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ON INTERTWINING DILATIONS. VII.

by

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ON INTERTWINING DILATIONS. VII

by

Gr.Arsene, Z.Ceauşescu, C.Foiaş

I n t r o d u c t i o n. Contractive intertwining dilations constitute an interesting object in Operator theory occuring implicitely or explicitely in several branches of the analysis (see e.g. [3], [4], [7], [8], [11], [12], [14], [15], [20], [22], etc.). Therefore the explicite descriptions of the set of all contractive intertwining dilations of an intertwining contraction (for the definitions see the next section) constitute a problem of wide interest. The existence of Schur type, extending the Adamjan-Arov-Krein description (see [1], [2], [3]) to the case of an arbitrary contraction, was established in [9], Proposition 4.1. In the present Note we shall give the explicite formulas for this description, which labels, let us recall, the set of all contractive intertwining dilations by contractive analytic functions. On the way we shall give also an algorithm for computing the choice sequences of any contractive intertwining dilations (see Definition 1.1 below), which as was proved in [9], yields also another label for contractive intertwining dilations. This label was suggested to us by a type of problem occuring in geophysics and briefly described in (4), Sec. 6.7. The proofs of the formulas leading to the algorithm will be given in full detail (see Sections 6,7,8), but those of the explicite Schur-type labelling (see Section 9), exploiting these formulas, will be only sketched.

These are many further developments connecting explicitely our results to [12], [19], [21], [22], [7], , and . These, as well as different concrete illustrations will be given in subsequent papers.

1. We start by recalling notations and simple facts concerning contractive intertwining dilations.

Let H and H' be some (complex) Hilbert spaces and let $L(H, H')$ be the algebra of all (linear, bounded) operators from H into H'. The space $L(H,H)$ will be denoted simply by $L(H)$. In the sequel T (resp. T') will be a contraction on H (resp. H') and we fix $U\in L(K)$ (resp. $U'\in L(K')$) to be its minimal isometric dilation. We will use freely the results from [17] concerning minimal isometric (and unitary) dilation of a given contraction.

Let $P=P_o=P^K_H$ (¹), $P'=P'_o=P^{K'}_{H'}$ and for $n\geq 1$ $P_n=P^K_{H_n}$ and $P'_n=P^{K'}_{H'_n}$, for

$$H_n=H+L+UL+\ldots+U^{n-1}L$$

and

$$H'_n= H'+L'+U'L'+\ldots+U'^{n-1}L'$$

where $L=(U-T)(H)^-$ and $L'=(U'-T')(H')^-$. We denote by T_n the operator $P_nU|H_n$ on H_n (resp. $T'_n=P'_nU'|H'_n$) for every $n\geq 0$; from the properties of isometric dilation we have that $T_o=T$ and $T'_o=T'$ (of course $H_o=H$ and $H'_o=H'$).

Denote by $\hat{U}\in L(\hat{K})$ the minimal unitary dilation of T containing U; we have

$$\hat{K}=\ldots+\hat{U}^{*2}L^*+\hat{U}^*L^*+L^*+K,$$

where

$$L^*=(\hat{U}^*-T^*)(H)^- .$$

We define

$$K_*=\ldots+\hat{U}^*L^*+L^*+H$$

and

$$U_*=\hat{U}^*|K_* ;$$

then $U_* \in L(K_*)$ is a minimal isometric dilation of T^*. Because $U \in L(K)$ and $\hat{U} \in L(\hat{K})$ are the minimal isometric (resp. unitary) dilation of T_n, it follows that $\hat{U}$ contains a minimal isometric dilation of T_n^*, denoted in the sequel by $U_{*n} \in L(K_{*n})$, $(n \geq 1)$. It is easy to see that

$$K_{*n} = K_* + L + \ldots + U^{n-1} L \quad ,$$

$$U_{*n} = \hat{U}^* | K_{*n} \quad .$$

The notations $\hat{U}'$, $\hat{K}'$, K'_*, U'_*, K'_{*n} , U'_{*n} , $(n \geq 1)$ are now clear.

By $I(T', T)$ we denote the set of all operators A in $L(H, H')$ intertwining T' and T (i.e. $T'A = AT$). In the sequel A will be a fixed contraction in $I(T', T)$. A contractive intertwining dilation (CID), respectively a n-partial contractive intertwining dilation (n-PCID) of A is a contraction $\hat{A} \in I(U', U)$, respectively A_n $I(T'_n, T_n)$ such that

$$P' \hat{A} = AP,$$

respectively

$$P' A_n = AP | H_n \quad ,$$

$(n \geq 0)$. It is clear that $A_0 = A$. A chain of PCID of A is a sequence $\{A_n\}_{n=0}^{\infty}$, such that for every $n \geq 0$, A_n is a n-PCID of A and $P'_n A_{n+1} = A_n P_n | H_{n+1}$. The applications

$$\hat{A} \longmapsto \{P'_n \hat{A} | H_n\}_{n=0}^{\infty}$$

$$\{A_n\}_{n=0}^{\infty} \longmapsto (s)\text{-}\lim_{n \to \infty} A_n P_n$$

establish a one-to-one correspondence between all CID's of A and all chains of PCID of A.

As it was pointed out in [5], [6] and [9] the following spaces

$$(1.1)\qquad \begin{cases} F_A(T)=F_A=\{D_A Th+(U-T)h \;:\; h\in H\}^- \\ R_A(T)=R_A=(\mathcal{D}_A+L)\ominus F_A \end{cases}$$ ([2])

and

$$(1.1)'\qquad \begin{cases} F^A(T')=F^A=\{D_A h\oplus(U'-T')Ah \;:\; h\in H\}^- \\ R^A(T')=R^A=(\mathcal{D}_A\oplus L')\ominus F^A \end{cases}$$

are very important for the structure of all CID's of A. We use also the following notations:

$$(1.2)\qquad \begin{cases} p_A(T)=p_A=P^{\mathcal{D}_A+L}_{F_A} \\ q_A(T)=q_A=P^{\mathcal{D}_A+L}_{L} \end{cases},$$

$$(1.2)'\qquad \begin{cases} p^A(T')=p^A=P^{\mathcal{D}_A\oplus L'}_{F_A} \\ q^A(T')=q^A=P^{\mathcal{D}_A\oplus L'}_{\{0\}\oplus L'} \end{cases}$$

and

$$(1.3)\qquad \begin{aligned} &\sigma(A;T',T)=\sigma_A : F_A \longmapsto F^A \\ &\sigma_A(D_A Th+(U-T)h)=D_A h\oplus(U'-T')Ah, \qquad h\in H \,. \end{aligned}$$

All the definitions from (1.1) to (1.3) make sense for $(A_n;T'_n, T_n)$ instead of $(A;T',T)$, so the notations F_{A_n}, R_{A_n}, F^{A_n}, R^{A_n}, p_{A_n}, q_{A_n}, p^{A_n}, q^{A_n}, σ_{A_n} are clear for all $n\geq 0$.

For the sake of completeness we prove here the following facts, which result from [17], Sec. II.1 and [9], Lemma 4.4.

Lemma 1.1. (a) σ_A is unitary.

(b) $((1-p_A)\mathcal{D}_A)^{-}=\mathcal{R}_A$.

(c) $((1-p^A)(\{0\}\oplus L'))^{-}=\mathcal{R}^A$.

Proof. (a) is obvious from the fact that:

$$||D_ATh+(U-T)h||^2=||D_ATh||^2+||(U-T)h||^2=||Th||^2-||ATh||^2+$$
$$+||D_Th||^2=||h||^2-||T'Ah||^2=||D_Ah||^2+||D_{T'}Ah||^2=||D_Ah||^2+$$
$$+||(U'-T')Ah||^2=||D_Ah\oplus(U'-T')Ah||^2 , \qquad (h\in H).$$

(b) Let $r=d+l$, where $r\in\mathcal{R}_A$, $d\in\mathcal{D}_A$ and $l\in L$; suppose that $\langle r, (1-p_A)\tilde{d}\rangle=0$, for every $\tilde{d}\in\mathcal{D}_A$. This implies that $\langle r,\tilde{d}\rangle=0$, so $\langle d,\tilde{d}\rangle=0$ for every $\tilde{d}\in\mathcal{D}_A$, which means that $d=0$. Because r is in $\mathcal{R}_A$, we have that $\langle r,D_ATh+(U-T)h\rangle=0$, for every $h\in H$. But $d=0$, so $\langle l,(U-T)h\rangle=0$, for every $h\in H$, which implies that $l=0$, so $r=0$.

(c) The proof is analogous to (b).

Let us recall the following definition from [9], Definition 3.1.

Definition 1.1. A sequence $\{\Gamma_k\}_{k=1}^{\infty}$ (resp. a string $\{\Gamma_k\}_{k=1}^{n}$, $n\geq 1$) of contractions will be called an A-choice sequence (resp. an A-choice string) if $\Gamma_1\in L(\mathcal{R}_A, \mathcal{R}^A)$ and $\Gamma_k\in L(\mathcal{D}_{\Gamma_{k-1}}, \mathcal{D}_{\Gamma^*_{k-1}})$ for every $k\geq 2$ (resp. for every $2\leq k\leq n$).

One of the main result of [9] (see Propositions 2.1, 2.2 and 3.1) is the following:

T h e o r e m 1.1. *There exists a one-to-one correspondence between all* CID's (*resp.*, n-PCID's, $n\geq 1$) *of* A *and all* A-*choice sequences* (*resp.*, A-*choice strings of length* n).

In order to make self-contained this paper, we will give an alternate proof of this theorem; the objects involved in this proof will be also useful in the sequel.

2. In this section we will study only the so-called "first step", which means the structure of all 1-PCID's of A.

Let Γ_1 be an arbitrary contraction in $L(\mathcal{R}_A, \mathcal{R}^A)$ and define

$$A_1 = AP|\mathcal{H}_1 + q^A(\sigma_A p_A + \Gamma_1(1-p_A))\,(D_A P + I - P)|\mathcal{H}_1 \tag{2.1}$$

We will prove that $A_1 \in L(\mathcal{H}_1, \mathcal{H}_1')$ is an 1-PCID of A. Indeed from (2.1) it is clear that $P'A_1 = AP|\mathcal{H}_1$. Moreover

$$A_1T_1 = A_1(TP+(U-T)P)|\mathcal{H}_1 = ATP|\mathcal{H}_1 + q^A(\sigma_A p_A + \Gamma_1(1-p_A))\cdot$$

$$\cdot(D_A TP + (U-T)P)|\mathcal{H}_1 = T'AP|\mathcal{H}_1 + q^A\sigma_A(D_A TP + (U-T)P)|\mathcal{H}_1 =$$

$$= T'AP|\mathcal{H}_1 + (U'-T')AP|\mathcal{H}_1 = T'AP|\mathcal{H}_1 = T_1'A_1 \quad ,$$

so A_1 is in $\mathcal{I}(T_1', T_1)$. Finally, A_1 is a contraction because

$$\|A_1(h+l)\|^2 = \|Ah\|^2 + \|q^A(\sigma_A p_A + \Gamma_1(1-p_A))\,(D_A h + l)\|^2 \leq$$

$$\leq \|Ah\|^2 + \|\sigma_A p_A(D_A h + l)\|^2 + \|\Gamma_1(1-p_A)(D_A h + l)\|^2 \leq$$

$$\leq \|Ah\|^2 + \|p_A(D_A h + l)\|^2 + \|(1-p_A)(D_A h + l)\|^2 =$$

$$= \|Ah\|^2 + \|D_A h + l\|^2 = \|h+l\|^2 \quad , \qquad h\in\mathcal{H}, \qquad l\in\mathcal{L},$$

so A_1 is an 1-PCID of A.

In particular we shall denote in the sequel by A_1^o the 1-PCID of A, associated by (2.1) to $\Gamma_1=0$; in other words

(2.2) $$A_1^o=AP|\mathcal{H}_1+q^A\sigma_A p_A(D_A P+I-P)|\mathcal{H}_1 .$$

Conversely, let A_1 be an arbitrary 1-PCID of A. Because $P'A_1=AP|\mathcal{H}_1$, we have

$$||h||^2+||l||^2=||h+l||^2\geq||A_1(h+l)||^2=||Ah||^2+||(I-P')A_1(h+l)||^2$$

which means that

$$||(I-P')A_1(h+l)|| \leq ||D_A h+l|| \qquad h\in\mathcal{H},\quad l\in\mathcal{L} .$$

This relation implies the existence of a contraction $B:\mathcal{D}_A+\mathcal{L} \longrightarrow \mathcal{L}'$, such that $(I-P')A_1=B(D_A P+I-P)|\mathcal{H}_1$. This contraction verifies

(2.3) $$Bp_A=q^A\sigma_A p_A .$$

Indeed

$$B(D_A Th+(U-T)h)=(I-P')A_1(Th+(U-T)h)=(I-P')A_1 Uh=$$

$$=(1-P')A_1T_1h=(I-P')T_1'A_1h=(U'-T')Ah=q^A\sigma_A(D_A Th+(U-T)h),$$

for every $h\in\mathcal{H}$.

From (2.3) we infer that $p_A B^*=\sigma_A^* p^A|(\{0\}\oplus\mathcal{L}')$, which implies that

$$||(1-p_A)B^*l'||^2=||B^*l'||^2-||\sigma_A^* p^A(0\oplus l')||^2\leq$$

$$\leq||l'||^2-||p^A(0\oplus l')||^2=||(1-p^A)(0\oplus l')||^2, \qquad l'\in L'.$$

This relation and Lemma 1.1 (c) imply that there exists a contraction Γ_1^* in $L(R^A, R_A)$, such that

$$(2.4) \qquad \Gamma_1^*(1-p^A)(0\oplus l')=(1-p_A)B^*l', \qquad l'\in L'.$$

We have

$$(2.5) \qquad q^A\Gamma_1(1-p_A)(D_A P+I-P)|H_1=A_1-A_1^0 \;;$$

indeed, because $(A_1-A_1^0)(H')\subset L'$ (A_1 and A_1^0 being both 1-PCID of A), we infer that

$$<q^A\Gamma_1(1-p_A)(D_Ah+l),\ l'>=<(1-p_A)(D_Ah+l),\ \Gamma_1^*(1-p^A)(0\oplus l')>=$$

$$=<(1-p_A)(D_Ah+l),\ (1-p_A)B^*l'>=<B(1-p_A)(D_Ah+l),\ l'>=$$

$$=<(I-P')A_1(h+l),\ l'>-<q^A\sigma_A p_A(D_Ah+l),\ l'>=$$

$$=<(I-P')(A_1-A_1^0)(h+l),\ l'>=<(A_1-A_1^0)(h+l),\ l'>,$$

for every $h\in H$, $l\in L$, $l'\in L'$, which is exactly (2.5). We use here in order (2.4), the definition of B, (2.3) and (2.2).
Taking into account (2.5) and (2.2), we have that A_1 and Γ_1 verify also (2.1). So we proved the following

L e m m a 2.1. The formulas (2.1) and (2.5) establish a one-to-one correspondence between all A-choice strings of lenght one and all 1-PCID's of A.

In order to emphasize that Γ_1 corresponds to A_1 by Lemma 2.1, we will write

(2.6) $$\Gamma_1=\Gamma(A, A_1).$$

We will prove now other useful facts concerning the previous correspondence. Let A be an 1-PCID of A and $\{\Gamma_1\}$ its A-choice string. Firstly we note that

(2.7) $$\mathcal{R}_{A_1}=\mathcal{D}_{A_1} \ominus (D_{A_1}U(\mathcal{H}))^{-}.$$

Indeed, from (1.1) it follows:

$$\mathcal{F}_{A_1}=\{D_{A_1}T_1(h+l)+(U-T_1)(h+l) \ : \ h\in\mathcal{H}, \ \ l\in\mathcal{L}\}^{-}=$$

$$=\{D_{A_1}Uh+Ul; \ \ h\in\mathcal{H}, \ \ l\in\mathcal{L}\}^{-}=(D_{A_1}U(\mathcal{H}))^{-}+U\mathcal{L},$$

so

$$\mathcal{R}_{A_1}=(\mathcal{D}_{A_1}+U\mathcal{L})\ominus\mathcal{F}_{A_1}=\mathcal{D}_{A_1}\ominus(D_{A_1}U(\mathcal{H}))^{-}.$$

Consider the operator

(2.8) $$\begin{cases}\tilde{\omega}(A_1;T_1', \ T_1)=\tilde{\omega}_{A_1} \ : \ \mathcal{D}_{A_1}\longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}\\ \tilde{\omega}_{A_1}D_{A_1}=\left[(1-q^A)(\sigma_A\rho_A+\Gamma_1(1-\rho_A))+D_{\Gamma_1}(1-\rho_A)\right]\cdot(D_AP+I-P)|\mathcal{H}_1\end{cases}$$

L e m m a 2.2. (a) $\tilde{\omega}_{A_1}$ is unitary.

(b) $\tilde{\omega}_{A_1}P_{A_1}|\mathcal{D}_{A_1}=P^{\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}}_{\mathcal{D}_A\oplus\{0\}}\tilde{\omega}_{A_1}.$

$$\text{(c)}\quad \tilde{\omega}_{A_1}(1-p_{A_1})|\mathcal{D}_{A_1} = P^{\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}}_{\{0\} \oplus \mathcal{D}_{\Gamma_1}} \tilde{\omega}_{A_1}.$$

$$\text{(d)}\quad \tilde{\omega}_{A_1}(1-p_{A_1})D_{A_1} = 0 \oplus D_{\Gamma_1}(1-p_A)(D_A P + I - P)|\mathcal{H}_1 .$$

P r o o f. Using (2.1) we have

$$\|D_{A_1}(h+l)\|^2 = \|h+l\|^2 - \|A_1(h+l)\|^2 = \|h\|^2 + \|l\|^2 - \|Ah\|^2 -$$

$$- \|q^A(\sigma_A p_A + \Gamma_1(1-p_A))(D_A h + l)\|^2 = \|D_A h + l\|^2 -$$

$$- \|(\sigma_A p_A + \Gamma_1(1-p_A))(D_A h + l)\|^2 + \|(1-q^A)(\sigma_A p_A + \Gamma_1(1-p_A))(D_A h + l)\|^2 =$$

$$= \|D_A h + l\|^2 - \|p_A(D_A h + l)\|^2 - \|\Gamma_1(1-p_A)(D_A h + l)\|^2 + \|(1-q^A)(\sigma_A p_A +$$

$$+ \Gamma_1(1-p_A))(D_A h + l)\|^2 = \|D_{\Gamma_1}(1-p_A)(D_A h + l)\|^2 + \|(1-q^A)(\sigma_A p_A +$$

$$+ \Gamma_1(1-p_A))(D_A h + l)\|^2 = \|\tilde{\omega}_{A_1} D_{A_1}(h+l)\|^2 , \qquad h \in \mathcal{H},\quad l \in \mathcal{L}.$$

So $\tilde{\omega}_{A_1}$ is isometric. Using (2.8), (1.1) and (1.3), we have

$$\text{(2.9)}\quad \tilde{\omega}_{A_1}(D_{A_1} Uh) = \tilde{\omega}_{A_1} D_{A_1}(Th + (U-T)h) = (1-q^A)\sigma_A(D_A Th + (U-T)h) \oplus 0 =$$

$$= (1-q^A)(D_A h + (U'-T')Ah) \oplus 0 = D_A h \oplus 0 , \qquad h \in \mathcal{H}.$$

From (2.9) we infer that

$$\text{(2.10)}\qquad \tilde{\omega}_{A_1}(D_{A_1} U(\mathcal{H})^-) = \mathcal{D}_A \oplus \{0\}.$$

The relation (2.10) and the fact that $\tilde{\omega}_{A_1}$ is isometric imply (b) and (c). Now (d) results from (c) and (2.8). By (c), we have

(2.10)' $$\tilde{\omega}_{A_1}(\mathcal{R}_{A_1})=\{0\}\oplus \mathcal{D}_{\Gamma_1},$$

so $\tilde{\omega}_{A_1}$ is unitary (see (2.10) and (2.10)') and the lemma is completely proved.

Denote by $\omega(A_1;T',T_1)=\omega_{A_1}$ the <u>unitary operator</u> $\tilde{\omega}_{A_1}|\mathcal{R}_{A_1}$, consider as an operator from $\mathcal{R}_{A_1}$ onto $\mathcal{D}_{\Gamma_1}$. Explicitely:

(2.11) $$\begin{cases} \omega_{A_1} : \mathcal{R}_{A_1} \longrightarrow \mathcal{D}_{\Gamma_1} \\ \omega_{A_1}(1-\mathcal{P}_{A_1})D_{A_1}=D_{\Gamma_1}(1-\mathcal{P}_A)(D_AP+I-P)|\mathcal{H}_1 . \end{cases}$$

This unitary operator gives a slight idea of how to use iteratively Lemma 2.1 in order to prove Theorem 1.1. This it will be done after the study of the connections between Lemma 2.1 and the adjoint operation ; this study will give, in particular, the analogous unitary operator between $\mathcal{R}^{A_1}$ and $\mathcal{D}_{\Gamma_1^*}$.

3. For the begining, let Z be a contraction in $L(\mathcal{H},\mathcal{H}')$. Then

$$\mathcal{H}=\ker D_Z \oplus \mathcal{D}_Z ,$$

$$\mathcal{H}'=\ker D_{Z^*} \oplus \mathcal{D}_{Z^*}$$

and the matrix of Z with respect to these decomposition is

(3.1) $$Z=\begin{pmatrix} Z_u & 0 \\ 0 & Z_c \end{pmatrix} .$$

The operator Z_u : ker $D_Z \longmapsto$ ker D_{Z^*} is unitary and it will be called the <u>unitary core</u> of Z; the operator $Z_c : \mathcal{D}_Z \longmapsto \mathcal{D}_{Z^*}$ is a pure contraction (i.e., $h \in \mathcal{D}_Z$, $h \neq 0$ implies $\|Z_c h\| < \|h\|$) and it will be called the <u>pure contractive core</u> of Z.

Note that $Z_c = 0$ if and only if Z is a partial isometry and that $Z_u = 0$ if and only if Z is a pure contraction.

We will have now a closer look to the proof of Proposition 3.2 (a), ch.VII, [17], in order to connect the factorization of a contraction with the correspondent factorization of its adjoint. For this, let $\mathcal{H}''$ be another Hilbert space, $B'' = B' \cdot B$ a factorization of the contraction B'' in $L(\mathcal{H}, \mathcal{H}'')$ by the contractions B in $L(\mathcal{H}, \mathcal{H}')$ and B' in $L(\mathcal{H}', \mathcal{H}'')$.

Define the spaces

$$(3.2) \qquad \begin{cases} F(B' \cdot B) = \{D_B h \oplus D_{B'} Bh : h \in \mathcal{H}\} \subset \mathcal{D}_B \oplus \mathcal{D}_{B'} , \\ R(B' \cdot B) = \mathcal{D}_B \oplus \mathcal{D}_{B'} \ominus F(B' \cdot B) . \end{cases}$$

Recall that by [17], Sec. VII.3, the factorization $B' \cdot B$ is called regular if $R(B' \cdot B) = \{0\}$.

In order to connect the factorization $B'' = B' \cdot B$ and the factorization $B''^* = B^* \cdot B'^*$, we define a contraction from $\mathcal{D}_B \oplus \mathcal{D}_{B'}$ into $\mathcal{D}_{B'^*} \oplus \mathcal{D}_{B^*}$. Because $B(\mathcal{D}_B) \subset \mathcal{D}_{B^*}$ and $B'(\mathcal{D}_{B'}) \subset \mathcal{D}_{B'^*}$, we choose this contraction to be of the form

$$(3.3) \qquad J \circ \begin{pmatrix} B & Y \\ 0 & B' \end{pmatrix} ,$$

where $Y : \mathcal{D}_{B'} \longmapsto \mathcal{D}_{B^*}$ and J is the operator which intertwines the terms in a direct sum. By [18], Théorème 1, the operator (3.3) is a contraction if and only if $Y = D_{B^*} X D_{B'}$, where $X : \mathcal{D}_{B'} \to \mathcal{D}_{B^*}$ is an arbitrary contraction. Define $Z(B' \cdot B) = Z$, the contraction obtained by (3.3) with $X=-P^{H'}_{\mathcal{D}_{B^*}}|\mathcal{D}_{B'}$. Explicitely

(3.4) $$Z(B'\cdot B)=Z=J\circ\begin{pmatrix} B & -D_{B^*}D_{B'} \\ 0 & B' \end{pmatrix}, \quad \text{or}$$

(3.4)' $$\begin{cases} Z(B'.B)=Z : \mathcal{D}_B\oplus\mathcal{D}_{B'} \longmapsto \mathcal{D}_{B'^*}\oplus\mathcal{D}_{B^*}, \\ Z(b\oplus b')=B'b'\oplus(Bb-D_{B^*}D_{B'}b'), \quad b\in\mathcal{D}_{B'}, \quad b'\in\mathcal{D}_{B'}. \end{cases}$$

From (3.4) it is easy to infer that

(3.4)" $$\begin{cases} Z(B'\cdot B)^*=Z^* : \mathcal{D}_{B'^*}\oplus\mathcal{D}_{B^*} \longmapsto \mathcal{D}_B\oplus\mathcal{D}_{B'}; \\ Z^*(b'_*\oplus b_*)=B^*b_*\oplus(B'^*b'_*-D_{B'}D_{B^*}b_*), \quad b_*\in\mathcal{D}_{B^*}, \quad b'_*\in\mathcal{D}_{B'^*}, \end{cases}$$

which means that

(3.4)''' $$Z(B'\cdot B)^*=Z(B^*\cdot B'^*).$$

L e m m a 3.1. <u>The unitary core of</u> Z <u>acts between</u> $R(B'.B)$ <u>and</u> $R(B^*\cdot B'^*)$.

P r o o f. Let $b\in\mathcal{D}_B$ and $b'\in\mathcal{D}_{B'}$; then

$$||Z(b\oplus b')||^2=||B'b'||^2+||Bb-D_{B^*}D_{B'}b'||^2=||B'b'||^2+||Bb||^2+$$
$$+||D_{B^*}D_{B'}b'||^2-2\mathrm{Re}\langle Bb, D_{B^*}D_{B'}b'\rangle=||B'b'||^2+||b||^2-||D_Bb||^2+$$
$$+||D_{B'}b'||^2-||B^*D_{B'}b'||^2-2\mathrm{Re}\langle D_{B^*}Bb, D_{B'}b'\rangle=$$
$$=||b\oplus b'||^2-(||D_Bb||^2+||B^*D_{B'}b'||^2+2\mathrm{Re}\langle D_Bb, B^*D_{B'}b'\rangle)=$$
$$=||b\oplus b'||^2-||D_Bb+B^*D_{B'}b'||^2.$$

This implies that

$$||D_Z(b \oplus b')||=||D_B b + B^* D_{B'} b'||,$$

which means that $b \oplus b' \in \ker D_Z$ if and only if $D_B b + B^* D_{B'} b' = 0$, therefore if and only if $b \oplus b'$ is orthogonal on $F(B' \cdot B)$. This implies that

$$\ker D_Z = R(B'.B) \quad ;$$

analogously (see (3.4)''')

$$\ker D_{Z^*} = R(B^*.B'^*)$$

and lemma is now completely proved (see (3.1)).

C o r o l l a r y 3.1. (see [17], Ch.VII, Proposition 3.2 (a)). The factorization $B' \cdot B$ is regular if and only if the factorization $B^*.B'^*$ is regular.

C o r o l l a r y 3.2. (a) The factorization $B' \cdot B$ is regular if and only if $Z(B'.B)$ is a pure contraction.

(b) $Z(B'.B)=0$ if and only if B and B' are partial isometries such that the final space of B includes the orthogonal of the initial space of B'.

P r o o f. (a) is an easy consequence of Lemma 3.1. Suppose now that $Z(B'.B)=0$; using (3.4)' it follows that $B'|\mathcal{D}_{B'}=0$ which means that B' is a partial isometry. Taking $b'=0$ in (3.4)', one obtains that $B|\mathcal{D}_B=0$, so B is also a partial isometry. The affirmation (b) results now from (3.4)', taking there $b=0$.

We return to the situation considered in the first section, namely $T \in L(\mathcal{H})$, $T' \in L(\mathcal{H}')$, $A \in \mathcal{I}(T',T)$ are contractions. We have that $A^* \in \mathcal{I}(T^*, T'^*)$.

Define the unitary operator

$$(3.5)\qquad \begin{cases} \alpha_A(T) = \alpha_A : \mathcal{D}_T \oplus \mathcal{D}_A \longmapsto \mathcal{D}_A + \mathcal{L} \\ \alpha_A = (I + \varphi) o J, \end{cases}$$

where $\varphi : \mathcal{D}_T \longmapsto \mathcal{L}$ is the unitary operator defined by $\varphi(D_T h) = (U-T)h$, $h \in \mathcal{H}$ (see [17], ch.II).

Consider also the unitary operator

$$(3.5)'\qquad \begin{cases} \alpha^A(T') = \alpha^A : \mathcal{D}_A \oplus \mathcal{D}_{T'} \longmapsto \mathcal{D}_A \oplus \mathcal{L}' \\ \alpha^A = I \oplus \varphi', \end{cases}$$

where φ' is analogous to φ.

We will need also the unitary operators $\alpha_{A^*} = \alpha_{A^*}(T'^*)$ and $\alpha^{A^*} = \alpha^{A^*}(T^*)$. Define the contractions:

$$(3.6)\qquad \begin{cases} Z_A(T',T) = Z_A : \mathcal{D}_A + \mathcal{L} \longrightarrow \mathcal{D}_{A^*} \oplus \mathcal{L}^* \\ Z_A = \alpha^{A^*} Z(A.T) \alpha_A^* \end{cases}$$

and

$$(3.6)'\qquad \begin{cases} Z^A(T',T) = Z^A : \mathcal{D}_A \oplus \mathcal{L}' \longmapsto \mathcal{D}_{A^*} + \mathcal{L}'^* \\ Z^A = \alpha_{A^*} Z(T'.A)(\alpha^A)^* \end{cases}$$

By virtue of (3.6), (3.5), (3.5)' and (3.4)', the explicite formulas for Z_A and $(Z_A)^*$ are

$$(3.7)\qquad Z_A(a + (U-T)h) = Aa \oplus (U_* - T^*)(Th - D_A a), \qquad a \in \mathcal{D}_A, \quad h \in \mathcal{H},$$

(3.7)′ $(Z_A)^*(a_* \oplus (U_*-T^*)h)=(A^*a_*-D_A D_T^2{}^*h)+(U-T)T^*h, \quad a_*\in\mathcal{D}_{A^*}, h\in H.$

The formulas for Z^A and $(Z^A)^*$ are similar, because (3.4)‴ implies that

$$(3.8) \qquad Z^A=(Z_{A^*})^*$$

Note also that $F_A=\alpha_A(F(A.T))$, $R_A=\alpha_A(R(A.T))$, $F^A=\alpha^A(F(T'.A))$, $R^A=\alpha^A(R(T'.A))$, so we infer from Lemma 3.1.

C o r o l l a r y 3.3. (a) <u>The unitary core of Z_A acts between R_A and R^{A^*}.</u>

(b) <u>The unitary core of Z^A acts between R^A and R_{A^*}.</u>

L e m m a 3.2. <u>The pure contractive cores of Z_A and Z^A verify the following relation</u>

$$(3.9) \qquad Z_A|F_A=\sigma_{A^*}(Z^A|F^A)\sigma_A.$$

P r o o f. The lemma follows from the equalities

$\sigma_{A^*}Z^A\sigma_A(D_ATh+(U-T)h)=\sigma_{A^*}Z^A(D_Ah \oplus (U'-T')Ah)=$

$=\sigma_{A^*}(AD_Ah-D_{A^*}D_{T'}^2Ah+(U'_*-T'^*)T'Ah)=$

$=\sigma_{A^*}(D_{A^*}T'^*(T'Ah)+(U'_*-T'^*)T'Ah)=$

$=D_A{}^*(T'Ah) \oplus (U_*-T^*)A^*T'Ah)=AD_ATh \oplus (U_*-T^*)(Th-D_A^2Th)=$

$=Z_A(D_ATh+(U-T)h).$

Let now $\{\Gamma_n\}_{n=1}^{\infty}$ be a choice sequence for A and define the sequence $\{\Gamma_{*n}\}_{n=1}^{\infty}$ by

$$(3.10)_n \qquad \Gamma_{*n} Z^A | \mathcal{D}_{\Gamma^*_{n-1}} = Z_A \Gamma_n^*$$

P r o p o s i t i o n 3.1. The formulas $(3.10)_n$, $n\geq 1$ give an explicit one-to-one correspondence between the choice sequences of A and the choice sequences of A^*.

P r o o f. We have to verify that $\{\Gamma_{*n}\}_{n=1}^{\infty}$ defined by $(3.10)_n$, $n\geq 1$ is a choice sequence for A^*. For this, note that Γ_{*1} is a contraction from R_{A^*} into R^{A^*} (because $Z_A(R_A)=R^{A^*}$). Moreover $\mathcal{D}_{\Gamma_{*1}}=Z^A\mathcal{D}_{\Gamma_1^*}$ and $\mathcal{D}_{\Gamma^*_{*1}}=Z_A\mathcal{D}_{\Gamma_1}$. The operator Γ_2^* is a contraction from $\mathcal{D}_{\Gamma_1^*}$ into $\mathcal{D}_{\Gamma_1}$, so Γ_{*2} is a contraction from $\mathcal{D}_{\Gamma_{*1}}$ into $\mathcal{D}_{\Gamma^*_{*1}}$. The lemma follows now easily by induction.

4. Proposition 3.1 and Lemma 2.1 rise the problem of finding the explicit bijection between 1-PCID's of A and 1-PCID's of A^* given by them. To this aim, define

$$(4.1) \qquad (A^*)_1 = \hat{U}^* A_1^* \hat{U}' \quad (H'+L'^*),$$

for A_1 an 1-PCID of A with $\Gamma(A,A_1)=\Gamma_1$. Consider also Γ_{*1} defined by $(3.10)_1$.

P r o p o s i t i o n 4.1. (a) $(A^*)_1$ is an 1-PCID of A^*.

(b) $\Gamma(A^*, (A^*)_1)=\Gamma_{*1}$.

P r o o f. Let $\tilde{B}$ the 1-PCID of A^* defined by (2.1) with $\Gamma(A^*,\tilde{B})=\Gamma_{*1}$. We have to prove that $\tilde{B}=(A^*)_1$, or, if we take $B=\hat{U}\tilde{B}\hat{U}'^*|H_1$,

(4.2) $$B=A_1^*$$

For proving (4.2), we use that $H_1=L_*+U(H)$, where $L_*=\hat{U}L^*=(I-UT^*)(H)^-$. Therefore (4.2) is equivalent to

(4.3) $$P_{UH}^{H_1}B=P_{UH}^{H_1}A_1^*$$

and

(4.4) $$P_{L_*}^{H_1}B=P_{L_*}^{H_1}A_1^* .$$

From (2.1) we have that

(4.5) $$\tilde{B}(h'+l'^*)=A^*h'+q^{A^*}(\sigma_{A_*}P_{A^*}+\Gamma_{*1}(1-P_{A^*}))(D_{A^*}h'+l'^*),$$

for every $h'\in H'$, $l'^*\in L'^*$.

Recall that T_1 is a partial isometry from H onto UH, so $UT_1^*=T_1T_1^*=P_{UH}^{H_1}$; analogously $U'T_1'^*=T_1'T_1'^*=P_{U'H'}^{H_1'}$. Now

$$P_{UH}^{H_1}A_1^*P_{U'H'}^{H_1'}=T_1T_1^*A_1^*T_1'T_1'^*=T_1A_1^*T_1'^*T_1'T_1'^*=$$

$$=T_1A_1^*T_1'^*=T_1T_1^*A_1^*=P_{UH}^{H_1}A_1^*$$

Using this, and the fact that $A_1^*|H=A^*$, we infer that

(4.6) $$P_{UH}^{H_1}A_1^*(l_*'+U'h')=P_{UH}^{H_1}A_1^*P_{U'H'}^{H_1'}(l_*+U'h')=P_{UH}^{H_1}A_1^*U'h'=$$

$$=UT_1^*A_1^*T_1'h'=UA_1^*T_1'^*T_1'h'=UA_1^*h'=UA^*h',$$

for every $h'\in H'$, $l'_*\in L'_*$.

On the other hand, by (4.5) we deduce

$$(4.7)\qquad P^{H_1}_{UH}B(l'_*+U'h')=P^{H_1}_{UH}\hat{U}\tilde{B}(\hat{U}'^*l'_*+h')=$$

$$=UP^{\hat{K}}_{H}\tilde{B}(h'+\hat{U}'^*l'_*)=UA^*h' \quad ,\qquad h'\in H',\ l'_*\in L'_*.$$

The relation (4.6) and (4.7) prove (4.3)'.

In order to obtain (4.4), we will prove

$$(4.4)'\qquad P^{H_1}_{L_*}Bh'=P^{H_1}_{L_*}A_1^*h' \qquad h'\in H' ,$$

and

$$(4.4)''\qquad P^{H_1}_{L_*}B(U'-T')h=P^{H_1}_{L_*}A_1^*(U'-T')h' \qquad h'\in H'.$$

For (4.4)', we have:

$$P^{H_1}_{L_*}Bh'=P^{H_1}_{L_*}\hat{U}\tilde{B}\hat{U}'^*h'=\hat{U}q^{A^*}\tilde{B}(T'^*h'+(U'_*-T'^*)h')=$$

$$=\hat{U}q^{A^*}(\sigma_{A^*}p_{A^*}+\Gamma_{*1}(1-p_{A^*})(D_{A^*}T'^*h'+(U'_*-T'^*)h')=$$

$$=\hat{U}q^{A^*}\sigma_{A^*}(D_{A^*}T'^*h'+(U'_*-T'^*)h')=\hat{U}(U_*-T^*)A^*h'=$$

$$=(I-UT^*)A^*h' \qquad h\in H,$$

and

$$<P^{H_1}_{L_*}A_1^*h',\ (I-UT^*)h>=<h',A_1(D^2_{T^*}h-(U-T)T^*h>=$$

$$=<h',\ AD^2_{T^*}h>=<(I-UT^*)A^*h',\ \ (I-UT^*)h>,$$

for every $h\in H$, $h'\in H'$. The last two relations prove (4.4)'.

The proof of (4.4)" will involve the whole construction preceding Proposition 3.1. First, we notice that

$$<P^{H_1}_{L_*}A_1^*(U'-T')h', \quad (I-UT^*)h>=$$

$$=<(U'-T')h', \quad A_1(D^2_{T^*}h-(U-T)T^*h)>=$$

$$=<0\oplus (U'-T')h', \quad (\sigma_A p_A+\Gamma_1(1-p_A))(D_A h-(D_A TT^*h+(U-T)T^*h))>=$$

$$=<0\oplus (U'-T')h', \quad \sigma_A p_A(D_A D^2_{T^*}h-(U-T)T^*h)>+$$

$$+<0\oplus (U'-T')h', \quad \Gamma_1(1-p_A)D_A h> \qquad h\in H, \quad h'\in H',$$

and

$$<P^{H_1}_{L_*}B(U'-T')h', \quad (I-UT^*)h>=$$

$$=<\hat{U}\tilde{B}(I-U'_*T')h', \quad (I-UT^*)h>=<\tilde{B}(D^2_{T'}-(U'_*-T'^*)T'h'), \quad (U_*-T^*)h =$$

$$=<(\sigma_{A^*}p_{A^*}+\Gamma_{*1}(1-p_{A^*}))\left(D_{A^*}h'-(D_{A^*}T'^*T'h'+(U'_*-T'^*)T'h'),0\oplus(U_*-T^*)h\right>=$$

$$=<\sigma_{A^*}p_{A^*}(D_{A^*}D^2_{T'}h'-(U'_*-T'^*)T'h'), \quad 0\oplus(U_*-T^*)h>+$$

$$+<\Gamma_{*1}(1-p_{A^*})D_{A^*}h', \quad 0\oplus (U_*-T^*)h>, \qquad h\in H, \quad h'\in H'.$$

These imply that (4.4)" is equivalent to

(4.8) $$<0\oplus (U'-T')h', \quad \sigma_A p_A\left(D_A D^2_{T^*}h-(U-T)T^*h\right)>=$$

$$=<\sigma_{A^*}p_{A^*}\left(D_{A^*}D^2_{T'}h'-(U'_*-T'^*)T'h'\right), \quad 0\oplus (U_*-T^*)h>,$$

and

(4.8)' $$<0\oplus (U'-T')h', \Gamma_1(1-p_A)D_A h>=<\Gamma_{*1}(1-p_{A^*})D_{A^*}h', \quad 0\oplus (U_*-T^*)h>$$

for every $h\in H$, $h'\in H'$.

For (4.8), the point is that from (3.7)' we have that

$$(4.9)\quad D_A D_T^2{}^* h-(U-T)T^* h=-Z_A^*(0 \oplus (U_*-T^*)h), \qquad h\in H.$$

Analogously

$$(4.9)'\quad D_{A^*} D_{T'}^2 h'-(U'_*-T'^*)T'h'=-Z^A(0 \oplus (U'-T')h'), \qquad h'\in H'.$$

These means that (4.8) is equivalent to

$$<0 \oplus (U'-T')h',\ \sigma_A p_A Z_A^*(0 \oplus (U_*-T^*)h)>=$$

$$=<\sigma_A{}^* p_A{}^* Z^A(0 \oplus (U'-T')h'),\ 0 \oplus (U_*-T^*)h>, \qquad h\in H, \quad h'\in H'.$$

But this follows from (3.9), because

$$<0 \oplus (U'-T')h',\ \sigma_A p_A Z_A^*(0 \oplus (U_*-T^*)h)>=$$

$$=<p^A(0 \oplus (U'-T')h',\ \sigma_A p_A Z_A^*(0 \oplus (U_*-T^*)h)>=$$

$$=<Z_A \sigma_A^* p_A(0 \oplus (U'-T')h'),\ 0 \oplus (U_*- T^*)h>=$$

$$=<\sigma_A{}^* Z^A p^A(0 \oplus (U'-T')h',\ 0 \oplus (U_*-T^*)h>, \quad h\in H, \quad h'\in H',$$

so (4.8) is proved.

For (4.8)', we have (using (4.6) and (4.6)'),

$$<\Gamma_{*1}(1-p_A{}^*)D_A{}^*h',\ \ 0 \oplus (U_*-T^*)h>=<Z_A\Gamma_1^*(Z^A)^*(1-p_A{}^*)D_A{}^*h',$$

$$0 \oplus (U_*-T^*)h>=<\Gamma_1^* Z_A{}^*(1-p_A{}^*)D_A{}^*h',\ \ (-D_A D_T^2{}^*h+(U-T)T^*h>=$$

$$=<\Gamma_1^*(1-p^A)Z_A{}^* D_A{}^*h',\ (1-p_A)\left[-D_A h+D_A T(T^*h)+(U-T)T^*h\right]>=$$

$$=\langle (1-p^A)(A^*D_{A^*}h' \oplus (U'-T')D^2_{A^*}h'), \qquad \Gamma_1(1-p_A)D_Ah\rangle=$$

$$=\langle 0 \oplus (U'-T')h', \ \Gamma_1(1-p_A)D_Ah\rangle, \qquad h\in H, \ h'\in H'.$$

We have now (4.8) and (4.8)′, so (4.4)″, which completes the proof of the proposition.

Note that, in particular,

$$(A^*)^o_1=\hat{U}^*(A^o_1)^*(\hat{U}'|H'+L'^*).$$

We will define now the unitary operator between R^A and $\mathcal{D}_{\Gamma^*}$, announced at the end of Section 2. First, note that from (4.1) it follows:

(4.10) $$\hat{U}'(R_{(A^*)_1})=R_{A_1^*}.$$

Indeed, by (2.7), for $(A^*)_1$ instead of A_1,

$$R_{(A^*)_1}=\mathcal{D}_{(A^*)_1} \ominus (D_{(A^*)_1}U'_*(H))^-$$

and from the definition of $R_{A_1^*}$,

$$R_{A_1^*}=(\mathcal{D}_{A_1^*}+L'^*) \ominus \{D_{A_1^*}T'_1 h'_1+(U'_{*1}-T'^*_1)h'_1 \ : \ h'_1\in H'_1\}^-=$$

$$=(\mathcal{D}_{A_1^*}+L'^*) \ominus\{D_{A_1^*}T'^*_1Uh'+(U'_{*1}-T'^*_1)l'_* \ : \ h'\in H', \ l'_*\in L'_*\}^- =$$

$$=(\mathcal{D}_{A_1^*}+L'^*) \ominus\{D_{A_1^*}h'+U'_{*1}l'_* \ : \ h'\in H', \ l'_*\in L'_*\}^- =$$

$$=\mathcal{D}_{A_1^*} \ominus \{D_{A_1^*}(H')\},$$

which completes the proof of (4.10).

Consider now the operator $\omega_{(A^*)_1}$; explicitely

(4.11) $$\begin{cases} \omega_{(A^*)_1} : \mathcal{R}_{(A^*)_1} \longmapsto \mathcal{D}_{\Gamma_{*1}} \\ \omega_{(A^*)_1}(1-p_{(A^*)_1})D_{(A^*)_1}(h'+l'^*)=D_{\Gamma_1^*}(1-p_{A^*})(D_{A^*}h'+l'^*), \end{cases}$$

$h'\in\mathcal{H}'$, $l'^*\in\mathcal{L}'^*$.

We finally define the operator ω^{A_1} by

(4.12) $$\begin{cases} \omega^{A_1}(T_1';T_1)=\omega^{A_1} : \mathcal{R}^{A_1} \longmapsto \mathcal{D}_{\Gamma_1^*} \\ \omega^{A_1}=(Z^A)^*\omega_{(A^*)_1}\hat{U}'^*Z^{A_1}|\mathcal{R}^{A_1}. \end{cases}$$

L e m m a 4.1. <u>The operator</u> ω^{A_1} <u>defined by</u> (4.12) <u>is unitary from</u> $\mathcal{R}^{A_1}$ <u>onto</u> $\mathcal{D}_{\Gamma_1^*}$ <u>and</u>

(4.13) $$\omega^{A_1}(1-p^{A_1})(0\oplus U'l')=D_{\Gamma_1^*}(1-p^A)(0\oplus l'), \qquad l'\in\mathcal{L}'.$$

P r o o f. From (4.12) and Corollary 3.3 it follows that ω^{A_1} is a unitary operator from $\mathcal{R}^{A_1}$ onto $\mathcal{D}_{\Gamma_1^*}$. In order to verify (4.13), we have:

$$(4.14)\quad \omega^{A_1}(1-p^{A_1})(0\oplus U'l')=(Z^A)^*\omega_{(A^*)_1}\hat{U}'^*Z^{A_1}(1-p^{A_1})(0\oplus U'l')=$$

$$=Z_{A^*}\omega_{(A^*)_1}\hat{U}'^*(1-p_{A_1^*})Z^{A_1}(0\oplus(U'-T_1')l')=$$

$$=Z_{A^*}\omega_{(A^*)_1}(1-p_{(A^*)_1})U'^*(-D_{A_1^*}D^2_{T_1'}l'+(U_{*1}-T_1'^*)T_1'l')=$$

$$=Z_{A^*}\omega_{(A^*)_1}(1-p_{(A^*)_1})\hat{U}'^*(-D_{A_1^*}l')=$$

$$=Z_{A^*}\omega_{(A^*)_1}(1-p_{(A^*)_1})D_{(A^*)_1}(-\hat{U}'^*l'),\qquad l'\in L',$$

where we used (4.12), Corollary 3.3 (for A_1 instead of A), (3.8) (for A_1), (3.7)' (for A_1^*), the facts that $D^2_{T_1'}=P^{H_1'}_{L'}$, that $T_1'|L'=0$ and (4.1). Suppose now that $l'=(U'-T')h'$, where $h'\in H'$.

From (4.14) it follows that

$$(4.15)\quad \omega^{A_1}(1-p_{A_1})(0\oplus U'(U'-T')h')=Z_{A^*}\omega_{(A^*)_1}(1-p_{(A^*)_1})D_{(A^*)_1}\cdot$$

$$\cdot(-D^2_{T'}h'+(U'^*-T'^*)T'h')=Z_{A^*}D_{\Gamma_{*1}}(1-p_{A^*})(-D_{A^*}D^2_{T'}h'+(U'^*-$$

$$-T'^*)T'h')=D_{\Gamma_1^*}Z_{A^*}(1-p_{A^*})(Z_{A^*})^*(0\oplus(U'-T')h')=$$

$$=D_{\Gamma_1^*}(1-p^A)(0\oplus(U'-T')h'),\qquad h'\in H',$$

where we used (4.11), $(3.10)_1$, (3.7)' (for A^*) and Corollary 3.3 (for A^*). The relation (4.15) is exactly (4.13) and the lemma is now completely proved.

As an application of the construction involved in the last two sections we will give the following result, which will be esentially used in Sections 7 and 8.

As we know (see (2.7)), the space $\mathcal{R}_{A_1}$ is included in $\mathcal{D}_{A_1}$, so it is possible to consider the operator $(1-p^{A_1})|\mathcal{R}_{A_1} : \mathcal{R}_{A_1} \mapsto \mathcal{R}^{A_1}$.

P r o p o s i t i o n 4.2. <u>The diagram</u>

$$\begin{array}{ccc} \mathcal{R}_{A_1} & \xrightarrow{(1-p^{A_1})|\mathcal{R}_{A_1}} & \mathcal{R}^{A_1} \\ \omega_{A_1} \downarrow & & \downarrow \omega^{A_1} \\ \mathcal{D}_{\Gamma_1} & \xrightarrow{-\Gamma_1} & \mathcal{D}_{\Gamma_1^*} \end{array}$$

<u>is commutative; that is</u>

(4.16) $$\Gamma_1 \omega_{A_1} = -\omega^{A_1}(1-p^{A_1})|\mathcal{R}_{A_1} .$$

P r o o f. From Lemma 1.1 (b) (for A_1 instead of A), we have that (4.16) is equivalent to

(4.16)′ $$\Gamma_1 \omega_{A_1}(1-p_{A_1})D_{A_1}h_1 = -\omega^{A_1}(1-p^{A_1})(1-p_{A_1})D_{A_1}h_1 ,$$

for every $h_1 \in \mathcal{H}_1$.

Because $(1-p_{A_1})D_{A_1}h_1$ is in $\mathcal{D}_{A_1}$, there exists a sequence $\{D_{A_1}h_1^n\}_{n=1}^{\infty}$, where h_1^n is in $\mathcal{H}_1$ for every $n \geq 1$, such that

(4.17) $$D_{A_1}h_1^n \longrightarrow (1-p_{A_1})D_{A_1}h_1 .$$

From (4.17) we have that

$$\omega_{A_1} D_{A_1} h_1^n \longrightarrow 0 \oplus \omega_{A_1}(1-p_{A_1}) D_{A_1} h_1 \quad ,$$

which implies by (2.8) that

$$(4.18) \qquad (1-q^A)(\sigma_A p_A+\Gamma_1(1-p_A))(D_A P+I-P)h_1^n \longrightarrow 0 \quad .$$

Now, from (2.11) it follows

$$(4.19) \qquad \Gamma_1 \omega_{A_1}(1-p_{A_1})D_{A_1} h_1^n = \Gamma_1 D_{\Gamma_1}(1-p_A)(D_A P+I-P)h_1^n \text{ , for every } n\geq 1.$$

On the other hand, the relation (1.1)' for A_1 gives that

$$F^{A_1} = \{D_{A_1}\tilde{h}_1 \oplus U' q^A A_1 \tilde{h}_1 \ : \ \tilde{h}_1 \in H_1\}^- \ ,$$

so

$$(1-p^{A_1})D_{A_1} h_1^n = -(1-p^{A_1})U' q^A A_1 h_1^n \ , \qquad n\geq 1.$$

This implies by (4.13) that

$$(4.20) \qquad \omega^{A_1}(1-p^{A_1})D_{A_1} h_1^n = -\omega^{A_1}(1-p^{A_1})U' q^A A_1 h_1^n =$$

$$= -D_{\Gamma_1}{}^*(1-p^A)q^A(\sigma_A p_A+\Gamma_1(1-p_A))(D_A P+I-P)h_1^n \ , \qquad n\geq 1.$$

Because $\omega^{A_1}(1-p^{A_1})D_{A_1} h_1^n \longrightarrow \omega^{A_1}(1-p^{A_1})(1-p_{A_1})D_{A_1} h_1$, from (4.20), (4.18) and (4.19) it follows that

$$\omega^{A_1}(1-p^{A_1})(1-p_{A_1})D_{A_1} h_1 = \lim_{n\to\infty} \omega^{A_1}(1-p^{A_1})D_{A_1} h_1^n =$$

$$= -\lim_{n\to\infty} D_{\Gamma_1}{}^*(1-p^A)q^A(\sigma_A p_A+\Gamma_1(1-p_A))(D_A P+I-P)h_1^n =$$

$$= -\lim_{n\to\infty} D_{\Gamma_1}{}^*(1-p^A)(\sigma_A p_A+\Gamma_1(1-p_A))(D_A P+I-P)h_1^n =$$

$$= -\lim_{n\to\infty} D_{\Gamma_1}{}^*\Gamma_1(1-p_A)(D_A P+I-P)h_1^n =$$

$$= -\lim_{n\to\infty} \Gamma_1 \omega_{A_1}(1-p_{A_1})D_{A_1} h_1^n = -\Gamma_1 \omega_{A_1}(1-p_{A_1})D_{A_1} h_1 \ .$$

As h_1 was arbitrary in $\mathcal{H}_1$ the proposition is completely proved.

5. In this section we shall prove Theorem 1.1. and the general form of Proposition 4.1.

We will fix $A \in \mathcal{I}(T', T)$ a contraction and $\hat{A}$ a CID of A; denote by $\{A_n\}_{n=0}^{\infty}$ the chain of PCID associated to $\hat{A}$. The basic way to use Sections 3 and 4 in this situation is that for every $n \geq 1$, A_n is an 1-PCID of A_{n-1}. Therefore it is possible to apply Lemma 2.1. to the pair (A_{n-1}, A_n), $(n \geq 1)$, in order to obtain a A_{n-1}-choice sequence of lenght one, namely a contraction

$$\Gamma(A_{n-1}, A_n) : \mathcal{R}_{A_{n-1}} \longmapsto \mathcal{R}^{A_{n-1}}, \quad \textit{such that}$$

$$(5.1)_n \quad A_n(h_{n-1}+l_{n-1}) = A_{n-1}h_{n-1} + q^{A_{n-1}}\left[\sigma_{A_{n-1}} P_{A_{n-1}} + \Gamma_1(A_{n-1}, A_n) \cdot (1-p_{A_{n-1}})\right](D_{A_{n-1}} h_{n-1} + l_{n-1}), \quad h_{n-1} \in \mathcal{H}_{n-1}, \quad l_{n-1} \in U^{n-1}\mathcal{L}.$$

In the same way we obtain the unitary operators $\omega_{A_n} = \omega_{A_n}(T'_n, T_n)$ and $\omega^{A_n} = \omega^{A_n}(T'_n, T_n)$ such that:

$$(5.2)_n \quad \begin{cases} \omega_{A_n} : \mathcal{R}_{A_n} \longmapsto \mathcal{D}_{\Gamma_1(A_{n-1}, A_n)} \\ \omega_{A_n}(1-p_{A_n}) D_{A_n} = D_{\Gamma_1(A_{n-1}, A_n)} (1-p_{A_{n-1}})(D_{A_{n-1}} P_{n-1} + I - P_{n-1}) \,|\, \mathcal{H}_n, \end{cases}$$

$$(5.2)'_n \quad \begin{cases} \omega^{A_n} : \mathcal{R}^{A_n} \longmapsto \mathcal{D}_{\Gamma_1^*(A_{n-1}, A_n)} \\ \omega^{A_n}(1-p^{A_n})(0 \oplus U' l'_{n-1}) = D_{\Gamma_1^*(A_{n-1}, A_n)}(1-p^{A_{n-1}}) l'_{n-1}, \end{cases}$$

$$l_{n-1} \in U^{n-1}\mathcal{L}.$$

We will define by induction a sequence of contractions $\{\Gamma_n\}_{n=1}^{\infty}$ and two sequences of unitary operators $\{\Omega_{A_n}\}_{n=1}^{\infty}$ and $\{\Omega^{A_n}\}_{n=1}^{\infty}$ as follows

$$(5.3)_1 \qquad \begin{cases} \Gamma_1=\Gamma_1(A,A_1), \\ \Omega_{A_1}=\omega_{A_1}, \\ \Omega^{A_1}=\omega^{A_1}, \end{cases}$$

and for $n\geq 1$,

$$(5.3)_n \qquad \begin{cases} \Gamma_n=\Omega^{A_{n-1}}\Gamma_1(A_{n-1}, A_n)\Omega^*_{A_{n-1}}, \\ \Omega_{A_n}=\Omega_{A_{n-1}}\circ\omega_{A_n}, \\ \Omega^{A_n}=\Omega^{A_{n-1}}\circ\omega^{A_n}. \end{cases}$$

L e m m a 5.1. The sequence $\{\Gamma_n\}_{n=1}^{\infty}$ <u>is an A-choice sequence; for every</u> $n\geq 1$ <u>the operator</u> Ω_{A_n} <u>(resp.</u> Ω^{A_n}<u>) is a unitary from</u> R_{A_n} (resp. R^{A_n}) onto $\mathcal{D}_{\Gamma_n}$ (resp. $\mathcal{D}_{\Gamma_n^*}$).

P r o o f. For $n=1$, the assertions of the lemma follow from the definitions (see $(5.3)_1$). Suppose now that $n>1$ and that $\{\Gamma_k\}_{k=1}^{n-1}$ is an A-choice string of lenght $n-1$, Ω_{A_k} (and Ω^{A_k}) are unitary operators ($1\leq k\leq n-1$) from R_{A_k} (resp. R^{A_k}) onto $\mathcal{D}_{\Gamma_k}$ (resp. $\mathcal{D}_{\Gamma_k^*}$) such that $\Gamma_k=\Omega^{A_{k-1}}\Gamma_1(A_{k-1}, A_k)\Omega^*_{A_{k-1}}$, for every $2\leq k\leq n-1$. Define $\Gamma_n=\Omega^{A_{n-1}}\Gamma_1(A_{n-1}, A_n)\Omega^*_{A_{n-1}}$; it follows that Γ_n is a contraction from $\mathcal{D}_{\Gamma_{n-1}}$ into $\mathcal{D}_{\Gamma^*_{n-1}}$, which means that $\{\Gamma_k\}_{k=1}^{n}$ is an A-choice string of lenght n. Take now

$$\Omega_{A_n} = \Omega_{A_{n-1}} \circ \omega_{A_n} \quad \text{and}$$

$$\Omega^{A_n} = \Omega_{A_{n-1}} \circ \omega^{A_n}$$

The previous definitions make sense because $\omega_{A_n}(R_{A_n}) = \mathcal{D}_{\Gamma_1(A_{n-1}, A_n)} \subset R_{A_{n-1}}$ and $\omega^{A_n}(R^{A_n}) = \mathcal{D}_{\Gamma_1^*(A_{n-1}, A_n)} \subset R^{A_{n-1}}$; moreover $\Omega_{A_n}(R_{A_n}) = \Omega_{A_{n-1}}(\mathcal{D}_{\Gamma_1(A_{n-1}, A_n)}) = \mathcal{D}_{\Gamma_n}$ and $\Omega^{A_n}(R^{A_n}) = \Omega^{A_{n-1}}(\mathcal{D}_{\Gamma_1^*(A_{n-1}, A_n)}) = \mathcal{D}_{\Gamma_n^*}$, just because $\Gamma_1(A_{n-1}, A_n) = \Omega_{A_{n-1}} \Gamma_n (\Omega^{A_{n-1}})^*$. The lemma is now completely proved by induction.

Note that the A-choice string $\{\Gamma_k\}_{k=1}^n$ and the strings $\{\Omega_{A_k}\}_{k=1}^n$, $\{\Omega^{A_k}\}_{k=1}^n$ depend only on the operator A_n, $(n \geq 1)$.

D e f i n i t i o n 5.1. The A-choice sequence $\{\Gamma_n\}_{n=1}^\infty$ and the sequences $\{\Omega_{A_n}\}_{n=1}^\infty$, $\{\Omega^{A_n}\}_{n=1}^\infty$ will be called <u>the A-choice sequences of $\hat{A}$</u>, resp. <u>the sequences of indetificators</u> of $\hat{A}$. For $n \geq 1$, the A-choice string $\{\Gamma_k\}_{k=1}^n$ and the sequences $\{\Omega_{A_k}\}_{k=1}^n$, $\{\Omega^{A_k}\}_{k=1}^n$ will be called <u>the A-choice string of A_n</u>, resp. <u>the string of identificators of A_n</u>.

The unitary operator ω_{A_1}, defined by (2.11), was obtained from a larger one, namely $\tilde{\omega}_{A_1}$, which maps $\mathcal{D}_{A_1}$ onto $\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}$ (see 2.8). It is clear that if we define reccurently

$$(5.4)_n \quad \begin{cases} \Omega_{A_n} : \mathcal{D}_{A_n} \longrightarrow \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_n} \\ \tilde{\Omega}_{A_n} = (\tilde{\Omega}_{A_{n-1}} \oplus \Omega_{A_{n-1}}) \circ \tilde{\omega}_{A_n} \qquad n \geq 2, \end{cases}$$

where $\tilde{\Omega}_{A_1}=\tilde{\omega}_{A_1}$, then the operators $\{\tilde{\Omega}_{A_n}\}_{n=1}^{\infty}$ are unitary; moreover we have, for every $n\geq 1$,

$$(5.5)_n \qquad \tilde{\Omega}_{A_n}(R_{A_n})=\mathcal{D}_{\Gamma_n} , \quad P^{\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_n}}_{\{0\}\oplus\{0\}\oplus\ldots\oplus\mathcal{D}_{\Gamma_n}}\tilde{\Omega}_{A_n}|R_{A_n}=\Omega_{A_n} ,$$

$$(5.6)_n \qquad \Omega^{*}_{A_n}(\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_{n-1}}\oplus\{0\})+U^{n}L=F_{A_n} .$$

We are now able to give

P r o o f o f T h e o r e m 1.1. Taking into account Lemma 5.1, we have to prove only that if $\{\Gamma_n\}_{n=1}^{\infty}$ is an A-choice sequence, then there exists a CID of A, $\hat{A}$, such that the A-choice sequence of $\hat{A}$ (see Definition 5.1) is exactly $\{\Gamma_n\}_{n=1}^{\infty}$. For this, we construct by induction a chain $\{A_n\}_{n=1}^{\infty}$ of PCID of A, such that the A-choice string of A_n is $\{\Gamma_k\}_{k=1}^{n}$. If n=1, starting with Γ_1 we define (by (2.1)) an 1-PCID, A_1, of A, such that $\Gamma_1(A,A_1)=\Gamma_1$. Suppose now that for $n\geq 1$ we have A_n, an n-PCID of A, such that the A-choice string of A_n is $\{\Gamma_k\}_{k=1}^{n}$; consider also the strings of identificators of A_n , namely $\{\Omega_{A_k}\}_{k=1}^{n}$ and $\{\Omega^{A_k}\}_{k=1}^{n}$. Define

$$(5.7)_n \qquad \tilde{\Gamma}_n=(\Omega^{A_n})^{*}\Gamma_{n+1}\Omega_{A_n}$$

It is clear that $\{\tilde{\Gamma}_n\}$ is an A_n-choise string of lenght one and so it defines (by (2.1)) an 1-PCID of A_n , A_{n+1} , such that $\tilde{\Gamma}_n=\Gamma_1(A_n,A_{n+1})$. Taking into account $(5.7)_n$ and $(5.3)_n$, it follows that the A-choice string A_{n+1} is $\{\Gamma_k\}_{k=1}^{n+1}$. Since moreover it is now plain that this A_{n+1} is uniquely determined, the theorem is completely proved.

R e m a r k 5.1. In [23] it is proved that the one-to-one correspondence described in Theorem 1.1 is exactly the same which results by Proposition 2.1, 2.2 and 3.1 from [9].

C o r o l l a r y 5.1. <u>For every</u> $n\geq 1$,

$$(5.8)_n \qquad \Gamma_1(A_{n-1}, A_n)\omega_{A_n} = -\omega^{A_n}(1-p^{A_n})|\mathcal{R}_{A_n} .$$

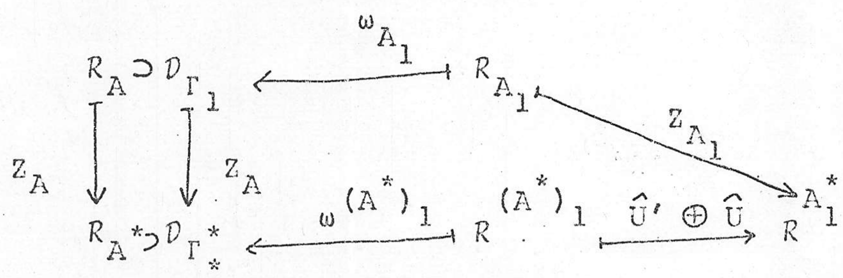
P r o o f. The corollary follows from Proposition 4.2, where A_n (as an 1-PCID of A_{n-1}) is used in place of A_1.

We will prove now the general form of Proposition 4.1. Let $\{\Gamma_n\}_{n=1}^{\infty}$ be an A-choice sequence and let $\{\Gamma_{*n}\}_{n=1}^{\infty}$ be the A^*-choice sequence defined by $(3.10)_n$, $(n\geq 1)$. Consider the CID of A, $\hat{A}$, defined by $\{\Gamma_n\}_{n=1}^{\infty}$ (see Theorem 1.1) and $\widehat{A^*}$ the CID of A^* defined by the chain of PCID of A^*, $\{(A^*)_n\}_{n=1}^{\infty}$, where

$$(5.9)_n \qquad (A^*)_n = \hat{U}^{*n} A_n^* \hat{U}'^n | (\mathcal{H}' + \mathcal{L}'^* + \dots + U_*'^{n-1}\mathcal{L}'^*),$$

$(n\geq 1)$. What we have to prove is that the A^*-choice sequence of $\widehat{A^*}$ is $\{\Gamma_{*n}\}_{n=1}^{\infty}$. For this, we give firstly the following result

L e m m a 5.2. <u>The diagram</u>



<u>is commutative; that is</u>

$$(5.10) \qquad Z_A \omega_{A_1} = \omega^{(A^*)_1}(\hat{U}' \oplus \hat{U})^* Z_{A_1} .$$

P r o o f. Because $R_{A_1} \subset \mathcal{D}_{A_1}$, the relation (5.10) will be proved if we show that

(5.10)′ $$Z_A \omega_{A_1} (1-p_{A_1}) D_{A_1} (h+(U-T)\tilde{h}) = \omega_{(A^*)_1} (\hat{U}'\oplus\hat{U}) Z_{A_1} (1-p_{A_1}) D_{A_1} (h+(U-T)\tilde{h}),$$

for every h, $\tilde{h}$ in H.

First, we have

$$Z_A \omega_{A_1} (1-p_{A_1}) D_{A_1} (h+(U-T)\tilde{h}) = Z_A D_{\Gamma_1} (1-p_A)(D_A h+(U-T)\tilde{h}) =$$

$$= D_{\Gamma^*_{*1}} Z_A (1-p_A)(D_A h+(U-T)\tilde{h}) = D_{\Gamma^*_{*1}} (1-p^{A^*}) Z_A (D_A h+(U-T)\tilde{h}) =$$

$$= D_{\Gamma^*_{*1}} (1-p^{A^*})(D_{A^*} Ah \oplus (U_* - T^*)(T\tilde{h} - D_A^2 h) =$$

$$= D_{\Gamma^*_{*1}} (1-p^{A^*})(0 \oplus (U_* - T^*)(T\tilde{h}-h)) \qquad (h,\ \tilde{h} \in H),$$

where we use, in order, (2.11), $(3.10)_1$, Corollary 3.3., (3.7) and the structure of $F^{\dot{A}^*}$ (see (1.1)′). On the other hand, we have

$$\Omega^{(A^*)_1} (\hat{U}'\oplus\hat{U})^* Z_{A_1} (1-p_{A_1}) D_{A_1} (h+(U-T)\tilde{h}) =$$

$$= \omega^{(A^*)_1} (\hat{U}'\oplus\hat{U})^* (1-p^{A_1^*}) Z_{A_1} D_{A_1} (h+(U-T)\tilde{h}) =$$

$$= \omega^{(A^*)_1} (\hat{U}\oplus\hat{U})^* (1-p^{A_1^*})(D_{A_1^*} A_1 (h+(U-T)\tilde{h}) \oplus (U_{*1} - T_1^*)(-D_{A_1}^2 (h+(U-T)\tilde{h})) =$$

$$= -\omega^{(A^*)_1} (1-p^{(A^*)_1})(\hat{U}'\oplus\hat{U})^* (0 \oplus (U^*_1 - T_1^*)(h+(U-T)\tilde{h})) =$$

$$= -\omega^{(A^*)_1} (1-p^{(A^*)_1})(0 \oplus U_* (U_{*1} - T_1^*)(h+(U-T)\tilde{h})) =$$

$$= -D_{\Gamma^*_{*1}} (1-p^{\dot{A}^*})(0 \oplus (U_{*1} - T_1^*)(h+(U-T)\tilde{h})) =$$

$$= D_{\Gamma^*_{*1}} (1-p^{A_*^*})(0 \oplus (U_* - T^*)(T\tilde{h}-h)) \qquad (h,\tilde{h} \in H),$$

where we used Corollary 3.3, (3.7), the structure of $F^{A_1^*}$, $(3.10)_1$, (4.13) and the properties of isometric dilation. The lemma is now completely proved.

P r o p o s i t i o n 5.1. <u>The choice sequence of</u> $\hat{A}^*$ <u>is</u> $\{\Gamma_{*n}\}_{n=1}^{\infty}$ <u>and</u>

$$(5.11)_n \qquad Z_{A_n}|R_{A_n}=(\hat{U}'\oplus\hat{U})^n(\Omega^{(A^*)_n})^* Z_{\Lambda}\Omega_{A_n},$$

$$(5.11)'_n \qquad Z^{A_n}|R^{A_n}=\hat{U}'^n(\Omega_{(A^*)_n}\quad Z^{A}\Omega^{A_n}$$

<u>for every</u> $n\geq 1$, <u>where</u> $\{\Omega_{(A^*)_n}\}_{n=1}^{\infty}$ <u>and</u> $\{\Omega^{(A^*)_n}\}_{n=1}^{\infty}$ <u>are the sequences of identificators of</u> A^*.

P r o o f. We proceed by induction for proving that the choice string of $(A^*)_n$ is $\{\Gamma_{*k}\}_{k=1}^{n}$ and that the formulas $(5.11)_n$ and $(5.11)'_n$ are true, for every $n\geq 1$. For $n=1$, the results follow from Proposition 4.1, Lemma 5.2 and (4.12). Suppose now that $n>1$ and that the A^*-choice string of $(A^*)_n$ is $\{\Gamma_{*k}\}_{k=1}^{n}$, the formulas $(3.8)_n$ and $(3.8)'_n$ being true with $\{\Omega_{(A^*)_k}\}_{k=1}^{n}$ and $\{\Omega^{(A^*)_k}\}_{k=1}^{n}$, the strings of identificators of $(A^*)_n$. Consider A_{n+1} as a 1-PCID of A_n with the A_n-choice string $\Gamma_1(A_n, A_{n+1})=(\Omega^{A_n})^*\Gamma_{n+1}(\Omega_{A_n})$. From Proposition 4.1 it follows that the A_n^*-choice string of $(A_n^*)_1=\hat{U}^{\wedge *}A_{n+1}^* U'|(H'_n+L'^*)$ is

$$\Gamma_1(A_n^*, (A_n^*)_1)=Z_{A_n}\Gamma_1^*(A_n,A_{n+1})(Z^{A_n})^*.$$

From $(5.11)_n$, $(5,11)'_n$ and the definition of $\Gamma_1(A_n, A_{n+1})$ it follows

$$(5.12)\qquad \Gamma_1(A_n^*, (A_n^*)_1) = (\hat{U}'\oplus\hat{U})^n (\Omega^{(A^*)_n})^* Z_A \Omega_{A_n} (\Omega_{A_n})^* \Gamma_{n+1}^* \Omega^{A_n} .$$

$$.(\Omega^{A_n})^* (Z^A)^* \Omega_{(A^*)_n} \hat{U}'^{*n} | \mathcal{R}_{A_n^*} =$$

$$= (\hat{U}'\oplus\hat{U})^n (\Omega^{(A^*)_n})^* \Gamma_{*,n+1} \Omega_{(A^*)_n} \hat{U}'^{*n} | \mathcal{R}_{A_n^*} .$$

But from $(5.9)_{n+1}$, it follows that

$$(5.13)\qquad (A^*)_{n+1} = \hat{U}^{*n+1} A_{n+1}^* \hat{U}'^{n+1} | (\mathcal{H}' + L'^* + \ldots + U_*'^n L'^*) =$$

$$= \hat{U}^{*n} (A_n^*)_1 \hat{U}'^n | (\mathcal{H}' + \ '^* + \ldots + U_*'^n L'^*),$$

wich implies (by (5.12)) that the $(A^*)_n$-choice string of $(A^*)_{n+1}$ is $(\Omega^{(A^*)_n})^* \Gamma_{*,n+1} \Omega_{(A^*)_n}$. This means that the A^*-choice string of $(A^*)_{n+1}$ is $\{\Gamma_{*,k}\}_{k=1}^{n+1}$. Using Lemma 5.2 for the pair (A_n, A_{n+1}), we have

$$Z_{A_{n+1}} = (\hat{U}'\oplus\hat{U})(\omega^{(A_n^*)_1})^* Z_{A_n} \omega_{A_n} ,$$

and then $(5.11)_{n+1}$ follows from $(5.11)_n$, (5.13) and Lemma 5.1.

From (4.12) (for the pair (A_n, A_{n+1})), we have

$$\omega^{A_{n+1}} = (Z^{A_n})^* \omega^{(A_n^*)_1} \cdot \hat{U}'^* Z^{A_{n+1}} ,$$

which easily implies $(5.11)'_{n+1}$ and therefore the proposition is completely proved.

6. Definition 1.1. shows that the notion of choice sequence involves operators defined on different spaces, constructed in an iterative way. This is not the case in the classical extrapolation theorems [16], [13], [2]. In order to obtain in our situation an analogue of Schur indexing, as well as for the sake of applications in the theory of wave propagation in layered media, we give now a " more computable " description of Theorem 1.1. To this aim we first define the " observable sequence " of a CID of A; then we give iterative formulas in order to connect observable sequence and choice sequence of a CID of A.

Consider again the typical situation of this paper (i.e. $T \in L(\mathcal{H})$, $T' \in L(\mathcal{H}')$, $A \in I(T',T)$ are fixed contraction). Let $\hat{A}$ be a CID of A and let $\{A_n\}_{n=1}^{\infty}$ be the chain of PCID of $\hat{A}$. For every $n \geq 1$, we have

$$(6.1)_n \qquad \mathcal{H}_n = \mathcal{L}_* + U\mathcal{L}_* + \ldots + U^{n-1}\mathcal{L}_* + U^n\mathcal{H},$$

and

$$(6.1)'_n \qquad \mathcal{H}'_n = \mathcal{H}' + \mathcal{L}' + \ldots + U^{n-1}\mathcal{L}'.$$

D e f i n i t i o n 6.1. For $n \geq 1$ and $1 \leq k \leq n$, the operator

$$S_k(A_n) = U'^{*k-1}(P'_k - P'_{k-1})A_n|\mathcal{L}_* : \mathcal{L}_* \longrightarrow \mathcal{L}'$$

will be called the <u>k-th observable operator of A_n</u>.

The string $\{S_k(A_n)\}_{k=1}^{n}$ will be called <u>the observable string of A_n</u>.

It is obviously that $\{S_k(A_n\}_{k=1}^{n}$ is uniquely determined by A_n.

L e m m a 6.1. For an $n\geq 1$, the observable string of A_n uniquely determines A_n.

P r o o f. The operator $A_n \in L(H_n, H_n')$ is uniquely determined by its matrix with respect to the decompositions $(6.1)_n$ and $(6.1)_n'$. From the definition of an n-PCID of A, we have that

$$P'A_n = AP|H_n,$$

which means that A_n is uniquely determined by the operator $(1-P')A_n$. On the other hand

$$A_nU^n|H=A_nT_n^n|H=T_n'^nA_n|H=P_n'U'^nA_n|H= \\ = U'^nP'A_n|H= T_n'^nA,$$

so A_n is uniquely determined by the operator

$$(I-P')\ A_n \mid (L_* + UL_* + \ldots + U^{n-1}L_*).$$

But, for $0\leq k\leq n-1$

$$(I-P')A_nU^k|L_*=(I-P')A_nT_n^k|L_*=(I-P')T_n'^kA_n|L_*= \\ =(I-P')T_n'^k\Big[AP|L_*+\sum_{\ell=1}^{n}U'^{\ell-1}S_\ell(A_n)\Big]=(I-P')T_n'^k\Big[AP|L_*+ \\ +\sum_{\ell=1}^{n}T'^{\ell-1}S_\ell(A_n)\ ,$$

and therefore A_n is uniquely determined by its observable string.

R e m a r k 6.1. Let n, m≥1 and $1\le k\le \min\{n,m\}$; then

$$S_k(A_n) = S_k(A_m),$$

just because $\{A_p\}_{p=1}^{\infty}$ is a <u>chain</u> of PCID of A.

This remark justifies the notations

$$(6.2)_n \qquad S_n(\hat{A}) = S_n(A_m), \qquad n\ge 1,$$

where $m\ge n$ is arbitrary.

D e f i n i t i o n 6.2. The sequence $\{S_n(\hat{A})\}_{n=1}^{\infty}$ will be called the <u>observable sequence of $\hat{A}$</u>.

C o r o l l a r y 6.1. <u>The observable sequence of $\hat{A}$ uniquely determines $\hat{A}$.</u>

R e m a r k 6.2. We do not know yet the conditions for a sequence $S_n : L_* \longmapsto L'$, n≥1, such that there (should) exists $\hat{A}$, a CID of A, such that $S_n = S_n(\hat{A})$ for every n≥1.

We will try now to understand a little more the observable sequence of $\hat{A}$. For this, note that from (2.1) it follows

$$(6.3) \quad S_1(\hat{A}) = q^A \sigma_A p_A (D_A P + I - P) | L_* + q^A \Gamma_1 (1 - p_A)(D_A P + I - P) | L_*.$$

Define the operators

$$(6.4)\qquad \begin{cases} R^{o}(A): L_{*} \longmapsto L' \\ R^{o}(A) = q^{A}\sigma_{A}p_{A}(D_{A}P + I-P)|L_{*} \end{cases},$$

$$(6.5)\qquad \begin{cases} R(A) : R_{A} \longmapsto L_{*} \\ R(A)^{*} = (1-p_{A})(D_{A}P + I-P)|L_{*} \end{cases},$$

$$(6.6)\qquad \begin{cases} R'(A): R^{A} \longmapsto L' \\ R'(A) = q^{A}| R^{A} \end{cases}.$$

These definitions make sense for every A_n instead of A, $n\geq 1$, therefore we can define the operators:

$$(6.7)_o\qquad R_o^{o}(\hat{A}) = R^{o}(A),$$

$$(6.8)_o\qquad R_o(\hat{A}) = R(A),$$

$$(6.9)_o\qquad R'_o(\hat{A}) = R'(A)$$

and for any $n\geq 1$,

$$(6.7)_n\qquad R_n^{o}(\hat{A}) = U'^{*n}R^{o}(A_n): L_{*} \longmapsto L'$$

$$(6.8)_n\qquad R_n(\hat{A}) = R(A_n)\Omega_{A_n}^{*} : \mathcal{D}_{\Gamma_n} \longmapsto L_{*}$$

$$(6.9)_n\qquad R'_n(\hat{A}) = U'^{*n}R(A_n)(\Omega^{A_n})^{*}$$

From (6.3) it follows that for $n\geq 1$ we have

$$(6.10)_n\qquad S_n(\hat{A}) = R_{n-1}^{o}(\hat{A}) + R'_{n-1}(\hat{A})\Gamma_n R_{n-1}^{*}(\hat{A}).$$

L e m m a 6.2. *The operators $R_n(\hat{A})$ and $R'_n(\hat{A})$ are injective for every $n\geq 0$.*

P r o o f. From $(6.8)_n$ and $(6.9)_n$, $n\geq 0$, it is clear that the lemma will be proved if we show that $R(A)$ and $R'(A)$ are injective. But for $l_*=(I-UT^*)h\in L_*$, where $h\in H$, we have that

$$R(A)^*l_*=(1-p_A)(D_AP+I-P)(h-T^*Th-(U-T)T^*h) =$$
$$=(1-p_A)(D_Ah-(D_ATT^*h+(U-T)T^*h)) = (1-p_A)D_Ah.$$

Therefore, from Lemma 1.1. (b) and (c) it follows that $R(A)^*$ and $R'(A)^*$ have dense range, which means that $R(A)$ and $R'(A)$ are injective.

C o r o l l a r y 6.2. *Let $n\geq 0$. a) The factorization $T'_n \cdot A_n$ is regular if and only if $R'_n(\hat{A}) = 0$.*

b) The factorization $A_n \cdot T_n$ is regular if and only if $R_n(\hat{A})=0$.

7. In the next sections $\hat{A}$ will be a fixed CID of A; we will denote

$$R_n^o=R_n^o(\hat{A}),$$
$$R_n=R_n(\hat{A}),$$
$$R'_n=R'_n(\hat{A}) \qquad \text{for } n\geq 0,$$

and

$$S_n = S_n(\hat{A}), \qquad \text{for } n\geq 1.$$

We will give now iterative formulas for the sequences $\{R_n^o\}_{n=0}^{\infty}$, $\{R_n\}_{n=0}^{\infty}$ and $\{R'_n\}_{n=0}^{\infty}$. Let start with the sequence $\{R_n\}_{n=0}^{\infty}$. For $(6.8)_1$, we infer that

$$R_1 = R(A_1)\Omega^*_{A_1},$$

which means

$$R_1^* = \Omega_{A_1} R(A_1)^* = \omega_{A_1}(1-p_1)D_{A_1}|L_* =$$

$$= D_{\Gamma_1}(1-p_A)(D_A P + I - P)|L_* = D_{\Gamma_1} R_o^*.$$

Therefore

$(7.1)_1$ $$R_1 = R_o D_{\Gamma_1}.$$

Suppose now that $n>1$ is given and that

$(7.1)_{n-1}$ $$R_{n-1} = R_o D_{\Gamma_1} \ldots D_{\Gamma_{n-1}}.$$

Then, from $(6.8)_n$ and $(5.3)_n$, it follows

$$R_n^* = \Omega_{A_n} \mathcal{R}(A_n)^* = \Omega_{A_{n-1}} \omega_{A_n} R(A_n)^* =$$

$$= \Omega_{A_{n-1}} \omega_{A_n} (1-p_{A_n}) D_{A_n} | L_* =$$

$$= \Omega_{A_{n-1}} D_{\Gamma_1(A_{n-1}, A_n)} (1-p_{A_{n-1}}) D_{A_{n-1}} | L_* =$$

$$= D_{\Gamma_n} \Omega_{A_{n-1}} R(A_{n-1})^* = D_{\Gamma_n} R_{n-1}^*,$$

so that

$(7.1)_n$ $$R_n = R_{n-1} D_{\Gamma_n} = R_o D_{\Gamma_1} \ldots D_{\Gamma_n}.$$

Therefore we proved $(7.1)_n$, $n\geq 1$, by induction.

A similar formula is valid for $\{R'_n\}_{n=0}^{\infty}$. Indeed, for $(6.9)_1$, (6.6) and (4.13) it follows that:

$$R_1'^{*}=\Omega^{A_1}R'(A_1)^{*}U'|L'=\Omega^{A_1}(1-p^{A_1})U'|L'=$$

$$=D_{\Gamma_1}^{*}(1-p^{A})|L'=D_{\Gamma_1}^{*}R_o'^{*},$$

so we have

$(7.2)_1$ $$R_1'=R_o'D_{\Gamma_1}^{*}.$$

A similar inductive argument (as for $(7.1)_n$) implies that

$(7.2)_n$ $$R_n'=R_{n-1}'D_{\Gamma_n}^{*}=R_o'D_{\Gamma_1}^{*}\dots D_{\Gamma_n}^{*}$$

We proved thus

P r o p o s i t i o n 7.1. <u>For every</u> $n\geq 0$

$$S_{n+1}=R_n^o+R_n'\Gamma_{n+1}R_n^{*},$$

<u>where</u>

$$R_o^{*}=(1-p_A)(D_AP+I-P)|L_{*}, \qquad R_n=R_oD_{\Gamma_1}\dots D_{\Gamma_n},$$

$$R_o'=q^{A}|R_A, \qquad R_n'=R_o'D_{\Gamma_1}^{*}\dots D_{\Gamma_n}^{*},$$

<u>and</u> R_n^o <u>depends only on</u> A_n.

The case of $\{R_n^o\}_{n=0}^{\infty}$ is a little more complicated. In fact, the iterative formulas of $\{R_n^o\}_{n=0}^{\infty}$ give also iterative formulas of the same type for $\{S_n\}_{n=1}^{\infty}$. From $(6.7)_n$ and (6.4) we have that

$$R_o^o=q^{A}\sigma_A p_A(D_AP+I-P)|L_{*} \quad \text{and}$$

$$R_n^o=U'^{*n}q^{A_n}\sigma_{A_n}p_{A_n}D_{A_n}|L_{*} \qquad n\geq 1.$$

Define the operators:

$$(7.3)_n \quad \begin{cases} X_n : L_* \longmapsto \mathcal{D}_A + \mathcal{D}_{\Gamma_1} + \dots + \mathcal{D}_{\Gamma_n} \\ X_n = \tilde{\Omega}_{A_n} D_{A_n} \mid L_* \ . \end{cases}$$

and

$$(7.4)_n \quad \begin{cases} Y_n : \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_n} \longmapsto L' \\ Y_n = U'^{*n} q^{A_n} (\sigma_{A_n} p_{A_n} + \Gamma_1(A_n, A_{n+1})(1-p_{A_n}))\tilde{\Omega}^*_{A_n} , \end{cases}$$

for every $n \geq 1$.

Taking into account the direct sum $\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_n}$ which appears in $(7.3)_n$ and $(7.4)_n$, we have

$$(7.5)_n \quad \begin{cases} X_n = \begin{pmatrix} X_n^o \\ X_n^1 \\ \vdots \\ X_n^n \end{pmatrix} , & \text{where} \\ X_n^o : L_* \longmapsto \mathcal{D}_A & \text{and} \\ X_n^j : L_* \longmapsto \mathcal{D}_{\Gamma_j} , & 1 \leq j \leq n \end{cases}$$

and

$$(7.6)_n \quad \begin{cases} Y_n = (Y_n^o , Y_n^1 , \dots , Y_n^n), & \text{where} \\ Y_n^o : \mathcal{D}_A \longmapsto L' \\ Y_n^j : \mathcal{D}_{\Gamma_j} \longmapsto L' & 1 \leq j \leq n , \end{cases}$$

for every $n \geq 1$.

From the definitions and the properties of the sequence $\{\tilde{\Omega}_n\}_{n=1}^{\infty}$ (see $(5.4)_n$, $(5.5)_n$, $(5.6)_n$, $n \geq 1$), we infer that:

$(7.7)_n$ $\qquad S_{n+1}=Y_nX_n = Y_n^oX_n^o+Y_n^1X_n^1+\ldots+Y_n^nX_n^n$

$(7.8)_n^o$ $\qquad R_n^o=Y_n^oX_n^o+Y_n^1X_n^1+\ldots+Y_n^{n-1}X_n^{n-1}$

$(7.8)_n$ $\qquad R_n^* = X_n^n$

$(7.8)_n'$ $\qquad R_n'\Gamma_{n+1}=Y_n^n$

for every $n\geq 1$.

The aim of the rest of this section is to give iterative formulas for the sequences $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$. Let start with the study of the sequence $\{Y_n\}_{n=1}^{\infty}$.

L e m m a 7.1. With the notations of $(7.6)_1$ and $(7.6)_2$,

a) $Y_2^o=A_1^o$

b) $Y_2^1=Y_1^1$

P r o o f. a) Let $D_Ah\in\mathcal{D}_A$, $(h\in H)$; using (2.9), we have

$$Y_1^o(D_Ah)=U'^*q^{A_1}(\sigma_{A_1}p_{A_1}+\Gamma_1(A_1, A_2)(1-p_{A_1}))\tilde{\Omega}_{A_1}^*(D_Ah)=$$

$$=U'^*q^{A_1}(\sigma_{A_1}p_{A_1}+\Gamma_1(A_1, A_2)(1-p_{A_1}))D_{A_1}Uh=$$

$$=U'^*q^{A_1}\sigma_{A_1}(D_{A_1}T_1h+(U-T_1)h)=U'^*q^{A_1}(D_{A_1}h\oplus(U'-T_1')A_1h)=$$

$$=U'^*U'(1-P')A_1h=(1-P')A_1h,$$

and

$$Y_2^o(D_Ah)=U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2}+\Gamma_1(A_2, A_3)(1-p_{A_2}))\tilde{\omega}_{A_2}^*\tilde{\omega}_{A_1}^*(D_Ah)=$$

$$=U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2}+\Gamma_1(A_2, A_3)(1-p_{A_2}))D_{A_2}U^2h=$$

$$=U'^{*2}q^{A_2}\sigma_{A_2}\left[D_{A_2}T_2(Uh)+(U-T_2)Uh\right]=$$

$$=U'^{*2}q^{A_2}\left[D_{A_2}Uh \oplus (U'-T_2')A_2Uh\right]=$$

$$=U'^{*2}(U'-T_2')T_2'A_2h=U'^{*}(1-P_1')T_2'A_2h=(1-P')A_1h$$

therefore

$$Y_2^o = Y_1^o .$$

b) Let $\gamma_1=D_{\Gamma_1}(1-p_A)(D_AP+I-P)h_1\in\mathcal{D}_{\Gamma_1}$ $(h_1\in H_1)$.

Then, using (2.11), we have that

$$\gamma_1=\omega_{A_1}(1-p_{A_1})D_{A_1}h_1 .$$

Because $(1-p_{A_1})D_{A_1}h_1$ is in $\mathcal{D}_{A_1}$, there exists a sequence $\{D_{A_1}h_1^n\}_{n=1}^{\infty}$ such that $D_{A_1}h_1^n \longrightarrow (1-p_{A_1})D_{A_1}h_1$, where $h_1^n\in H_1$ for every $n\geq 1$. Then, using (2.9), we have

$$Y_2^1\gamma_1=U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2}+\Gamma_1(A_2\ ,A_3)(1-p_{A_2}))\tilde{\Omega}^*_{A_2}\omega_{A_1}(1-p_1)D_{A_1}h_1=$$

$$=\lim_{n\to\infty} U'^{*2}q^{A_2}(\sigma_{A_2}p_{A_2}+\Gamma_1(A_2\ ,A_3)(1-p_{A_2}))\tilde{\omega}^*_{A_2}D_{A_1}h_1^n=$$

$$=\lim_{n\to\infty} U'^{*2}q^{A_2}\sigma_{A_2}D_{A_2}Uh_1^n=\lim_{n\to\infty} U'^{*2}(U'-T_2')A_2h_1^n=$$

$$=\lim_{n\to\infty} U'^{*}(1-P_1')A_2h_1^n=\lim_{n\to\infty} U'^{*}(1-P_1')(\sigma_{A_1}p_{A_1}+\Gamma_1(A_1\ ,A_2)(1-p_{A_1}))D_Ah_1^n$$

$$=U'^{*}(1-P_1')(\omega^{A_1})^*\Gamma_2\omega_{A_1}(1-p_1)D_{A_1}h_1=U'^{*}q^{A_1}(\omega^{A_1})^*\Gamma_2\gamma_1=$$

$$=R_1'\Gamma_2\gamma_1 .$$

Taking into account $(7.8)_1'$, the lemma is completely proved.

L e m m a 7.2. <u>For every</u> $n,m\geq 1$ <u>and</u> $0\leq j\leq \min\{n,m\}$ <u>we have</u>

$$Y_n^j = Y_m^j \ .$$

P r o o f. It is sufficient to prove that if $n \geq 1$, then

$$Y_n^j = Y_{n+1}^j \ , \qquad \text{for every } 0 \leq j \leq n.$$

To apply Lemma 7.1 for (A_{n-1}, A_n, A_{n+1}), instead of (A, A_1, A_2), denote by $\bar{Y}_i^j$ the operator "similar" to Y_i^j, which appears in this situation ($i=1,2$; $0 \leq j \leq i$). Then, by $(5.4)_n$, $(5.5)_n$ and $(5.6)_n$, we have

$$\bar{Y}_1^1 = Y_n^n \omega_{A_{n-1}} | \mathcal{D}_{\Gamma_1(A_{n-1}, A_n)},$$

$$\bar{Y}_2^1 = Y_{n+1}^n \omega_{A_{n-1}} | \mathcal{D}_{\Gamma_1}(A_{n-1}, A_n),$$

and

$$\bar{Y}_1^o \tilde{\Omega}_{A_{n-1}} = (Y_n^o, Y_n^1, \ldots, Y_n^{n-1}),$$

$$\bar{Y}_2^o \tilde{\Omega}_{A_{n-1}} = (Y_{n+1}^o, \ldots, Y_{n+1}^{n-1}) .$$

The lemma follows now immediately from Lemma 7.1.

This lemma enables us to define, for every $n \geq 0$,

$(7.9)_n$ $$Y^n = Y_m^n \ ,$$

where $m > n$ is arbitrary.

P r o p o s i t i o n 7.2. For all $m \geq n \geq 1$,

$(7.10)_n$ $$Y^n = Y_m^n = R'_o D_{\Gamma_1^*} \ldots D_{\Gamma_n^*} \Gamma_{n+1} \ .$$

P r o o f. From Lemma 7.2, we have for $n \geq 1$,

$$Y^n = Y_n^n$$

so that the proposition follows from $(7.8)'_n$ and $(7.2)_n$.

We fix now our attention to the sequence $\{X_n\}_{n=1}^{\infty}$.

Let $n\geq 1$ and $h_{n-1}\in \mathcal{H}_{n-1}$. Then, from (2.1) (for A_n instead of A_1), it follows that

$$||D_{A_n}h_{n-1}||^2=||h_{n-1}||^2-||A_nh_{n-1}||^2\leq ||h_{n-1}||^2-||A_{n-1}h_{n-1}||^2=$$

$$=||D_{A_{n-1}}h_{n-1}||^2.$$

This means that there exists a contraction

$(7.11)_n$
$$\begin{cases} B_n : \mathcal{D}_{A_{n-1}} \longmapsto \mathcal{D}_{A_n} \\ B_n D_{A_{n-1}} = D_{A_n} | \mathcal{H}_{n-1} . \end{cases}$$

Using $(7.3)_n$, $(7.11)_n$ and $(7.3)_{n-1}$, we infer that

$(7.12)_{n+1}$
$$X_{n+1}=\tilde{B}_{n+1}X_n ,$$

where

$(7.13)_{n+1}$
$$\begin{cases} \tilde{B}_{n+1}:\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus \ldots\oplus \mathcal{D}_{\Gamma_n} \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}\oplus \ldots \oplus \mathcal{D}_{\Gamma_{n+1}} , \\ \tilde{B}_{n+1}=\tilde{\Omega}_{A_{n+1}} B_{n+1}\tilde{\Omega}^*_{A_n} . \end{cases}$$

We will need also the operator

$(7.13)_1$
$$\begin{cases} \tilde{B}_1:\mathcal{D}_A \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \\ \tilde{B}_1=\tilde{\Omega}_1 B_1 . \end{cases}$$

The formulas $(7.12)_n$, $(n\geq 2)$ show that for understanding the structure of the sequence $\{X_n\}_{n=1}^{\infty}$, we must focus our attention on $\{B_n\}_{n=2}^{\infty}$. Denote by Q_n^0 the "projection" from $\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \cdots \oplus \mathcal{D}_{\Gamma_n}$ onto $\mathcal{D}_A$ and by Q_n^j the projection from the same space onto $\mathcal{D}_{\Gamma_j}$, $(1\le j\le n)$, $n\ge 1$. The key step is the following.

L e m m a 7.3. For every $n\ge 1$

$$(7.14)_{n+1} \qquad \tilde{B}_{n+1}=\left[\tilde{B}_n(1-Q_n^n)+\tilde{\Omega}_{A_n}(1-q^{A_n})(\Omega^{A_n})^*\Gamma_{n+1}Q_n^n\right]\oplus D_{\Gamma_{n+1}}Q_n^n .$$

P r o o f. As in the proof of Lemma 7.2, one can reduce the proof of $(7.14)_n$ (for $n\ge 3$) to that of $(5.14)_3$. So, we must proof only $(7.14)_3$ and $(7.14)_2$. It is easy to see that the proof of $(7.14)_2$ is a simplified version of the proof of $(5.14)_3$, which will be verified in the sequence.

Let $x\in\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}$ and $\{D_{A_1}h_1^n\}_{n=1}^{\infty}$ a sequence such that $D_{A_1}h_1^n\to\Omega_{A_1}^*x$, where $h_1^n\in\mathcal{H}_1$ for every $n\ge 1$. Then

$$\tilde{B}_3(x\oplus 0)=\tilde{\Omega}_{A_3}B_3\tilde{\Omega}_{A_2}^*(x\oplus 0)=\lim_{n\to\infty}\tilde{\Omega}_{A_3}B_3\tilde{\omega}_{A_2}^*(D_{A_1}h_1^n\oplus 0)=$$

$$=\lim_{n\to\infty}\tilde{\Omega}_{A_3}B_3D_{A_2}Uh_1^n=\lim_{n\to\infty}\tilde{\Omega}_{A_3}D_{A_3}Uh_1^n=\lim_{n\to\infty}(\tilde{\Omega}_{A_2}D_{A_2}h_1^n)\oplus 0=$$

$$=\lim_{n\to\infty}(\tilde{\Omega}_{A_2}B_2D_{A_1}h_1^n)\oplus 0=(\tilde{\Omega}_{A_2}B_2\tilde{\Omega}_{A_1}^*x)\oplus 0=\tilde{B}_2x\oplus 0,$$

where we used twice (2.9). This relation means that

$$(7.15) \qquad \tilde{B}_3|\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\{0\}=(1-Q_3^3)^*\tilde{B}_2(1-Q_2^2)|\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\{0\},$$

which also implies that

$$(7.16) \qquad Q_3^3\tilde{B}_3=Q_3^3\tilde{B}_3Q_2^2 .$$

Take now an element $\gamma_2\in\mathcal{D}_{\Gamma_2}$ such that

$$\Omega^*_{A_2}(0\oplus 0\oplus\gamma_2)=(1-p_{A_2})D_{A_2}h_1 \quad , \qquad (h_1\in H_1).$$

We have

$$Q_3^3\tilde{B}_3(0\oplus 0\oplus\gamma_2)=Q_3^3\tilde{\Omega}_{A_3}\tilde{\Omega}^*_{A_2}(0\oplus 0\oplus\gamma_2)=Q_3^3\tilde{\Omega}_{A_3}B_3(1-p_{A_2})D_{A_2}h_1=$$

$$=\Omega_{A_3}(1-P_{A_3})B_3(1-p_{A_2})D_{A_2}h_1=\Omega_{A_3}(1-p_{A_3})B_3D_{A_2}h_1=$$

$$=\Omega_{A_3}(1-p_3)D_{A_3}h_1=\Omega_{A_2}D_{\Gamma_1(A_2,A_3)}(1-p_{A_2})D_{A_2}h_1=$$

$$=D_{\Gamma_3}\Omega_{A_2}(1-p_{A_2})D_{A_2}h_1=D_{\Gamma_3}\gamma_2 ,$$

where we used that $B_3(D_{A_2}UH_1)^-\subset(D_{A_3}UH_2)^-$.

We proved that

(7.17) $$Q_3^3\tilde{B}_3=D_{\Gamma_3}Q_2^2 .$$

It remains to study $(1-Q_3^3)\tilde{B}_3Q_2^2$. For this, take again $\gamma_2\in\mathcal{D}_{\Gamma_2}$ such that $\Omega^*_{A_2}\gamma_2=(1-p_{A_2})D_{A_2}h_1$, $(h_1\in H_1)$ and a sequence $\{D_{A_2}h_2^n\}_{n=1}^\infty$, with $D_{A_2}h_2^n$ $\longrightarrow(1-p_2)D_{A_2}h_1$, (where $h_2^n\in H_2$ for every $n\geq 1$). Then

(7.18) $$(1-Q_3^3)\tilde{B}_3\gamma_2=(1-Q_3^3)\tilde{\Omega}_{A_3}B_3\tilde{\Omega}^*_{A_2}\gamma_2 =$$

$$=\lim_{n\to\infty}\tilde{\Omega}_{A_3}p_{A_3}B_3D_{A_2}h_2^n=\lim_{n\to\infty}\tilde{\Omega}_{A_2}\tilde{\omega}_{A_3}p_{A_3}D_{A_3}h_2^n=$$

$$=\lim_{n\to\infty}\tilde{\Omega}_{A_2}P_{\mathcal{D}_{A_2}\oplus\mathcal{D}_{\Gamma_1(A_2,A_3)}}^{\mathcal{D}_{A_2}}\tilde{\omega}_{A_3}D_{A_3}h_2^n=$$

$$=\lim_{n\to\infty}\tilde{\Omega}_{A_2}(1-q^{A_2})(\sigma_{A_2}p_{A_2}+\Gamma_1(A_2,A_3)(1-p_{A_2}))D_{A_2}h_2^n=$$

$$=\tilde{\Omega}_{A_2}(1-q^{A_2})\Gamma_1(A_2,A_3)(1-p_{A_2})D_{A_2}h_1=$$

$$=\tilde{\Omega}_{A_2}(1-q^{A_2})(\Omega^{A_2})^*\Gamma_3\gamma_2 ,$$

where we used $(5.4)_3$, $(5.6)_3$, Lemma 2.2, (2.8) and $(5.3)_3$. The relations (7.15), (7.17) and (7.18) imply $(5.14)_3$ and the lemma is completely proved.

The final step is to study the operators:

$$(7.19)_n \qquad \begin{cases} C_n : \mathcal{D}_{\Gamma_n^*} \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_n} \\ C_n = -\tilde{\Omega}_{A_n}(1-q^{A_n})(\Omega^{A_n})^*, \end{cases}$$

(see $(7.14)_n$), $n \geq 2$.

L e m m a 7.4. For every $n \geq 2$,

$$(7.20)_n \qquad C_n = \begin{pmatrix} C_{n-1} D_{\Gamma_n^*} \\ \Gamma_n^* \end{pmatrix}$$

P r o o f. We have that

$$\begin{cases} C_n^* : \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_n} \longmapsto \mathcal{D}_{\Gamma_n^*} \\ C_n^* = -\Omega^{A_n}(1-p^{A_n})(\Omega_{A_n})^* \end{cases}$$

From $(5.8)_n$, it follows immediatly that

$$(7.21)_n \qquad C_n^* | \mathcal{D}_{\Gamma_n} = -\Omega^{A_n}(1-p^{A_n})\Omega_{A_n}^* = \Gamma_n .$$

On the other hand, take $x \in \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \dots \oplus \mathcal{D}_{\Gamma_{n-1}}$, such that $x = \tilde{\Omega}_{A_{n-1}} D_{A_{n-1}} h_{n-1}$, $(h_{n-1} \in \mathcal{H}_{n-1})$. Then:

$$C_n^*(x\oplus 0)=-\Omega^{A_n}(1-p^{A_n})\tilde{\Omega}^*_{A_n}(\tilde{\Omega}_{A_{n-1}}D_{A_{n-1}}h_{n-1}\oplus 0)=$$

$$=-\Omega^{A_n}(1-p^{A_n})D_{A_n}Uh_{n-1}=\Omega^{A_n}(1-p^{A_n})(U'-T'_n)A_nT_nh_{n-1}=$$

$$=\Omega^{A_n}(1-p^{A_n})U'(1-P'_{n-1})T'_nA_nh_{n-1}=$$

$$=\Omega^{A_{n-1}}D_{\Gamma_1^*(A_{n-1},\,A_n)}(1-p^{A_{n-1}})U'(1-P'_{n-2})A_{n-1}h_{n-1}=$$

$$=D_{\Gamma_n^*}\Omega^{A_{n-1}}(1-p^{A_{n-1}})(U'-T'_{n-1})A_{n-1}h_{n-1}=$$

$$=-D_{\Gamma_n^*}\Omega^{A_{n-1}}(1-p^{A_{n-1}})D_{A_{n-1}}h_{n-1}=D_{\Gamma_n^*}C^*_{n-1}x\ ,$$

where we used $(5.4)_n$, (2.9), (1.1), $(5.2)'_n$, (5.3) and (1.1). We proved that

$(7.22)_n$ $$C_n^*\,|\,\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_{n-1}}\oplus\{0\}=D_{\Gamma_n^*}C^*_{n-1}(1-Q_n^n)\,|\,\mathcal{D}_A\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_{n-1}}\oplus\{0\}$$

The relations $(7.21)_n$ and $(7.22)_n$ imply the lemma.

Lemma 7.3 and Lemma 7.4 suggest the following discussion. Let $\Gamma:\mathcal{G}\longmapsto\mathcal{G}'$ be a contraction between the Hilbert spaces $\mathcal{G}$ and $\mathcal{G}'$. Define

(7.23) $$J(\Gamma):\mathcal{G}\oplus\mathcal{D}_{\Gamma^*}\longmapsto\mathcal{G}'\oplus\mathcal{D}_\Gamma$$

$$J(\Gamma)=\begin{pmatrix}-\Gamma & D_{\Gamma^*}\\ D_\Gamma & \Gamma^*\end{pmatrix}.$$

A matrix computation shows that $J(\Gamma)$ is unitary.

Returning to our situation, we define for $n\geq 2$ and $2\leq k\leq n$, the operator

$$(7.24)^k_n \qquad J_n(\Gamma_k): F_A \oplus R_A \oplus \mathcal{D}_{\Gamma_1} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{k-1}} \oplus \mathcal{D}_{\Gamma_k^*} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n} \longmapsto$$

$$\longmapsto F_A \oplus R_A \oplus \mathcal{D}_{\Gamma_1} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{k-1}^*} \oplus \mathcal{D}_{\Gamma_k} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n},$$

which acts as the identity on each component, excepting those framed, on which acts as $J(\Gamma_k).(^3)$. We will need also the operators

$$(7.24)^1_n \qquad J_n(\Gamma_1): F_A \oplus R_A \oplus \mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n} \longmapsto$$

$$\longmapsto F_A \oplus R^A \oplus \mathcal{D}_{\Gamma_1} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n},$$

defined as above, (for $n \geq 1$).

Consider the operators

$$(7.25)_1 \qquad \Delta_1^1 = \tilde{B}_1 \oplus C_1 : \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1^*} \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1}$$

and, for $n \geq 2$,

$$(7.25)_n \qquad \Delta_1^n : \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1^*} \oplus \mathcal{D}_{\Gamma_2} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n} \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \ldots \oplus \mathcal{D}_{\Gamma_n}$$

which acts as the identity on each component, excepting the first two on which acts as Δ_1^1.

L e m m a 7.5. <u>For every</u> $n \geq 2$

$$(7.26)_n \qquad \tilde{B}_n = \Delta_1^n J_n(\Gamma_2) \ldots J_n(\Gamma_n) C_n,$$

<u>where</u>

$$i_n: \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{n-1}} \longmapsto \mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1} \oplus \ldots \oplus \mathcal{D}_{\Gamma_{n-1}} \oplus \mathcal{D}_{\Gamma_n^*}$$

<u>is the canonical inclusion.</u>

P r o o f. The lemma follows from $(7.14)_n$ and $(7.20)_n$, $n\geq 2$; note that the right side of $(7.26)_n$ makes sense because $J_n(\Gamma_k)$ is the identity on $F_A \oplus R_A$, which will be identified with $\mathcal{D}_A + L$, (for $k\geq 2$).

C o r o l l a r y 7.1. For every $n\geq 2$

$$(7.27)_n \qquad \begin{pmatrix} X_n^o \\ X_n^1 \\ \vdots \\ X_n^n \end{pmatrix} = \Delta_1^n J_n(\Gamma_2)\dots J_n(\Gamma_n) \begin{pmatrix} X_{n-1}^o \\ X_{n-1}^1 \\ \vdots \\ X_{n-1}^{n-1} \\ {}^o\mathcal{D}_{\Gamma_n^*} \end{pmatrix}.$$

We conclude this section in

P r o p o s i t i o n 7.3. For every $n\geq 2$

$$(7.28)_n \qquad S_{n+1} = Y^o X_n^o + Y_1 X_n^1 + \dots + Y^n X_n^n ,$$

where Y^j is given by $(7.10)_j$, $(0\leq j\leq n)$, and $(X_n^j)_{j=0}^n$ are obtained recurently by $(7.27)_n$.

8. The recurent formulas obtained in Section 7 requires the further study of the first objects involved in the construction. This will be done in this section. We start by analysing the connections between S_1 and S_2; these turn out to imply a slightly different formulas for (7.27), $n\geq 2$.

The whole idea is to use the fact that

$$\mathcal{D}_A + L = F_A \oplus R_A$$

From $(6.10)_1$, it follows that

$$S_1=R_o^o+R_o'\Gamma_1 R_o^*=q^A\sigma_A p_A(D_A P+I-P)|L_*+$$

$$+q^A\Gamma_1(1-p_A)(D_A P+I-P)|L_*$$

Define

(8.1)'
$$\begin{cases} X'=p_A(D_A P+I-P)|L_*:\mathcal{L}_*\mapsto F_A \\ Y'=q^A\sigma_A:F_A\longmapsto L' \end{cases},$$

and

(8.1)"
$$\begin{cases} X''=(1-p_A)(D_A P+I-P)|L_*:L_*\longmapsto R_A \\ Y''=q^A\Gamma_1=R_o'\Gamma_1:R_A\longmapsto L' \end{cases}.$$

With these definitions we have

(8.2)
$$\begin{cases} R_o^o=Y'X' \\ S_1=Y'X'+Y''X'' \end{cases}$$

In order to connect S_2 with S_1 it is possible to use the same way as in proving $(7.27)_n$, $n\geq 2$; the final result can be stated as follows:

Let consider the operators

$(8.3)_o$
$$\Delta_o=\begin{pmatrix} p_A(1-q^A)\sigma_A & -p_A(1-q^A) \\ (1-p_A)(1-q^A)\sigma_A & -(1-p_A)(1-q^A) \end{pmatrix}: \begin{matrix}F_A\\ \oplus\\ R_A\end{matrix}, \longmapsto \begin{matrix}F_A\\ \oplus\\ R_A\end{matrix}$$

and for any $n\geq 1$.

$(8.3)_n$
$$\Delta_o^n:F_A\oplus R_A'\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_n}\longmapsto F_A\oplus R_A\oplus\mathcal{D}_{\Gamma_1}\oplus\ldots\oplus\mathcal{D}_{\Gamma_n}$$

where Δ_o^n acts as the identity on all the components, excepting the first two on which acts as Δ_o.

Put also

$$\begin{cases} X_n' = p_A X_n^o & \text{and} \\ X_n'' = (1-p_A) X_n^o \quad , & \text{for every } n \geq 1 \;. \end{cases}$$

Then we have:

L e m m a 8.1. With the previous notations

$$(8.4)_1 \qquad \begin{pmatrix} X_1' \\ X_1'' \\ X_1^1 \end{pmatrix} = \Delta_o^1 J_1(\Gamma_1) \begin{pmatrix} X' \\ X'' \\ {}^o\mathcal{D}_{\Gamma_1^*} \end{pmatrix}$$

P r o o f. Let $1_* \in L_*$; then

$$\begin{pmatrix} X_1' \\ X_1'' \\ X_1^1 \end{pmatrix} 1_* = \begin{pmatrix} p_A(1-q^A)\tilde{\omega}_{A_1} D_{A_1} 1_* \\ (1-p_A)(1-q^A)\tilde{\omega}_{A_1} D_{A_1} 1_* \\ q^A \tilde{\omega}_{A_1} D_{A_1} 1_* \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q_A)(\sigma_A p_A + \Gamma_1(1-p_A))(D_A P + I - P) 1_* \\ (1-p_A)(1-q_A)(\sigma_A p_A + \Gamma_1(1-p_A))(D_A P + I - P) 1_* \\ D_{\Gamma_1}(1-p_A)(D_A P + I - P) 1_* \end{pmatrix}$$

On the other hand

$$\Delta_o^1 J_1(\Gamma_1)\begin{pmatrix} X' \\ X'' \\ O_{\mathcal{D}_{\Gamma_1^*}} \end{pmatrix} 1_* = \Delta_o^1 \begin{pmatrix} I & 0 & 0 \\ 0 & -\Gamma_1 & D_{\Gamma_1^*} \\ 0 & D_{\Gamma_1} & \Gamma_1^* \end{pmatrix} \begin{pmatrix} p_A(D_A P+I-P)1_* \\ (1-p_A)(D_A P+I-P)1_* \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q^A)\sigma_A & -p_A(1-q^A) & 0 \\ (1-p_A)(1-q^A)\sigma_A & -(1-p_A)(1-q^A) & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} p_A(D_A P+I-P)1_* \\ -\Gamma_1(1-p_A)(D_A P+I-P)1_* \\ D_{\Gamma_1}(1-p_A)(D_A P+I-P)1_* \end{pmatrix} =$$

$$= \begin{pmatrix} p_A(1-q^A)\cdot(\sigma_A p_A+\Gamma_1(1-p_A))\cdot(D_A P+I-P)1_* \\ (1-p_A)(1-q^A)\cdot(\sigma_A p_A+\Gamma_1(1-p_A))\cdot(D_A P+I-P)1_* \\ D_{\Gamma_1}(1-p_A)\cdot(D_A P+I-P)1_* \end{pmatrix},$$

which proves the proposition.

In order to connect $(8.4)_1$ with $(7.27)_n$, we note that

L e m m a 8.2.

$$\Delta_1^1 = {}_o^1 J_1(\Gamma_1)\,|\,\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1^*} .$$

P r o o f. A matrix computation shows that

$$\Delta_0^1 J_1(\Gamma_1)\,|\,\mathcal{D}_A \oplus \mathcal{D}_{\Gamma_1^*} = \begin{pmatrix} (1-q^A)\left[\sigma_A p_A+\Gamma_1(1-p_A)\right] & -(1-q^A)D_{\Gamma_1^*} \\ D_{\Gamma_1}(1-p_A) & \Gamma_1^* \end{pmatrix}$$

therefore we have to prove that

$$\tilde{B}_1 = \begin{pmatrix} (1-q^A)\left[\sigma_A p_A + \Gamma_1(1-p_A)\right] \\ \\ D_{\Gamma_1}(1-p_A) \end{pmatrix}$$

and that

$$C_1 = \begin{pmatrix} -(1-q^A)D_{\Gamma_1^*} \\ \\ \Gamma_1^* \end{pmatrix}.$$

The first equality is a direct consequence of (2.8) the second follows from (4.16) and from

$$(C_1^*|(\mathcal{D}_A \oplus \{0\}))\ (D_A h) = -\omega^{A_1}(1-p^{A_1})(\tilde{\omega}^{A_1})^*(D_A h) =$$

$$= -\omega^{A_1}(1-p^{A_1})D_{A_1}Uh = \omega^{A_1}(1-p^{A_1})(U'-T_1')A_1T_1h =$$

$$= \omega^{A_1}(1-p^{A_1})U'(1-P')T_1'A_1h = D_{\Gamma_1}(1-p^A)(U'-T')Ah =$$

$$= -D_{\Gamma_1}(1-p^A)D_A h_1 .$$

(where we used (2.9), (1.1)′, (4.13) and (1.1)′).

T h e o r e m 8.1. _For every_ $n \geq 1$

$$(8.5)_n \qquad S_n = Y'X'_{n-1} + Y''X''_{n-1} + Y^1X^1_{n-1} + \dots + Y^{n-1}X^{n-1}_{n-1} ,$$

where Y', Y'', $X'_o = X'$, $X''_o = X''$ _are given by_ (8.1)′ _and_ (8.1)″ ,

$$Y^n = q^A D_{\Gamma_1} \dots D_{\Gamma_n} \Gamma_{n+1} \qquad \text{for } n \geq 1,$$

and

$$(8.4)_n \qquad \begin{pmatrix} X'_n \\ X''_n \\ X^1_n \\ \vdots \\ X^n_n \end{pmatrix} = \Delta_n \begin{pmatrix} X'_{n-1} \\ X''_{n-1} \\ X^1_{n-1} \\ \vdots \\ X^{n-1}_{n-1} \\ 0_{\mathcal{D}_{\Gamma_n^*}} \end{pmatrix},$$

with

$$(8.6)_n \qquad \Delta_n = \Delta_o^n J_n(\Gamma_1)\ldots J_n(\Gamma_n)$$

R e m a r k 8.1. From (8.1)′ and (8.1)″, we have that

$$(8.7) \qquad \begin{cases} X' = W'(D_A P + I - P) \mid L_* \\ X'' = W''(D_A P + I - P) \mid L_* \end{cases},$$

where

$$(8.8) \qquad \begin{cases} W' = W_o = p_A & : \mathcal{D}_A + L \longmapsto F_A \\ W'' = W_o'' = (1 - p_A) & : \mathcal{D}_A + L \longmapsto R_A \end{cases}.$$

From this and from $(8.4)_n$, $(n \geq 1)$, it is easy to obtain the existence the operators

$$W_n' : \mathcal{D}_A + \mathcal{L} \longmapsto F_A$$

$$W_n'' : \mathcal{D}_A + L \longmapsto R_A , \qquad n \geq 0$$

and for $n \geq 1$

$$W_n^k : \mathcal{D}_A + L \longmapsto \mathcal{D}_{\Gamma_k} \qquad 1 \leq k \leq n ,$$

such that

$$(8.9)_n \qquad \begin{pmatrix} X_n' \\ X_n'' \\ X_n^1 \\ \vdots \\ X_n^n \end{pmatrix} = \begin{pmatrix} W_n' \\ W_n'' \\ W_n^1 \\ \vdots \\ W_n^n \end{pmatrix} (D_A P + I - P) \mid L_* \qquad (n \geq 0)$$

,

and

$$(8.10)_n \qquad \begin{pmatrix} W'_n \\ W''_n \\ W^1_n \\ \vdots \\ W^n_n \end{pmatrix} = \Delta_n \begin{pmatrix} W'_{n-1} \\ W''_{n-1} \\ W^1_{n-1} \\ \vdots \\ W^{n-1}_{n-1} \\ 0_{\mathcal{D}_{\Gamma^*_n}} \end{pmatrix} , \qquad (n \geq 1) .$$

9. In this section we shall make explicite the one-to-one correspondence between the CID of A and the contractive analytic $L(R_A, R^A)$ - valued functions (see [9], Proposition 4.1).

To this aim, let us recall the construction made there. Let $\hat{A}$ be a fixed CID of A with the A-choice sequence $\{\Gamma_n\}_{n=1}^{\infty}$; it is easy to see that from $\{\Gamma_n\}_{n=2}^{\infty}$ it is possible to construct a Γ_1 - choice sequence (where $\Gamma_1 \in L(R_A, R^A)$ is considered to be in $I(O_{R_A}, O_{R^A})$; $(9.4)_n$ see below). This Γ_1 - choice sequence gives a CID, $\hat{\Gamma}_1$ of Γ_1, which, because intertwines the unilateral shifts of multiplicity R_A and R^A, is the multiplication by a contractive analytic $L(R_A, R^A)$-valued function s(z) with $s(0)=\Gamma_1$; the function s(z) will be called the contractive analytic $L(R_A, R^A)$- valued function associated to A. This correspondence is the one described in [9], Proposition 4.1. With obvious identifications, the Taylor coefficients of s(z) are Γ_1 and the terms of observable sequence of $\hat{\Gamma}_1$. This means that we must make explicite the connections between the observable sequence of $\hat{A}$ and $\hat{\Gamma}_1$.

We start that by describing the objects which appear in the study of CID's of $\Gamma_1 \in I(O_{R_A}, O_{R^A})$. For the contraction O_{R_A} we will fix the minimal isometric dilation V_A being the unilateral shift of multiplication R_A, that is the operator

$$(9.1)\qquad \begin{cases} V_A : R_A^{\infty} \longmapsto R_A^{\infty} \\ V_A(r_1 \oplus r_2 \oplus r_3 \oplus \dots) = (0 \oplus r_1 \oplus r_2 \oplus \dots), \end{cases}$$

$r_n \in R_A$ $(n \geq 1)$, where

$$R_A^{\infty} = R_A \oplus R_A \oplus R_A \oplus \dots$$

The space R_A will be identified with $R_A \oplus \{0\} \oplus \{0\} \oplus \dots$; in this case the role of the spaces L_* and L' will be played ba R_A and, respectively,

$V_A R_A = \{0\} \oplus R_A$. The same considerations can be made with the minimal isometric dilation of 0_{R^A}, nalely $V^A \in L((R^A)^\infty)$. It is easy to check that

$$(9.2)\quad \begin{cases} F_{\Gamma_1} = V_A R_A \\ R_{\Gamma_1} = D_{\Gamma_1} \\ F^{\Gamma_1} = \{D_{\Gamma_1} r \oplus V^A \Gamma_1 r : r \in R_A\} \\ R^{\Gamma_1} = j(\Gamma_1) D_{\Gamma_1^*} \end{cases},$$

where

$$j(\Gamma_1)r' = (I \oplus V^A) J(\Gamma_1^*)(r' \oplus 0) = (-\Gamma^* r') \oplus (V^A D_{\Gamma^*} r') \quad (r' \in R^A)$$

(see (7.23)), and

$$(9.3)\qquad \sigma_{\Gamma_1}(V_A r) = D_{\Gamma_1} r \oplus V^A \Gamma_1 r \quad , \quad (r \in R_A).$$

Consider the sequence $\{\gamma_n\}_{n=1}^\infty$, where

$$(9.4)_n \qquad \gamma_n = j(\Gamma_1)\ \Gamma_{n+1}\ ,$$

for every $n \geq 1$.

From the fact that $\{\Gamma_n\}_{n=1}^\infty$ is an A-choice sequence and from (9.2) it follows that $\{\gamma_n\}_{n=1}^\infty$ is a Γ_1 - choice sequence. Denote by $\hat{\Gamma}_1$ the CID of Γ_1 corresponding to $\{\gamma_n\}_{n=1}^\infty$ by Theorem 1.1. We identify now the objects of $\hat{\Gamma}_1$, analogous to those of $\hat{A}$ described in Sections 6,7 and 8. We denote by $\{s_n\}_{n=1}^\infty$ the observable sequence of $\hat{\Gamma}_1$ and by $\{y', y'', \{y^n\}_{n=1}^\infty\}$, $\{x'_n\}_{n=0}^\infty$, $\{x''_n\}_{n=0}^\infty$, $\{\{x^j_n\}_{j=1}^n\}_{n=1}^\infty$, $\{\delta_n\}_{n=1}^\infty$, the operators analogously defined for Γ_1 as $\{Y', Y'', \{Y^n\}_{n=1}^{\infty}\}$, $\{X'_n\}_{n=0}^{\infty}$, $\{X''_n\}_{n=0}^{\infty}$, $\left\{\{X_n^j\}_{j=1}^{n}\right\}_{n=1}^{\infty}$, $\{\Delta_n\}_{n=1}^{\infty}$ in Theorem 8.1 for $\hat{A}$. From Theorem 8.1, (and more precisely (8.1)', (8.1)", $(7.10)_n(n\geq 1)$, $(8.3)_n(n\geq 0)$) one can easily infer

L e m m a 9.1. *We have:*

$$(9.5)' \quad \begin{cases} X'_0 = 0 \\ Y' = V^A \Gamma_1 (V^A)^*, \end{cases}$$

$$(9.5)'' \quad \begin{cases} X''_0 = D_{\Gamma_1} \\ Y'' = q^{\Gamma_1} \gamma_1 \end{cases}$$

$$(9.6)_n \quad Y_n = R'_0 (V^A)^* Y_{n-1}$$

$$(9.7)_0 \quad \delta_0 = \begin{pmatrix} 0 & 0 \\ D_{\Gamma_1} V_A^* | V_A R_A & -(1-q^{\Gamma_1}) | R_A \end{pmatrix}$$

$$(9.7)_n \quad \delta_n = \left(\begin{array}{cc|c} 0 & 0 & \\ D_{\Gamma_1} V_A^* | V_A R_A & \Gamma_1^* D_{\Gamma_1^*} & 0 \\ \hline & 0 & I_n \end{array}\right) J_n(\Gamma_2) \dots J_n(\Gamma_n).$$

$$\left(\begin{array}{c|cc} I_n & 0 & \\ \hline 0 & I & 0 \\ & 0 & j(\Gamma_1)^* D_{\gamma_n} \end{array}\right),$$

where I_n is the identity operator on n components prescribed by the type of the matrix.

Consider also the analytic $L(\mathcal{D}_A + L, L')$ - valued function on $\{z \; ; \; |z| < 1\}$

$$\Psi(z) = \Psi_1 + z\Psi_2 + z^2\Psi_3 + \dots,$$

where

$$\Psi_n = Y'W'_n + Y''W''_n + Y^1 W^1_n + \dots + Y^n W^n_n, \quad (n \geq 1).$$

(see $(7.10)_n$ and $(8.9)_n$).

The key step in obtaining a Schur-type labelling is the following

L e m m a 9.1. <u>For every</u> $n \geq 2$

$$(9.8)_n \quad \Psi_n = \left[\Psi(z)(1-q^A)s(z)(1-p_A)\right]_{n-1} + \Psi_{n-1}(1-q^A)\sigma_A p_A \; + $$
$$+ \; q^A s_{n-1}(1-p_A),$$

<u>where</u>

$$s(z) = s_o + zs_1 + z^2 s_2 + \dots, \qquad s_o = \Gamma_1,$$

<u>and the first term in the right side means the</u> n^{th} <u>coefficient of the analytic function in the brackets</u>.

The proof of this lemma implies a descend induction using the previous formulas; we will omit it here because of its lenght (thougt is rather straightforward).

Denote by $S(z)$ the analytic $L(L_*, L')$-valued function (on $\{z: |z|<1\}$ defined by the observable sequence of $\hat{A}$ (i.e $S(z) = S_1 + zS_2 + z^2 S_3 + \dots$). The <u>Schur-type labelling</u> for CID's of A is given in

T h e o r e m 9.1. <u>We have</u>

$$(9.9)\quad S(z)= q^{\Lambda}(\sigma_A p_A+s(z)(1-p_A))(I-z(1-q^{\Lambda})(\sigma_A p_A+s(z)(1-p_A)))^{-1}\cdot$$
$$\cdot\ (D_A P + I - P)\ L_*.$$

P r o o f. Using $(9.8)_n$ for $n\geq 1$ in the definition of $\Psi(z)$, we have

$$\Psi(z)=\Psi_1+z\Psi_2+z^2\Psi_3+\ldots=\Psi_1+z\Big(\big[\Psi(z)(1-q^{\Lambda})s(z)(1-p_A)\big]_1 +$$
$$+\ \Psi_1(1-q^{\Lambda})\sigma_A p_A + q^{\Lambda}s_1(1-p_A)\Big) + \ldots =$$
$$= \Psi_1+z\Big(\big[\Psi(z)(1-q^{\Lambda})s(z)(1-p_A)\big]_1+z\big[\Psi(z)(1-q^{\Lambda})s(z)+1-p_A)\big]_2+\ldots\Big)$$
$$+\ z(\Psi_1+z\Psi_2\ldots)(1-q^{\Lambda})\sigma_A p_A)+(zq^{\Lambda}s_1(1-p_A)+z^2q^{\Lambda}s_2(1-p_A)+\ldots)$$
$$= \Psi_1+z\Psi(z)(1-q^{\Lambda})s(z)(1-p_A)+z\Psi(z)(1-q^{\Lambda})\sigma_A p_A +$$
$$+\ q^{\Lambda}s(z)(1-p_A) - q^{\Lambda}s_0(1-p_A),$$

whence

$$(9.10)\quad \Psi(z)(I-z(1-q^{\Lambda})(\sigma_A p_A+s(z)(1-p_A))=q^{\Lambda}s(z)(1-p_A)\ +q^{\Lambda}\sigma_A p_A,$$

because $\Psi_1=q^{\Lambda}\big[\sigma_A p_A+\Gamma_1(1-p_A)\big]$.

From (9.10) it is clear that

$$\Psi(z)=q^{\Lambda}(\sigma_A p_A+s(z)(1-p_A))(I-z(1-q^{\Lambda})(\sigma_A p_A+s(z)(1-p_A)))^{-1},$$

and (9.9) follows by $S(z)=\Psi(z)(D_A P+I-P)L_*$.

The formula (9.9) can be seen as a generalization of the formula of the characteristic function (see [17], Ch.VI, (1.1)). Indeed, take $H'=H$, $T'=T=0$; in this case $R_A=\mathcal{D}_A$ and $R^{\Lambda}=j(\Lambda)\mathcal{D}_A{}^*$.

[illegible]

Consider $\Gamma_1 = -j(A)P^H_{\mathcal{D}_{A^*}} | \mathcal{D}_A$ and $\Gamma_n = 0$ for every $n \geq 2$; then (9.9) implies in this case that

$$(9.11) \quad j(A)^* S(z) = -D_{A^*}(I - zA^*)^{-1} D_A,$$

This means that

$$\Theta_A(z) = -(A + j(A)^* S(z)) | \mathcal{D}_A ,$$

where $\Theta_A(z)$ is the characteristic function of A.

On the other hand, (9.9) suggests a cascade transform [12]. Indeed, consider the $L(L_*, L')$ (resp. $L(R^A, L')$, $L(L_*, R_A)$), $L(R^A_*, R_A)$ - valued function, analytic for $\{z : |z| \leq 1\}$, defined by

$$a_A(z) = q^A \sigma_A p_A (I - z(1-q^A)\sigma_A p_A)^{-1} (D_A P + I - P) | L_* ,$$

(resp.

$$b_A(z) = q^A (I + z\sigma_A p_A (I - z(1-q_A)\sigma_A p_A)^{-1}),$$

$$c_A(z) = (1-p_A)(I - z(1-q^A)\sigma_A p_A)^{-1} (D_A P + I - P) | L_* ,$$

$$d_A(z) = (1-p_A)\ (I - z(1-q^A)\sigma_A p_A)^{-1} (1-q^A) | R^A. \quad)$$

C o r o l l a r y 9.1 <u>With the following notations,</u>

$$S(z) = a(z) + b(z)s(z)(I - zd(z)s(z))^{-1} c(z).$$

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Footnotes

(1) For G a subspace (linear, closed) of H, the notation $P=P_G^H$ means the orthogonal projection of H on G; in this case $1-P$ will be the orthogonal projection of H on $H \ominus G$.

(2) For a contraction A in $L(H,H')$, we put $D_A=(I-A^*A)^{1/2}$ and $\mathcal{D}_A=(D_A(H))^-$, I being the identity operator (on any Hilbert space).

(3) The role of these operators in the study of CID of pointed out in [10].

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