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A LINEAR FILTERING PROBLEM IN COMPLETE CORRELATED ACTIONS

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1. Introduction

In this paper we shall concern with the following problem. Suppose that a mesage is described by a stationary process $\{x_n\}_{-\infty}^{+\infty}$ in a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$. We received this mesage under the form of a signal described also by a stationary process $\{z_n\}_{-\infty}^{+\infty}$ in $\{\mathcal{E}, \mathcal{H}, \Gamma\}$, which is the result of the transmision and measurement perturbations of the mesage. The problem is to determine the best information about the mesage $\{x_n\}_{-\infty}^{+\infty}$ at the moment $t = 0$, from the knowledge of the signal $\{z_n\}_{-\infty}^{+\infty}$ up to the moment $t = 0$.

In order to give clear meaning of the notions used above, we shall briefly repeat, in Section 2, the basic facts about stationary processes in complete correlated actions presented in [5] , [6] . In addition we construct the " Γ - orthogonal projection on " a submodule of the right $L(\mathcal{E})$ - module $\mathcal{H}$ which gives a more clear meaning of the term " the best information " used in the paper.

In Section 3 we formulate the Γ- optimum linear filtering problem and, under some theoretical relations between the processes $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$, we construct the solution of this problem.

In Section 4 we shall discus the problem of determination of filter coefficients. We shall show that, as in the classical filtering theory, the coefficients of the Γ- optimum linear filter satisfy the system of normal equations in which, the correlation function of the input and the cross-correlation function between the input and the desired output apear as known. Since, in general, the cross-correlation function between masage and signal is not known, this method to determine the filter coeficients is not available (even theoretically). But, the formula of filter coefficients sugests that they can be constructed recursively using the techniques in construction of exact intertwining dilations delivered in [1] . In some special situations (for example when the mesage and the signal have the some autocorrelation function) we succed to show that the autocorrelation of the signal and an exact intertwining dilation of a contraction which intertwines two contractions, canonicaly related to the processes $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{\infty}$, determine the coefficients of the Γ- optimum filter.

We hope that this idea will permit to use the labeling of the exact intertwining dilations given in [1] , in order to obtain recursive methods in filtering theory.

2. Complete correlated actions

As was introduced in [5] , a correlated action is a triplet $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ where $\mathcal{E}$ is a separable Hilbert space, $\mathcal{H}$ is a right $L(\mathcal{E})$ - module, and Γ a map from $\mathcal{H} \times \mathcal{H}$ into $L(\mathcal{E})$ with the properties:

(i) $\Gamma[h,h]$ is a positive operator for any $h \in \mathcal{H}$, and $\Gamma[h,h] = 0$ implies $h = 0$.

(ii) $\Gamma[h,g] = \Gamma[g,h]^*$, $\qquad (h,g \in \mathcal{H})$

(iii) $\Gamma\left[\sum_{k=1}^{m} A_k h_k, \sum_{j=1}^{n} B_j g_j\right] = \sum_{k=1}^{m}\sum_{j=1}^{n} A_k^* \Gamma\,[h_k, g_j]\, B_j$.

The separable Hilbert space $\mathcal{E}$ is called the parameter space. The action of $L(\mathcal{E})$ onto the state space H is the map $L(\mathcal{E}) \times H \to H$ which arise from the fact that H is a right $L(\mathcal{E})$-module. The correlation of the action of $L(\mathcal{E})$ onto H is given by the map

(2.1) $(h,g) \to \Gamma[h,g]$, $\qquad (h,g \in H)$.

To any correlated action $\{\mathcal{E}, H, \Gamma\}$ we can attach its measuring space as the Hilbert space K obtained using Aronszajn's way in construction of the reproducing kernel Hilbert space, starting from the operatorial kernel Γ. The Hilbert space K is uniquely determined by the following facts: there exists an injective morphism $h \to V_h$ from the right $L(\mathcal{E})$ - module H into the right $L(\mathcal{E})$ - module $L(\mathcal{E}, K)$ such that

(2.2) $\Gamma[h_1,h_2] = V_{h_1}^* V_{h_2}$, $\qquad (h_1, h_2 \in H)$,

and

(2.3) $K = \bigvee_{h \in H} V_h \mathcal{E}$.

More precisely, the generators of K have the form

(2.4) $V_h a = \gamma_{(a,h)}$,

where $\gamma_{(a,h)}$ is a map from $\mathcal{E} \times H$ into $\mathbb{C}$ defined by

(2.5) $\gamma_{(a,h)}(b,g) = (\Gamma[g,h]\,a,b)_{\mathcal{E}}$.

We say that the correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is *complete*, if this injective morphism V is onto.

If $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ is a complete correlated action, we can define on $\mathcal{H}$ a "Γ-orthogonal projection on" a submodule $\mathcal{H}_1$ in $\mathcal{H}$. Indeed, we have

PROPOSITION. *Let $\mathcal{H}_1$ be a submodule in the right $L(\mathcal{E})$- module $\mathcal{H}$. For any $h \in \mathcal{H}$ there exists a unique element $h_1 \in \mathcal{H}$ such that for any $a \in \mathcal{E}$*

$$(2.6)\qquad V_{h_1} a \in \bigvee_{x \in \mathcal{H}_1} V_x \mathcal{E} = K_1 \text{ and } V_{h-h_1} a \in K_1^{\perp}.$$

Moreover, we have

$$(2.7)\qquad \Gamma\left[h-h_1,\ h-h_1\right] = \inf_{x \in \mathcal{H}_1} \Gamma\left[h-h_1, h-h_1\right] = \inf_{x \in \mathcal{H}_1} \Gamma\left[h-x, h-x\right] ,$$

where the infimum is taken in the set of positive operators in $L(\mathcal{E})$.

Proof. If $K_1 = \bigvee_{x \in \mathcal{H}_1} V_x \mathcal{E}$, and P_{K_1} is the orthogonal projection of K on K_1, putting

$$(2.8)\qquad V_{h_1} = P_{K_1} V_h$$

then clearly $V_{h_1} a \in K_1$, for any $a \in \mathcal{E}$, and

$$V_{h-h_1} a = V_h a - V_{h_1} a = V_h a - P_{K_1} V_h a = (I - P_{K_1}) V_h a \in K_1^{\perp}.$$

Let h_2 be another element in $\mathcal{H}$ with the properties (2.6). Then for any $a \in \mathcal{E}$ we have

$$V_h a = V_{h_2} a + V_{h-h_2} a \cdot$$

It follows that $V_{h_2} a = P_{K_1} V_h a = V_{h_1} a$, hence $h_2 = h_1$.

We have also

$$\begin{aligned}
(\Gamma[h-h_1, h-h_1]\, a, a) &= \| V_{h-h_1} a \|^2 = \\
&= \| (I-P_{K_1}) V_h a \|^2 = \inf_{k \in K_1} \| V_h a - k \|^2 = \\
&= \inf_{\Sigma} \| V_h a - \sum_1^n V_{x_k} a_k \|^2 = \inf_{\Sigma} \| V_h a - V_{\sum_1^n A_k x_k} a \|^2 = \\
&= \inf_{\Sigma} (\Gamma[h - \sum_1^n A_k x_k,\ h - \sum_1^n A_k x_k]\, a, a) = \\
&= \inf_{x \in \mathcal{H}_1} (\Gamma[h-x, h-x]\, a, a),
\end{aligned}$$

where for any finite sistem $a_1, \dots, a_n \in \mathcal{E}$ we choose $A_1, \dots, A_n \in L(\mathcal{E})$ such that $A_k a = a_k$, $k = 1, \dots, n$.

The Proposition is proved.

If we put $\mathcal{P}_{H_1} h = h_1$, then clearly we obtain an endomorphism of H such that $\mathcal{P}_{H_1}^2 = \mathcal{P}_{H_1}$, $\Gamma[\mathcal{P}_{H_1} h, g] = \Gamma[h, \mathcal{P}_{H_1} g]$, and one can interpret $\mathcal{P}_{H_1}$ as a "orthogonal projection on" H_1.

In the context of a correlated action $\{\mathcal{E}, H, \Gamma\}$ we define a Γ-<u>stationary process</u> as a sequence $\{f_n\}_{-\infty}^{+\infty}$ of elements in H, such that $\Gamma[f_n, f_m]$ depends only of the difference $m-n$ and not on m and n separately. In the measuring space K we consider now the following subspaces, relative to the stationary process $\{f_n\}_{-\infty}^{+\infty}$:

$$(2.9) \quad K_n^f = \bigvee_{-\infty}^{n} V_{f_n} \mathcal{E},$$

$$(2.10) \quad K_\infty^f = \bigvee_{-\infty}^{+\infty} V_{f_n} \mathcal{E}.$$

Also, for the stationary process $\{f_n\}_{-\infty}^{+\infty}$ we exhibid in the state space H the linear manifold:

(2.11) $\mathcal{H}_n^f = \{h \in \mathcal{H};\ h = \sum_{k=0}^{m} A_k f_{n-k}, A_k \in L(\mathcal{E})\}$.

Hence we have $K_n^f = \bigvee_{h \in \mathcal{H}_n^f} V_h \mathcal{E}$, or using (2.4) $K_n^f = \bigvee_{\substack{a \in \mathcal{E} \\ h \in \mathcal{H}}} \gamma(a,h)$.

It is known that for any stationary process $\{f_n\}_{-\infty}^{+\infty}$ there exists a unitary operator U_f on K_∞^f, so called the shift operator attached to the process $\{f_n\}_{-\infty}^{+\infty}$, such that

$$U_f^m V_{f_n} = V_{f_{n+m}}.$$

Such a way, in a complete correlated action $\{\mathcal{E}, \mathcal{H}, \Gamma\}$ we can express the process $\{f_n\}_{-\infty}^{+\infty}$, as:

(2.12) $f_n = U_f^n V_f$,

where $V_f = V_{f_0}$.

For the stationary process $\{f_n\}_{-\infty}^{+\infty}$ we define $\{g_n\}_{-\infty}^{+\infty}$ as

(2.13) $g_n = (I - \mathcal{P}_{\mathcal{H}^f_{n-1}}) f_n$,

so called the innovation part of the process $\{f_n\}_{-\infty}^{+\infty}$.

A stationary process $\{h_n\}_{-\infty}^{+\infty}$ is called a white noise process if $\Gamma[h_n, h_m] = 0$ for $m \neq n$. It is easy to see that the innovation part $\{g_n\}_{-\infty}^{+\infty}$ of $\{f_n\}_{-\infty}^{+\infty}$, defined by (2.13), is a white noise process and

(2.14) $\Gamma[g_n, g_n] = \inf_{h \in \mathcal{H}^f_{n-1}} \Gamma[f_n - h, f_n - h] = G_f$.

G_f is called the prediction -error operator of the stationary process $\{f_n\}_{-\infty}^{+\infty}$.

If the prediction - error operator G_f is invertible, then setting

(2.15) $h_n = G_f^{-1/2} g_n$

we obtain a white noise stationary process $\{h_n\}_{-\infty}^{+\infty}$ such that

(2.16) $\Gamma[h_n, h_n] = I_{\mathcal{E}}$.

The process $\{h_n\}_{-\infty}^{+\infty}$ is called the <u>normalised innovation process</u> of $\{f_n\}$.

Let $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ be stationary processes. If $\Gamma[f_n, g_m]$ depends only of the difference m-n and not on n and m separately, then $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$ are called <u>stationary cross-correlated</u> processes, and the map $n \longmapsto \Gamma_{fg}(n)$ from $\mathbb{Z}$ into $L(\mathcal{E})$ given by $\Gamma_{fg}(n) = \Gamma[f_k, g_{k+n}]$ is called the <u>cross-correlation</u> function of $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$. On the space $\mathcal{K}_\infty^{fg} = \mathcal{K}_\infty^f \vee \mathcal{K}_\infty^g$ there exists a unitary operator U, so called the <u>extended shift</u> of the stationary processes $\{f_n\}_{-\infty}^{+\infty}$ and $\{g_n\}_{-\infty}^{+\infty}$, such that $U_f = U|\mathcal{K}_\infty^f$, $U_g = U|\mathcal{K}_\infty^g$.

For a stationary process $\{f_n\}_{-\infty}^{+\infty}$, the $L(\mathcal{E})$ - valued positive definite function on $\mathbb{Z}$ given by $n \longmapsto \Gamma(n) = \Gamma[f_o, f_n]$, is called the <u>autocorrelation function</u> of $\{f_n\}_{-\infty}^{+\infty}$. Using Naimark dilation theorem, there exists an $L(\mathcal{E})$ - valued semi-spectral measure F on the unit torus $\mathbb{T}$ such that

(2.17) $\Gamma(n) = \int_0^{2\pi} e^{-int} dF(t)$.

This semi-spectral measure F is called the <u>spectral distribution</u> of the process $\{f_n\}_{-\infty}^{+\infty}$. In [6] was proved a factorization theorem by analytic functions and was found a characterization of

G_f in this terms.

If the spectral distribution F of $\{f_n\}_{-\infty}^{+\infty}$ verifies the condition:

$$(2.18)\quad c\,dt \leq dF \leq c^{-1}dt,$$

where c is a positive constant, then (see [4]) there exists a unique bounded outer analytic function $\{\mathcal{E},\mathcal{E}, \theta(\lambda)\}$, so called the <u>maximal outer function</u> of the process $\{f_n\}_{-\infty}^{+\infty}$, which has a bounded analytic inverse $\{\mathcal{E},\mathcal{E},\Omega(\lambda)\}$ and

$$(2.19)\quad dF = \theta(e^{it})^{*}\theta(e^{it})dt,$$

$$(2.20)\quad G_f^{1/2} = \theta(0).$$

Moreover, if

$$(2.21)\quad \theta(\lambda) = G^{1/2} + \sum_{k=1}^{\infty}\lambda^k\theta_k$$

and

$$(2.22)\quad \Omega(\lambda) = G^{-1/2} + \sum_{k=1}^{\infty}\lambda^k\Omega_k$$

are the Taylor expansions of the functions $\{\mathcal{E},\mathcal{E},\theta(\lambda)\}$ and $\{\mathcal{E},\mathcal{E},\Omega(\lambda)\}$, respectively, then between the initial process $\{f_n\}_{-\infty}^{+\infty}$ and his normalised innovation process $\{h_n\}_{-\infty}^{+\infty}$ there exist the following relations:

$$(2.23)\quad f_n = \sum_{k=0}^{\infty}\theta_k h_{n-k}$$

and

(2.24) $h_n = \sum_{k=0}^{\infty} \Omega_k f_{n-k}$,

where the series are supposed to be convergent in the strong topology of $L(\mathcal{E},K)$.

As a remark, in this case we have the following identification for the geometrical model of prediction $[K,V,U]$:

$$K = L^2(\mathcal{E})$$

(2.25) $(Va)(t) = \theta(e^{it})a, \qquad a \in \mathcal{E}$

$(Uk)(t) = e^{-it}k(t), \qquad k \in L^2(\mathcal{E})$.

Hence the process $\{f_n\}_{-\infty}^{+\infty}$ and his normalised innovation $\{h_n\}_{-\infty}^{+\infty}$ can be seen as operators from $\mathcal{E}$ into $L^2(\mathcal{E})$, respectively:

(2.26) $(f_n a)(t) = e^{-int}\theta(e^{it})a$

and

(2.27) $(h_n a)(t) = e^{-int}a$.

3. Γ- optimum linear filter

An input - output system in a correlated action $\{\mathcal{E},H,\Gamma\}$ is called a <u>linear filter</u>, if there exists a sequence $\{A_n\}_0^{\infty}$ of operators in $L(\mathcal{E})$ such that the outputs $\{x_n\}_{-\infty}^{+\infty}$ are related with the inputs $\{z_n\}_{-\infty}^{+\infty}$ by the formula

(3.1) $x_n = \sum_{k=0}^{\infty} A_k z_{n-k}$.

The series in (3.1) being strongly convergent in $L(\mathcal{E},K)$.

In this paper we shall be concerned with the following filtering problem: consider that the mesage model is given by a stationary process $\{x_n\}_{-\infty}^{+\infty}$ in the correlated action $\{\mathcal{E},\mathcal{H},\Gamma\}$ and the observation or the measurement model is given by the stationary process $\{z_n\}_{-\infty}^{+\infty}$ in $\{\mathcal{E},\mathcal{H},\Gamma\}$. Under some theoretical relations between $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$, to determine the sequence $\{A_n\}_0^{\infty}$ of operators in $L(\mathcal{E})$ such that using $\{z_n\}$ as inputs, in the linear filter given by $\{A_n\}_0^{\infty}$, the obtained outputs

$$(3.2)\quad \hat{x}_n=\sum_{k=0}^{\infty}A_k z_{n-k}$$

are the best information we can obtain about $\{x_n\}$ acting on the observation model up to the moment n. This means that

$$(3.3)\quad \Gamma[x_n-\hat{x}_n,x_n-\hat{x}_n]=\inf_{h\in\mathcal{H}_n^z}\Gamma[x_n-h,x_n-h].$$

The infimum in (3.3) is taken in the partialy ordered set of the positive operators in $L(\mathcal{E})$ with the meaning that: for any $h\in\mathcal{H}_n^z$ we have

$$\Gamma[x_n-\hat{x}_n,x_n-\hat{x}_n]\leq\Gamma[x_n-h,x_n-h]$$

as operators in $L(\mathcal{E})$, and if B is a positive operator in $L(\mathcal{E})$, such that for any $h\in\mathcal{H}_n^z$ we have $B\leq\Gamma[x_n-h,x_n-h]$, then $B\leq\Gamma[x_n-\hat{x}_n,x_n-\hat{x}_n]$.

The positive operator

$$(3.4)\quad G_{x,z}=\Gamma[x_n-\hat{x}_n,x_n-\hat{x}_n]$$

is called the filtering - error operator.

The existence of $\hat{x}_n$ in $\mathcal{H}$ which verifies (3.3) arise from the existence of the Γ- orthogonal projection on H . Indeed, if we take

$$(3.5)\quad \hat{x}_n = \mathcal{P}_{H_n^z}\, x_n,$$

then according to (2.7) $\hat{x}_n$ verifies (3.3).

Since, moreover, $V_{\hat{x}_n} \in K_n^z$ we have a kind of closness of $\hat{x}_n$ to H_n^z, but the problem to describe this closness by an approximation procedure, or more precisely to construct the linear filter of action $\{A_k\}_0^\infty$ such that $\{\hat{x}_n\}$ to arise as response of this filter to the inputs $\{z_n\}$, seems to be in general very difficult.

We shal determine this filter under some theoretical relations which we impose to $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$.

Firstly we suppose that the processes $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$ are stationary cross-correlated. Hence they have a common operatorial model of the form:

$$x_n = U^n V_x$$

$$z_n = U^n V_z \quad ,$$

where U is a unitary operator (the extended shift) on the subspace $K_\infty^x \bigvee K_\infty^z$ of K. Without lossing the generality, we can suppose that $K_\infty^x = K_\infty^z = \mathcal{K}$.

Let F_x and F_z be the spectral distributions of $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$, respectively. We shall suppose that F_x and F_z are Harnack equivalent with the normalized Lebesgue measure dt on $\mathbb{T}$, i.e., there exist the positive constants c_x, c_z such that:

$$(3.6)\quad c_x dt \leq dF_x \leq c_x^{-1} dt$$

and

(3.7) $c_z dt \leq dF_z \leq c_z^{-1} dt.$

It follows [5] that

(3.8) $\mathcal{K}=M(F_x)=M(F_z)$,

(3.9) $\mathcal{K}_o^x = M_+(F_x)$

and

(3.10) $\mathcal{K}_o^z = M_+(F_z)$,

where

(3.11) $F_x=\mathcal{K}_o^x \ominus U^* \mathcal{K}_o^x$

and

(3.12) $F_z=\mathcal{K}_o^z \ominus U^* \mathcal{K}_o^z$

are the <u>innovation spaces</u> of the processes $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$, respectively. As in [7], for a wandering subspace $\mathcal{F}$ of U^*, we denoted by $M(F)=\overset{+\infty}{\underset{-\infty}{\oplus}} U^{*n}F$ and $M_+(F)=\overset{\infty}{\underset{o}{\oplus}} U^{*n}F$.

From (3.6) and (3.7) clearly it results the existence of a positive constant c_{xz} such that

(3.13) $c_{xz} dF_x \leq dF_z \leq c_{xz}^{-1} dF_x$.

Then [4] there exists a linear bounded invertible operator S on $\mathcal{K}$ such that

(3.14) $SU^* = U^*S$

and

(3.15) $SV_z = V_x$.

Since clearly $SK_o^z = K_o^x$ we have

(3.16) $SM_+(F_z) \subset M_+(F_x)$

Suppose now that the innovation space F_x and F_z are related by

(3.17) $F_x = UF_z$

Then denoting

(3.18) $B = U^*S$

we have $BU^* = U^*B$, $BM_+(F_z) \subset M_+(F_z)$ and

(3.19) $V_x = UBV_z$.

Let now $\{\mathcal{E}, \mathcal{E}, \theta(\lambda)\}$ be the maximal outer function of the process $\{z_n\}_{-\infty}^{+\infty}$. Identifying K with $L^2(\mathcal{E})$ as in (2.25), we have:

$$K = L^2(\mathcal{E}),\ F_z = \mathcal{E}\ ,\ K_n^z = \bigoplus_{-n}^{\infty} e^{ikt}\mathcal{E}.$$

$$(Uk)(t) = e^{-it}k(t), \qquad k \in L^2(\mathcal{E}).$$

$$(V_z a)(t) = \theta(e^{it})a, \qquad a \in \mathcal{E}.$$

Clearly B appears as an operator on $L^2(\mathcal{E})$ which commutes with the multiplication by e^{it} in $L^2(\mathcal{E})$ and

$$BL_+^2(\mathcal{E}) \subset L_+^2(\mathcal{E}).$$

It results that B can be represented as the pointwise multiplication by the boundary function $B(e^{it})$ of a bounded analytic function $\{\mathcal{E},\mathcal{E},B(\lambda)\}$.Let

$$(3.20) \quad B(\lambda) = \sum_{k=0}^{\infty} \lambda^k B_k, \qquad (\lambda \in \mathbb{D})$$

be the Taylor expansion of B.

Now, using (3.19) we have

$$\begin{aligned}(V_x a)(t) &= (UBV_z a)(t) = e^{-it} B(e^{it})\theta(e^{it})a = \\ &= e^{-it}\sum_{k=0}^{\infty} e^{ikt} B_k \sum_{s=0}^{\infty} e^{ist}\, \theta_s a = \\ &= e^{-it} B_o\theta_o a + \sum_{p=0}^{\infty} e^{ipt} \sum_{k+s=p+1} B_k\, \theta_s a = \\ &= e^{-it} B_o\theta_o a + \sum_{p=0}^{\infty} e^{ipt} \Big(\sum_{k=0}^{p+1} B_k \theta_{p-k+1}\Big) a = \\ &= e^{-it} B_o\theta_o a + \sum_{p=0}^{\infty} e^{ipt} E_p a,\end{aligned}$$

where

$$(3.21) \quad E_p = \sum_{k=0}^{\infty} B_k \theta_{p-k+1}.$$

Hence

$$(V_{x_n}a)(t)=e^{-int}(V_x a)(t) =$$
$$= e^{-i(n+1)t}B_o\theta_o a+\sum_{p=0}^{\infty}e^{i(p-n)t}E_p a.$$

Clearly then

$$(V_{\hat{x}_n}a)(t)=(P_{K_n}zV_{x_n}a)(t)=\sum_{p=0}^{\infty}e^{i(p-n)t}E_p a.$$

If $\{h_n\}_{-\infty}^{+\infty}$ is the normalized innovation process of $\{z_n\}_{-\infty}^{+\infty}$ then the last relation can be written in time domain as:

$$(3.22)\quad \hat{x}_n=\sum_{p=0}^{\infty}E_p h_{n-p},$$

and using (2.24) for $\{z_n\}_{-\infty}^{+\infty}$, we obtain

$$\hat{x}_n=\sum_{p=0}^{\infty}E_p(\sum_{q=0}^{\infty}\Omega_q z_{n-p-q})=$$
$$=\sum_{j=0}^{\infty}(\sum_{s=0}^{j}\Omega_{j-s}E_s)z_{n-j}.$$

We conclude that

$$(3.23)\quad \hat{x}_n=\sum_{j=0}^{\infty}A_j z_{n-j},$$

where

$$A_j=\sum_{s=0}^{j}\Omega_{j-s}\sum_{k=0}^{s+1}B_k\Theta_{s-k+1}.$$

In this way we constructed a linear filter $\{A_j\}_0^{\infty}$ with the coefficients given by

$$(3.24)\quad A_j = \sum_{s=0}^{j}\sum_{k=0}^{s+1} \Omega_{j-s} B_k \Theta_{s-k+1},$$

which solve our Γ - optimum filtering problem.

The filtering - error operator G_{xz} is given by

$$(3.25)\quad G_{xz} = \Theta(0) B(0)^* B(0) \Theta(0).$$

REMARK 1. The prediction problem for a stationary process $\{f_n\}_{-\infty}^{+\infty}$, whose spectral distribution F is Harnack equivalent with the normalized Lebesgue measure on $\mathbb{T}$, can be solved as a particular case of the filtering problem here considered. Indeed, if we take $x_n = f_n$ and $z_n = f_{n-1}$, then it is easy to verify that $\{x_n\}_{-\infty}^{+\infty}$ and $\{z_n\}_{-\infty}^{+\infty}$ satisfy all the conditions imposed above.

Moreover, in this case we have S=U and consequently $B = I_K$. Hence $B_o = I_{\mathcal{E}}$, $B_k = 0$ for $k \neq 0$. Thus the coefficients of the prediction filter are given by

$$(3.26)\quad A_j = \sum_{s=0}^{j} \Omega_{j-s} \Theta_{s+1},$$

which are the coefficients of the Wiener filter for prediction obtained in [5] .

4. The computation of filter coefficients

In the classical filtering theory, the coefficients of the optimum filter are obtained as the solution of the linear system of the normal equation. The unknowns of this system are the coefficients of the filter, the known data being the coefficients of the autocorrelation function of the input and the coefficients of the cross-correlation function between input and desired output.

The coefficients of our Γ- optimum filter satisfies also the system of normal equations. Indeed, in the precedent section we obtained the formula:

$$(4.1) \quad x_n = B_o \textcircled{H}_o h_{n-1} + \sum_{j=0}^{\infty} A_j z_{n-j} \quad ,$$

where $\{h_n\}_{-\infty}^{+\infty}$ is the normalised innovation process of $\{z_n\}_{-\infty}^{+\infty}$.

Since $\Gamma[z_k, h_{n+1}] = 0$ for $k \leq n$, from (4.1) we obtain:

$$\Gamma[z_k, x_n] = \sum_{j=0}^{\infty} \Gamma[z_k, z_{n-j}] A_j \ .$$

Thus the coefficients $\{A_j\}_0^{\infty}$ satisfy the following (infinite) system of normal equations:

$$(4.2) \begin{cases} \Gamma_z(0)A_o + \Gamma_z(-1)A_1 + \Gamma_z(-2)A_2 + \dots = \Gamma_{z,x}(0) \\ \Gamma_z(1)A_o + \Gamma_z(0)A_1 + \Gamma_z(-1)A_2 + \dots = \Gamma_{z,x}(1) \\ - - - - - - - - - - - - - - - - - - - \\ \Gamma_z(m)A_o + \Gamma_z(m-1)A_1 + \Gamma_z(m-2)A_2 + \dots = \Gamma_{z,x}(m) \\ - - - - - - - - - - - - - - - - - - - \end{cases}$$

where $\Gamma_z(k) = \Gamma[z_n, z_{n+k}]$ and $\Gamma_{z,x}(k) = \Gamma[z_n, x_{n+k}]$ are the autocorrelation function of the input $\{z_n\}_{-\infty}^{+\infty}$ and the cross-correlation function of the input $\{z_n\}_{-\infty}^{+\infty}$ and the desired output $\{x_n\}_{-\infty}^{+\infty}$.

In the univariate case, the celebrated Levinson algorithm [2] , [8] permits us to determine the filter coefficients $\{A_j\}_0^{\infty}$ in a recursive way, the computational work involved being less than usual from the special form of the matrix of the system (all the elements of any given diagonal are equal). Similar algorithm was obtained for the matrix valued case by Robinson and Treitel (cf. [3]), but here the task became already more difficult because the algorithm involves matrix inversions. Complicated problems arise also relativ to the stability of the solution, because the system is infinite.

In the operator valued case, such considerations have, however, only theoretical signifiance. But even from this point of wiew, the normal equations (4.2) are not satisfactory in our filtering problem, since the values of cross-correlation function $\Gamma_{z,x}(k)$ are supposed to be known, while we have no informations about them. In such situations the solutions of the filtering problem is determined recursively, starting with an initial estimation based on a prior statistics. Following this idea we shall show that the pure operatorial method used in [1] for the recursive construction of intertwining dilation can be aplied here to determine recursively, in some special cases, the solution of our filtering problem. This particular case sugests that the labeling of contractive intertwining dilation presented in [1], and the recursive method used there, can be aplied to construct our Γ- optimum filter in more general situations.

Firstly, let us remark that the coefficients Ω_k and Θ_k which apear in (3.24) are well determined by the autocorrelation function $\Gamma_z(n)$ of the known input $\{z_n\}$ and, they can be obtained (at least theoreticaly) by the standard deconvolution methods. We shall concern on the determination of the coefficients B_k in (3.24), or, of the operator S.

Our supplementary assumption is

(4.3) $\quad S^*K_o^x \subset K_o^z.$

Let us denote $U_{x+} = U^*|K_o^x$, $U_{z+}=U^*|K_o^z$ and $H_x=\bigvee_0^\infty U_{x+}^* V_x \mathcal{E}$, $H_z=\bigvee_0^\infty U_{z+}^* V_z \mathcal{E}$. Using (4.3), for any $k \in K_o^x$

we have:

$$\begin{aligned}(S\sum_{n\geq 0} U_{z+}^{*n} V_z a_n, k) &= (\sum_{n\geq 0} U_{z+}^{*n} V_z a_n, S^* k) = \\ &= (\sum_{n\geq 0} U^n V_z a_n, S^* k) = (S\sum_{n\geq 0} U^n V_z a_n, k) = \\ &= (\sum_{n\geq 0} U_{x+}^{\ n} S V_z a_n, k) = (\sum_{n\geq 0} U^n V_x a_n, k) = \\ &= (\sum_{n\geq 0} U_{x+}^{*n} V_x a_n, k).\end{aligned}$$

We used (4.3) and the fact that $U_{x+}^* = P_{K_o^z} U | K_o^x$.

Thus

$$S\sum_{n\geq 0} U_{z+}^{*n} V_z a_n = \sum_{n\geq 0} U_{x+}^{*n} V_x a_n$$

and it follows that

$$(4.4) \quad S H_z = H_x$$

It results that $S^* H_x^{\perp} \subset H_z^{\perp}$. Putting

$$(4.5) \quad S_+^* = S^* | K_o^x$$

we have

$$(4.6) \quad S_+^* U_{x+} = U_{z+} S_+^*$$

and

$$(4.7) \quad S_+^* (K_o^x \ominus H_x) \subset K_o^z \ominus H_z.$$

Let now T_x and T_z be the contractions defined by

$$(4.8)\quad T_x = P_x U_{x+} \big| H_x$$

and

$$(4.9)\quad T_z = P_z U_{z+} \big| H_z$$

where P_x and P_z are the projections from K_o^x onto H_x and from K_o^z onto H_z, respectively.

If A is the operator defined from H_x into H_z by

$$(4.10)\quad A = P_z S_+^* \big| H_x,$$

then

$$(4.11)\quad AP_x = P_z S_+^*.$$

Indeed, for any $k \in K_o^x$ we have

$$AP_x k = P_z S_+^* P_x k = P_z S_+^* k - P_z S_+^* (I - P_x) k = P_z S_+^* k.$$

Moreover, for any $h \in H_x$ we have

$$AT_x h = AP_x U_{x+} h = P_z S_+^* U_{x+} h = P_z U_{z+} S_+^* h =$$

$$= P_z U_{z+} P_z S_+^* h = P_z U_{z+} AP_x h = T_z Ah.$$

Hence

(4.12) $AT_x = T_z A.$

Thus S_+^* appear as an exact intertwining dilation of the operator A which intertwines T_x and T_z (see [1]).

If we suppose that S is a unitary operator, which is the case when the mesage model $\{x_n\}$ and the observation model $\{z_n\}$ have the same spectral distribution, then S itself appears as the unique exact intertwining dilation of the contraction A defined on H_z by

(4.13) $A = S \mid H_z.$

Let A_o be the operator on $\mathcal{E}$ defined by

(4.14) $A_o = V_z^* A V_z.$

Then we have

$$V_z A_o = V_z V_z^* A V_z = V_z V_z^* S V_z = V_z V_z^* V_x$$

Such a way, in the time domain, we have

$$A_o z_o = \mathcal{P}_o x_o,$$

where $\mathcal{P}_o$ is the " Γ- orthogonal projection on " the submodule generated in $\mathcal{H}$ by z_o.

This gives to A_o a clear meaning of initial estimator.

Hence, at least in this particular case, we can determine the coefficients of the Γ- optimum filter, based on the autocorrelation function of the signal $\{z_n\}$ and prior statistics which produce the initial estimator A_o.

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