 0$ such that

$$\frac{c_o\,\alpha(2er)}{1+r} < \frac{1}{8e} , \quad r > 0 .$$

For $r>0$, let us put

$$\beta(r) = 6\alpha(2er)\ln\frac{1+r}{c_0\alpha(2er)} + 8\sum_{j=1}^{\infty}\frac{\alpha(2^j er)}{4^j} .$$

Then $\beta : (0,+\infty) \longrightarrow (0,+\infty)$ is an increasing function such that

$$(1.12) \qquad \beta(r) \geq \alpha(2er) \quad , \quad r>0 .$$

By the assumptions on the function α, we successively have

$$\int_1^{+\infty} \frac{\alpha(2er)}{r^2}\ln\frac{1+r}{c_0\alpha(2er)}\,dr < +\infty ,$$

$$\int_1^{+\infty}\Big[\sum_{j=1}^{+\infty}\frac{\alpha(2^j er)}{4^j r^2}\Big]dr = e\Big(\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big)\int_1^{+\infty}\frac{\alpha(r)}{r^2}dr < +\infty ,$$

so that

$$\int_1^{+\infty}\frac{\beta(r)}{r^2}dr < +\infty .$$

Let f be an entire function satisfying (1.11) and $r_0>0$; we can obviously suppose that $f(0)=1$. Apply Theorem 1.1. to f, for $\tau = \frac{c_0\alpha(2er_0)}{1+r_0}$; then, by (1.12), there is an r' with $r_0 \leq r' \leq r_0 + c_0\beta(r_0)$, such that

$$\inf_{|z|=r'} \ln|f(z)| \geq -c\beta(r_0) - 6c'(\ln 8+2) .$$

Then the Corollary results with $d = \max(c_0,c)$ and $d' = 6c'(\ln 8+2)$.

q.e.d.

§ 2. Invertible ultradifferential operators.

Let $0<t_1\leq t_2 \ldots$ be such that $t_1<+\infty$ and $\sum_{k=1}^{\infty}\frac{1}{t_k} < +\infty$.
For $r>0$ we define

the distribution function of the sequence $\{t_k\}$

$n(r) = n_{\{t_k\}}(r) =$ the number of t_k with $t_k \leq r$;

the associated function of the sequence $\{t_k\}$

$$N(r) = N_{\{t_k\}}(r) = \ln \max \left\{1, \sup_{k\geq 1} \frac{r^k}{t_1\cdots t_k}\right\} .$$

We further define the entire function of exponential type zero $\omega = \omega_{\{t_k\}}$ by

$$\omega(z) = \prod_{k=1}^{\infty} \left(1 + \frac{iz}{t_k}\right) , \quad z \in \mathbb{C} .$$

We note that (see [12],Ch.I.or [13],Ch.II,Section 1) n,N and ω satisfy the non-quasianalyticity conditions

(2.1) $$\int_1^{+\infty} \frac{n(r)}{r^2}\, dr < +\infty, \quad \int_1^{+\infty} \frac{N(r)}{r^2}\, dr < +\infty, \quad \int_1^{+\infty} \frac{\ln|\omega(r)|}{r^2}\, dr < +\infty.$$

Moreover we have

(2.2) $$\lim_{r\to+\infty} \frac{n(r)}{r} = 0 , \quad \lim_{r\to+\infty} \frac{N(r)}{r} = 0 , \quad \lim_{r\to+\infty} \frac{\ln|\omega(r)|}{r} = 0 .$$

As

(2.3) $$N(r) = \int_0^r \frac{n(\lambda)}{\lambda}\, d\lambda ,$$

it follows

(2.4) $$n(r) \leq N(er) .$$

Obviously

(2.5) $$N(r) \leq \ln|\omega(r)| .$$

On the other hand, for every $r>0$

$$\ln|\omega(r)| = \frac{1}{2}\int_0^{+\infty} \ln\left(1 + \frac{r^2}{\lambda^2}\right) dn(\lambda) =$$

$$= \frac{1}{2} n(\lambda)\ln\left(1 + \frac{r^2}{\lambda^2}\right)\Big|_{\lambda=0}^{\lambda=+\infty} + \int_0^{+\infty} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda =$$

$$= \int_0^{2r} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda + \sum_{j=2}^{+\infty} \int_{2^{j-1}r}^{2^j r} n(\lambda)\frac{r^2}{\lambda(\lambda^2+r^2)}\, d\lambda$$

$$\leq \int_0^{2r} \frac{n(\lambda)}{\lambda}\, d\lambda + \sum_{j=2}^{+\infty} \int_{2^{j-1}r}^{2^j r} \frac{n(\lambda)}{\lambda} \cdot \frac{1}{4^{j-1}}\, d\lambda$$

$$\leq \sum_{j=1}^{+\infty} \frac{1}{4^{j-1}} \int_0^{2^j r} \frac{n(\lambda)}{\lambda} d\lambda \quad .$$

hence by (2.3), we finally get

$$\ln|\omega(r)| \leq 4 \sum_{j=1}^{+\infty} \frac{N(2^j r)}{4^j} \quad . \tag{2.6}$$

We shall give now the

2.1. Lemma. *Let n, N, ω be as above. Then the following statements are equivalent* :

(i) $$\sum_{k=1}^{+\infty} \frac{\ln \frac{t_k}{k}}{t_k} < +\infty \ ;$$

(ii) $$\int_1^{+\infty} \frac{n(r)}{r^2} \ln \frac{r}{n(r)} dr < +\infty \ ;$$

(iii) $$\int_1^{+\infty} \frac{N(r)}{r^2} \ln \frac{r}{N(r)} dr < +\infty \ ;$$

(iv) $$\int_1^{+\infty} \frac{\ln|\omega(r)|}{r^2} \ln \frac{r}{\ln|\omega(r)|} dr < +\infty \quad .$$

Proof. (i) $\Longleftrightarrow$ (ii). Let $k \geq 1$ integer: integrating by parts, we get :

$$\int_{t_k}^{t_{k+1}} \frac{n(r)}{r^2} \ln \frac{r}{n(r)} dr = \int_{t_k}^{t_{k+1}} \frac{k}{r^2} \ln \frac{r}{k} dr =$$

$$= k \int_{t_k}^{t_{k+1}} \ln r \, d\left(-\frac{1}{r}\right) - k\ln k \int_{t_k}^{t_{k+1}} \frac{dr}{r^2} =$$

$$= k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}}\right) + (k - k\ln k) \int_{t_k}^{t_{k+1}} \frac{dr}{r^2} =$$

$$= k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}}\right) + (k - k\ln k)\left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) \ .$$

Hence $$\int_1^{+\infty} \frac{n(r)}{r^2} \ln \frac{r}{n(r)} dr =$$

$$= \sum_{k=1}^{+\infty} \left[k\left(\frac{\ln t_k}{t_k} - \frac{\ln t_{k+1}}{t_{k+1}} \right) + (k-k\ln k)\left(\frac{1}{t_k} - \frac{1}{t_{k+1}} \right) \right] =$$

$$= \sum_{k=1}^{+\infty} \left[\frac{\ln t_k}{t_k} + \frac{1 - \ln \frac{k^k}{(k-1)^{k-1}}}{t_k} \right] =$$

$$= \sum_{k=1} \left[\frac{\ln \frac{t_k}{k}}{t_k} + \frac{1 + \ln k - \ln \frac{k^k}{(k-1)^{k-1}}}{t_k} \right]$$

Using the Stirling formula, it is easy to get

$$\lim_{k \to +\infty} \left[1 + \ln k - \ln \frac{k^k}{(k-1)^{k-1}} \right] = 0 \quad ,$$

hence there exists $c > 0$ such that

$$\left| 1 + \ln k - \ln \frac{k^k}{(k-1)^{k-1}} \right| \leq c \quad , \quad k \geqslant 1 \quad .$$

This yields the desired equivalence.

Let us further examine the condition (iv). By the inequality

$$a^2 \ln\left(1 + \frac{u^2}{c^2}\right) \geqslant \ln(1+u^2) \quad , \quad a,u > 0 \quad ,$$

we have

$$a^2 \ln\left|\omega\left(\frac{t}{a}\right)\right| \geqslant \ln|\omega(t)| \quad , \quad a,t > 0 \quad . \tag{2.7}$$

Using (2.6) and (2.7), we obtain

$$\int_1^{+\infty} \frac{\ln|\omega(r)|}{r^2} \ln \frac{r}{\ln|\omega(r)|} dr \leq$$

$$\leq 4 \sum_{j=1}^{+\infty} \frac{1}{4^j} \int_1^{+\infty} \frac{N(2^j r)}{r^2} \ln \frac{r}{\ln|\omega(r)|} dr =$$

$$= 4 \sum_{j=1}^{+\infty} \frac{1}{4^j} \cdot 2^j \int_{2^j}^{+\infty} \frac{N(t)}{t^2} \ln \frac{t}{2^j \ln\left|\omega\left(\frac{t}{2^j}\right)\right|} dt \leq$$

$$\leq 4\sum_{j=1}^{+\infty}\frac{1}{2^j}\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{2^j a^2 t}{4^j a^2\ln|\omega(\frac{t}{2^j})|}\,dt \leq$$

$$\leq 4\sum_{j=1}^{+\infty}\frac{1}{2^j}\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{2^j a^2 t}{\ln|\omega(at)|}\,dt =$$

$$4\Big(\sum_{j=1}^{+\infty}\frac{1}{2^j}\Big)\int_1^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{\ln|\omega(at)|}\,dt+4\Big(\sum_{j=1}^{+\infty}\frac{\ln 2^j a^2}{2^j}\Big)\int_1^{+\infty}\frac{N(t)}{t^2}dt$$

Thus, by (2.1), (iv) holds if

$$(2.8)\qquad \int_{t_1}^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{\ln|\omega(at)|}\,dt < +\infty \quad \text{for some } a>0 .$$

(iii) $\Longrightarrow$ (iv). This implication results immediately from (2.5) which imply (2.8) with a=1.

(ii) $\Longrightarrow$ (iv). By (2.4) and (2.5), we have

$$\ln|\omega(er)| \geq n(r) .$$

Hence, from (2.8) for a=e, it is enough to prove that

$$(2.9)\qquad \int_{t_1}^{+\infty}\frac{N(t)}{t^2}\ln\frac{t}{n(t)}\,dt < +\infty .$$

Let $\lambda > t_1$ and $n=n(\lambda)$. Then

$$\int_{\lambda}^{+\infty}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt = \int_{\lambda}^{t_{n+1}}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt + \sum_{k\geq n+1}\int_{t_k}^{t_{k+1}}\frac{1}{t^2}\ln\frac{t}{n(t)}\,dt =$$

$$= \int_{\lambda}^{t_{n+1}}\frac{1}{t^2}\ln\frac{t}{n}\,dt + \sum_{k\geq n+1}\int_{t_k}^{t_{k+1}}\frac{1}{t^2}\ln\frac{t}{k}\,dt .$$

Since

$$\int\frac{1}{t^2}\ln\frac{t}{c}\,dt = -\frac{1+\ln\frac{t}{c}}{t} + \text{constant}, \quad c,t>0 ,$$

we deduce

$$\int_{\lambda}^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt = \left(\frac{1}{\lambda} - \frac{1}{t_{n+1}}\right) + \sum_{k \geq n+1} \left(\frac{1}{t_k} - \frac{1}{t_{k+1}}\right) +$$

$$+ \left(\frac{\ln \frac{\lambda}{n}}{\lambda} - \frac{\ln \frac{t_{n+1}}{n}}{t_{n+1}}\right) + \sum_{k \geq n+1} \left(\frac{\ln \frac{t_k}{k}}{t_k} - \frac{\ln \frac{t_{k+1}}{k}}{t_{k+1}}\right) =$$

$$= \frac{1}{\lambda} + \frac{\ln \frac{\lambda}{n}}{\lambda} + \sum_{k \geq n+1} \frac{\ln \frac{k-1}{k}}{t_k} ,$$

consequently

$$(2.10) \qquad \int_{\lambda}^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt \leq \frac{1}{\lambda} + \frac{\ln \frac{\lambda}{n(\lambda)}}{\lambda} .$$

Further, by (2.3) and (2.10) we have

$$\frac{N(t)}{t^2} \ln \frac{t}{n(t)}\, dt = \int_{t_1}^{+\infty} \frac{1}{t^2} \left(\int_{t_1}^{t} \frac{n(\lambda)}{\lambda}\, d\lambda \right) \ln \frac{t}{n(t)}\, dt =$$

$$= \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda} \left(\int_{\lambda}^{+\infty} \frac{1}{t^2} \ln \frac{t}{n(t)}\, dt \right) d\lambda \leq$$

$$\leq \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda^2}\, d\lambda + \int_{t_1}^{+\infty} \frac{n(\lambda)}{\lambda^2} \ln \frac{\lambda}{n(\lambda)}\, d\lambda .$$

Therefore (2.9) holds.

(iv) $\Longrightarrow$ (ii). By (2.7), (2.5) and (2.4), we have

$$e^2 \ln |\omega(r)| \geq \ln |\omega(er)| \geq n(r) , \qquad r > 0 ,$$

so that

$$\frac{n(r)}{r} \leq e^2 \, \frac{\ln |\omega(r)|}{r} .$$

The function $t \longrightarrow t \ln \frac{1}{t}$ is increasing for $0 < t < 1$, so that by (2.2

$$\frac{n(r)}{r^2} \ln \frac{r}{n(r)} \leq e^2 \cdot \frac{\ln |\omega(r)|}{r^2} \cdot \ln \frac{r}{e^2 \ln |\omega(r)|}$$

if r is sufficiently large; this yields the desired implication.

By a similar reasoning we deduce that (iv) $\Longrightarrow$(iii) and this ends the proof of the lemma.

q.e.d.

We can now give our main result :

2.2. Theorem. Let $0 < t_1 \leq t_2 \leq \dots$ be such that

$$t_1 < +\infty , \sum_{k=1}^{+\infty} \frac{1}{t_k} < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{\ln \frac{t_k}{k}}{t_k} < +\infty .$$

Then there exists $0 < s_1 \leq s_2 \leq \dots$, $s_1 < +\infty$, $\sum_{k=1}^{+\infty} \frac{1}{s_k} < +\infty$, with the property :

if f is an entire function with

$$|f(z)| \leq c_0 |\omega_{\{t_k\}}(|z|)|^{n_0} \tag{2.11}$$

for some $c_0 > 0, n_0 \geq 1$ integer, then there exist $a, M > 0$ such that

$$\sup_{|\eta| \leq a\ln M|\omega_{\{s_k\}}(\xi)|} \ln f(\xi+\eta) \geq -a\ln M|\omega_{\{s_k\}}(\xi)|, \ \xi \in \mathbb{C} ,$$

where the supremum can be taken over R, when $\xi \in R$.

Proof. Let us denote $\alpha(r) = \ln|\omega(r)|$, $r > 0$. Then by Lemma 2.1., the conditions from Corollary 1.2. are fullfilled for the function Hence there exists an increasing function $\beta : (0,+\infty) \longmapsto (0,+\infty)$ with $\int_1^{+\infty} \frac{\beta(r)}{r^2} dr < +\infty$ such that for every function satisfying (2.11) there are $d, d' > 0$ such that for each $r_0 > 0$

$$\sup_{r_0 \leq r \leq r_0 + d.\beta(r_0)} \inf_{|z|=r} |f(z)| \geq -d\beta(r_0) - d' .$$

But from the above inequality we easily get

$$\sup_{|\eta| \leq d\beta(|\xi|)} \ln|f(\xi+\eta)| \geq -d\beta(|\xi|) - d' , \ \xi \in \mathbb{C}, \xi \neq 0 , \tag{2.12}$$

where for real ξ , the supremum can be taken over $\eta \in R, |\eta| \leq d\beta(|\xi|)$ as a simple reasoning shows.

Further, by a result of O.I.Izonemcev and V.A.Marcenko [11], for the function β there is a sequence $0 < s_1 \leq s_2 \leq \dots, \sum_{k=1}^{+\infty} \frac{1}{s_k} < +\infty$, such that

$$\beta(t) \leq \ln |\omega_{\{s_k\}}(t)| \quad , \quad t > 0$$

for some constant $\gamma > 0$ .Then the statement results from (2.12) for suitable positive constants a and M,enabling us also to remove the restriction $\xi \neq 0$.

q.e.d.

We shall give now the

2.3.Definition. Let $0 < t_1 \leq t_2 \leq \dots, t_1 < +\infty$, $\sum_{k=1}^{+\infty} \frac{1}{t_k} < +\infty$ and $\omega = \omega_{\{t_k\}}$ .An entire function f is called ω-slowly decreasing if there are a and $M > 0$ such that

$$\sup_{\substack{|\eta| \leq a \ln M |\omega(\xi)| \\ \eta \in R}} |f(\xi + \eta)| \geq M^{-a} |\omega(\xi)|^{-a} \quad , \quad \forall \xi \in R .$$

Let us remark that all kinds of slowly decreasing functions considered in [1], [8] and [9] are ω-slowly decreasing for some suitable function ω ,as the above mentioned result on entire majorants from [11] shows.

From Theorem 2.2. we immediately get

2.4.Corollary. Let $\omega_{\{t_k\}}, \omega_{\{s_k\}}$ and f be as in Theorem 2.2.;then f is $\omega_{\{s_k\}}$ -slowly decreasing .

In order to apply the above results to ultradifferential operators, we recall some facts about the ω -ultradistributions considered in [3], [4], [5] and [6] .

For $K \subset R$ compact,we define the functions space :

$$\mathcal{D}_\omega(K) = \left\{ \varphi \in C_0^\infty \; ; \; \begin{array}{l} p_{L,n}(\varphi) = \sup_{t \in R} |\hat{\varphi}(t)\, \omega(Lt)^n| < +\infty \\ \text{for every } L > 0, n \geq 1 \text{ integer} \end{array} \right\}$$

($\hat{\varphi}$ denotes the Fourier transform of φ)

$$\mathcal{D}_\omega = \underset{K \subset R}{\underrightarrow{\lim}} \mathcal{D}_\omega(K) .$$

The elements of the dual $\mathcal{D}'_\omega$ are called ω -ultradistributions.

The "union" over ω of all ω-ultradistributions coïncide with the "union" of all Roumieu (or Beirling) ultradistributions (see [6]); the parametrization over ω presents a series of advantages among which we mention the stability under ultradifferential operators of each $\mathcal{D}'_\omega$.

As for distributions, one can define the notion of support for ultradistributions. If $S \in \mathcal{D}'_\omega$ has a compact support, then its Fourier transform $\hat{S}$ is defined by $\hat{S}(z) = \langle S, e^{-izt} \rangle$, $z \in \mathbb{C}$.

2.5. Definition. We say that the ω-ultradistribution S with compact support is invertible in $\mathcal{D}'_\omega$ if $S * \mathcal{D}'_\omega = \mathcal{D}'_\omega$.

In [3] the following result is proved :

S is invertible in $\mathcal{D}'_\omega$ iff S is ω-slowly decreasing.

In [5], [6] we called ω-ultradifferential operators all linear operators on $\mathcal{D}_\omega$ preservin the support.

Among the ω-ultradifferential operators we distinguish a particular class defined by

2.6. Definition. An operator of the form

$$f(D) = \sum_{k=1}^{+\infty} c_k D^k \quad , \quad c_k \in \mathbb{C} \quad ,$$

is called an ω-ultradifferential operator with constant coefficients if there are $c_o, n_o > 0$ such that

$$|f(z)| \le c_o |\omega(|z|)|^{n_o} \quad , \quad z \in \mathbb{C} \quad .$$

(we mention that in [5],[6] we called ω-ultradifferential operators with constant coefficients a more general class of operators, but in applications those from the above definition are more significant).

ω-ultradifferential operators with constant coefficients have the property that the series $\sum_{k=1}^{+\infty} c_k D^k$ converges in $\mathcal{L}(\mathcal{D}'_\omega)$, without requiring additional properties for the function ω (as the condition (M.2) of stability under ultradifferential operators in the frame of Roumieu or Beurling ultradistributions in [10]). Moreover, every ultradifferential operator of class (M_k) or $\{M_k\}$ (see [10]) is an

ω-ultradifferential operator for a suitable function ω.

With the above considerations we directly get from Theorem 2.2.:

2.7. Proposition. Let $0 < t_1 \leq t_2 \leq \ldots$, be such that

$$(2.13) \qquad t_1 < +\infty, \quad \sum_{k=1}^{+\infty} \frac{1}{t_k} < +\infty \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{\ln \frac{t_k}{k}}{t_k} < +\infty .$$

Then there exist $0 < s_1 \leq s_2 \leq \ldots, s_1 < +\infty$, $\sum_{k=1}^{+\infty} \frac{1}{s_k} < +\infty$ such that every $\omega_{\{t_k\}}$ -ultradifferential operator with constant coefficients is invertible in $\mathcal{D}'_{\omega_{\{s_k\}}}$.

We shall apply this invertibility result in some particular cases.

I. It is clear that the Gevrey sequence $t_k = k^{\alpha}$, $\alpha > 1$, satisfies the conditions (2.13).

II. Let $t_k = k(\ln k)^{\alpha}$, $\alpha > 1$; an easy computation shows that also this sequence satisfies the conditions (2.13). So we can positively answer to the question asked in [1] concearning the invertibility of ultradifferential operators of class $\{ k!(\prod_{j=2}^{k} \ln j)^{\alpha} \}$, for arbitrary $\alpha > 1$. In [1] only the case $\alpha > 2$ was solved.

III. Finally we remark that the above proposition improve Theorem III 2-3., from [1].

Namely, let $\{M_k\} \in \mathcal{M}$, where $\mathcal{M}$ is the space of sequences from [1]. If there is $\{Q_k\} \in \mathcal{M}$ such that the associated functions $M(r)$ and $Q(r)$, to the sequences $t_k = M_k / M_{k-1}$ and $s_k = Q_k / Q_{k-1}$, satisfy

i) $r \longmapsto \frac{M(2r)}{Q(r)}$ is decreasing and $\int_1^{+\infty} \frac{M(2t)}{tQ(t)} dt = O\left(\frac{Q(r)}{r}\right)$,

ii) $\frac{Q(r)}{rM(r)} e^{\frac{Q(r)}{M(r)}} \gg 1$,

then by Theorem III.2-3. from [1], each $\{M_k\}$ -ultradifferential operator with constant coefficients is invertible in a $\mathcal{D}'_{\{R_k\}}$, for a suitable sequence $\{R_k\} \in \mathcal{M}$.

As for r sufficiently large, $Q(r) \geq r$, from ii) we get

$$e^{\frac{Q(r)}{M(r)}} \geq \frac{rM(r)}{Q(r)} \geq M(r) \quad ,$$

so that

$$M(r)\ln M(r) \leq Q(r) \quad .$$

Consequently

$$(2.14) \qquad \int_1^{+\infty} \frac{M(r)\ln M(r)}{r^2}\,dr \leq \int_1^{+\infty} \frac{Q(r)}{r^2}dr < +\infty \quad .$$

(we used the non-quasianalyticity of the sequences from $\mathcal{M}$)
By Theorem 4,Ch.II,§ 2 from [13], (2.14) is equivalent to

$$(2.15) \qquad \sum_{k=1}^{+\infty} \frac{\ln t_k}{t_k} < +\infty .$$

It is obvious that condition (2.15) implies condition (2.13) and thus Proposition 2.7. extends Chou's result.Let us still remark that for the sequence $\{t_k\}$ from II, (2.15) holds only for $\alpha > 2$;the case $1 < \alpha \leq 2$ can be solved only in the frame of the less restrictive condition (2.13).

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