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DERIVATIONS OF NEST ALGEBRAS WITH VALUES IN $K(H)$

by

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ABSTRACT. It is shown that every derivation of a nest algebra $\mathcal{B} \subset B(H)$ into $\mathcal{K}(H)$ is ineer.

1. Introduction and statement of the results

Let H be a separable Hilbert space, B(H) the algebra of all bounded, linear operators on H and $\mathcal{K}(H)$ the two sided ideal of compact operators.

In [2] there are considered the following properties which a von Neumann algebra $\mathcal{A} \subset B(H)$ may have:

(P_1) If $b \in B(H)$ and $ab - b.a \in \mathcal{K}(H)$ for all $a \in \mathcal{A}$, then $b \in \mathcal{A}' + \mathcal{K}(H)$.

(P_2) Every derivation from $\mathcal{A}$ into $\mathcal{K}(H)$ is inner.

It is easy to see that $(P_2) \Rightarrow (P_1)$. In [2] it is shown that any von Neumann algebra $\mathcal{A} \subset B(H)$ which does not contains some Type II_1 factors as a direct summand has (P_2) and so it has (P_1) too.

In this paper (Theorem 1) we show that every derivation from a nest algebra $\mathcal{B}$ into $\mathcal{K}(H)$ is inner. If $\mathcal{B} \subset B(H)$ is a subalgebra, we denote by Lat $\mathcal{B}$ the lattice of all invariant projections of $\mathcal{B}$. If $\mathcal{L}$ is a lattice of projections, we denote $\mathrm{alg}\,\mathcal{L} = \{a \in B(H) \mid \mathcal{L} \subset \mathrm{Lat}\, a\}$. A subalgebra $\mathcal{B} \subset B(H)$ is called reflexive if $\mathcal{B} = \mathrm{alg\,Lat}\,\mathcal{B}$ [4]. A reflexive algebra $\mathcal{B}$ is called a nest algebra if Lat $\mathcal{B}$ is totally ordered by inclusion [5]. A derivation from $\mathcal{A}$ into $\mathcal{K}(H)$ is a linear map $D: \mathcal{A} \longrightarrow \mathcal{K}(H)$ with $D(a_1.a_2) = a_1 D(a_2) + D(a_1).a_2$ for all a_1, a_2 in $\mathcal{A}$. If $c \in \mathcal{K}(H)$ then $\delta c: \mathcal{A} \longmapsto \mathcal{K}(H)$, defined by $\delta c(a) = ac - ca$, $a \in \mathcal{A}$ is a derivation; such derivations are called inner.

<u>Theorem 1</u>. Let $\mathcal{B} \subset B(H)$ be a nest algebra. Then, every derivation $D: \mathcal{B} \longmapsto \mathcal{K}(H)$ is inner.

This result will be deduced from the following theorem. Recall that an operator $b \in B(H)$ essentially commutes with $\mathcal{B}$ if $ba - ab \in \mathcal{K}(H)$ for all $a \in \mathcal{B}$.

We shall denote by $\mathcal{B}'_{ess}$ the essential commutant of $\mathcal{B}$.

<u>Theorem 2</u>. Let $\mathcal{B} \subset B(H)$ be a nest algebra. Then

$$\mathcal{B}'_{ess} = \{\lambda \cdot 1 + c \mid \lambda \in \mathbb{C},\ c \in \mathcal{K}(H)\}.$$

<u>Proof of Theorem 1</u>. Since $\mathcal{B}$ is a nest algebra, and hence reflexive, it follows that the maximal abelian von Neumann subalgebra $\mathcal{R}$ of $B(H)$ which contains Lat $\mathcal{B}$ is included in $\mathcal{B}$. Let now $D: \mathcal{B} \longmapsto \mathcal{K}(H)$ be a derivation. Then $D|_{\mathcal{R}}$ is a derivation from $\mathcal{R}$ into $\mathcal{K}(H)$. By [2] Theorem 2.1, there exists $c_o \in \mathcal{K}(H)$ such that $D|_{\mathcal{R}} = \delta c_o|_{\mathcal{R}}$. Let $D_o = D - \delta c_o$. Then $D_o|_{\mathcal{R}} = 0$. We show that $D_o(\mathcal{B}) \subset \mathcal{B}$. Indeed, let $a \in \mathcal{B}$ and $p \in \mathrm{Lat}\,\mathcal{B} \subset \mathcal{R}$. Then $ap = pap$, and we have:

$$D_o(a.p)=a.D_o(p)+D_o(a)p=D_o(a).p$$

$$D_o(p.a.p)=pD_o(a)p+D_o(p)a.p+paD_o(p)=pD_o(a)p$$

Hence $D_o(a)p=p.D_o(a)p$ and so $D_o(a)\in \text{alg Lat}\,\mathcal{B}=\mathcal{B}$. Therefore $D_o:\mathcal{B}\mapsto\mathcal{B}\cap\mathcal{K}(H)$. By [1] Corollary 3.11 it follows that there exists $a_o\in\mathcal{B}$ such that $D_o=\delta a_o$. Since $D_o:\mathcal{B}\mapsto\mathcal{K}(H)$, we have $a_o\in\mathcal{B}'_{ess}$. By Theorem 2 $a_o=\lambda 1+c_1$ for some $\lambda\in\mathbb{C}$ and $c_1\in\mathcal{K}(H)$. Thus $D=\delta(c_o+a_o)=\delta c$ where $c=c_1+c_o\in\mathcal{K}(H)$ which completes the proof.

The proof of Theorem 2 occupies the whole of this paper. In the § 2 we prove Theorem 2 in the special case when the diagonal $\mathcal{B}\cap\mathcal{B}^*$ is non-atomic, the § 3 is concerned with the totally atomic case and in the § 4 we give the proof of Theorem 2.

2. The non-atomic case

Let $R_+^*=(0,+\infty)$ be the multiplicative group of positive reals and μ the Haar measure on R_+^*.

2.1. Lemma. Let $h:(0,+\infty)\longmapsto\mathbb{C}$ be an essentially bounded, measurable function such that for every ε, $0<\varepsilon<1$, we have $h(x)=h(\varepsilon x)$ μ-a.e. Then h is constant a.e.

Proof. It is easy to see that $h(x)=h(d.x)$ μ-a.e. for every $d\in(0,+\infty)$. Let M_h be the multiplication by h on $L^2(0,+\infty)$. Then M_h commutes with all operators $T\in B(L^2(0,+\infty))$ defined by:

$$(T_\varepsilon f)(x)=f(\varepsilon x) \qquad (\forall) \quad \varepsilon>0.$$

By Fuglede's Theorem, the operators T_ε commute with the spectral projections of M_h. On the other hand, by the uniqueness of the Haar measure, it follows that the set of operators $\{T_\varepsilon\}_{\varepsilon>0}$ has no nontrivial invariant projections in $L^\infty(0,+\infty)$. Therefore, h is constant a.e.

For the definition and properties of triangular operator algebras we send to [3].

2.2. <u>Proposition</u>. Let $\mathcal{T}\subset B(H)$ be a weakly closed, maximal triangular operator algebra with non-atomic diagonal $\mathcal{R}$. Then $\mathcal{T}'_{ess}=\{\lambda.1+k \mid \lambda\in\mathbb{C},\ k\in\mathcal{K}(H)\}$.

<u>Proof</u>. The hypotesis on $\mathcal{T}$ and [6] implies that $\mathcal{T}$ is hyperreducible. By [3] Theorem 3.3.1, all hyperreducible, maximal triangular algebras with non-atomic diagonals are unitarily equivalent. So we can assume that $H=L^2[(0,+\infty),\mu]$ and $\mathcal{T}=\mathrm{alg}\,\mathcal{L}$ where $\mathcal{L}=\{P_{(\lambda,+\infty)} \mid \lambda\in[0,+\infty)$ and $P_{(\lambda,+\infty)}$ is the projection due to multiplication by the characteristic function χ_λ of $(\lambda,+\infty)$. Obviously $\mathcal{R}=\mathcal{T}\cap\mathcal{T}^*=L^\infty(0,+\infty)$. Let now $b\in\mathcal{T}'_{ess}$. In particular $b\in\mathcal{R}'_{ess}$. Since $\mathcal{R}$ is a maximal abelian von Neumann algebra, by [2] Theorem 2.1 it follows that $b=r_1+k_1$ where $r_1\in\mathcal{R}$ and $k_1\in\mathcal{K}(H)$. It is obvious that $r_1\in\mathcal{T}'_{ess}$. Let $h_1\in L^\infty(0,+\infty)$ be such that r_1 is equal to the multiplication M_{h_1} by h_1. We show that h_1 is constant a.e. and therefore $r_1=\lambda_1.1$ for some $\lambda_1\in\mathbb{C}$.

Suppose by contradiction that h_1 is not constant a.e. Then, by Lemma 2.1 there exists ε, $0<\varepsilon<1$ such that the function $h_2(x)=h_1(x)-h_1(\varepsilon x)$ is not negligeable. We consider the following operator $T_\varepsilon\in B[L^2(0,+\infty)]$:

$$(T_\varepsilon f)(x)=f(\varepsilon x)$$

It is easy to see that $T_\varepsilon\in\mathcal{T}$. Moreover:

$$\left[(r_1T_\varepsilon - T_\varepsilon r_1)f\right](x) = \left[(M_{h_1}T_\varepsilon - T_\varepsilon M_{h_1})f\right](x) = (h_1(x) - h_1(\varepsilon x))f(\varepsilon x) =$$

$$= h_2(x)f(\varepsilon x).$$

The operator $S = r_1T_\varepsilon - T_\varepsilon r_1$ is not compact. Indeed, since h_2 is not negligeable, there exists $\delta_0 > 0$ such that the set $A_0 = \{|h_2(x)| \geqslant \delta_0$ is not negligeable. Then the set Λ_0 is also not negligeable. An easy computation shows that the range of the operator $SM_{\chi_{\varepsilon A_0}}$ is $M_{\chi_{A_0}}.L^2(0,\infty)$. Therefore S cannot be compact.

This contradiction shows that h_1 is constant a.e., so $r_1 = \lambda_1.1$ for some $\lambda_1 \in \mathbb{C}$, which completes the proof.

2.3. <u>Lemma</u>. Let $\mathcal{B} \subset B(H)$ be a reflexive algebra. Then $(\mathrm{Lat}\,\mathcal{B})' = \mathcal{B} \cap \mathcal{B}^*$.

<u>Proof</u>. The inclusion $\mathcal{B} \cap \mathcal{B}^* \subset (\mathrm{Lat}\,\mathcal{B})'$ is obvious. Let now $b \in (\mathrm{Lat}\,\mathcal{B})'$. Then $\mathrm{Lat}\,\mathcal{B} \subset (\mathrm{Lat}\ b) \cap (\mathrm{Lat}\ b^*)$. Since $\mathcal{B}$ is reflexive, it follows that $b, b^* \in \mathcal{B}$. Therefore $b \in \mathcal{B} \cap \mathcal{B}^*$.

2.4. <u>Lemma</u>. Let $\mathcal{B} \subset B(H)$ be a reflexive algebra, and $p \in \mathcal{B}$ a projection. Then $p\mathcal{B}p = \mathrm{alg}\{pep \mid e \in \mathrm{Lat}\,\mathcal{B}\}$.

<u>Proof</u>. By Lemma 2.3. we have $pe = ep$ for every $e \in \mathrm{Lat}\,\mathcal{B}$. Let $b \in \mathcal{B}$ and $e \in \mathrm{Lat}\,\mathcal{B}$. We have:

$pbpep = pbep = pebep = pepbpep$. Hence
$p\mathcal{B}p \subset \mathrm{alg}\{pep \mid e \in \mathrm{Lat}\,\mathcal{B}\}$.

Let now $c \in B(H)$ such that $pcp \in \mathrm{alg}\{pep \mid e \in \mathrm{Lat}\,\mathcal{B}\}$. Then, we have:

$pcpep = pepcpep$ for every $e \in \mathrm{Lat}\,\mathcal{B}$.

Therefore

$$pcpe=epcpe, \quad \text{so } pcp \ \ p\mathcal{B}p.$$

2.5. Lemma. Let $\mathcal{N}\subset B(H)$ be a von Neumann algebra with commutative commutant. Then $\mathcal{N}$ has minimal projections if and only if $\mathcal{N}'$ has atoms.

Proof. Let $e\in\mathcal{N}$ be a minimal porjection, and $z(e)$ the central support of e. It is easy to see that $z(e)$ is an atom in $\mathcal{N}'$. Conversely let $z\in\mathcal{N}'$ be an atom. Then $z\mathcal{N}'z=\mathbb{C}z$. Hence $z\mathcal{N}z=B(zH)\subset\mathcal{N}$, so $\mathcal{N}$ has minimal projections.

2.6. Proposition. Let $\mathcal{B}\subset B(H)$ be a nest algebra such that its diagonal $\mathcal{B}\cap\mathcal{B}^*$ has no minimal projections. Then $\mathcal{B}'_{ess}=\{\lambda\cdot 1+c \mid \lambda\in\mathbb{C},\ c\in\mathcal{K}(H)\}$.

Proof. Since $\mathcal{B}$ is reflexive and Lat $\mathcal{B}$ is commutative, it follows that $\mathcal{B}$ contains the maximal abelian von Neumann algebra $\mathcal{R}$ generated by Lat $\mathcal{B}$. Obviously, $\mathcal{R}\subset\mathcal{B}\cap\mathcal{B}^*$. Let $b\in\mathcal{B}'_{ess}$. In particular, $b\in(\mathcal{B}\cap\mathcal{B}^*)'_{ess}$. By [2] §§ 2, 3 there exist $r_1\in(\mathcal{B}\cap\mathcal{B}^*)'$ and $c\in\mathcal{K}(H)$ such that $b=r_1+c$. Since $b\in\mathcal{B}'_{ess}$ it follows easily that $r_1\in\mathcal{B}'_{ess}$. Let $\mathcal{R}_1$ be the von Neumann algebra generated by Lat $\mathcal{B}$, $x_o\in H$ a cyclic vector for $\mathcal{R}$ and $p\in\mathcal{R}_1'$ the projection onto $\overline{sp}\,\{rx_o \mid r\in\mathcal{R}_1\}$. By Lemma 2.3, $\mathcal{R}_1'=\mathcal{B}\cap\mathcal{B}^*$. Therefore, by Lemma 2.5 $p\mathcal{R}_1p$ has no atoms. Since x_o is cyclic for $p\mathcal{R}_1p$, it follows that $p\mathcal{R}_1p$ is maximal abelian in $B(pH)$. It is obvious that the von Neumann algebra generated by $\{pep \mid e\in \text{Lat}\,\mathcal{B}\}$ is $p\mathcal{R}_1p$. By Lemma 2.4 $p\mathcal{B}p=\text{alg}\,\{pep \mid e\in\text{Lat}\,\mathcal{B}\}$. Using [3] Theorem 3.1.1 it follows that $p\mathcal{B}p$ is a weakly closed maximal triangular algebra in $B(pH)$ with diagonal $p\mathcal{R}_1p$. Now, we show that $r_1=\lambda\cdot 1$ for some $\lambda\in\mathbb{C}$ Suppose by contradiction that r_1 is not a multiple of the [illegible]. Indeed if $r_1.p=\lambda.p$ for some $\lambda\in\mathbb{C}$, then in particular $r_1x_o=\lambda x_o$. Since $r_1\in\mathcal{R}_1\subset\mathcal{R}$ and x_o is cyclic for $\mathcal{R}$ it results that $r_1=\lambda.1$. Hence $r_1.p\notin\mathbb{C}.p$.

By Proposition 2.2, there exists $c\in p\mathcal{B}p$ such that $pr_1pcp-pcpr_1p\notin\mathcal{K}(pH)$. Therefore $r_1(pcp)-(pcp)r_1\notin\mathcal{K}(H)$. This last fact contradicts $r_1\in\mathcal{B}'_{ess}$. Hence $r_1=\lambda.1$ for some $\lambda\in\mathbb{C}$.

3. The totally atomic case

3.1. Proposition. Let $\mathcal{T}\subset\mathcal{B}(H)$ be a weakly closed, maximal triangular algebra with totally atomic diagonal $\mathcal{R}$. Then $\mathcal{T}'_{ess}=\{\lambda.1+k \mid \lambda\in\mathbb{C},\ k\in K(H)\}$.

For the proof of this propostion we need the following two Lemmas:

3.2. Lemma. Let F be an infinite, countable, totally ordered set. Suppose that I, J are infinite, disjoint subsets of F. Then, there exist two infinite subsets $I_1\subset I$, $J_1\subset J$ and a one-to-one mapping $\varphi:I_1\longmapsto J_1$ such that wither $\varphi(i)\leq i$ for every $i\in I_1$, or $\varphi(i)\geq i$ for every $i\in I_1$.

Proof. Let $\Gamma=\{(f_1,f_2)\in F\times F \mid f_1\leq f_2\}$ We denote:

$$\mathcal{F}=\{F'\subset(I\times J)\cap\Gamma \mid (\forall)\ (i,j),(i',j')\in F' \text{ with } (i,j)\neq(i',j') \text{ implies } i\neq i' \text{ and } j\neq j'\}.$$

If $\mathcal{F}=\emptyset$ then $J\times I\subset\Gamma$ and hence every one-to-one mapping

$\varphi: I \longmapsto J$ satisfies $\varphi(i) \geq i$ for every $i \in I$. In this case $I_1 = I$, $J_1 = J$. Suppose now $\mathcal{F} \neq \emptyset$. We show that $\mathcal{F}$ is inductively ordered by inclusion. Let $\{F'_\alpha\}_{\alpha \in A} \subset \mathcal{F}$ be a totally ordered family. Since $F'_\alpha \subset (I \times J) \cap \Gamma$, we have $\bigcup_\alpha F'_\alpha \subset (I \times J) \cap \Gamma$. Let (i,j), $(i',j') \in \bigcup_\alpha F'_\alpha$, with $(i,j) \neq (i',j')$. Since $\{F'_\alpha\}$ is totally ordered, there exists $\alpha_0 \in A$ such that (i,j), $(i',j') \in F'_{\alpha_0}$. Therefore $i \neq i'$ and $j \neq j'$ whence $\bigcup_\alpha F'_\alpha \in \mathcal{F}$, so $\mathcal{F}$ is inductively ordered. By Zorn's Lemma, $\mathcal{F}$ has a maximal element F'_0. If F'_0 is infinite, then it is the graph of a function φ with the required properties. In this case $I_1 = pr_I F'_0$ and $J_1 = pr_J F'_0$.

If F'_0 is finite, then every one-to-one mapping $\varphi: \complement pr_I F'_0 \longmapsto \complement pr_J F'_0$ satisfies $\varphi(i) \geq i$ for every $i \in \complement pr_I F'_0$. In this case $I_1 = pr_I F'_0$ and $J_1 = pr_J F'_0$.

3.3. <u>Lemma</u>. Let H be a separable Hilbert space and $\{x_s\}_{s \in \Sigma}$ an orthonormal basis. Let $\{r_s\}_{s \in \Sigma}$ be a bounded sequence of complex numbers. Then the operator r defined by $r x_s = r_s x_s$ is compact if and only if the set $\{r_s\}$ has zero as the only limit point.

<u>Proof</u>. Let $\theta: \mathbb{N} \longmapsto \Sigma$ be a one-to-one mapping. Then the operator $U: \ell^2 \longmapsto H$ defined by $U(0,\ldots,\underset{k}{1},0,\ldots) = x_{\theta(k)}$ is unitary and $U^{-1} r U$ is the multiplication by the bounded sequence $\{r_{\theta(n)}\}$ on ℓ^2. In this case the result is well-known.

Let now $\mathcal{T}$ be a weakly closed, maximal triangular algebra with totally atomic diagonal $\mathcal{R}$. If $f \in B(H)$ is a projection, we denote by $h(f)$ ("the hull of f in $\mathcal{T}$" [3]) the intersection of all $e \in \mathrm{Lat}\,\mathcal{T}$ such that $f \leq e$.

By [3] Theorem 3.2.1 the total ordering of $\mathrm{Lat}\,\mathcal{T}$ induces a total ordering on the atoms $\{e_s\}_{s \in \Sigma}$ of $\mathcal{R}$ by means of the mapping from projections to their hulls, which is one-to-one on the atoms.

Let $\ll$ this ordering on Σ . From the proof of [3] Theorem 3.2.1 we have $h(e_s)=\sum_{s'\ll s} e_s$, for every $s\in\Sigma$. Since Lat$\mathcal{T}\subset\mathcal{R}$, it follows that every $e\in$Lat$\mathcal{T}$ is a supremum of a family $\{h(e_s)\}$.

<u>Proof of Proposition 3.1</u>. Let $\{e_s\}_{s\in\Sigma}$ be the atoms of $\mathcal{R}$. If $b\in\mathcal{T}'_{ess}$, then in particular $b\in\mathcal{R}'_{ess}$. By [2] Theorem 2.1, $b=r+k_1$ where $r\in R$ and $k_1\in\mathcal{K}(H)$. Obviously, $r\in\mathcal{T}'_{ess}$. We show that $r=\lambda\cdot 1+k_2$ where $\lambda\in\mathbb{C}$ and $k_2\in\mathcal{K}(H)$. Suppose by contradiction that r is not of this form. Then, by Lemma 3.3, the set $\{r_s\}_{s\in\Sigma}$ where $re_s=r_se_s$ for every $s\in\Sigma$, has at least two different limit points $\lambda_1\neq\lambda_2$. Then, there exists two infinite subsets $\Sigma_1=\{s_n^1\}\subset\Sigma$, $\Sigma_2=\{s_n^2\}\subset\Sigma$ and $\varepsilon_0>0$ such that:

(i) $\Sigma_1\cap\Sigma_2=\emptyset$

(ii) $\lim_{n\to\infty} r_{s_n^1}=\lambda_1$ and $\lim_{n\to\infty} r_{s_n^2}=\lambda_2$

(iii) $|s_n^1-s_k^2|\geq\varepsilon_0$ for every $n,k\in\mathbb{N}$.

By Lemma 3.2, there exist two infinite subsets $\Sigma_{11}\subset\Sigma_1$, $\Sigma_{22}\subset\Sigma_2$ and a one-to-one mapping $\varphi:\Sigma_{11}\longmapsto\Sigma_{22}$ such that $\varphi(s)\ll s$ for every $s\in\Sigma_{11}$. (If the situation $\varphi(S)\gg S$ hold, then we consider the inverse map $\varphi^{-1}:\Sigma_{22}\longmapsto\Sigma_{11}$).

We now define the mapping $:\tilde{\varphi}:\Sigma\longmapsto\Sigma$ in the following way:

$$\tilde{\varphi}(s)=\begin{cases}\varphi(s) & \text{if } s\in\Sigma_{11}\\ s & \text{if } s\notin\Sigma_{11}\end{cases}$$

Obviously: $\tilde{\varphi}(s)\leqq s$ for every $s\in\Sigma$, $\tilde{\varphi}(s)\ll s$ if $s\in\Sigma_{11}$, $\tilde{\varphi}|_{\Sigma_{11}}$ is injective and $\tilde{\varphi}(\Sigma_{11})=\Sigma_{22}$.

Let $x_s\in e_sH$, $\|x_s\|=1$ for every $s\in\Sigma$. We define the operator $T_{\tilde{\varphi}}\in B(H)$ by:

$$T_{\tilde{\varphi}}\left[(\lambda_s x_s)\right] = \left(\lambda_{\tilde{\varphi}(s)} x_{\tilde{\varphi}(s)}\right) \quad \text{for every } (\lambda_s x_s)_{s\in\Sigma} \in H.$$

We show that $T_{\tilde{\varphi}} \in \mathcal{T}$. Since $\mathcal{T}$ is reflexive, then it suffices to show that $\mathrm{Lat}\,\mathcal{T} \subset \mathrm{Lat}\, T_{\tilde{\varphi}}$. Since every $e \in \mathrm{Lat}\,\mathcal{T}$ is a supremum of a family $\{h(e_s)\}$, we must show that $h(e_s) \in \mathrm{Lat}\, T_{\tilde{\varphi}}$ for every $s \in \Sigma$. By the remark before the proof of Proposition 3.1 we have $h(e_s) = \sum_{s' \ll s} \varrho_{s'}$. Since $\tilde{\varphi}(s) \lll s$ for every $s \in \Sigma$, it is easy to see that $h(e_s) \in \mathrm{Lat}\; T_{\tilde{\varphi}}$. Therefore $T_{\tilde{\varphi}} \in \mathcal{T}$.

We show, now that the operator $rT_{\tilde{\varphi}} - T_{\tilde{\varphi}}r$ is not compact. Indeed, we have:

$$\left(rT_{\tilde{\varphi}} - T_{\tilde{\varphi}}r\right)\left[(\lambda_s x_s)\right] = \left((r_s - r_{\tilde{\varphi}(s)})\,\lambda_{\tilde{\varphi}(s)}\, x_{\tilde{\varphi}(s)}\right).$$

Let $S = rT_{\tilde{\varphi}} - T_{\tilde{\varphi}}.r$ and $p_s = r_s - r_{\tilde{\varphi}(s)}$ for every $s \in \Sigma$. By the property (iii) above $|p_s| \geq \varepsilon_0 > 0$ for every $s \in \Sigma_{11}$. Let p_1 be the projection of H onto the subspace generated by $\{e_s H\}_{s\in\Sigma_{11}}$ and p_2 be the projection of H onto the subspace generated by $\{e_s H\}_{s\in\Sigma_{22}}$.

Since Σ_{11} and Σ_{22} are infinite subsets of Σ, it results that $p_1 H$ and $p_2 H$ are infinite dimensional. It is easy to see that the range of the operator Sp_2 is $p_1 H$ and so S cannot be compact. This contradiction shows that r is of the form $r = \lambda \cdot 1 + k_2$ and hence $b = \lambda 1 + k$ where $\lambda \in \mathbb{C}$ and $k = k_1 + k_2 \in \mathcal{K}(H)$.

4. The general case

The proof of the following Lemma is similar with the proof of [3] Lemma 2.3.4.

4.1. Lemma. Let $\mathcal{L}$ be a maximal, totally ordered lattice of projections in $B(H)$ and $\mathcal{R}_1$ the von Neumann algebra generated by $\mathcal{L}$. If $e \in \mathcal{R}_1$ is an atom and $h(e)$ the hull of e in $\mathcal{L}$, then:

(i) $h(e)-e \in \mathcal{L}$

(ii) $h(e)-e$ immediately precedes $h(e)$ in $\mathcal{L}$.

Proof. We prove at first the following assertion:

a) If $e_1 \in \mathcal{L}$ and if there exists $e_2 \in \mathcal{L}$, $e_2 < e_1$ such that e_2 immediately precedes e_1, then e_1-e_2 is an atom of $\mathcal{R}_1$. Indeed, suppose that e_1-e_2 is not minimal in $\mathcal{R}_1$. Let $e_3 \in \mathcal{R}_1$ be a projection such that $0 \neq e_3 < e_1-e_2$. Then $e_2 < e_2+e_3 < e_1$. Since $\mathcal{L}$ is maximal; we have that $e_2+e_3 \in \mathcal{L}$. This contradicts the fact that e_2 immediately precedes e_1 in $\mathcal{L}$. Therefore e_1-e_2 is an atom of $\mathcal{R}_1$.

Let now $e \in \mathcal{R}_1$ be an atom and $h(e)$ the hull of e in $\mathcal{L}$. If $f \in \mathcal{L}$ is a projection which does not contains e, then from minimality of e, we have $f.e=0$, Since $\mathcal{L}$ is totally ordered it follows that $f \leq h(e)$. Therefore, $f \leq h(e)-e$. Let $f_o = \vee\{f \in \mathcal{L} \mid f \leq h(e)-e\}$. Then, $f_o \in \mathcal{L}$ and $f_o \leq h(e)-e$. Obviously f_o immediately precedes $h(e)$ in $\mathcal{L}$. By a) it follows that $h(e)-f_o$ is an atom of $\mathcal{R}_1$. Since $f_o \leq h(e)-e$, we have $e \leq h(e)-f_o$. Therefore $e=h(e)-f_o$, so $f_o=h(e)-e \in \mathcal{L}$ and immediately precedes $h(e)$ in $\mathcal{L}$.

4.2. Lemma. Let $\mathcal{B} \subset B(H)$ be a nest algebra, $\mathcal{R}$ a maximal abelian von Neumann algebra which contains $\mathrm{Lat}\,\mathcal{B}$ and $\mathcal{L} \supset \mathrm{Lat}\,\mathcal{B}$, a maximal totally ordered lattice of projections in $\mathcal{R}$. Suppose that the von Neumann algebra $\mathcal{R}_1$ generated by is totally atomic. Then $\mathcal{R}_1=\mathcal{R}$.

Proof. Since the inclusion $\mathcal{R}_1 \subset \mathcal{R}$ is obvious, we must show that $\mathcal{R} \subset \mathcal{R}_1$. Let $\{e_n\}_{n=1}^{\infty}$ be the atomis of $\mathcal{R}_1$. We show that

e_n is an atom of $\mathcal{R}$ for every $n\in\mathbb{N}$. Let $h(e_n)$ be the hull of e_n in $\mathcal{L}$. If $h(e_n)=e_n$ then e_n is minimal in $\mathcal{R}$ because of the maximality of $\mathcal{L}$ in $\mathcal{R}$. Suppose $h(e_n)>e_n$. By Lemma 4.1, $h(e_n)-e_n\in\mathcal{L}$ and immediately precedes $h(e_n)$ in $\mathcal{L}$. ~~By Lemma 4.1, $h(e_n)-e_n\in\mathcal{L}$ and immediately precedes $h(e_n)$ in $\mathcal{L}$~~. From the maximality of $\mathcal{L}$ in $\mathcal{R}$, it follows that e_n is minimal in $\mathcal{R}$. If $e\in\mathcal{R}$ is an arbitrary projection, then $e=\sum_{\{e_n \mid ee_n\neq 0\}} e_n$. Hence $e\in\mathcal{R}_1$. Therefore $\mathcal{R}\subset\mathcal{R}_1$.

The following Lemma is an easy consequence of the fact that a nest algebra is reflexive.

4.3. Lemma. Let $\mathcal{B}\subset B(H)$ be a nest algebra. If $e\in\mathrm{Lat}\,\mathcal{B}$, then $eB(H)(1-e)\subset\mathcal{B}$.

4.4. Lemma. Let $\mathcal{R}\subset B(H)$ be a maximal abelian von Neumann algebra, and $\mathcal{L}\subset\mathcal{R}$ a maximal, totally ordered lattice of projections. If $p\in\mathcal{R}$ is a projection, then $\mathcal{L}.p=\{e.p \mid e\in\mathcal{L}\}$ is a maximal totally ordered lattice of projections in $\mathcal{R}.p$.

Proof. Suppose by contradiction that there exists $e_o.p\in\mathcal{R}_p$, $e_o.p\notin\mathcal{L}.p$ such that $\{e_o.p\}\cup\mathcal{L}.p$ be totally ordered. We denote:

$$\mathcal{L}_1=\{e\in\mathcal{L} \mid e.p\geq e_o.p\} \quad , \quad e_1=\wedge\mathcal{L}_1$$

$$\mathcal{L}_2=\{e\in\mathcal{L} \mid e.p\leq e_o.p\} \quad , \quad e_2=\vee\mathcal{L}_2$$

Since $e_o.p\notin\mathcal{L}.p$ it follows that $e_1p>e_o.p$ and $e_2.p<e_o.p$. On the other hand, since $\{e_o.p\}\cup\mathcal{L}.p$ is totally ordered, we have $\mathcal{L}=\mathcal{L}_1\cup\mathcal{L}_2$. Let $f\in\mathcal{L}_1$ and $e\in\mathcal{L}_2$. Then $e<f$. Indeed, if $f\leq e$, then $fp\leq e.p<e_o.p$ which is a contradiction. Hence $f>e$.

Further, we show that $e_o.p+e_2(1-p)$ is comparable with every projection $f\in\mathcal{L}$:

a) If $f \in \mathcal{L}_1$ then $f.p > e_o.p$ and, by a remark above $f > e_2$ so $f(1-p) \geq e_2(1-p)$. Hence $f > e_o.p+e_2(1-p)$.

b) If $f \in \mathcal{L}_2$, then $f.p < e_o.p$ and $f(1-p) \leq e_2(1-p)$. Hence $f < e_o.p+e_2(1-p)$.

Hence $\{e_o.p+e_2(1-p)\} \cup \mathcal{L}$ is totally ordered. It is obvious that $e_o.p+e_2(1-p) \notin \mathcal{L}$. This contradicts the maximality of . Therefore .p is a maximal, totally ordered lattice of projections in $\mathcal{R}.p$.

<u>Proof of Theorem 2</u>. Let $\mathcal{R} \subset \mathcal{B}$ be a maximal abelian von Neumann algebra which contains Lat $\mathcal{B}$, $\mathcal{L} \supset$ Lat $\mathcal{B}$ a maximal totally ordered lattice of projections in $\mathcal{R}$ and $\mathcal{R}_1 \subset \mathcal{R}$ the von Neumann algebra generated by $\mathcal{L}$. We denote by $p \in \mathcal{R}_1$ the sum of all atoms of $\mathcal{R}_1$ and $q=1-p$. Obviously $\mathcal{R}_1.p$ is totally atomic, and $\mathcal{R}_1.q$ is non atomic. Let $\mathcal{B}_o = \mathrm{alg}\,\mathcal{L}$; Since $\mathcal{L} \supset$ Lat $\mathcal{B}$, we have $\mathcal{B}_o \subset \mathcal{B}$. Let now $b \in \mathcal{B}'_{ess}$. Then $b \in \mathcal{B}'_{oess}$. In particular $b \in (\mathcal{B}_o \cap \mathcal{B}_o^*)'_{ess}$. From [2] §§ 2, 3 it follows that there exist $r \in (\mathcal{B}_o \cap \mathcal{B}_o^*)'$ and $k \in \mathcal{K}(H)$ such that $b=r+k$. By Lemma 2.3, $(\mathcal{B}_o \cap \mathcal{B}_o^*)' = \mathcal{R}_1$. Hence $r \in \mathcal{R}_1$. Since $b \in \mathcal{B}'_{oess}$, it follows that $r \in \mathcal{B}'_{oess}$. Using Lemma 2.4, Lemma 2.5 and Proposition 2.6 it follows that $r.q=\lambda_1.q$ for some $\lambda_1 \in \mathbb{C}$.

On the other hand, by Lemma 4.4, $\mathcal{L}.p$ is a maximal totally ordered lattice of projections in $\mathcal{R}.p$. Using Lemma 4.2 we obtain $\mathcal{R}_1.p=\mathcal{R}.p$. Therefore the algebra $p\mathcal{B}_o p$ is a weakly closed, maximal triangular algebra with totally atomic diagonal $\mathcal{R}.p$. By Proposition 3.1, $r.p=\lambda_2 p+k_2$ where $\lambda_2 \in \mathbb{C}$ and $k_2 \in \mathcal{K}(H)$.

We consider the two possible conditions on the dimension of pH:

I. $\dim pH < \infty$. Then we have:

$$r=rp+rq=\lambda_2 p+k_2+\lambda_1 q=\lambda_1 1+(\lambda_2-\lambda_1)p+k_2=\lambda_1 1+k,$$

where $k=(\lambda_2-\lambda_1)p+k_2 \in \mathcal{K}(H)$.

II. dim $pH=$. Since $r=\lambda_1.1+(\lambda_2-\lambda_1)p+k_2=\lambda_2.1+(\lambda_1-\lambda_2)q+k_2$, from $r\in\mathcal{B}'_{oess}$ it follows that $(\lambda_2-\lambda_1)p$, $(\lambda_1-\lambda_2)q\in\mathcal{B}'_{oess}$. If $\lambda_1\neq\lambda_2$ then, obviously p, $q\in\mathcal{B}'_{oess}$. We shall show that this is not the case and so $\lambda_1=\lambda_2$. We examine the following two possibilities:

1) For every $e\in\mathcal{L}$ with $qe<q$ we have $p.e=0$.

In this case, we denote $e_o=\vee\{e\in\mathcal{L}\mid e<q\}\in\mathcal{L}$. Obviously $e_o\leq q$ From the maximality of $\mathcal{L}$ it follows that $e_o\neq 0$. Since $e_o\in\mathcal{R}_1 q$ and $\mathcal{R}_1 q$ is non-atomic, it follows that dim $e_oH=\infty$. We have also dim $(1-e_o)H=\infty$ (for $1-e_o\geq p$ and dim $pH=\infty$).

By Lemma 4.3, $e_oB(H)(1-e_o)\subset\mathcal{B}_o$. Let $c\in e_oB(H)(1-e_o)$ be a non-compact operator (for exemple the partial isometry $c:pH\mapsto e_oH$). We have:

$$qcp=qe_oc(1-e_o)p=e_ocp=c\in\mathcal{K}(H)$$

Hence:

$$q.c-c.q=q.c-q.c.q=qc.p\notin\mathcal{K}(H).$$

Therefore $q\notin\mathcal{B}'_{oess}$.

2) There exists $e_o\in\mathcal{L}$ with $qe_o<q$ and $p.e_o\neq 0$.

2a) Suppose dim $pe_oH<\infty$. Then, since dim $pH=\infty$, we have dim $p(1-e_o)H=\infty$. By Lemma 4.3, $e_oB(H)(1-e_o)\subset\mathcal{B}_o$. Let $c\in e_oqB(H)p(1-e_o)$ be a non compact operator (for example the partial isometry $c:p(1-e_o)H\longmapsto qe_oH$). We have:

$$qcp=qe_oc(1-e_o)p=c\notin\mathcal{K}(H)$$

Hence:

$$qc-c.q=q.c-qcq=qcp\notin\mathcal{K}(H).$$

Therefore $q \notin \mathcal{B}'_{oess}$.

2b) Suppose $\dim pe_oH=\infty$. Let $c \in e_o pB(H)q(1-e_o)$ be a non compact operator. We have:

$$pcq=pe_o c(1-e_o)q=c \notin \mathcal{K}(H)$$

Hence

$$pc-c.p=pc-p.cp=pcq \notin \mathcal{K}(H)$$

Therefore $p \notin \mathcal{B}'_{oess}$. So, we have shown that in any case, the situation $p, q \in \mathcal{B}'_{oess}$ is impossible, such that $\lambda_1=\lambda_2$. Then, $r=\lambda_1.1+k_2$ where $\lambda_1 \in \mathbb{C}$ and $k_2 \in \mathcal{K}(H)$, whence $b=\lambda_1.1+k+k_2$ which completes the proof.

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