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FACTORIZATION THEOREMS FOR OPERATOR VALUED FUNCTIONS ON MULTIPLY CONNECTED DOMAINS

by

Ion SUCIU

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Introduction

The classical problem of factorization by means of analytic functions was insistently studied in the operator valued case due to its relevance in many aspects of operator theory as well as in the applications of operatorial methods in scattering theory, prediction and filtering theory, etc. If we refer to the unit disk, where, in the scalar valued case, the famous theorems of Szegö and Kolmogorov-Krein give a complete description of this problem, in the operator valued case, as Kolomogorov and Wiener remarked at the begining, the principal tool in abording this problem is the Wold decomposition of an isometry. Wold obtained its decomposition in the context of the prediction theory for stationary stochastic process [35] but, in operator terms, the result is essentially the same as von Neumann characterization of an isometry by a direct sum of a unitary operator with a unilateral shift of an appropriate multiplicity [23]. Using this result various types of factorization theorems for operator valued functions on the unit circle were obtained [7], [8], [13], [18], [30], [35]. In [27], [28] quite general factorization theorems of Szegö, Kolmogorov-Krein type for semispectral measures supported on the unit circle were proved.

There are not but technical difficulties to extend such results from the unit disk to a simply connected Jordan domain, or to a semi-plane. When the domain has holes the problem becames complicated even in the scalar valued case [3]. The difficulties are amplified in the operator valued case by the absence of an adequate Wold decomposition associated with such domains.

Recently M.B. Abrahamse and R.G.Douglas [2] prouved a Wold decomposition for the subnormal operator whose spectrum is contained in the closure of a bounded, open, and connected subset Ω of the complex plane such that $\partial\Omega$ consists of finite number of nonintersecting analytic Jordan curves, and the normal spectrum is contained in $\partial\Omega$. They proved that any such operator is a direct sum of a normal operator having spectrum in $\partial\Omega$ with an appropiate bundle shift. Since they give also a functional model for the bundle shift as the multiplication by coordinate function on a space of analytic cross sections of a fibre bundle over Ω, their Wold decomposition furnishes a quite satisfactory quantity of analyticity which can be exploited in factorization theorems.

In this paper, using Wold decomposition of Abrahamse and Douglas, we shall prove a Szegö, Kolmogorov-Krein type factorization theorem for semi-spectral measures supported by $\partial\Omega$. Following 2 in Section 1 we shortly describe the basic facts about bundle shifts. In Section 2 we discuss the operator valued automorphic analytic function. We introduce the notion of L^2-boundedness for such functions and give an intrinsec characterizations of the operators (bounded or not) which appear as pointwise multiplication on H^2-spaces by an L^2-bounded operator-valued analytic function (Theorem 1). The intertwining property which appear in this characterization extends to the non-bounded case the usual intertwining property of a bounded operators relative to bundle shifts (cf. [2], [20], [21]). In section 3 we prove our factorization theorems. We follow a way similar to that adopted by B.Sz.-Nagy - C.Foiaş [30], [32] in proving Lowdenslager factorization theorem [18] for operator-valued bounded function on the unit disk or circle, working, instead of usual Wold decomposition, with the Wold decomposition of Abrahamse and Douglas. The main result is Theorem 3 which attaches to any semi-spectral measure F on $\partial\Omega$ its maximal outer function. Inner-outer factorization (Theorem 4) comes out from this result in a standard way. Theorem 5 which relates the maximal outer function with the Szegö operator of F, contains some elements from Szegö-Kolmogorov-Krein factorization theorem.

The scalar valued case, in a litle more general and abstract setting of hypo-Dirichlet algebras, was successfully studied by Ahern and Sarason in [3] (see also [33]). We hope that other aspects of the last quotted paper, especially the connection between the dual extremal problem and factorization (see also [28]) can be extended to the operator-valued case. Some elements of such study were already given by J.A. Ball in [4].

Finally, we want to recall the major reason for interest in these results in preparing a way to study the compression to a semi-invariant subspace of a normal operator with spectrum in the boundary of Ω. The factorization theorem for semi-spectral measure may be a step in the natural attempt to generalize the Sz.Nagy - C.FOiaş model theory for contractions to this setting.

1. Bundle Shift and Wold decomposition

Let Ω be a bounded, open, and connected subset of the complex plane whose boundary $\partial\Omega$ consists of n+1 nonintersecting analytic Jordan curves. In all what follows z_o will be a fixed point in Ω and m the harmonic measure supported by $\partial\Omega$ of the point z_o. Denote by A the algebra of all complex valued functions which are analytic in Ω and continuous on $\partial\Omega$. Any element of A can be uniformly approximated on $\partial\Omega$ by rational functions with poles off $\bar{\Omega}$ (Mergelyan theorem). If we consider A as function algebra on $\partial\Omega$ then it is an hypo-Dirichlet algebra (cf. [3]) and m is the unique logmodular representing measure for the complex homomorphism of A given by the evaluations on the point z_o. We shall denote $A_o = \{f \in A \; : \; f(z_o)=0\}$.

Let $\pi_o(\Omega)$ be the fundamental group for Ω. It is known that $\pi_o(\Omega)$ is a free abelian group whith n generators. For a separable Hilbert space $\mathcal{F}$ let Hom $(\pi_o(\Omega), \mathcal{U}(\mathcal{F}))$ be the group of all group homomorphisms of $\pi_o(\Omega)$ into the group $\mathcal{U}(\mathcal{F})$ of unitary operators on $\mathcal{F}$.

Let $C_1, \ldots, C_n$ be n cuts in Ω such that if C is the union of C_i, $\Omega - C$ is simply connected. For a function h which is holomorphic in $\Omega - C$, having analytic continuations along any curves in Ω, and $A \in \pi_o(\Omega)$ we shall denote by $(h \circ A)(z)$ the values in $\gamma(1)$ of the analytic continuation of h along the closed curve γ in A which begins and ends in $z \in \Omega - C$. Let $\alpha \in$ Hom$(\pi_o(\Omega),$ $\mathcal{U}(\mathcal{F}))$. We say that an $\mathcal{F}$-valued function h which is holomorphic in Ω - C and admits analytic continuation along any curve in Ω produces an α-automorphic multiform function on Ω if for any $A \in \pi_o(\Omega)$ and $z \in \Omega - C$ we have

(1.1) $$(h\circ A)(z) = \alpha(A)h(z)$$

From (1.1) it results that $\|h(z)\|$ is a well defined subharmonic function on Ω. We shall denote by $H^2_{\mathcal{F}}(\alpha)$ the space of all $\mathcal{F}$ -valued functions h on $\Omega - C$ which produces an α-automorphic multiform function on Ω such that there exists a pozitive harmonic function u in Ω verifying $\|h(z)\|^2 \leq u(z)$, $z \in \Omega$. If we put for any $h \in H^2_{\mathcal{F}}(\alpha)$

(1.2) $$\|h\|^2 = \inf\left\{ u(z_0),\ u \text{ harmonic in } \Omega,\ u(z) \geq \|h(z)\|^2,\ z \in \Omega \right\},$$

we obtain a norm on $H^2_{\mathcal{F}}(\alpha)$ with respect to which $H^2_{\mathcal{F}}(\alpha)$ becomes a Hilbert space. It can be showed that any element $h \in H^2_{\mathcal{F}}(\alpha)$ has well-defined non-tangential boundary limits at $\partial\Omega$, almost everywhere with respect to the measure m. These limits define a function h_α in $L^2_{\mathcal{F}}(m)$ and $h \longrightarrow h_\alpha$ is an isometric imbedding of $H^2_{\mathcal{F}}(\alpha)$ into $L^2(dm)$:

(1.3) $$\|h\|^2 = \int_{\partial\Omega} \|h_\alpha(z)\|^2 dm(z), \qquad h \in H^2_{\mathcal{F}}(\alpha)$$

Whenever it is necessar, we shall consider $H^2_{\mathcal{F}}(\alpha)$ as a subspace of $L^2_{\mathcal{F}}(m)$, via above described imbedding. If α is the identity 1 of Hom $(\pi_0(\Omega), \mathcal{U}(\mathcal{F}))$ we shall write $H^2_{\mathcal{F}}(\Omega)$ instead of $H^2_{\mathcal{F}}(1)$. The elements in $H^2_{\mathcal{F}}(\Omega)$ of the form h(z)=f(z) a, when f runs over A and a runs over $\mathcal{F}$, span a dense subspace in $H^2_{\mathcal{F}}(\Omega)$.

In [2] M.B. Abrahamse and R.G.Douglas realized $H^2_{\mathcal{F}}(\alpha)$ as a space of analytic cross sections of a flat unitary vector bundle, with fiber $\mathcal{F}$, over Ω, cannonicaly attached to α. They introduced the <u>bundle shift operator</u> T_α on $H^2_{\mathcal{F}}(\alpha)$ as the operator given by multiplication with the identical function z on $H^2_{\mathcal{F}}(\alpha)$. The operator T_α is uniquely determined (up to a unitary equivalence) by the unitary equivalence class of α. Moreover, using the Grauert-Bungart theorem on the triviality of analytic bundles over Ω, they proved that for any $\alpha \in \mathrm{Hom}(\pi_0(\Omega), \mathcal{U}(\mathcal{F}))$, T_α is similar with $T_{\Omega} = T_1$. Each T_α is the restriction to $H^2_{\mathcal{F}}(\alpha)$ of the normal operator N_Ω defined as the multiplication by z on $L^2_{\mathcal{F}}(m)$. T_α is pure subnormal operator having $N_{\mathcal{F}}$ as minimal extension. If T is pure sub-normal operator on a separable Hilbert space H, such that $\sigma(T) \subset \overline{\Omega}$ and the spectrum $\sigma(N)$ of its minimal normal extension N is contained in $\partial\Omega$, then there exist a Hilbert space $\mathcal{F}$ and an element $\alpha \in \mathrm{Hom}(\pi_0(\Omega), \mathcal{U}(\mathcal{F}))$ such that T is unitarily equivalent to T_α. As a consequence they obtained the Wold decomposition we state here in a form which will be convenient further.

<u>Theorem 0</u>. <u>Let</u> N <u>be a normal operator on a separable Hilbert space</u> $\mathcal{K}$ <u>such that</u> $\sigma(N) \subset \partial\Omega$. <u>Let</u> $\mathcal{K}_+ \subset \mathcal{K}$ <u>be an invariant subspace for</u> N. <u>Suppose that</u> N <u>is the minimal normal extension of</u> $N_+ = N|\mathcal{K}_+$. <u>Then there exists a unitary representation</u> α <u>of</u> $\pi_0(\Omega)$ <u>on a separable Hilbert space</u> $\mathcal{F}$ (possible 0) <u>such that</u> $\mathcal{K}$ <u>can be isometricaly identified with a direct sum</u> $L^2(m) \oplus \mathcal{K}_1$ <u>in such that</u> N <u>becames a direct sum of</u> N_Ω <u>with a normal operator</u> N_1 <u>on</u> $\mathcal{K}_1$ <u>having spectrum in</u> $\partial\Omega$, $\mathcal{K}_+$ <u>becomes</u> $H^2_{\mathcal{F}}(\alpha) \oplus \mathcal{K}_1$ <u>and</u> N_+ <u>becomes</u> $T_\alpha \oplus N_1$.

In [2] a functional model for T_α in terms of automorphic functions on the universal covering space, was also given. The construction of the covering space for Ω produces the following:

1. A group G of linear feactional transformations that map the unit disk D onto D.

2. An open G-invariant subset Γ of ∂D of zero Lebesg[ue]
measure.

3. A simply connected open G-invariant subset D' conta[i]ning $D \cup \Gamma$.

4. An open set Ω' containing $\overline{\Omega}$.

5. A holomorphic covering map π from D' onto Ω' such that $\pi(D)=\Omega$, $\pi(\Gamma)=\partial\Omega$, and G is the group of all linear fractional transformations A having the property $\pi \circ A=\pi$.

We can suppose also $\pi(0)=z_o$.

In fact the group G is isomorphic with $\pi_o(\Omega)$. This isomorphism is given by the so called lifting to the universal cover procedure which we briefly describe. For $\lambda \in D$ and $A \in \pi_o(\Omega)$ let γ be a representant of A which begins and ends in $z=\pi(\lambda)$. Define $A\lambda$ to be $(\pi^{-1} \circ \gamma)(z)$ where $\pi^{-1} \circ \gamma$ is the analytic continuation of π^{-1} along the curve γ.

The normalized Lebesgue measure μ on ∂D lifts to the universal cover the measure m in the sense that for any $f \in L^1(m)$ we have

$$(1.4) \qquad \int_{\partial D} (f \circ \pi)(\lambda)\, d\mu(\lambda) = \int_{\partial\Omega} f(z)\, dm(z).$$

Consider now the subspace $H^2_{\mathcal{F}}(D;\alpha)$ of α-automorphic functions in $H^2_{\mathcal{F}}(D)$:

$$(1.5) \qquad H^2(D;\alpha) = \left\{ h \in H^2_{\mathcal{F}}(D) \ : \ h(A\lambda)=\alpha(A)h(\lambda), \quad \lambda \in D,\ A \in G \right\}$$

If $\alpha=1$ we shall denote $H^2_{\mathcal{F}}(D,\alpha)$ by $H^2_{1\mathcal{F}}(D:G)$. Let $\pi^{-1}_{z_o}$ be the holomorphic function in $\Omega - C$ obtained by the analytic continuation of π^{-1} along the curves in $\Omega - C$ which begin in z_o. For any $h \in H^2_{\mathcal{F}}(D,\alpha)$ the function $h \circ \pi^{-1}_{z_o}$ produces an α-automorph[ic] multiform holomorphic function on Ω . Using (1.4) it results that $h \circ \pi_{z_o}^{-1} \in H^2_{\mathcal{F}}(\alpha)$ and $\|h\| = \|h \circ \pi_{z_o}^{-1}\|$. In fact the map $h \to h \circ \pi_{z_o}^{-1}$ is an izometric isomorphism between $H^2_{\mathcal{F}}(D;\alpha)$ and $H^2_{\mathcal{F}}(\alpha)$ which make T_α unitarily equivalent to the multiplication by π on $H^2_{\mathcal{F}}(D;\alpha)$.

In that follows we shall use freely one or the other functional model for T_α in a way which will be convenient in the context, having in mind the identifications briefly described in this section.

2. Operator valued (α,β) automorphic analytic functions

Let now $\mathcal{E}$, $\mathcal{F}$ be two separable Hilbert spaces. Following B.Sz.-Nagy - C.Foiaş [26] we shall denote by the triplet $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ an $\mathcal{L}(\mathcal{E},\mathcal{F})$-valued analytic function on D. Let α be a representation of G on $\mathcal{F}$ and β a representation of G on $\mathcal{E}$. The analytic function $\{\mathcal{E},\mathcal{F},\Theta(\;)\}$ on D will be called (α,β)-automorphic if for any $\lambda\in$ D and A in G we have

$$\Theta(A\lambda)=\alpha(A)\Theta(\lambda)\beta(A)^{*} \tag{2.1}$$

We shall write α-automorphic instead of (α,β)-automorphic in case $\beta=1$, and β^{*}-automorphic in case $\alpha=1$.

The result about similarity in [2] can be formulated in terms of automorphic functions as follows: There exists a bounded analytic function $\{\mathcal{F},\mathcal{F},\Phi^{\alpha}(\lambda)\}$ which is α-automorphic, having as inverse a bounded analytic function $\{\mathcal{F},\mathcal{F},\Psi^{\alpha}(\lambda)\}$ which is α^{*}-automorphic, such that:

$$\Phi^{\alpha}H^2_{\mathcal{F}}(D;G)=H^2_{\mathcal{F}}(D;\alpha) \tag{2.2}$$

$$\Psi^{\alpha}H^2_{\mathcal{F}}(D;\alpha)=H^2_{\mathcal{F}}(D;G)$$

where Φ^{α} and Ψ^{α} are the operators of pointwise multiplications by $\Phi^{\alpha}(\lambda)$ and $\Psi^{\alpha}(\lambda)$ on $H^2_{\mathcal{F}}(D)$.

The existence of the functions $\{\mathcal{F},\mathcal{F},\Phi^{\alpha}(\lambda)\}$ and $\{\mathcal{F},\mathcal{F},\Psi^{\alpha}(\lambda)\}$ comes from the Grauert-Bungart theorem on the triviality of analytic vector bundles over Ω, and it seems to be very difficult to precise more about them. The operators Φ^{α}

and Ψ^{α} realise a similarity between bundle shifts T_{α} and T_{Ω}. We shall call them an α-pair of similarity.

Remark 1. For any $a \in \mathcal{F}$ there exists $h \in H^2_{\mathcal{F}}(D;\alpha)$ such that $h(0)=a$. Indeed, $\Psi^{\alpha}(0)a$, considered as constant function on D, belongs to $H^2_{\mathcal{F}}(D;\mathcal{G})$. Then $h=\Phi^{\alpha}\Psi^{\alpha}(0)a$ belongs to $H^2_{\mathcal{F}}(D;\alpha)$ and $h(0)=\Phi^{\alpha}(0)\Psi^{\alpha}(0)a=a$

If $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is (α,β)-automorphic in D then it lifts to the covering space the (α,β)-automorphic multiform function defined by

$$\Theta'(z)=(\Theta\circ\pi^{-1})(z), \qquad (z\in\Omega). \tag{2.3}$$

Conversely, any (α,β)-automorphic multiform holomorphic function on Ω with values in $\mathcal{L}(\mathcal{E},\mathcal{F})$ gives rise by lifting procedure, to an (α,β)-automorphic function analytic on D.

We shall work only with (α,β)-automorphic function on D but using above considerations, we obtain interpretations for some results in terms of multiform functions on Ω, or in terms of bundle maps.

There are no difficulties to define the boundedness for the (α,β)-automorphic functions. But, when we want to define L^2-boundedness (in the strong sense) some difficulties arise.

In the case $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is α-automorphic the corresponding α-automorphic multiform function $\Theta'(z)$ on Ω produces a subharmonic (uniform) function $z \to \|\Theta'(z)\|$ on Ω. We say that $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is L^2-bounded if for any $a\in\mathcal{E}$ there exists an harmonic function u_a on Ω such that for any $z\in\Omega$

$$\|\Theta'(z)a\|\le u_a(z)\|a\|^2.$$

It we put

(2.4) $$(V_{\Theta}a)(\lambda)=\Theta(\lambda)a \qquad (\lambda\in D,\ a\in\mathcal{E})$$

we obtain a bounded operator V_{Θ} from $\mathcal{E}$ into $H^2_{\mathcal{F}}(\alpha)$. Indeed

$$\|V_{\Theta}a\|^2_{H^2_{\mathcal{F}}(\alpha)}=\int_{\partial D}\|\Theta(\lambda)a\|^2_{\mathcal{F}}d\mu(\lambda)=\int_{\partial\Omega}\|\Theta'(z)a\|^2_{\mathcal{F}}dm(z)\leq$$

$$\leq\int_{\partial\Omega}u_a(z)\|a\|^2_{\mathcal{E}}dm(z)\leq M_a\|a\|^2_{\mathcal{E}}.$$

Standard arguments imply that V_{Θ} is bounded.

In case $\beta\neq 1$ such arguments are not more applicable. But they sugest the following deffinitions.

We say that an (α,β)-automorphic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is L^2-bounded if there exists a β-pair of similarity $\Phi^{\beta},\Psi^{\beta}$ such that the formula

(2.5) $$(V_{\Theta}a)(\lambda)=\Theta(\lambda)\Phi^{\beta}(\lambda)a$$

defines a bounded operator V_{Θ} from $\mathcal{E}$ into $H^2_{\mathcal{F}}(\alpha)$.

If $\beta=1$ clearly this definition is the preceding one.

Let $D(\Theta_+)$ be the subspace of $H^2_{\mathcal{E}}(D;\beta)$ consisting from all functions $h\in H^2_{\mathcal{E}}(D;\beta)$ for which $\lambda\longrightarrow\Theta(\lambda)h(\lambda)$ belongs to $H^2_{\mathcal{F}}(D,\alpha)$. Define Θ_+ on $D(\Theta_+)$ by

(2.6) $$\left[\Theta_+h\right](\lambda)=\Theta(\lambda)h(\lambda) \qquad (\lambda\in D,\ h\in D(\Theta_+))$$

We shall see that Θ_+ is a dense domain closed operator from $H^2_{\mathcal{E}}(D;\beta)$ into $H^2_{\mathcal{F}}(D;\alpha)$. Our aim is to characterize the operators from $H^2_{\mathcal{E}}(D;\beta)$ into $H^2_{\mathcal{F}}(D;\alpha)$ which are representable as Θ_+ for an L^2-bounded (α,β) automorphic analytic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$.

Let Q be an operator defined on a subspace $D(Q)$ of

$H^2_{\mathcal{E}}(D;\beta)$ with values into $H^2_{\mathcal{F}}(D;\alpha)$. We say that Q <u>intertwines the point-evaluations</u> on $H^2_{\mathcal{E}}(D;\beta)$ and $H^2_{\mathcal{F}}(D;\alpha)$ if the following conditions are satisfied.

a) $\Phi^{\beta}\mathcal{E}\subset D(Q)$ and $Q|\Phi^{\beta}\mathcal{E}$ is bounded.

b) For any $h\in H^2_{\mathcal{E}}(D;\beta)$ for which the function $\lambda\rightarrow[Q\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda)$ belongs to $H^2_{\mathcal{F}}(D;\alpha)$ we have $h\in D(Q)$.

c) For any $h\in D(Q)$ and $\lambda\in D$ we have:

$$[Qh](\lambda)=[Q\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda). \tag{2.7}$$

<u>Proposition 1</u>. <u>If</u> Q <u>intertwines the point-evaluations on</u> $H^2_{\mathcal{E}}(D;\beta)$ <u>and</u> $H^2_{\mathcal{F}}(D;\alpha)$ <u>then</u> Q <u>is dense domain closed operator from</u> $H^2_{\mathcal{E}}(D;\beta)$ <u>into</u> $H^2_{\mathcal{F}}(D;\alpha)$. <u>If</u> Q <u>is bounded then</u> (2.7) <u>is equivalent to</u>

$$QT_{\beta} = T_{\alpha}Q. \tag{2.8}$$

<u>Proof</u>. Since the set of all functions h of the form $h(\lambda)=f(\pi(\lambda))a$ when f runs over all complex valued functions which are analytic in a neigborhood of $\overline{\Omega}$ and a runs over $\mathcal{E}$, spans a dense subspace in $H^2_{\mathcal{E}}(D;G)$, in order to prove $\overline{D(Q)}=H^2_{\mathcal{E}}(D;\beta)$ it is sufficient to show that, for such h, $\Phi^{\beta}h\in D(Q)$. But

$$[Q\Phi^{\beta}\psi^{\beta}(\lambda)(\Phi^{\beta}h)(\lambda)](\lambda)=[Q\Phi^{\beta}\psi^{\beta}(\lambda)\Phi^{\beta}(\lambda)h(\lambda)](\lambda)=$$

$$=[Q\Phi^{\beta}h(\lambda)](\lambda)=f(\pi(\lambda))[Q\Phi^{\beta}(\lambda)a](\lambda)=[f\circ\pi\, Q\Phi^{\beta}a](\lambda).$$

We used the properties a), b) and c) of Q. Since $f\circ\pi$ is bounded on D then clearly $(f\circ\pi)\,Q\Phi^{\beta}a$ belongs to $H^2_{\mathcal{F}}(D;\alpha)$. Hence, from b) it results that $\Phi^{\beta}h\in D(Q)$.

Suppose now that $h_n\in D(Q)$, $h_n\rightarrow h$ in $H^2_{\mathcal{E}}(D;\beta)$ and $Qh_n\rightarrow q$

in $H^2_{\mathcal{F}}(D;\alpha)$. Then

$$[Q\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda) = [Q\Phi^{\beta}\psi^{\beta}(\lambda)\lim h_n(\lambda)](\lambda) =$$

$$= \lim [Q\Phi^{\beta}\psi^{\beta}(\lambda)h_n(\lambda)](\lambda) = \lim [Qh_n](\lambda) = g(\lambda).$$

We used a), b), c) and the continuity of the point-evaluations operators on H^2 spaces. From b) it results that $h\in D(Q)$ and from c) $Qh=g$.

Suppose now that Q is bounded operator from $H^2_{\mathcal{E}}(D;\beta)$ into $H^2_{\mathcal{F}}(D;\alpha)$. We have

$$[QT_{\beta}h](\lambda) = [Q\pi h](\lambda) = [Q\Phi^{\beta}\psi^{\beta}(\lambda)\pi(\lambda)h(\lambda)](\lambda) =$$

$$= \pi(\lambda)[Q\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda) = \pi(\lambda)[Qh](\lambda) = [T_{\alpha}Qh](\lambda).$$

We used $D(Q)=H^2_{\mathcal{E}}(D;\beta)$ and c). Hence $QT_{\beta}=T_{\alpha}Q$.

Conversely if (2.8) holds, then for any h of the form $h(\lambda)=\Phi^{\beta}(\lambda)f(\pi(\lambda))a$, with f complex valued and analytic in a neigborhood of $\overline{\Omega}$, we have:

$$[Qh](\lambda) = [Q\Phi^{\beta}(f\circ\pi)a](\lambda) = [Qf(T_{\beta})\Phi^{\beta}a](\lambda) =$$

$$= [f(T_{\alpha})Q\Phi^{\beta}a](\lambda) = (f\circ\pi)(\lambda)[Q\Phi^{\beta}a](\lambda) =$$

$$= [Q\Phi^{\beta}(f\circ\pi)(\lambda)a](\lambda) = [Q\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda).$$

We used usual properties of functional calculus with functions which are analytic in a neigborhood of the spectrum of T_{β} and T_{α}. Clearly now that (2.8) results for any h $H^2_{\mathcal{E}}(D;\beta)$.

The proof of the proposition is complete.

Intertwining point-evaluations on $H^2_{\mathcal{E}}(D;\beta)$ and $H^2_{\mathcal{F}}(D;\alpha)$ appears as an extension to the non-bounded operators of usual intertwining of bundle shifts property.

Theorem 1. The operator Q from $D(Q)\subset H^2_{\mathcal{E}}(D;\beta)$ into $H^2_{\mathcal{F}}(D;\alpha)$ is representable as Θ_+ for an L^2-bounded (α,β)-automorphic analytic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ if and only if Q intertwines the point-evaluations on $H^2_{\mathcal{E}}(D;\beta)$ and $H^2_{\mathcal{F}}(D;\alpha)$. The function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is bounded if and only if Q is bounded.

Proof. Let $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ be an L^2-bounded (α,β) automorphic analytic function. From the definitions of L^2-boundedness and $D(\Theta_+)$ it results that Θ_+ verifies a) and b). For any $h\in D(\Theta_+)$ we have

$$[\Theta_+\Phi^\beta\varphi^\beta(\lambda)h(\lambda)](\lambda)=\Theta(\lambda)[\Phi^\beta\varphi^\beta(\lambda)](\lambda)=$$

$$=\Theta(\lambda)\Phi^\beta(\lambda)\varphi^\beta(\lambda)h(\lambda)=\Theta(\lambda)h(\lambda)=[\Theta_+h](\lambda).$$

i.e. Θ_+ verifies c) too.

Let now Q be an operator which intertwiness point-evaluations on $H^2_{\mathcal{E}}(D;\beta)$ and $H^2_{\mathcal{F}}(D;\alpha)$.

Define $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ by

$$\Theta(\lambda)a=[Q\,\Phi^\beta\varphi^\beta(\lambda)a](\lambda)\qquad(\lambda\in D,\ a\in\mathcal{E}) \tag{2.9}$$

For a fixed $\lambda\in D$, $\Theta(\lambda)$ is a bounded operator from $\mathcal{E}$ into $\mathcal{F}$. This comes from the boundedeness of Q and of the point evaluation operators on $\Phi^\beta\mathcal{E}$ and $H^2(D)$ respectively. The analyticity comes from the analiticity of φ^β and the boundedness of $Q\Phi^{\beta}$ on the constant functions. For any $A\in G$ we have

$$\Theta(A\lambda)a=[Q\Phi^{\beta}\psi^{\beta}(A\lambda)a](\lambda)=\alpha(A)[Q\Phi^{\beta}\psi^{\beta}(\lambda)\beta(A)^{*}a](\lambda)$$
$$=\alpha(A)\,\Theta(\lambda)\,\beta(A)^{*}a$$

Hence $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is $(\alpha;\beta)$-automorphic. Since

$$\Theta(\lambda)\,\Phi^{\beta}(\lambda)a=[Q\Phi^{\beta}a](\lambda)$$

the L^2-boundedeness of $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ results from the property a) of Q. Now, clearly, from b) and c) which hold for both Θ_{+} and Q it results that $Q=\Theta_{+}$. If $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is bounded then $Q=\Theta_{+}$ is bounded. Suppose now that Q is bounded.

For any $a\in\mathcal{E}$ let h_a be the function in $H^2_{\mathcal{F}}(\Omega)$ which comes from the function $\lambda\to[Q\Phi^{\beta}\psi^{\beta}(\lambda)a](\lambda)$ in $H^2_{\mathcal{F}}(D;\alpha)$ by applying ψ^{α} and the described isomorphism between $H^2_{\mathcal{F}}(D;G)$ and $H^2_{\mathcal{F}}(\Omega)$. For any complex valued function f which is analytic in a neighborhood of $\overline{\Omega}$ we have

$$\int_{\partial\Omega}|f(z)|^2\|h_a(z)\|^2_{\mathcal{F}}\,dm(z)=\int_{\partial\Omega}\|f(z)h_a(z)\|^2_{\mathcal{F}}\,dm(z)=$$
$$=\int_{\partial D}\|(f\circ\pi)(\lambda)\psi^{\alpha}(\lambda)[Q\Phi^{\beta}\psi^{\beta}(\lambda)a](\lambda)\|^2_{\mathcal{F}}\,d\mu(\lambda)\le$$
$$\le\|\psi^{\alpha}Q\|^2\int_{\partial D}\|f(T_{\beta})\Phi^{\beta}\psi^{\beta}(\lambda)a](\lambda)\|^2_{\mathcal{E}}\,d\mu(\lambda)\le$$
$$=\|\psi^{\alpha}Q\|^2\int_{\partial D}|(f\circ\pi)(\lambda)|^2d\mu(\lambda)\|a\|^2_{\mathcal{E}}=\|\psi^{\alpha}Q\|^2\int_{\partial\Omega}|f(z)|^2dm(z)\|a\|^2_{\mathcal{E}}$$

It results that for any f on $\partial\Omega$ which is uniform limit on $\partial\Omega$ of rational functions with poles off $\overline{\Omega}$ we have

$$\int_{\partial\Omega}|f(z)|^2\|h_a(z)\|^2dm(z)\le M\int_{\partial\Omega}|f(z)|^2dm(z)\|a\|^2,\quad(a\in\mathcal{E})$$

Since any positive continuous function φ on $\partial\Omega$ is uniform limit of modulus of such functions f (approximating in modulus property of $R(\bar{\Omega})$ (cf. [49]) we have

$$\int_{\partial\Omega} \varphi(z)\,\|h_a(z)\|^2\,dm(z) \le M \int_{\partial\Omega} \varphi(z)\,dm(z)\,\|a\|^2, \quad (\varphi \in C(\partial\Omega),\ \varphi \ge 0).$$

It results that

$$\|h_a(z)\| \le M\,\|a\| \qquad \text{m-a.e. on } \partial\Omega,$$

i.e.

$$\|h_a(z)\| \le M\,\|a\|, \qquad (a \in \mathcal{E},\ z \in \Omega).$$

Hence, for any $\lambda \in D$ and $a \in \mathcal{E}$ we have

$$\|\Theta(\lambda)a\| \le \|[Q\,\Phi^\beta\,\Psi^\beta(\lambda)a](\lambda)\| \le$$

$$\le \|\Phi^\alpha(\lambda)\|\,\|\Psi^\alpha(\lambda)[Q\,\Phi^\beta\Psi^\beta(\lambda)a](\lambda)\| \le \|\Phi^\alpha\|\,\|h_a(\pi(\lambda))\| \le M_1\|a\|,$$

i.e. $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is bounded.

The proof of the theorem is complete.

The correspondence between the non bounded operator Θ_+ and the function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ depends, in general, by the β - paire of similarity Φ^β, Ψ^β. In the bounded case, however, this correspondence does not depend of Φ^β. Moreover, in this case there exist boundary strong limits, in the Fatou sense, for $\Theta(\lambda)$ and Θ_+ can be realized as pointwise multiplication on $H^2_{\mathcal{E}}(\beta)$ considered as the space of functions on ∂D or $\partial\Omega$. In the non bounded case, in general, we have not such a boundary limit for $\Theta(\lambda)$.

We say that the (α,β)-automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$

is inner if it is bounded and the corresponding Θ_+ is an isometry from $H^2_{\mathcal{E}}(\beta)$ into $H^2_{\mathcal{F}}(\alpha)$. Clearly then the boundary limits of $\Theta(\lambda)$ is μ -a.e. isometric on ∂D (or m-a.e. on $\partial\Omega$).

The L^2-bounded (α,β)-automorphic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is called outer if the corresponding Θ_+ has dense range in $H^2_{\mathcal{F}}(\alpha)$.

Since for any $f\in A$ and $a\in\mathcal{E}$, $\bar{\Phi}^{\beta}(f\circ\pi)a\in D(\Theta_+)$ and $\Theta_+\Phi^{\beta}(f\circ\pi)a=fV_{\Theta}a$ it results that $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is outer if and only if

(2.10) $$\bigvee_{f\in A} fV_{\Theta}\mathcal{E} = H^2_{\mathcal{F}}(\alpha)$$

Remark 2. If $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is an (α,β)-automorphic outer function then for any $\lambda\in D$, $\Theta(\lambda)\mathcal{E}$ is dense in $\mathcal{F}$. Indeed let $b\in\mathcal{F}$ such that $(\Theta(0)a, b)_{\mathcal{F}}=0$ for any $a\in\mathcal{E}$. Then for any $a\in\mathcal{E}$ and $f\in A$ we have:

$$(fV_{\Theta}a,b)_{L^2_{\mathcal{F}}(m)}=f(z_0)(\Theta(0)\ \psi^{\beta}(0)a,b)_{\mathcal{F}}=0$$

From (2.10) it results $b\perp H^2_{\mathcal{F}}(\alpha)$. If $h\in H^2_{\mathcal{F}}(\alpha)$ such that $h(0)=b$ (see Remark 1) then

$$0=(h,b)_{L^2_{\mathcal{F}}(\partial D)}=(h(0),b)_{\mathcal{F}}=\|b\|^2$$

i.e. $b=0$.

Since for any $\lambda\in D$ there exists $A\in G$ such that $A0=\lambda$ and $\Theta(\lambda)=\alpha(A)\Theta(0)\beta(A)^*\mathcal{E}$ .it results $\Theta(\lambda)\mathcal{E}$ is dense in $\mathcal{F}$ for any $\lambda\in D$.

Proposition 2. An L^2-bounded (α,β)-automorphic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is simultaneously inner and outer if and only if

<u>it is a unitary constant function. Its constant value U realizes a unitary equivalence between</u> β <u>and</u> α.

<u>Proof</u>. In both cases Θ_+ is a unitary operator from $H^2_{\mathcal{E}}(\beta)$ onto $H^2_{\mathcal{F}}(\alpha)$ such that

$$\Theta_+ T_\beta = T_\alpha \Theta_+$$

As in the proof of Theorem 6 in [2] we can show that Θ_+ is the multiplication with a unitary constant function.

3. Factorization theorems

We shall recall firstly some basic facts about semi-spectral measures (cf. [9], [25]).

Let X be a compact Hansdorff space and H be a separable Hilbert space. An $\mathcal{L}(H)$-valued semi-spectral measure on X is a map $\sigma \to F(\sigma)$ of the family B(X) of Borel subsets of X such that for any $h \in H$ the map $\sigma \to (F(\sigma)h,h)$ is a (positive) Radon measure on X. The semi-spectral measure F is called spectral if $F(X)=I_H$, and $F(\sigma_1 \cap \sigma_2)=F(\sigma_1)F(\sigma_2)$ for any $\sigma_1, \sigma_2 \in B(X)$.

Let $\mathcal{K}$ be a Hilbert space, V a bounded operator from H into $\mathcal{K}$ and E an $\mathcal{L}(K)$-valued spectral measure on Y. Setting for any $\sigma \in B(X)$

$$F(\sigma)=V^{*}E(\sigma)V \tag{3.1}$$

we obtain an $\mathcal{L}(H)$-valued semi-spectral measure on X. Conversely, the celebrated Naymark dilation theorem asserts that for any $\mathcal{L}(H)$-valued semi-spectral measure on X there exist a Hilbert space $\mathcal{K}$, a bounded linear operator $V:H \to \mathcal{K}$, and an $\mathcal{L}(\mathcal{K})$-valued spectral measure on X such that for any $\sigma \in B(X)$

$$F(\sigma)=V^{*}E(\sigma)V \tag{3.2}$$

The triplet $\{\mathcal{K}, V, E\}$ is called a spectral dilation of F. Under some natural condition of minimality this triplet is well defined by F up to a unitarity which preserves the operators V.

From Naimark theorem it results that any semi-spectral measure F gives rise to a completely positive map from C(X) into $\mathcal{L}(H)$ i.e. for any $f_1 \dots f_n \in C(X)$ and $h_1 \dots h_n \in H$ we have

$$\sum_{j,k} \int f_j(x)\overline{f_k(x)}\, d(F(x)h_j, h_k) \geq 0 . \tag{3.3}$$

If F and F' are two $\mathcal{L}(H)$-valued semi-spectral measure on X then we shall write $F \leq F'$ if $F'-F$ is a semi-spectral measure on X. From (3.3) it results that the inequality $F \leq F'$ implies the inequality in completely positivity sense:

$$\sum_{j,k} \int f_j(x)\overline{f_k(x)}\, d(F(x)h_j, h_k) \leq \sum_{j,k} \int f_j(x)\overline{f_k(x)}\, d(F'(x)h_j, h_k) \tag{3.4}$$

In case X is a compact set in complex plane then the spectral theory for normal operators establishes a correspondence between the $\mathcal{L}(\mathcal{K})$-valued spectral measures E on X and the normal operators N on $\mathcal{K}$ having spectrum in X via the formula

$$f(N) = \int f(x)\, dE(x), \qquad f \in C(X). \tag{3.5}$$

Let now Ω and m be as in the preceding sections. For a Hilbert space $\mathcal{F}$ we shall denote by $E^*_{\mathcal{F}}$ the spectral measure on $\partial\Omega$ whose values are projections on $L^2_{\mathcal{F}}(dm)$ defined by

$$E^*_{\mathcal{F}}(\sigma) = \chi_\sigma h, \qquad h \in L^2_{\mathcal{F}}(m),\ \sigma \in B(\partial\Omega) \tag{3.6}$$

where χ_σ denotes the characteristic function of σ.

For an (α,β)-automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which is L^2-bounded the corresponding operator V_Θ from $\mathcal{E}$ into $H^2_{\mathcal{F}}(\alpha)$ can be considered as operator from $\mathcal{E}$ into $L^2_{\mathcal{F}}(dm)$. We shall define an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure F_Θ on $\partial\Omega$ by

$$F_\Theta(\sigma) = V^*_\Theta E^x(\sigma) V_\Theta, \qquad (\sigma \in B(\partial\Omega)). \tag{3.7}$$

The semi-spectral measure F_Θ admits the triplet $\{L^2_{\mathcal{F}}(m), V_\Theta, E^x_{\mathcal{F}}\}$ as spectral dilation. Conversely, if F is an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\partial\Omega$ which admits as spectral dilation the triplet $\{L^2_{\mathcal{F}}(m), V, E^x_{\mathcal{F}}\}$ such that $V\mathcal{E} \subset H^2_{\mathcal{F}}(\alpha)$ then for any $\beta \in \mathrm{Hom}(\pi_0(\Omega), \mathcal{U}(\mathcal{E}))$ there exists an $(\alpha;\beta)$-automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which is L^2-bounded such that $V=V_\Theta$ $F=F_\Theta$. We have only to choose an β-pair of similarity, Φ^β, ψ^β and define

$$\Theta(\lambda)a = \left[V\psi^\beta(\lambda)a\right](\lambda), \qquad (a\in\mathcal{E},\ \lambda\in D) \tag{3.8}$$

Indeed, with $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ defined as in (3.8) we have

$$\Theta(A\lambda) = \left[V\psi^\beta(A\lambda)a\right](A\lambda) = \alpha(A)\psi^\beta(\lambda)\beta(A)^*a = \alpha(A)\Theta(\lambda)\beta(A)^*a,$$

and

$$\left[V_\Theta a\right](\lambda) = \Theta(\lambda)\Phi^\alpha(\lambda)a = \left[Va\right](\lambda)$$

We shall call F_Θ the semi-spectral measure <u>attached</u> to $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$.

<u>Theorem 2</u>. Let $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ be <u>(α_1,β)-automorphic and</u> $\{\mathcal{E}, \mathcal{F}_2, \Theta_2(\lambda)\}$ <u>be (α_2,β)-automorphic and outer. Suppose that</u> $F_{\Theta_1} \le F_{\Theta}$ <u>. Then there exists a contractive</u> $\{\mathcal{F}_2, \mathcal{F}_1, \Theta(\lambda)\}$ <u>which is (α_1, α_2)-automorphic, such that</u>

$$\Theta_1(\lambda) = \Theta(\lambda)\Theta_2(\lambda), \qquad (\lambda\in D). \tag{3.9}$$

<u>If</u> $F_{\Theta_1} = F_{\Theta_2}$, <u>then</u> $\{\mathcal{F}_2, \mathcal{F}_1, \Theta(\lambda)\}$ <u>is inner</u>. <u>If moreover</u>

$\{\mathcal{E}, \mathfrak{F}_1, \Theta_1(\lambda)\}$ is outer then $\Theta(\lambda)$ is unitary constant. Its constant value U verifies $\alpha_1(A)U = U\alpha_2(A)$ for any $A \in G$.

Proof. Let $f_1, \ldots, f_n \in A$ and $a_1, \ldots, a_n \in \mathcal{E}$. We have

$$\|\sum_{k=1}^{n} f_k \bar{V}_{\Theta_1} a_k\|^2_{L^2_{\mathfrak{F}_1}(m)} = \sum_{k,j} \int_{\partial\Omega} f_j(z)\overline{f_k(z)}\, d(F_{\Theta_1}(z)a_j, a_k)_{\mathcal{E}} \leq$$

$$\leq \sum_{k,j} f_j(z)\overline{f_k(z)}\, d(F_{\Theta_2}(z)a_j, a_k)_{\mathcal{E}} = \|\sum_{k=1}^{n} f_k \bar{V}_{\Theta_2} a_k\|^2_{L^2_{\mathfrak{F}_2}(m)}.$$

Hence

$$\|\sum_{k=1}^{n} f_k \bar{V}_{\Theta_1} a_k\| \leq \|\sum_{k=1}^{n} f_k \bar{V}_{\Theta_2} a_k\|. \tag{3.10}$$

Since $\{\mathcal{E}, \mathfrak{F}_2, \Theta_2(\lambda)\}$ is outer it results that we can define a contraction Q from $H^2_{\mathfrak{F}_2}(\alpha_2)$ into $H^2_{\mathfrak{F}_1}(\alpha_1)$ by

$$Q\sum_{k=1}^{n} f_k \bar{V}_{\Theta_2} a_k = \sum_{k=1}^{n} f_k \bar{V}_{\Theta_1} a_k. \tag{3.11}$$

Since for any $h \in D(\Theta_{2+})$ we clearly have $Q\Theta_{2+}h = \Theta_{1+}h$ and

$$[Q\Theta_{2+}h](\lambda) = [\Theta_{1+}h](\lambda) = \Theta_1(\lambda)h(\lambda) =$$

$$= [\Theta_{1+}\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda) = [Q\Theta_{2+}\Phi^{\beta}\psi^{\beta}(\lambda)h(\lambda)](\lambda) =$$

$$= [Q\Phi^{\alpha_2}\psi^{\alpha_2}(\lambda)\Theta_{2+}h](\lambda),$$

it results that Q intertwines the point evaluations on $H^2_{\mathfrak{F}_2}(D,\alpha_2)$ and $H^2_{\mathfrak{F}_1}(D;\alpha_1)$. Hence $Q = \Theta_+$ where $\{\mathfrak{F}_2, \mathfrak{F}_1, \Theta(\lambda)\}$ is a contractive

(α_1, α_2)-automorphic function on D. We have

$$\Theta_1(\lambda)a = [\Theta_{1+} \Phi^{\beta} \psi^{\beta}(\lambda) a](\lambda) = [Q\Theta_{2+} \Phi^{\beta} \psi^{\beta}(\lambda) a](\lambda) =$$

$$= \Theta(\lambda)[\Theta_+ \Phi^{\beta} \psi^{\beta}(\lambda) a](\lambda) = \Theta(\lambda)\,\Theta_2(\lambda) a.$$

If $F_{\Theta_1} = F_{\Theta_2}$ then clearly in (3.10) we have equality.

It results that Q is an isometry and consequently Θ_+ is isometry. Hence $\{\mathcal{F}_2, \mathcal{F}_1, \Theta(\lambda)\}$ is inner. If moreover $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ is outer then clearly $Q = \Theta_+$ has dense range and consequently $\{\mathcal{F}_2, \mathcal{F}_1, \Theta(\lambda)\}$ is outer. From Proposition 2 we conclude that $\Theta(\lambda)$ is unitary constant and its constant value U verifies $\alpha_1(A)U = U\alpha_2(A)$ for any $A \in G$.

The proof at the Theorem 2 is complete.

Let now F be an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\partial\Omega$. We are interested in the following problem. Does there exist an (α, β) automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $F_{\Theta} \leq F$? Let us remark that we can suppose $\beta = 1$. Indeed, if $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is an α-automorphic function such that $F_{\Theta} \leq F$ then the (α, β)-automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta'(\lambda)\}$ defined by $\Theta'(\lambda) = \Theta(\lambda)\,\psi^{\beta}(\lambda)$ verifies $V_{\Theta'} = V_{\Theta}$, $F_{\Theta'} = F_{\Theta}$.

Let $\{\mathcal{K}, V, E\}$ be the minimal spectral dilation of F and N be the normal operator corresponding to E. Denote by $\mathcal{K}_+$ the subspace of $\mathcal{K}$ defined by

$$\mathcal{K}_+ = \bigvee_{f \in A} f(N) V \mathcal{E}. \tag{3.12}$$

Then $\mathcal{K}_+$ is an invariant subspace for N. The operator $N_+ = N|\mathcal{K}_+$ is a subnormal operator with spectrum in Ω and normal spectrum in $\partial\Omega$. We call N_+ the subnormal operator attached to F.

We prove now the main theorem of the paper:

Theorem 3. Let F be an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\partial\Omega$. There exists an L^2-bounded α-automorphic analytic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ which is outer such that

(i) $F_\Theta \leq F$.

(ii) For any other L^2-bounded β-automorphic function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ which verifies $F_{\Theta_1} \leq F$ we have $F_{\Theta_1} \leq F_\Theta$.

The outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is uniquely determined by (i) and (ii) up to a unitary constant factor from the left. In order that the equality holds in (i) it is necessary and sufficient that the subnormal operator attached to F is pure.

Proof. Let $\{\mathcal{K}, V, E\}$ be the minimal spectral dilation of F, N the associated normal operator and $\mathcal{K}_+$ defined by (3.12). Applying Theorem 0 to N and $\mathcal{K}_+$ we obtain a representation α of G on a separable Hilbert space $\mathcal{F}$ such that

$$\mathcal{K} = L^2_{\mathcal{F}}(m) \oplus \mathcal{K}_1 ,$$

$$\mathcal{K}_+ = H^2_{\mathcal{F}}(\alpha) + \mathcal{K}_1 ,$$

$L^2_{\mathcal{F}}(m)$ reduces N to the multiplication by z and $H^2_{\mathcal{F}}(\alpha)$ reduces N_+ to the bundle shift T_α. Let P_α be the orthogonal projection of $\mathcal{K}$ onto $H^2_{\mathcal{F}}(\alpha)$. Denote

(3.13) $$V_\Theta = P_\alpha V.$$

Then $V_{\Theta}\mathcal{E}\subset H^2_{\mathcal{F}}(\alpha)$. Hence there exists an α-automorphic function $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ such that

$$\Theta(\lambda)a=\left[V_{\theta}a\right](\lambda), \qquad (\lambda\in D). \tag{3.14}$$

We have

$$\bigvee_{f\in A} fV_{\Theta}\mathcal{E}=\bigvee_{f\in A} fP_{\alpha}V\mathcal{E}=P_{\alpha}\bigvee_{f\in A} f(N)V\mathcal{E}=P_{\alpha}\mathcal{K}_{+}=H^2_{\mathcal{F}}(\alpha).$$

Hence $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ is outer.

For any function $f\in A$, and $a\in\mathcal{E}$, we have

$$\int_{\partial\Omega}|f|^2 d(F_{\Theta}a,a)=\int_{\partial\Omega}|f(z)|^2\|(V_{\Theta}a)(z)\|^2 dm=$$

$$=\|f(N)V_{\Theta}a\|^2_{L^2_{\mathcal{F}}(m)}=\|f(N)P_{\alpha}Va\|^2_{\mathcal{K}}=\|P_{\alpha}f(N)Va\|^2_{\mathcal{K}}\leq$$

$$\leq\|f(N)Va\|^2_{\mathcal{K}}=\int_{\partial\Omega}|f(z)|^2 d(F(z)a,a).$$

Using the approximation modulus property of A on $\partial\Omega$ (cf. [43]), for any positive function $\varphi\in C(\partial\Omega)$, we obtain

$$\int_{\partial\Omega}\varphi(z)d(F_{\Theta}(z)a,a)\leq\int_{\partial\Omega}\varphi(z)d(F(z)a,a)$$

i.e. $F_{\Theta}\leq F$.

Let now $\{\mathcal{E},\mathcal{F}_1,\Theta_1(\lambda)\}$ be an L^2-bounded β-automorphic function on D verifying $F_{\Theta_1}\leq F$. For any $f_1\dots f_n\in C(\partial\Omega)$ and $a_1\dots a_n\in\mathcal{E}$ let us put

$$X\sum_{k=1}^{n} f_k(N)Va_k=\sum_{k=1}^{n} f_kV_{\Theta_1}a_k.$$

We have

$$\|\sum_{k=1}^{n} f_k V_{\Theta_1} a_k\|^2 = \sum_{k,j} \int f_k \bar{f}_j \, d(F_{\Theta_1} a_k, a_j) \leq$$

$$\leq \sum_{k,j} \int f_k \bar{f}_j \, d(F a_k, a_j) = \|\sum_{k=1}^{n} f_k(N) V a_k\|^2.$$

Hence X is a contraction from $\mathcal{K}$ into $L^2_{\mathcal{F}_1}(m)$. Clearly

$$XN = N_{\mathcal{F}_1} X \tag{3.15}$$

Since $X\mathcal{K}_1 \subset H^2_{\mathcal{F}}(\beta)$ and $\mathcal{K}_1 = \{k \in K_+ : f(N)k \in K_+ \ (\forall)\ f \in C(\partial\Omega)\}$ we have $X\mathcal{K}_1 = X\{k \in \mathcal{K}_+ : f(N)k \in \mathcal{K}_+,\ (\forall)\ f \in C(\partial\Omega)\} \subset \{h \in H^2_{\mathcal{F}_1}(\beta) : f(N_{\mathcal{F}_1})h \in H^2_{\mathcal{F}_1}(\beta)\ (\forall)\ f \in C(\partial\Omega)\} = \{0\}$, because T_β is pure subnormal. Hence $X\mathcal{K}_1 = \{0\}$ which implies $XP_\alpha k = Xk$ for any $k \in \mathcal{K}_+$.

Then for any $f \in A$ we have

$$\int_{\partial\Omega} |f|^2 d(F_{\Theta_1} a, a) = \|fV_{\Theta} a\|^2 = \|Xf(N)Va\|^2 =$$

$$= \|XP_\alpha f(N)Va\|^2 \leq \|P_\alpha f(N)Va\|^2 =$$

$$= \|f(N)P_\alpha Va\|^2 = \|fV_\Theta a\|^2 = \int_{\partial\Omega} |f|^2 d(F_\Theta a, a)$$

Using again the approximation modulus property of A we conclude $F_{\Theta_1} \leq F_\Theta$.

Now, for any L^2-bounded outer function $\{\mathcal{E}, \mathcal{F}', \Theta'(\lambda)\}$ which is α'-automorphic and verifies (i) and (ii) we have $F_\Theta = F_{\Theta'}$. Using Theorem 2 we obtain $\Theta'(\lambda) = Z\Theta(\lambda)$ where Z is a unitary operator from $\mathcal{F}$ onto $\mathcal{F}'$ such that $\alpha'(A)Z = Z\alpha(A)$ for any $A \in G$. The proof of the theorem is complete.

As the corollary we obtain the following inner-outer factorization for α-automorphic functions.

Theorem 4. Let $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an α-automorphic L^2-bounded analytic function on D. There exist an β-automorphic function $\{\mathcal{E}, \mathcal{F}_1, \Theta_1(\lambda)\}$ which is outer and an (α, β)-automorphic function $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ which is inner such that

$$\Theta(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda), \qquad (\lambda \in D). \tag{3.16}$$

This factorization is unique in the following sense: If $\{\mathcal{E}, \mathcal{F}_1', \Theta_1'(\lambda)\}$ is an β'-automorphic outer function and $\{\mathcal{F}_1', \mathcal{F}, \Theta_2'(\lambda)\}$ is an (α, β')-automorphic inner function such that

$$\Theta(\lambda) = \Theta_2'(\lambda)\Theta_1'(\lambda), \qquad (\lambda \in D)$$

then there exists a unitary operator U from $\mathcal{F}_1$ onto $\mathcal{F}_1'$ such that

$$U\beta(A) = \beta'(A)U, \qquad (A \in G) \tag{3.17}$$

$$U\Theta_1(\lambda) = \Theta_1'(\lambda), \quad \Theta_2(\lambda) = \Theta_2'(\lambda)U, \qquad (\lambda \in D). \tag{3.18}$$

Proof. Applying Theorem 3 to F_Θ we obtain an β-automorphic outer function $\{\mathcal{E}, \mathcal{F}, \Theta_1(\lambda)\}$ such that $F_{\Theta_1} = F_\Theta$. Using Theorem 2, we obtain an inner (α, β)-automorphic function $\{\mathcal{F}_1, \mathcal{F}, \Theta_2(\lambda)\}$ such that

$$\Theta(\lambda) = \Theta_2(\lambda)\Theta_1(\lambda), \qquad (\lambda \in D). \tag{3.19}$$

Suppose now that

(3.20) $\Theta(\lambda) = \Theta_2'(\lambda)\,\Theta_1'(\lambda)$, $(\lambda \in D)$

where $\{\mathcal{E}, \mathcal{F}_1', \Theta_1'(\lambda)\}$ is an β'-automorphic outer function and $\{\mathcal{F}_1', \mathcal{F}, \Theta_2'(\lambda)\}$ is an (α,β)-automorphic inner function. Then it is easy to see that $F_{\Theta_1'} = F_{\Theta_1}$. Using Theorem 2 we obtain a unitary operator U from $\mathcal{F}_1$ onto $\mathcal{F}_1'$ such that (3.17) is satisfied and

(3.21) $\Theta'(\lambda) = U\,\Theta_1(\lambda)$ $(\lambda \in D)$

From (3.19), (3.20) and (3.21) it results

$$\Theta_2(\lambda)\,\Theta_1(\lambda) = \Theta_2'(\lambda)\,U\,\Theta_1(\lambda),$$

and using Remark 2 we obtain

$$\Theta_2(\lambda) = \Theta_2'(\lambda)\,U.$$

The proof of the Theorem 4 is complete.

Let now $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ be an (α,β)-automorphic function on D and let E be the flat unitary bundle over $\bar{\Omega}$ corresponding to α and F the flat unitary bundle corresponding to β. Then $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ gives rise to a holomorphic bundle map Θ from E to F. Suppose that Θ has nontangential limits m-a.e. on $\partial\Omega$. The limit at a point $z \in \partial\Omega$ can be regarded as an operator from the fiber $\mathcal{E}$ of E at z to the fiber $\mathcal{F}$ of F at z. Suppose that for any $a \in \mathcal{E}$, $z \to \|\Theta(z)a\|$ belongs to $L^2_{\mathcal{F}}(m)$. Then the function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is L^2-bounded and we have

(3.22) $$(V_{\Theta} a)(z) = \Theta(z)a \quad , \qquad (m\text{-a.e on } \partial\Omega)$$

If the function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ in Theorem 3 has these properties, and, moreover in (i) we have equality then we obtain

(3.23) $$dF(z) = \Theta(z)^* \Theta(z) dm(z),$$

which corresponds to the usual meaning of factorability of measures by means of analytic functions.

Generally, it is difficult to give simple characterizations of the fact that we have equality in (i), or at least that $\Theta(\lambda)$ is not a null function. Even in the scalar valued case the Szegö-Kolmogorov-Krein problems of factorability on multiply connected domains become more complicate (cf. [3]).

In the remainder we shall make some considerations on this problem.

For an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure F on $\partial\Omega$ let us define

(3.24) $$(\Delta[F]a, a) = \inf \int_{\partial\Omega} \sum_{i,j} f_j(z) \overline{f_i(z)}\, d(F(z)a_j, a_i)_{\mathcal{E}},$$

where the infimum is taken over all finite systems $f_0 = 1$ and $f_1, \ldots, f_n \in A_0$, $a_0 = a$, and $a_1, \ldots, a_n \in \mathcal{E}$.

If $\{\mathcal{K}, V, E\}$ is the minimal spectral dilation of F and N the normal operator on $\mathcal{K}$ corresponding to E let $\mathcal{K}_0 = \bigvee_{f \in A_0} f(N) V \mathcal{E}$.

We have

$$(\Delta[F]a, a) = \inf \sum_{i,j=0}^{n} \int f_j(z) \overline{f_i(z)}\, d(E(z) V a_j, V a_i) =$$

$$= \inf \| \sum_{j=0}^{n} f_j(N) V a_j \|^2 = \inf_{k \in \mathcal{K}_0} \| Va - k \|^2 = \| (I - P_0) V a \|^2,$$

where P_o is orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_o$.

Hence

$$(3.25) \qquad \left(\Delta[F]a, a\right)_{\mathcal{E}} = \left(V^{*}(I-P_o)Va, a\right)_{\mathcal{E}}$$

It results that (3.24) defines a positive operator $\Delta[F]$ on $\mathcal{E}$. We call $\Delta[F]$ the Szegö operator of F. The name is justified because in case F is scalar valued we have

$$\Delta[F] = \inf_{f \in A_o} \int_{\partial\Omega} |1-f|^2 dF .$$

Theorem 5. Let F be an $\mathcal{L}(\mathcal{E})$-valued semi-spectral measure on $\partial\Omega$ and $\Delta[F]$ be its Szegö operator. Then we have:

(i) $\Delta[F]=0$ if and only if there is no L^2-bounded α-automorphic function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$, such that $\Theta(\lambda)\neq 0$ and $F_{\Theta} \leq F$.

(ii) If $\Delta[F]\neq 0$ then there exists a unique maximal (in the sense of Theorem 3) L^2-bounded α-automorphic outer function $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $F_{\Theta} \leq F$, $\dim \mathcal{F} = \dim \Delta[F]\mathcal{E}$ and

$$(3.26) \qquad \Delta[F] = \Delta[F_{\Theta}].$$

Proof. Suppose that there exists $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ such that $\Theta(\lambda)\neq 0$ and $F_{\Theta} \leq F$. Then according to Theorem 4 we can suppose that $\{\mathcal{E}, \mathcal{F}, \Theta(\lambda)\}$ is outer. Then

$$\left(\Delta[F]a, a\right) \geq \left(\Delta[F_{\Theta}]a, a\right) = \inf_{f_1 \cdots f_m \in A_o} \| V_{\Theta} a - \sum_{j=1}^{m} V_{\Theta} a_j \|^2 \geq$$

$$\geq \inf_{\substack{h \in H^2_{\mathcal{F}}(D) \\ h(0)=0}} \| V_{\Theta} a - h \|^2 = \| \Theta(0) a \|^2 .$$

which implies $\Delta[F]\neq 0$.

Suppose now $\Delta[F]\neq 0$ and let $\{\mathcal{E},\mathcal{F},\Theta(\lambda)\}$ be the outer function given by Theorem 3. Recall that $V_\Theta = P_\alpha V$ where P_α is the orthogonal projection of $\mathcal{K}_+$ on $H^2_{\mathcal{F}}(\alpha)$ in Wold decomposition

$$\mathcal{K} = L^2_{\mathcal{F}}(m) \oplus \mathcal{K}_1, \quad \mathcal{K}_+ = H^2_{\mathcal{F}}(\alpha) \oplus \mathcal{K}_1 .$$

If $k \in \mathcal{K}_+$ is orthogonal to $\mathcal{K}_0$ then it is orthogonal on $(N-z_0)\mathcal{K}_1 = \mathcal{K}_1$. Hence

$$P_\alpha (I-P_0)V = (I-P_0)P_\alpha V = (I-P_0)V_\Theta,$$

and according to (3.25) we have

$$\Delta[F_\Theta] = \Delta[F].$$

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