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# SCHEMAS WITH CLOSED 1 - CODIMENSIONAL POINTS

by

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INTRODUCTION

We shall present in this paper some remarks about the proper morphisms of noetherian schemes and about the noetherian schemes which are dominated by algebraic varieties.

In chapter I, we prove a converse of the Grothendieck-Serre Theorem about the coherence of the direct image of a coherent sheaf via a proper morphism (Theorem 1). An analogous result holds also in the case of complex-analytic spaces (see [3] ), namely: a morphism $f:X \to Y$ of complex-analytic spaces is proper if $f_*(F)$ is coherent for every coherent $\mathcal{O}_X$ - module $F$ whose support is zero-dimensional. In the case where $f:X \to Y$ is a separated morphism of finite type of noetherian schemes, $f$ is proper if and only if $f_*(F)$ is coherent for every $\mathcal{O}_X$-coherent module $F$. The case where $X$ is an algebraic scheme over a field and $Y$ is a noetherian scheme is also considered (see Corollary 4). Thus, one gets a generalization of a result of J.E.Goodman and A.Landman ( [6] ). Another criterion of properness may be found in Corrolary 1.

The method of proof of the above results consists in applying either compactification theorem of Nagata, or. Chow's Lemma to reduce oneself to the situation when $f$ is quasi-projective. The conditions imposed to the morphism $f:X \to Y$ lead to a compactification $\bar{f}:\bar{X} \to Y$ with the property that $Z = \bar{X}-X$ is a set of closed points of codimension one in $X$. Note that the classical valuative criterion of properness (in the form of EGA II) may be also proved in the same way.

In the second chapter some properties of noetherian schemes which are dominated by algebraic varieties (which come from the analysis of a special case of noetherian scheme X with closed points of codimension one) are given. We prove that such X is a Jacobson scheme (Corollary 5) and " connected by curves " if the base field is algebraically closed (proposition 1). One also proves that X is catenarian and equicodimensional (Corollary 7).

In chapter III one introduces a special class of schemes the so-called universal 1-equicodimensional schemes (Definition 1) Such a scheme Z is defined in algebraic terms and caracterized by the following property: every separated morphism $f:X \to Y$ of schemes of finite type over Z is proper if every closed subscheme $C \subset X$ of dimension 1 is proper over Y . In the case where Z=spec k (with k a filed), this is always true (see [9] ). But in general this fails, (see Example 3). One gives various characterizations of these schemes and one shows that they have some "good" general properties. A special property of these schemes says that in an universal 1-equicodimensional scheme the catenarian + equicodimensional property "propagates" well (Proposition 5). As an application of the theory of universally 1-equicodimensional schemes, one proves that every noetherian scheme dominated by an algebraic variety has the same properties of dimension as for an algebraic variety (Corollary 15).

<u>N o t e: Throughout this paper we shall follow the terminology and notations of EGA I-IV except the term of "prescheme" which is replaced here by the term "scheme".</u>

# I

This section is devoted to the proof of a converse of the Serre-Grothendieck Theorem on the coherence of the image of a coherent module by a proper morphism of noetherian schemes. Some new criteria for proper morphisms are also derived from here.

The aim of this section is the following:

**Theorem 1.** *Let $X, Y$ be two noetherian schemes and $f: X \rightarrow Y$ a separated morphism of finite type. Then the following conditions are equivalent:*

(i) $f$ *is proper*.

(ii) $f_*(F)$ *is a coherent* $\mathcal{O}_Y$- *module for every coherent* $\mathcal{O}_X$-*module* $F$

(iii) $f_*(\mathcal{O}_X/I)$ *is coherent for every coherent ideal* $I$ *defining an integral closed subscheme of* $X$.

*Proof*. (iii) $\Rightarrow$ (i).

We may assume that $X, Y$ are integral schemes. Indeed, let $(Y_i)_{1 \leqslant i \leqslant n}$ be the irreducible components of $Y$ and $(X_{ij})_{1 \leqslant j \leqslant n_i}$ the irreducible components of $f^{-1}(Y_i)$; taking the reduced structures on $Y_i$ and $X_{ij}$, $(f_{ij})_{red}$ is proper $\Rightarrow f$ is proper (EGA II,5.4.5) and $f_{ij}: X_{ij} \rightarrow Y_j$ verifies the condition (iii) if $f$ does.

If $\dim X = 0$, $X$ contains a single closed point $x$ and $X = \operatorname{Spec} k(x)$ ($k(x)$ being the residual field of the point $x$). Let $V$ be an affine open set containing $f(x)$ and $A = \Gamma(V, \mathcal{O}_Y)$. By hypothesis, $k(x)$ is a finite $A$-algebra, hence $f(x)$ is a closed point of $Y$ and $k(x)$ is a finite extension of $k(f(x))$; hence $f$ factors : $X \xrightarrow{f'} \operatorname{Spec} k(y) \xhookrightarrow{i} Y$, where $f'$ is finite and $i$ is a closed immersion. It results that $f$ is proper.

If $\dim X > 0$, let $f: X \rightarrow Y$ be a separated morphism of finite type of noetherian schemes, which satisfies the condition (iii), such that f is not a proper morphism. We may suppose that every integral closed subscheme $X' \subsetneq X$ is proper over Y. Indeed, otherwise, there exists a minimal closed integral subscheme $\widetilde{X} \subset X$ such that $f|_{\widetilde{X}}: \widetilde{X} \to Y$ is not proper; by the minimality hypothesis, every integral closed subscheme $\widetilde{X}' \subsetneq \widetilde{X}$ is proper over Y and $f|_{\widetilde{X}}$ satisfies the condition (iii).

We may assume that f is a quasiprojective morphism. Indeed, by <u>Chow's</u> lemma (EGA II, 5.6.1) we find a projective birational morphism $p: \widetilde{X} \to X$, such that $f \circ p$ is quasiprojective; then f is proper iff $f \circ p$ is proper. Every integral closed subscheme $\widetilde{X}' \subsetneq X$ is proper over Y; since in the exact sequence of coherent $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \to p_*\mathcal{O}_{\widetilde{X}} \to p_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X \to 0$$

the last module has the support $\neq X$ (hence it is proper over Y), from the coherence of $f_*(\mathcal{O}_X)$ and of $f_*(p_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X)$ and from the exact sequence:

$$0 \to f_*(\mathcal{O}_X) \to f_*p_*\mathcal{O}_{\widetilde{X}} \to f_*(p_*\mathcal{O}_{\widetilde{X}}/\mathcal{O}_X)$$

it follows that $(f \circ p)_*\mathcal{O}_{\widetilde{X}}$ is $\mathcal{O}_Y$-coherent.

Thus, we may take f a quasiprojective morphism. Let $\overline{f}$ be a compactification of f

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & \overline{X} \arrow[dl, "\overline{f}"] \\ & Y & \end{tikzcd}$$

with $\overline{f}$ a projective morphism and i an open dense immersion. Because f is not proper, the closed set $Z = \overline{X} - X$ is not empty.

For any point $x \in Z$, let $\overline{X}_x$ be a closed integral subscheme of $\overline{X}$, passing through $x$, such that $\overline{X}_x \neq \overline{X}$. If $\overline{X}_x \cap X \neq \phi$, then $\overline{X}_x \cap X$ is proper over $Y$ by hypothesis; then $\overline{X}_x \cap X$ being proper over $Y$ , it is closed in $\overline{X}_x$, which contradicts the connectedness of $\overline{X}_x$. Hence, for every point $x \in Z$ and every integral closed subscheme $\overline{X}_x \subsetneqq X$ containing $x$, we see that $\overline{X}_x \subset Z$. If $Z_1, .., Z_n$ are the irreducible components of $Z$ and $x \in Z$, it follows that every non-zero prime ideal $\mathfrak{p} \subset \mathcal{O}_{\overline{X},x}$ contains any prime ideal $\mathfrak{q}_j \subset \mathcal{O}_{\overline{X},x}$, corresponding to any irreducible component $Z_j$ passing through $x$. Thus $\dim \mathcal{O}_{\overline{X},x} = 1$ and so $Z$ is a finite set of closed points of codimension 1 in $\overline{X}$.

We can prove that the open immersion $i : X \hookrightarrow \overline{X}$ is an affine morphism. For this, let $\overline{U} \subset \overline{X}$ be an affine open set of $X$, $U = \overline{U} \cap X$, $F$ a coherent $\mathcal{O}_U$ - module on $U$ and $\overline{F}$ a coherent $\mathcal{O}_{\overline{U}}$ -module on $\overline{U}$, extending $F$ (EGA I, 9.4.8); in the exact sequence:

$$H^1(\overline{U}, \overline{F}) \longrightarrow H^1(U, F) \longrightarrow H^2_{Z \cap \overline{U}}(\overline{U}, \overline{F})$$

we have $$H^2_{Z\cap\overline{U}}(\overline{U}, \overline{F}) = \bigoplus_{x \in Z \cap \overline{U}} H^2(\operatorname{Spec} \mathcal{O}_{\overline{X},x}, \overline{F}_x) = 0$$

because $\dim \operatorname{Spec} \mathcal{O}_{\overline{X},x} = \dim \mathcal{O}_{\overline{X},x} = 1$.
From $H^1(\overline{U}, \overline{F}) = 0$, it follows that $H^1(U, F) = 0$. From Serre's Criterion, it results that $U$ is an affine scheme.

From the exact sequence :

$$0 \to \mathcal{O}_{\overline{X}} \longrightarrow i_* \mathcal{O}_X \longrightarrow i_* \mathcal{O}_X / \mathcal{O}_{\overline{X}} \to 0$$

we obtain the exact sequence

$$f_* \mathcal{O}_X \longrightarrow \overline{f}_*(i_* \mathcal{O}_X / \mathcal{O}_{\overline{X}}) \longrightarrow R^1 \overline{f}_* \mathcal{O}_{\overline{X}}$$

Because $f_*\mathcal{O}_X$ is coherent, by condition (iii), and $R^1 f_* \mathcal{O}_{\bar{X}}$ is coherent because $\bar{f}$ is proper, we have that $\bar{f}_* (i_*\mathcal{O}_{\bar{X}}/\mathcal{O}_{\bar{X}})$ (which apriori is quasicoherent) is a coherent module. Because $i_*\mathcal{O}_X/\mathcal{O}_{\bar{X}}$ has the support in Z, which is a discrete closed subset of $\bar{X}$, and $\bar{f}_*(i_*\mathcal{O}_X/\mathcal{O}_{\bar{X}})$ is a coherent $\mathcal{O}_Y$ - module, it is easy to see that $i_*\mathcal{O}_X/\mathcal{O}_{\bar{X}}$ is a coherent $\mathcal{O}_{\bar{X}}$ - module.

Hence $i_*\mathcal{O}_X$ is a coherent $\mathcal{O}_{\bar{X}}$ - module. Because i is an affine morphism, then $\bar{X} = \operatorname{Spec} i_*\mathcal{O}_X$ and so, it follows that X is a finite scheme over $\bar{X}$. Thus, i being a closed dominant morphism, it is surjective. It follows that $Z = \bar{X} - X = \emptyset$, which contradicts the fact that f is not a proper morphism.

<u>Remark 1</u> In the proof of Theorem 1, we have used and proved the following property (which will be used in other proofs):

<u>Lemma A.</u> <u>Let</u> $i: X \hookrightarrow \bar{X}$ <u>be an open dense immersion of integral noetherian schemes and</u> $Z = \bar{X} - X$. <u>Then the following are equivalent:</u>

a) <u>every closed integral subscheme</u> $X' \subsetneqq \bar{X}$ <u>such that</u> $X' \cap Z \neq \emptyset$ <u>is contained in</u> Z.

b) <u>every closed integral subscheme</u> $X' \subsetneqq \bar{X}$ <u>such that</u> $X' \cap X \neq \emptyset$ <u>is contained in</u> X.

c) Z <u>is a (finite) set of closed points of codimension 1 in</u> $\bar{X}$.

From Lemma A it results the following property:

<u>Lemma B.</u> <u>Let</u> $i: X \hookrightarrow \bar{X}$ <u>be an open dense immersion of integral noetherian schemes. Suppose that for every closed point</u> $x \in X$, <u>i(x) is a closed point of</u> $\bar{X}$. <u>Then</u> $\dim \bar{X} = 1$ <u>iff</u> $\dim X = 1$.

Indeed, if $\dim X = 1$ and $Z \subset \bar{X}$ is an integral closed subscheme containing a point of $\bar{X} - X$, then $Z \cap X = \emptyset$ (otherwise, $Z \cap X$ is a closed point $x \in X$; x being a closed point of $\bar{X}$, then

$\{x\}$ is a isolated component of Z, which contradicts the fact that Z is irreducible). Hence, by Lemma A, for every point $x \in \overline{X}-X$, dim $\mathcal{O}_{\overline{X},x} = 1$ and so dim $\overline{X} = 1$.

We remark that in Lemma B, the condition that i(x) is a closed point of $\overline{X}$ if x is a closed point of X, is satisfied in the following two cases:

1) $\overline{X}$ is a Jacobson (or Hilbert) scheme

2) i is an immersion of schemes of finite type over a scheme S and, if f: $X \to S$ is the canonical morphism, for every closed point $x \in X$ f(x) is closed and the residual field k(x) is algebraic over k(f(x)).

For analytic spaces an analog of Theorem 1 is also true (see [3] ). In [3] it is shown that for analytic spaces only the coherence of $f_*(\mathcal{O}_X/I)$, for every coherent ideal I of $\mathcal{O}_X$ defining an analytic subspace of dimension 0, is enough to ensure the properness of f.

In the following, we want to obtain weaker conditions for proper morphisms, using the same method of compactification of morphisms by a finite set of closed points of codimension 1, as the method used in the proof of Theorem 1.

First we may point out:

Corollary 1. Let X,Y be two noetherian schemes, $f:X \to Y$ a separated morphism of finite type and $n > 0$ an integer number. Suppose that for every closed subscheme $X' \subset X$ of dimension $> n$ and for every closed point $y \in Y$, the closed subset $X' \cap f^{-1}(y)$ is either empty or of dimension $> 0$. Then the following assertions are equivalent

(i) f is proper

(iii') $f_*(\mathcal{O}_X/I)$ is a $\mathcal{O}_Y$- coherent module for every coherent ideal I, which defines an integral closed subscheme of X of dimension $\leq n$.

(v") every integral closed subscheme $X' \subset X$ with dim $X' \leq n$ is proper over Y.

Proof. Obviously, (i) $\Rightarrow$ (iii') $\Rightarrow$ (v"), by Theorem 1. We shall prove (v") $\Rightarrow$ (i).

As in the proof of Theorem 1, we may suppose that $X$ and $Y$ are integral schemes. Suppose $X$ is not proper over $Y$. Let $X' \subset X$ be a minimal integral closed subscheme such that it is not proper over $Y$; then $\dim X' > n$ and every integral closed subscheme $Z \subsetneqq X'$ is proper over $Y$. If we choose a dense compactification $g$ of $f|_{X'}$:

$$\begin{tikzcd} X' \arrow[r, hook, "i"] \arrow[dr, "f|_{X'}"'] & \overline{X}' \arrow[d, "g"] \\ & Y \end{tikzcd}$$

with $\overline{X}'$ an integral scheme; by Lemma A, it follows that $Z' = \overline{X}' - X'$ is a nonempty finite set of closed points of codimension 1 in $\overline{X}'$. In the Stein factorisation of $g$:

$$\begin{tikzcd} \overline{X}' \arrow[r, "h"] \arrow[dr, "g"'] & \overline{Y} \arrow[d, "p"] \\ & Y \end{tikzcd}$$

$p$ is finite and $\overline{Y}$ is an integral scheme.

If $\dim \overline{Y} = 0$, then $g(\overline{X}')$ is a closed point $y \in Y$ and $\overline{X}'$ will be a complete algebraic scheme over the residual field $k(y)$. Because $Z' \neq \emptyset$, it follows that $\dim \overline{X}' = \dim X' = 1$. But, because of condition (v'), we have that $X'$ is proper over $Y$, so that $Z' = \emptyset$, which is a contradiction.

If $\dim \overline{Y} > 0$, $h$ is surjective with connected fibres; because $\operatorname{codim}_{X'} \{x\} = 1$ for every $x \in Z'$, it follows that $h^{-1}h\{x\} = \{x\}$. Let us consider the set $Z'' = \{x \in \overline{X}' \mid x$ is not isolated in $h^{-1}h(x)\}$; by Zariski's Main Theorem [EGA III ], $Z''$ is closed and $h(Z'') \neq Y$, because $h(Z'') \cap h(Z') = \emptyset$. Then the set $D = \overline{Y} - h(Z' \cup Z'')$ is nonempty and open in $\overline{Y}$, and $h|_{h^{-1}(D)} : h^{-1}(D) \to D$

is a finite morphism, because it is a proper morphism with finite fibres; moreover, $h^{-1}(D) \subset X'$. The set $U = Y - p(\overline{Y} - D)$ is nonempty and open in Y, $g|_{g^{-1}(U)} : g^{-1}(U) \longrightarrow U$ is a finite morphism and $g^{-1}(U) \subset X'$; hence $f|_{X'}$ is generically a finite morphism. Then, for every closed point $y \in f(X') \cap U$, the set $X' \cap f^{-1}(y)$ is a nonempty finite set of closed points of X, which, together with $\dim X' > n$, contradicts the condition of the above corollary. Corollary 1 is proved.

In the following, we shall remark some cases when the condition (i) of Theorem 1 is equivalent to:

(iv) $f_*(\mathcal{O}_X/I)$ is an $\mathcal{O}_Y$- coherent module, for every coherent ideal I defining an integral closed subscheme of X of dimension $\leq 1$.

By Theorem 1, the condition (iv) is equivalent to the condition:

(v) every integral closed subscheme $X' \subset X$ of dimension $\leq 1$ is proper over Y.

In the case when for every closed point $x \in X$, $f(x)$ is closed and the residual field $k(x)$ is algebraic over $k(f(x))$, (in particular if Y is a Jacobson (Hilbert) scheme), the properties (iv) and (v) are equivalent to (iv') and (v') which follow:

(iv') $f_*(\mathcal{O}_X/I)$ is an $\mathcal{O}_Y$- coherent module for every coherent ideal I defining an integral closed subscheme of X of dimension 1.

(v') every integral closed 1-dimensional subscheme $C \subset X$ is proper over Y.

Remark 2. The equivalence (iv) $\Leftrightarrow$ (v) follows from Theorem 1, for the case when $\dim X = 1$. We may prove this case in other manner. We may suppose that $f : X \longrightarrow Y$ is a dominant morphism of integral schemes.

If f is not constant, then f is quasifinite; hence by [10] it is quasifinite; so in the Stein's decomposition of f:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \overline{X} = \operatorname{Spec} f_*\mathcal{O}_X \arrow[d, "\overline{f}"] \\ & Y \end{tikzcd}$$

i is an open immersion, f is finite (because $f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_Y$- algebra), $i_* \mathcal{O}_X = \mathcal{O}_{\overline{X}}$ (hence i is dominant) and, by Lemma B, $\dim \overline{X} = 1$. Then $\underline{H}^0_Z(X,\mathcal{O}_X) = \underline{H}^1_Z(X,\mathcal{O}_X) = 0$ as the kernel and cokernel of the morphism $\mathcal{O}_{\overline{X}} \longrightarrow i^* i_* \mathcal{O}_X$. It follows that $\operatorname{depth}_Z(\mathcal{O}_{\overline{X}}) \geq 2$ (cf. [8] ), where $Z = \overline{X} - X$. It results $Z = \emptyset$ and so f is proper.

If f is constant, because $f_* \mathcal{O}_X$ is $\mathcal{O}_Y$-coherent and f is dominant, it follows that $f(X)=Y=\operatorname{Spec} k$, where k is a field, and X is an algebraic curve over k such that $\dim_k H^0(X,\mathcal{O}_X) < \infty$. It must prove that X is proper over k.

We may assume that k is algebraically closed. Indeed, let $\overline{k}$ be an algebraic closure of k and $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$. We have $H^0(\overline{X},\mathcal{O}_{\overline{X}}) = H^0(X,\mathcal{O}_X) \otimes_k \overline{k}$, $\dim_{\overline{k}} H^0(\overline{X},\mathcal{O}_{\overline{X}}) = \dim_k H^0(X,\mathcal{O}_X) < \infty$ and X is proper over k iff $\overline{X}$ is proper over $\overline{k}$. But $\overline{X}$ is proper over $\overline{k}$ iff every irreducible component $\overline{X}_i$ of X, with the structure of reduced subscheme, is proper over $\overline{k}$. It remains to prove that $\dim_{\overline{k}} H^0(\overline{X}_i,\mathcal{O}_{\overline{X}_i}) < \infty$. Indeed, let $I_i \subset \mathcal{O}_{\overline{X}}$ be the ideal defining $\overline{X}_i$; in the exact sequence of $\overline{k}$ -spaces:

$$H^0(\overline{X},\mathcal{O}_{\overline{X}}) \longrightarrow H^0(\overline{X}_i,\mathcal{O}_{\overline{X}_i}) \longrightarrow H^1(\overline{X},I_i)$$

we have $\dim_{\overline{k}} H^1(\overline{X},\mathcal{I}) < \infty$ (by Lichtenbaum's Theorem) and $\dim_{\overline{k}} H^0(\overline{X},\mathcal{O}_X) < \infty$; hence $\dim_{\overline{k}} H^0(\overline{X}_i,\mathcal{O}_{\overline{X}_i}) < \infty$

Therefore, we assume that X is an integral 1-dimensional scheme over an algebraically closed field k. Let $p : \tilde{X} \rightarrow X$ be the normalisation morphism of X. In the exact sequence of k-spaces:

$$0 \longrightarrow H^0(X,\mathcal{O}_X) \longrightarrow H^0(\tilde{X},\mathcal{O}_{\tilde{X}}) \longrightarrow H^0(X, p_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$$

we have $\dim_k H^0(X, p_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) < \infty$ (because dim supp $p_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X = 0$) and $\dim_k H^0(X,\mathcal{O}_X) < \infty$

Then $\dim_k H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) < \infty$ and X is proper iff $\tilde{X}$ is proper.

Therefore, we may assume that X is a normal scheme. Then X is either affine or proper scheme over k. Because $\dim_k H^0(X, \mathcal{O}_X) < \infty$, it results that X is proper over k.

In the classical case, when $f:X \to Y$ is a morphism of algebraic varieties over an algebraically closed field, it is easy to prove (see [9]) that f is proper iff f satisfies the equivalent conditions (iv') or (v'). This fact is no longer true for the morphism of schemes; in fact, we have the following

Example 1. Let A be an integral noetherian local ring of dimension 1 and A[T] the algebra of polynomials in one indeterminate over A. Let $\underline{m} \subset A$ be the maximal ideal of A and t a non-zero element of $\underline{m}$; the principal ideal (tT-1) has height 1 and it is maximal because $A[T]/(tT-1)$ is isomorphic to the field of quotients of the ring A. Let us denote $x = (tT-1) \in \operatorname{Spec} A[T]$, $X = \operatorname{Spec} A[T] - \{x\}$, $Y = \operatorname{Spec} A[T]$ and $i: X \hookrightarrow Y$ the open immersion of X into Y. We remark that x is a closed point of codimension 1 in Y and for every integral closed subscheme $C \subset X$, $\dim C = 1$, the set $i(C)$ is closed in Y and hence C is a proper scheme over Y; indeed, if $i(C)$ is not closed in Y, then $x \in \overline{i(C)}$; hence either $\overline{i(C)} = \{x\}$ or $\overline{i(C)} = Y$; in the first case $C = \emptyset$, which is impossible, and in the second case we have $C = X$, which contradicts $\dim X = \dim Y = \dim A[T] = 2$. It follows that the separated morphism of finite type i satisfies the condition (v), but it is not proper.

The following case is an extension of the classical case

Corollary 2. Let $f:X \to Y$ be a separated a morphism of schemes over the field k. Suppose that X is an algebraic scheme over k, and Y is a noetherian scheme. Then f is proper iff the equivalent conditions (iv) or (v) are satisfied.

Proof. As in the proof of Theorem 1, we may suppose that $X$ and $Y$ are integral schemes.

If $\dim X = 0$, then $X$ has only one point $x$ and the point $f(x) = y$ is closed in $Y$. Then it is easy to see that $f$ is proper.

If $\dim X = 1$, the assertion is clear, by Theorem 1.

If $\dim X > 1$ and $f$ is not proper, by noetherian induction, we may assume that every integral closed subscheme $X' \subsetneqq X$ is proper over $Y$. If $X$ were not proper over $Y$ let $\overline{f}: \overline{X} \to Y$ be a dense compactification of $f$; then, by Lemma A, it results that the set $Z = \overline{X} - X$ is a finite set of closed points in $\overline{X}$ of codimension 1, since, as in the proof of Theorem 1, or of Corollary 1, every closed integral subscheme $X' \subsetneqq \overline{X}$ intersecting $X$ is contained in $X$. Then the Corollary 2 is proved if we prove Lemma 2, which follows; by this lemma, we have that $\dim \overline{X} = 1$, which contradicts the assumption that $\dim X > 1$.

We shall anticipate a consequence of Lemma 2 (see Corollary 5, which follows): in the situation given in Corollary 2, we have that $\overline{f(X)} \subset Y$ is a Jacobson scheme. Thus for $f$ the conditions (IV) or (V) are equivalent to (IV') or (V'). Therefore we have the following:

Corollary 3. Let $f: X \to Y$ be a morphism of schemes as in Corollary 2. Then $f$ is proper iff the equivalent condition (IV') or (V') are satisfied.

It remains to prove Lemma 2. We shall use in the proof of Lemma 2 the following result of Nagata ([13], Lemma 4.1):

Lemma 1 - Let $K$ be a function field over the field $k$. Every discrete valuation subring $\mathcal{O} \subset K$, containing $k$ and whose quotient field is $K$, is essentially of finite type.

Lemma 2. Let $\overline{X}$ be a noetherian integral scheme over a field $k$ and $X \subset \overline{X}$ an open nonempty subset of $\overline{X}$. Suppose that:

a) $X$ is algebraically over $k$

b) $\overline{X}-X$ is a nonempty finite set of closed points of codimension 1 in $X$.

Then $\overline{X}$ is an algebraic scheme over $k$ of dimension 1.

Proof. We may suppose that $\overline{X}$ is a normal scheme. Indeed, the normalisation morphism $p : \overline{X}^n \to \overline{X}$ has finite fibres (because $\overline{X}$ is noetherian), $p^{-1}(X)$ is an algebraic scheme over $k$ and $\overline{X}^n - p^{-1}(x)$ is a finite set of closed points of codimension 1 in $\overline{X}$, whose local rings are noetherian (because they are localisations of the integral closures of the rings $\mathcal{O}_{\overline{X},x}$, $x \in \overline{X}-X$, in the field of rational functions; these are noetherian rings because $\dim \mathcal{O}_{\overline{X},x} = 1$); it results that $\overline{X}^n$ is a noetherian scheme. If we prove that $\overline{X}^n$ is an algebraic scheme over $k$, it results that $\overline{X}$ is algebraically over $k$, because $p$ is an integral morphism. We may suppose that $\overline{X}-X$ has only one point $x$ and the local ring $\mathcal{O}_{\overline{X},x}$ is a discrete valuation ring.

By a Nagata's Theorem [13], there is an complete normal algebraic scheme $Y$ over $k$, having the same field of rational functions as $\overline{X}$, such that the discrete valuation ring $\mathcal{O}_{\overline{X},x}$ has a center $y \in Y$ with $\operatorname{codim}_Y \{y\} = 1$. Let $f : \overline{X} \to Y$ be the natural birational map; it results that $f$ is defined in a neighbourhood of $x$. Let $Z \subset Y$ be the topological closure of $y$ in $Y$; by restricting $\overline{X}$ to an open neighbourhood of $x$, we may suppose that $f$ is a morphism, $f(\overline{X}) \cap Z = \{y\}$ while $f|_X : X \to f(x)$ is an isomorphism of algebraic schemes over $k$.

Let us suppose, that $\dim Y > 1$; then the point $y$ is not closed in $Y$ and, by restricting $X$ to an open neighbourhood of $x$, we may suppose that $f$ is injective. Then $f$ is a homeomorphism of $\overline{X}$ onto $f(\overline{X})$; indeed, let $U$ be an open set in $\overline{X}$ which contains the

point $x$; it suffices to prove that there is an open set $V$ in $Y$, such that $f(U)=V\cap f(\overline{X})$; $f(U-\{x\})$ being an open set in $Y$, and $Z$ being a closed set of $Y$ of codimension 1 such that $f(U-\{x\})\cap Z=\emptyset$, it results that $Y-f(U-\{x\})=Z\cup Z_1\cup\ldots\cup Z_n$ with $\{Z_i\}_i$ the irreducible components of $Y-f(U-\{x\})$ different of $Z$; then, the set $V=Y-\bigcup_{i=1}^{n} Z_i$ satisfies the equality $f(U)=V\cap f(\overline{X})$. It follows that we have an isomorphism of ringed spaces $f:\overline{X}\longrightarrow f(\overline{X})$, if we take for $f(\overline{X})$ the structure of topological subspace of $Y$ and the local rings $\mathcal{O}_{f(\overline{X}),z}=\mathcal{O}_{Y,z}$, for every $z\in f(\overline{X})$.

Let $U$ be an affine open subset of $\overline{X}$ containing the point $x$. Then the restriction homomorphism $\rho_x^U:\Gamma(U,\mathcal{O}_{\overline{X}})\longrightarrow \mathcal{O}_{X,x}/m_x=k(x)$ is surjective, because $\{x\}$ is a closed subset in $U$. Passing through the isomorphism $f$ of ringed spaces, we see that the homomorphism $\rho_y^{f(U)}:\Gamma(f(U),\mathcal{O}_{f(\overline{X})})\longrightarrow \mathcal{O}_{Y,y}/\underline{m}_y=k(y)$ is surjective. Because $\Gamma(f(U),\mathcal{O}_{f(\overline{X})})$ is the ring of regular functions on the open subset $f(U)-\{y\}\subset Y$, which are defined on $Z$ (i.e. $Z$ is not present in the polar divisors of these functions) and $k(y)=k(Z)=$ = the function field of the variety $Z$, we have that every rational function on $Z$ is the restriction to $Z$ of a regular function on $f(U)-\{y\}$. If $Y-(f(U)-\{y\})=Z\cup Z_1\cup\ldots\cup Z_m$, $\{Z_i\}_i$ being the irreducible components of $Y-(f(U)-\{y\})$ different of $Z$, it results that for every $g\in k(Z)$ the polar divisor of $g$ in $Z$ has the support included in $\bigcup_{i=1}^{n}(Z_i\cap Z)$. This fact contradicts the supposition that $\dim Z=\dim Y-1>0$.

Therefore $\dim Y=1$ and $\dim.al._k k(Y)=\dim.al._k k(\overline{X})=\dim.al._k k(X)=\dim X=1$; hence $\dim \overline{X}=1$. Then the birational map $f:\overline{X}\longrightarrow Y$, by restricting $\overline{X}$ to an open neighbourhood of $x$, may be reduced, as above, to an isomorphism of ringed spaces of $\overline{X}$ on $f(\overline{X})$, but in this case $f(\overline{X})$ will be an open subset of $Y$, because $\dim Y=1$. Then $\overline{X}$ is an algebraic scheme over $k$ and Lemma 2 is proved.

Remark 3. Corollary 4 applied to the canonical morphism $\pi : X \to \mathrm{Spec}\,\Gamma(X)$ gives the following result, due to J.E. Goodman and L.Landman, obtained using the algebraic convexity [6] :

Let X be an algebraic variety over an algebraically closed field such that $\Gamma(X)$ is noetherian. Then the following assertions are equivalent:

(i) X is a semiaffine variety (i.e. $\pi$ is proper)

(ii) for every closed integral curve $C \subset X$ the restriction homomorphism $\rho^X_C : \Gamma(X) \longrightarrow \Gamma(C)$ is finite.

II

We may remark the following consequence of Lemma 2, which seems to be an extension of a form of the classical Hilbert's "Nullstellensatz ":

Corollary 4. Let $f:X \longrightarrow Y$ be a dominant separated morphism of schemes over a field k. Suppose that X is an algebraic scheme and Y is noetherian. Then for every closed point $y \in Y$, the residual field $k(y)$ is a finite extension of k.

Proof. We may suppose that X and Y are integral schemes. If we choose a dense compactification $\overline{f}:\overline{X} \longrightarrow Y$, then it suffices to prove the following:

(*) for every algebraic scheme X over a field k and every open immersion of X into a noetherian scheme $\overline{X}$ over k, the residual field $k(x)$ of every closed point $x \in \overline{X}-X$ is a finite extension of k.

If $\dim \overline{X}=0$, then $\overline{X}=X$.

If dim $\overline{X}=1$, then $\overline{X}-X$ is a finite set of closed points of codimension 1 and, by Lemma 2, $\overline{X}$ is an algebraic scheme; hence $k(x)$ is a finite extension of $k$ for every $x \in \overline{X}-X$.

If dim $\overline{X} > 1$ and $x \in \overline{X}-X$ is a closed point of $\overline{X}$, by noetherian induction, we may assume that every integral closed subscheme $\overline{Y} \subsetneqq \overline{X}$ passing through $x$, such that $Y=\overline{Y} \cap X \neq \emptyset$ has the property (*) (i.e. $k(y)$ is a finite extension of $k$ for every closed point $y \in \overline{Y} - Y$).

If there is such a subscheme $\overline{Y}$, then, because $x \in \overline{Y}$, it follows that $k(x)$ is a finite extension of $k$. If there is not such a subscheme $\overline{Y}$, then every integral closed subscheme $\overline{Y} \subsetneqq \overline{X}$ passing through $x$ is included in $\overline{X}-X$. Then, by Lemma A, it results that dim $\mathcal{O}_{\overline{X},x} = 1$ and $x$ will be a closed point of dimension 1 in $\overline{X}$. It follows that $\{x\}$ is a connected component of the closed subset $\overline{X}-X$ of $\overline{X}$ and then $X \cup \{x\}$ is an open subscheme of $\overline{X}$. By Lemma 2, applied to the scheme $X \cup \{x\}$ containing the open algebraic scheme $X$, it results that $X \cup \{x\}$ is an algebraic scheme over $k$. Therefore, $k(x)$ is a finite extension of $k$.

<u>Corollary 5.</u> <u>Let</u> $f:X \to Y$ <u>be a separated dominant morphism of schemes over a field</u> $k$. <u>If</u> $X$ <u>is an algebraic scheme over</u> $k$ <u>and</u> $Y$ <u>is noetherian then</u> $Y$ <u>is a Jacobson (or Hilbert) scheme.</u>

For the notion of a Jacobson (or Hilbert) scheme, see the paragraph after the enouncement of Proposition 4.

<u>Proof.</u> It suffices to prove Corollary 5 in the case when $f$ is an open immersion because, if $\overline{f}:\overline{X} \to Y$ is a dense compactification of $f$, and if $\overline{X}$ is a Jacobson scheme, it results that $Y$ is Jacobson scheme (indeed, let $V \subset Y$ be an open affine subset and $\mathfrak{q} \subset \Gamma(V,\mathcal{O}_Y)$ a prime ideal; because $\overline{f}$ is surjective, there is an open affine subset $U \subset \overline{X}$ and a prime ideal $\mathfrak{p} \subset \Gamma(U,\mathcal{O}_{\overline{X}})$ such that

$\Gamma(U,\mathcal{O}_{\bar{X}}) \supset \Gamma(V,\mathcal{O}_Y)$ and $\quad \mathfrak{q} = \mathfrak{p} \cap \Gamma(V,\mathcal{O}_Y)$ ; for every maximal ideal $\underline{m} \subset \Gamma(U,\mathcal{O}_{\bar{X}})$, the ideal $\underline{m} \cap \Gamma(V,\mathcal{O}_Y)$ is maximal in $\Gamma(V,\mathcal{O}_Y)$ because f is proper; because $\Gamma(U,\mathcal{O}_{\bar{X}})$ is a Jacobson ring, $\mathfrak{p}$ is an intersection of maximal ideals of $\Gamma(U,\mathcal{O}_X)$; then it results that $\mathfrak{q}$ is an intersection of maximal ideals of $\Gamma(V,\mathcal{O}_Y)$.

Therefore, we may suppose that f is an open immersion. We shall prove that Y is a Jacobson scheme by noetherian induction on Y.

If dim $Y \leq 1$, by Lemma 2, it results that Y is an algebraic scheme over k; hence it is a Jacobson scheme.

Suppose that dim $Y > 1$ and that every integral closed subscheme $Y' \subsetneq Y$, such that $X' = Y' \cap X \neq \emptyset$, is a Jacobson scheme. If Y is not a Jacobson scheme, then there is an affine open subscheme $U \subset Y$ such that $A = \Gamma(U,\mathcal{O}_Y)$ is not a Jacobson ring; obviously $U \cap \cap (Y-X) \neq \phi$. We may suppose that Y = U is an affine scheme. As in the proof of Proposition 4, a) $\Rightarrow$ b), which follows, there is a prime ideal $\mathfrak{p} \subset A$, such that $A/\mathfrak{p}$ is a semilocal ring of dimension 1. Therefore there is a semilocal closed subscheme Z = Spec $A/\mathfrak{p} \subset$ Y of dimension 1; let x be the generic point of Z and $x_1, \dots x_n$ the closed points of Z. Because X is a Jacobson scheme, then $x \notin X$ and for every i, $x_i \notin X$. Thus $Z \subset Y-X$. We have $\operatorname{codim}_Y Z > 1$; otherwise, $\operatorname{codim}_Y Z = 1$ and Z would be an irreducible component of Y-X; then $X \cup \{x\}$ would be an open subset of Y and x — a closed point of the scheme $X \cup \{x\}$, of codimension 1 in $X \cup \{x\}$; by Lemma 2, dim $X \cup \{x\} = 1$; by Lemma B, it follows dim Y = 1, which contradicts the fact that dim $Y > 1$. Therefore, because $\operatorname{codim}_Y Z > 1$, it results dim $\mathcal{O}_{Y,x} > 1$. Then there is an integral closed subscheme $Y' \subsetneq Y$ passing through x and such that $X' = Y' \cap X \neq \phi$ ; otherwise, every prime ideal $\mathfrak{q}' \subset \mathcal{O}_{Y,x}$ $\mathfrak{q}' \neq 0$ contains one of the prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_m$, corresponding to the irreducible components of Y-X, passing through x; it results dim $\mathcal{O}_{Y,x} = 1$, which contradicts the fact that dim $\mathcal{O}_{Y,x} > 1$. (see Lemma A)

Therefore, by the inductive hypothesis, the subscheme $Y' \subsetneq Y$ passing through x is a Jacobson scheme. If $\mathfrak{q}$ is the ideal of A corresponding to the integral closed subscheme $Y' \subset Y$ then $\mathfrak{p}/\mathfrak{q} \subset A/\mathfrak{q}$ is an intersection of maximal ideals; hence in $A/\mathfrak{p}$ the intersection of the maximal ideals is zero; this contradicts the fact that $A/\mathfrak{p}$ is a semilocal ring of dimension $> 0$.

Corollary 6. Let X be a separated algebraic scheme over a field such that $\Gamma(X)$ is noetherian. Then the following conditions are equivalent :

(i) the canonical morphism $\pi : X \to \operatorname{Spec} \Gamma(X)$ is surjective

(ii) the canonical map $\pi_{max} : X_{max} \to (\operatorname{Spec} \Gamma(X))_{max}$ between the sets of closed points of X and of Spec $\Gamma(X)$ is surjective.

Proof. By Ohi's result (see [15]), (i) $\Longleftrightarrow$ (ii) iff $\Gamma(X)$ is a Jacobson ring.

Remark 4. The problem of the equivalence of (i) and (ii) for the varieties X over algebraically closed fields k was enounced by J.E.Goodman and A.Landman [6].

Corollary 7. Let $i : X \hookrightarrow \overline{X}$ be an open dominant immersion of schemes over a field k such that X is an integral algebraic scheme over k and $\overline{X}$ is noetherian. Then for every closed point $x \in \overline{X}$, dim $\mathcal{O}_{\overline{X},x}$ = dim X = dim $\overline{X}$. More, $\overline{X}$ is a catenarian scheme.

Proof. If x is a closed point of $\overline{X}$, then dim $\mathcal{O}_{\overline{X},x} \geq$ dim X. Indeed, let $x \in \overline{X} - X$. If dim $\mathcal{O}_{\overline{X},x} = 1$, then, by Lemma 2, dim $\overline{X}$

$= \dim X = 1$. If $\dim \mathcal{O}_{\overline{X},x} > 1$, then there is a prime ideal $\mathfrak{p} \subset \mathcal{O}_{\overline{X},x}$ of coheight 1 and such that the corresponding integral subscheme $\overline{Y}_1 \subset \overline{X}$ passing through x has the property $Y_1 = \overline{Y}_1 \cap X \neq \emptyset$; by Lemma 2, $\dim \overline{Y}_1 = 1$, because x is a closed point of codimension 1 in $Y_1$. If $Y_1 \subsetneqq Y_2 \subsetneqq \dots \subsetneqq Y_m = X$ is a saturated chain of integral closed subschemes of X, then we have: $\{x\} \subsetneqq \overline{Y}_1 \subsetneqq \overline{Y}_2 \subsetneqq \dots \subsetneqq \overline{Y}_n = \overline{X}$, where $\overline{Y}_i$ is the closure in $\overline{X}$ of $Y_i$. Hence $\dim \mathcal{O}_{\overline{X},x} \geq n = \dim X$.

We shall prove that for every closed point $x \in \overline{X} - X$, $\dim \mathcal{O}_{\overline{X},x} \geq \dim X$, by noetherian induction on X. If $\dim X = 1$ then $\dim \overline{X} = 1$, and $\dim \mathcal{O}_{\overline{X},x} = 1$. If $\dim X > 1$, we may suppose, by the inductive hypothesis, that for every closed subscheme $\overline{Y} \subset \overline{X}$, such that $Y = \overline{Y} \cap X \neq \emptyset$, we have $\dim Y = \dim \overline{Y}$. We have $\dim \mathcal{O}_{\overline{X},x} > 1$; otherwise, x is a closed point of codimension 1 in $\overline{X}$ and then $X \cup \{x\}$ is an open subscheme of $\overline{X}$; by Lemma 2, it follows $\dim X = 1 = \dim \overline{X}$, which is not possible, because $\dim X > 1$. Let $\underline{m} = \mathfrak{p}_0 \supsetneqq \mathfrak{p}_1 \supsetneqq \dots \supsetneqq \mathfrak{p}_{n-1} \supsetneqq \mathfrak{p}_n = 0$ be a saturated chain of prime ideals of lenght $n = \dim \mathcal{O}_{\overline{X},x}$ and let $\overline{Y}_0 = \{x\} \subsetneqq \overline{Y}_1 \subsetneqq \dots \subsetneqq \overline{Y}_m = \overline{X}$ the corresponding chain of integral closed subschemes of $\overline{X}$. If $Y_{n-1} = \overline{Y}_{n-1} \cap X \neq \emptyset$ then $Y_{n-1} \subsetneqq X$ and so $n \leq \dim X$. If $Y_{n-1} = \overline{Y}_{n-1} \cap X = \emptyset$, then, for every $i < n$, $\overline{Y}_i \subset \overline{X} - X$. Because $n \geq 2$, $\overline{Y}_{n-1} \neq \{x\}$ and it is anirreducible component of $\overline{X} - X$. If $n=2$, then $\dim \mathcal{O}_{\overline{X},x} = 2 \geq \dim X > 1$. Hence $\dim \mathcal{O}_{\overline{X},x} = \dim X$. If $n > 2$, let $y \in \overline{Y}_{n-2}$ be the generic point; then $\dim \mathcal{O}_{\overline{Y},y} \geq 2$ and there is a prime ideal $\mathfrak{p} \neq \mathfrak{p}_{n-1}$ such that $\mathfrak{p}_{n-2} \supsetneqq \mathfrak{p} \supsetneqq \mathfrak{p}_n = 0$. Then the corresponding integral closed subscheme $\overline{Y} \subset \overline{X}$ passing through x, has the property that $Y = \overline{Y} \cap X \neq \emptyset$. Hence $\dim \overline{Y}_{n-2} < \dim Y < \dim X$, and so $\dim \mathcal{O}_{\overline{X},x} = n \leq \dim X$

We shall prove that $\overline{X}$ is a catenarian scheme. It suffices to see that, if x is a closed point of $\overline{X} - X$ and $0 = \mathfrak{p}_n \subsetneqq \dots \subsetneqq \mathfrak{p}_0 = \underline{m}$ is a saturate chain of prime ideals of $\mathcal{O}_{\overline{X},x}$, then $n = \dim \overline{X}$. We shall prove it by induction on $\dim \overline{X}$. Let $\overline{X} = Y_n \supsetneqq Y_{n-1} \supsetneqq \dots \supsetneqq$

$\overline{Y}_1 \supsetneqq \overline{Y}_0 = \{x\}$ be the corresponding chain of integral closed subschemes.

If $\dim \overline{X} \leq 2$, it is clear that $n = \dim \overline{X}$. If $\dim \overline{X} > 2$, suppose, by the inductive hypothesis, that every integral closed subscheme $\overline{Y} \subset \overline{X}$, such that $Y = \overline{Y} \cap X \neq \emptyset$, is a catenarian scheme. There are two possibilities:

a) $\overline{Y}_{n-1}$ is included in $\overline{X}-X$. If $y$ is the generic point of $\overline{Y}_{n-2}$, then in the local ring $\mathcal{O}_{\overline{X},y}$ there is a saturated chain $0 \subsetneqq \mathfrak{p}_{n-1}\mathcal{O}_{\overline{X},y} \subsetneqq \mathfrak{p}_{n-2}\mathcal{O}_{\overline{X},y}$; by a result of McAdam [1], there are infinitely many prime ideals $\mathfrak{p} \subset \mathcal{O}_{\overline{X},y}$, such that $\operatorname{ht} \mathfrak{p} = \operatorname{cht} \mathfrak{p} = 1$. We may choose such a prime ideal with the property $\mathfrak{p} \neq \mathfrak{p}_{n-1}\mathcal{O}_{\overline{X},y}$. Then the corresponding closed subscheme $\overline{Y}$, has the property $\overline{X} \supsetneqq \overline{Y} \supsetneqq \overline{Y}_{n-2}$ and $\overline{Y} \neq \overline{Y}_{n-1}$. Because $\overline{Y}_{n-1}$ is an irreducible component of $\overline{Y}$, we have $Y = \overline{Y} \cap X \neq \emptyset$. By the inductive hypothesis, $\overline{Y}_0 \subsetneqq \overline{Y}_1 \subset \cdots \subsetneqq \overline{Y}_{n-2} \subsetneqq \overline{Y}$ being a saturated chain in $Y$, we have $\dim \overline{Y} = n-1$ and, because $\overline{Y}$ is of codimension 1 in $\overline{X}$, we have $\dim \overline{X} = n$.

b) $Y_{n-1} = \overline{Y}_{n-1} \cap X \neq \emptyset$. Then we may apply the inductive hypothesis to $\overline{Y}_{n-1}$; because $\overline{Y}_0 \subsetneqq \overline{Y}_1 \subsetneqq \cdots \subsetneqq \overline{Y}_{n-2} \subsetneqq \overline{Y}_{n-1}$ is a saturated chain, then $n-1 = \dim \overline{Y}_{n-1} = \dim \overline{X} - 1$. Thus $n = \dim \overline{X}$.

Corollary 7 is proved.

We shall give another consenquence of Lemma 2.

Proposition 1. Let $f: X \longrightarrow Y$ be a dominat morphism of schemes over an algebraically closed field k. Suppose that X is a connected separated algebraic scheme over k and Y is noetherian of dimension $\geq 0$. Then for every closed points $y, y' \in Y$ there is a connected closed reduced subscheme $C \subset Y$ of dimension 1 such that $y, y' \in C$. More, C can be taken an algebraic scheme over k.

Proof. We may suppose that X and Y are integral schemes and $\dim Y > 0$. Choosing a dense compactification $\overline{f}: \overline{X} \to Y$ of f, it suffices to prove the following:

(**) for every open immersion $i:X \hookrightarrow \overline{X}$ of a separated algebraic scheme $X$ over an algebraically closed field $k$ into a noetherian scheme over $k$ and for every closed point $x \in \overline{X}-X$, there is an integral algebraic closed subscheme $\overline{Z} \subset \overline{X}$, such that $x \in \overline{Z}$, $\overline{Z}=Z \cap X \neq \emptyset$ and $\dim \overline{Z}=1$.

Indeed, if (**) is true, then for every closed points $y,y' \in Y$ there are two closed points $x,x' \in \overline{X}$ such that $f(x)=y$ and $f(x')=y'$, and a connected closed algebraic subscheme $\overline{Z} \subset \overline{X}$ of dimension 1, such that $x,x' \in \overline{Z}$. Then $C=\overline{f}(\overline{Z}) \subset Y$, with the reduced structure of subscheme, is algebraic over $k$, because $\overline{f}|_{\overline{Z}}: \overline{Z} \to C$ is a finite morphism and so $C$ is a connected scheme of dimension 1.

Let us prove (**) by noetherian induction on $\overline{X}$.

If $\dim \overline{X}=1$, then $\overline{X}$ is algebraic scheme over $k$, by Lemma 2.

If $\dim \overline{X}>1$, we may assume by the induction hypothesis that every integral closed subscheme $\overline{X}' \subsetneq \overline{X}$ such that $x \in \overline{X}'$ and $X'=\overline{X}' \cap X \neq \emptyset$ has the property (**) (i.e. there is an integral algebraic closed subscheme $\overline{Z} \subset X'$, such that $x \in \overline{Z}$, $Z = \overline{Z} \cap X \neq \emptyset$ and $\dim Z = 1$). If there is an integral closed subscheme $\overline{X}' \subsetneq \overline{X}$ such that $x \in \overline{X}'$ and $X' = \overline{X}' \cap X \neq \emptyset$, we find a closed integral subscheme $Z$ of $\overline{X}'$, hence of $\overline{X}$, with the required properties. If there is not such a subscheme, then, by Lemma A, it results that $\dim \mathcal{O}_{\overline{X},x} = 1$. Because $x$ will be a closed point of codimension 1, $\{x\}$ is a connected component of $\overline{X}-X$ and so $X \cup \{x\}$ is an open subscheme of $\overline{X}$. By Lemma 2, applied to the noetherian scheme $X \cup \{x\}$, containing the algebraic subscheme $X$ and the closed point $x$ of codimension 1 in $X \cup \{x\}$, it results $\dim X=1$ and so $\dim \overline{X}=1$ (by Lemma B). This contradicts the fact that $\dim \overline{X}>1$.

Proposition 1 is proved.

Corollary 8. a) Every integral noetherian subalgebra of dimension 1 of an algebra of finite type over an algebraically closed field k is of finite type over k.

b) Let X be a separated algebraic scheme over an algebraically closed field k with $\Gamma(X)$ noetherian. If dim $\Gamma(X) = 1$, then it is an algebra of finite type over k.

## III

First of all, we shall discuss the following problem; if Z is a noetherian scheme, what conditions must carry out Z, such that every separated morphism $f: X \rightarrow Y$ of schemes of finite type over Z, verifying the equivalent conditions (iv) or (v), is proper.

We shall introduce the following:

Definition 1. A ring A is called universally 1-equicodimensional if it is a noetherian ring and if every integral A - algebra of finite type, which has a maximal ideal of height 1, has dimension 1.

A scheme X is called universally 1-equicodimensional if there is a finite covering $(U_i)_{i \in I}$ with affine open subsets of X, such that, for every $i \in I$, the ring $\Gamma(U_i, \mathcal{O}_X)$ is universally 1 - equicodimensional.

Example 2. It is obvious that every k - algebra of finite type over a field k is universally 1 - equicodimensional.

Then every algebraic scheme over a field k is universally 1-equicodimensional. Later, we shall give other (some of these more general) examples of such objects.

Example 3. A noetherian semilocal ring A, of dimension $> 0$, is not universally 1 - equicodimensional. Indeed, because there is $f \in A$, $f \neq 0$, such that the ring of quotients $A_f$ is local, we may suppose A local ring. Then in the polynomial algebra $A[T]$, the principal ideal $\underline{m}_t = (tT - 1)$ is maximal for every non-zero element $t$ of the maximal ideal of A; then $ht\ \underline{m}_t = 1$, but dim $A[T] = 2$.

First of all, we shall give some elementary properties of the universally 1 - equicodimensional schemes:

Proposition 2. a) *If X is universally 1 - equicodimensional scheme, then, for every open affine subset $U \subset X$, the ring $\Gamma(U, \mathcal{O}_X)$ is universally 1 - equicodimensional.*

b) *Every scheme of finite type over an universally 1-equicodimensional scheme is universally 1 - equicodimensional (more particular, every locally closed subscheme of such a scheme or every fibred product of two schemes of finite type over such a scheme is universally 1 - equicodimensional).*

c) *X is universally 1 - equicodimensional if and only if every irreducible component of X, with the reduced structure of subscheme, is universally 1 - equicodimensional.*

d) *If $f: X \to Y$ is a surjective proper morphism (resp. surjective finite morphism, surjective integer morphism) of schemes with Y noetherian (resp. Y any scheme, Y noetherian) and X an universally 1-equicodimensional scheme, then Y is an universally 1-equicodimensional scheme.*

<u>Proof.</u> a) Because every ring of quotients of the type $A_f$ is universally 1-equicodimensional if A is such a ring, by Definition 1 it results that X has a topological basis $(U_i)_{i\in I}$ with affine open subsets of X, such that for every $i \in I$, $\Gamma(U_i, \mathcal{O}_X)$ is an universally 1-equicodimensional ring. For every open affine set $U \subset X$ and every integral $\Gamma(U, \mathcal{O}_X)$ - algebra of finite type A which contains a maximal ideal $\underline{m}$ of height 1, from the canonical morphism of affine schemes $\rho_{A,U}$: Spec $A \longrightarrow U$, there is $i \in I$ such that $\rho_{A,U}(\underline{m}) \in U_i$; then $\underline{m} \in \rho^{-1}_{A,U}(U_i)$ and $\rho^{-1}_{A,U}(U_i)$ is an affine scheme universally 1-equicodimensional, because it is of finite type over $U_i$ (see b)). Then dim $\rho^{-1}_{A,U}(U_i) = 1$. Hence dim Spec A= = dim A = 1 (by Lemma B), because X is a Jacobson scheme (see Proposition 4).

b) results from Definition 1 and from the fact that every algebra of finite type over an universally 1-equicodimensional ring is universally 1 - equicodimensional.

c) From b) it results that every irreducible component of X, with the reduced structure, is universally 1 - equicodimensional if X is so. Conversely, let U be an affine open subset of X and A an integral algebra of finite type over $\Gamma(U, \mathcal{O}_X)$, containing a maximal ideal $\underline{m}$ of height 1; then A is an algebra of finite type over $\Gamma(U, \mathcal{O}_X)_{red} = \Gamma(U, \mathcal{O}_{X_{red}})$. Let the canonical morphism of affine schemes be: $\rho_{A,U}$ : Spec $A \longrightarrow U_{red}$ and let $X_1, \ldots, X_n$ be the integral components of X; because A is integral and $U = \bigcup_{i=1}^{n}(X_i \cap U)$, there is an i, $1 \leq i \leq n$, such that $\rho_{A,U}$ factors through $X_i \cap U$. Then A is a $\Gamma(X_i \cap U, \mathcal{O}_{X_i})$ - algebra of finite type and, because $X_i$ is universally 1-equicodimensional scheme, it results dim A=1. Hence $\Gamma(U, \mathcal{O}_X)$ is an universally 1-equicodimensional ring and, by Definition 1, it follows that X is an universally 1-equicodimensional scheme.

d) Suppose that f is a surjective proper morphism with Y noetherian and X universally 1-equicodimensional. We may suppose that Y is an affine scheme. Because for every $n\geq 0$, $f_n$: $X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T_1,\dots,T_n] \to \mathrm{Spec}\, Y \times_{\mathrm{Spec}\,\mathbb{Z}[T_1,\dots,T_n]}$ is a surjective proper morphism and $X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T_1 \dots T_n]$ is universally 1-equicodimensional, it suffices to prove that, if $Z \subset Y$ is an integral closed scheme and $z \in Z$ is a closed point such that $\dim \mathcal{O}_{Z,z} = 1$, then $\dim Z=1$. Let z be such a point; f being a surjective proper morphism, there is an irreducible component W of $f^{-1}(Z)$, such that $f|_W : W \to Z$ is surjective. (We consider W with the reduced structure as a subscheme). Let w be a closed point of W, such that $f(w)=z$ and $\mathfrak{p} \subset \mathcal{O}_{W,w}$ a prime ideal of coheigt 1, such that by the canonical injective homomorphism $(f|_W)^*: \mathcal{O}_{Z,z} \to \mathcal{O}_{W,w}$ we have $(f|_W)^*(\underline{m}_z) \not\subset \mathfrak{p}$. If C is the closed subscheme of X passing through w, which corresponds to the prime ideal $\mathfrak{p}$, by b) it results that $\dim C = 1$. Then $f(C)$ is closed, contained in Z and contains strictly the set $\{z\}$. Hence $f(C) = Z$, because z is closed of codimension 1 in Z. The morphism $f|_C : C \to Z$ is finite and surjective. It follows: $\dim Z = \dim C = 1$.

If f is a surjective finite morphism, it results Y noetherian, by Nagata-Eakin Theorem . In this case and in the case when f is a surjective integer morphism, the proofs are similar with the above one. Some other direct proofs, based on definitions, can be given.

In the following, we shall characterize in more "geometrical" terms the universally 1-equicodimensional schemes, algebraically defined above, and we shall prove that these are the solution of the problem in discussion (see c) of the following proposition).

Proposition 3. Let Z be a noetherian scheme. The following conditions are equivalent:

a) $Z$ is universally 1-equicodimensional.

b) for every integral scheme $X$ of finite type over $Z$ of dimension $> 0$ and for every closed point $x \in X$, the set of closed points $x' \in X$ such that there is an integral closed subscheme $C \subset X$ (resp. a connected closed subscheme) of dimension 1, passing through $x$ and $x'$, is dense in $X$.

c) for any schemes $X, Y$ of finite type over $Z$ and every separated morphism $f: X \to Y$ over $Z$, $f$ is proper iff the equivalent conditions (iv') or (v') are satisfied.

d) for any schemes $X, Y$ of finite type over $Z$ and every separated morphism $f: X \to Y$ over $Z$, $f$ is proper iff is satisfied the following condition:

(vi) $f$ is a closed morphism with proper fibres and for every integral closed subscheme $C \subset X$ of dimension 1, noncontained in some fibre of $f$, the morphism $f|_C : C \to f(C)$ is finite.

e) for any schemes $X, Y$ of finite type over $Z$ and every separated morphism $f: X \to Y$ over $Z$, $f$ is finite iff every integral closed subscheme $C \subset X$ of dimension 1 is finite over $Y$.

Proof. a) $\Rightarrow$ b) We shall proceed by noetherian induction on $X$. If $\dim X = 1$, b) is satisfied. Suppose that $\dim X > 1$ and that every integral closed subscheme $X' \subsetneqq X$ has the property b). If there is a closed point $x \in X$, such that the set $Y$ of closed points $x' \in X$, which may be "joined" to $x$ by an integral closed subscheme of dimension 1, is not dense in $X$, then there is an open nonempty subset $U \subset X$, such that $U \cap Y = \emptyset$; hence $x \notin U$ and every integral closed subscheme $X' \subsetneqq X$ passing through $x$ has the property $X' \cap U = \emptyset$. It results, by Lemma A, that $\dim \mathcal{O}_{X,x} = 1$; by a), it follows that $\dim X = 1$, which contradicts the supposition $\dim X > 1$.

b) $\Rightarrow$ a) Let U be an affine open subset of Z and A an integral $\Gamma(U, \mathcal{O}_Z)$-algebra of finite type, which has a maximal ideal $\underline{m} \subset A$ of height 1. Then X=Spec A is a scheme of finite type over Z and $x=\underline{m}$ is a closed point of codimension 1 in X. If $Y = \{x' \in X \mid x'$ closed point such that there is a connected closed subscheme of dimension 1 "joing" x to $x'\}$, we have $x \in Y$. The equality: $Y = \{x\}$ is not possible (because, by b), $\{x\}$ would be dense in X and then it results that A is a field); so $Y \supsetneqq \{x\}$ and then dim X = 1 (because if C is an integral closed subscheme of dimension 1 passing through x, then C = X from the fact that $\operatorname{codim}_X \{x\} = 1$).

b) $\Rightarrow$ c) We may suppose that X and Y are integral schemes.

If dim X = 0, then X = Spec K with K a field and we may suppose that Y and Z are affine schemes; hence Y = Spec A and Z = Spec B; A is a B-algebra of finite type universally 1-equicodimensional. By Proposition 4 (which follows), it results that A is a Jacobson (or Hilbert) ring; then, because there is a prime ideal $\mathfrak{p} \subset A$, such that $K \supset A/\mathfrak{p}$ and K is a $A/\mathfrak{p}$ - algebra of finite type, it results that $A/\mathfrak{p}$ is a field and K is a finite extension of $A/\mathfrak{p}$ ; hence f is proper.

If dim X > 0, suppose that f is not proper. Lef $\bar{f} : \bar{X} \to Y$ be a dense compactification of f and $x \in \bar{X}-X$ a closed point; by b), it results that there is a connected closed subscheme $\bar{C} \subset \bar{X}$ of dimension 1, such that $x \in \bar{C}$ and $C = \bar{C} \cap X \neq \emptyset$. Because C is proper over Y, it results $C = \bar{C}$, which contradicts the fact $x \notin X$.

c) $\Rightarrow$ a) If Z is not an universally 1-equicodimensional scheme, then there exists an affine open subset $U \subset Z$ and an integral algebra A of finite type over $\Gamma(U, \mathcal{O}_X)$ such that dim A > 1 and A has a maximal ideal $\underline{m} \subset A$ of height 1. Let Y = Spec A, $X = Y - \{\underline{m}\}$ and $f : X \hookrightarrow Y$ the canonical open immersion . Then X and Y are schemes of finite type over Z and for every integral closed subscheme $C \subset X$ of dimension 1, $f(C)$ is closed in $Y$. Condition (v') is satisfied, but $f$ is not proper.

c)$\Leftrightarrow$d). It results easily, because (vi)$\Rightarrow$(v) and every proper morphism is closed.

c)$\Rightarrow$e) Indeed, if every closed integral subscheme $C \subset X$ of dimension 1 is finite over $Y$, by c), it results that $f$ is proper and it is easy to see that the dimension of all fibres of $f$ is zero; hence $f$ is finite.

e)$\Rightarrow$a). In the same manner as c)$\Rightarrow$a).

Proposition 3 is proved.

<u>Proposition 4</u>. <u>Let</u> $Z$ <u>be a noetherian scheme. The following assertions are equivalent</u>:

a) $Z$ <u>is universally</u> <u>1-equicodimensional</u>.

b) $Z$ <u>is a Jacobson (or Hilbert) scheme and every integral scheme</u> $X$ <u>finite over</u> $Z$, <u>which has a closed point of codimension</u> 1, <u>is of dimension</u> 1.

c) $Z$ <u>is a Jacobson (or Hilbert) scheme and for every integral scheme</u> $X$ <u>finite over</u> $Z$ <u>and for every closed point</u> $x \in X$, <u>the set of closed points</u> $x' \in X$ <u>such that there exists an integral closed subscheme</u> $C \subset X$ (<u>resp. connected closed subscheme</u> $C \subset X$) <u>of dimension</u> 1, <u>passing through</u> $x$ <u>and</u> $x'$, <u>is dense in</u> $X$.

Recall that a Jacobson ring is a ring with the property that every prime ideal is an intersection of maximal ideals (see [4] or [17]); it seems that in other terminology these are called Hilbert rings. In an obvious manner, it defines the "Jacobson (or Hilbert) schemes: a scheme $X$ is called Jacobson (or Hilbert) scheme if there is a covering $(U_i)_{i \in I}$ with open affine subsets such that $\Gamma(U_i, \mathcal{O}_X)$ is a Jacobson (or Hilbert) ring for every $i \in I$. Then, for every open affine subset $U$, the ring $\Gamma(U, \mathcal{O}_X)$ is a Jacobson ring (see EGA IV).

Proof. a) $\Rightarrow$ b) It suffices to prove that Z is a Jacobson scheme ; then there exists an open affine subset $U \subset Z$, such that the ring $A = \Gamma(U, \mathcal{O}_Z)$ is not a Jacobson ring; there is a prime ideal $\mathfrak{p}_1 \subset A$ such that $\mathfrak{p}_1$ is not the intersection of maximal ideals which contain it; there are two possibilities: either $A/\mathfrak{p}_1$ is of dimension 1, or there exists a prime ideal $\mathfrak{p}_2 \supsetneqq \mathfrak{p}_1$ such that $\mathfrak{p}_2/\mathfrak{p}_1$ has the height 1 in the ring $A/\mathfrak{p}_1$ and $\mathfrak{p}_2/\mathfrak{p}_1$ is not the intersection of the maximal ideals of $A/\mathfrak{p}_1$ which contain it. In the first case $A/\mathfrak{p}_1$ is a semilocal ring (otherwise, in the 1-dimensional ring $A/\mathfrak{p}_1$, there exists an infinite family of maximal ideals and the intersection of the maximal ideals is non-zero; this is not possible); then $A/\mathfrak{p}_1$ is not universally 1-equicodimensional (see Example 3) and this contradicts the fact that A is such a ring (by a) and Prop.1 a)). Repeating this procedure, we find an infinite sequence of prime ideals $\mathfrak{p}_1 \subsetneqq \mathfrak{p}_2 \subsetneqq \cdots \subsetneqq \mathfrak{p}_i \subsetneqq \cdots$ such that $A/\mathfrak{p}_i$ is not of dimension 1, for every $i \geqslant 1$, which contradicts the fact that A is a noetherian ring.

b) $\Rightarrow$ a) We may suppose that Z is an affine scheme; suppose Z= =Spec A, with A a noetherian ring. Let B be an integral A-algebra of finite type, such that it has a maximal ideal $\underline{n}$ of height 1. We have to prove that dim B=1. If $\mathfrak{p} \subset A$ is the kernel of the canonical morphism of A-algebras: $A \to B$, because Spec $A/\mathfrak{p}$ has the property b), we may suppose $A \subset B$. Let $f: X = \operatorname{Spec} B \to Z = \operatorname{Spec} A$ be the coresponding morphism of affine schemes and $U \subset X$ the set of point $x \in X$ isolated in $f^{-1}f(x)$; because $\underline{m} \in \operatorname{Spec} B$ is closed of codimension 1, it results that $\underline{m} \in U$ and the open set U is nonempty. By Zariski's Main Theorem [EGA III], there is an finite scheme Y over Z and an open immersion $U \hookrightarrow Y$ over Z. Because $U \hookrightarrow Y$ is a morphism of finite type of Jacobson schemes, $\underline{m}$ is closed in Y and it is of codimension 1 in

Y. By b), it results dim Y=1 and then dim U=1. Hence, dim Spec B= dim B=1.

a) $\Rightarrow$ c) Z is Jacobson because a) $\Rightarrow$ b). Because every closed subscheme of X is finite over Y. We proceed by noetherian induction on X, as in the proof of Proposition 3, a) $\Rightarrow$ b).

c) $\Rightarrow$ b) as in the proof of Proposition 3, b) $\Rightarrow$ a), taking X finite over Z.

<u>Example 4</u>. Let A be a ring. Then:

0) if dim A = 0, A is universally 1-equicodimensional iff A is an artinian ring.

1) if dim A=1, A is universally 1-equicodimensional iff A is a noetherian Jacobson ring.

2) if dim A=2 and if A is a normal ring, then A is universally 1-equicodimensional iff it is a noetherian Jacobson ring with all maximal ideals of heitgt 2.

3) a noetherian k-algebra A over a field k, which is integral over a k-algebra of finite type is universally 1-equicodimensional.

<u>Proof.</u> 0) and 1) result from Prop.3,b).

2) If B is an integral finite A-algebra with the maximal ideal $\underline{m} \subset B$ of height 1, and if $\mathfrak{p}$ is the kernel of the canonical morphism of A-algebras $\rho : A \longrightarrow B$, then $\mathfrak{p}$ can not be zero because, otherwise, $\underline{n} = \rho^{-1}(\underline{m})$ would be of height 1 (A is normal!); but $\underline{n}$ is a maximal ideal, because B is finite over A, which contradicts the hypothesis of 2). Hence $\mathfrak{p} \neq 0$. Then ht $\mathfrak{p}$ = 1 and dim $A/\mathfrak{p}$ = 1. Because B is finite over $A/\mathfrak{p}$ and contains it, it results dim B=1. The conditions of Prop.3,b) are satisfied.

3) We may suppose that A is an integral ring.

A is a Jacobson ring ( see [4] ). Let $B \subset A$ be a k-subalgebra of finite type, such that A is integral over B, and C an integral finite A-algebra which contains a maximal ideal $\underline{m}$ such that ht $\underline{m}$=1. We may suppose that C contains A. If $C'=C\otimes_B B'$, B' is the integral closure of B, then C' is integral over B' and C. If $\underline{m}'\subset C'$ is a maximal ideal which lies over $\underline{m}$ and $\mathfrak{p}\subset C'$ is a minimal prime ideal such that $\mathfrak{p}\subset\underline{m}'$, then ht $\underline{m}'/\mathfrak{p}$ = 1 in $C'/\mathfrak{p}$ and $\mathfrak{p}$ lies over the zero ideal in B' and C'. It follows, that $\underline{m}'/\mathfrak{p}$ lies over a maximal ideal of height 1 of B', because B' is normal. It results that dim B'= 1 (B' is an algebra of finite type over k) and then dim B=1=dim C. Hence A is universally 1-equicodimensional.

For the universally japanese schemes, we have the following properties:

Corollary 9. 1) Let Z be an universally japanese scheme. Then the following conditions are equivalent:

a) Z is universally 1-equicodimensional.

b) for every integral closed subscheme $Z'\subset Z$, if the normalisation scheme $\overline{Z}'$ of Z' has a closed point of codimension 1, then dim $\overline{Z}'$ = 1.

2) Let $f:X\longrightarrow Y$ an integer morphism of schemes such that Y is universally japanese scheme and X is noetherian. If Y is universally 1-equicodimensional, then X is universally 1-equicodimensional.

Proof. 1) a) $\Rightarrow$ b) is clear because $\overline{Z}'$ is of finite type over Z'. For b) $\Rightarrow$ a) we apply Proposition 4, b) $\Rightarrow$ a): if X is an integral dominant finite scheme over a closed integral subscheme $Z'\subset Z$, which contains a closed point of codimension 1, then $X\times_{Z'}\overline{Z}'$ is integral over $\overline{Z}'$ and contains a closed point of codimension 1; $\overline{Z}'$ being a normal scheme, it contains also a such point; by b) it follows dim $\overline{Z}'$ = 1 and so dim $Z'$=1=dim X

2) The same proof as for Example 4, 3).

Corollary 10. Let Z be an universally catenarian scheme.

Then the following are equivalent:

a) Z is universally 1-equicodimensional

b) Z is a Jacobson scheme.

Proof. Let Z be a noetherian universally catenarian Jacobson scheme and X an integral finite scheme over Z which contains a closed point x of codimension 1. If $f:X \to Z$ is the canonical morphism of Z-schemes, by EGA IV, Prop.5.6.10, we have $\dim \overline{f(X)} = \dim \mathcal{O}_{Z,x} = 1$; f being a finite morphism, it follows dim X=1. By Prop.4 (b) ⇒ a)), it results that Z is universally 1-equicodimensional.

Corollary 11. Let $f:X \longrightarrow Y$ be a separated morphism of finite type. Suppose that X is a Jacobson scheme and Y is a noetherian universally catenarian scheme. Then f is proper iff f satisfies the equivalent conditions (iv) or (v).

Proof. As in the proof of Corollary 1 or Corollary 2, it suffices to prove that if $\overline{f}:\overline{X} \longrightarrow Y$ is a dense compactification of f, such that $\overline{X}-X$ is a finite set of closed points of codimension 1, then $\dim \overline{X}=1$. Indeed, because X is a Jacobson scheme, then it is easy to verify that $\overline{X}$ is a Jacobson scheme. By hypothesis, Y is universally catenarian; hence $\overline{X}$ is universally catenarian. By Corollary 11, it results that $\overline{X}$ is universally 1-equicodimensional. Thus, it follows that $\dim \overline{X}=1$.

Corollary 12. Let A be a universally 1-equicodimensional ring and $S \subset A$ a multiplicative system such that for every maximal ideal $\underline{m} \subset A_S$, $\underline{m} \cap A$ is a maximal ideal. Then the following conditions are equivalent:

a) $A_S$ is universally 1-equicodimensional.

b) $A_S$ is a Jacobson ring.

Proof. a) $\Rightarrow$b) by Proposition 4. We shall prove b)$\Rightarrow$a). We may suppose A integral. Let B be a finite integral A-algebra, such that there exists a maximal ideal $\underline{m} \subset B$ of height 1. It is easy to see that there is a non-zero $f \in A$, and a finite $A_f$-subalgebra $B' \subset B$ such that $B = B' \otimes_{A_f} A_S$. Then $\underline{n} = B' \cap \underline{m}$ is a prime ideal of height 1 and it is maximal because, if $\varphi: A_f \longrightarrow B'$ is the canonical morphism of rings, $\varphi^{-1}(\underline{n})$ is maximal. Hence dim $(A_f/\ker f) = 1$, because $A_f$ is universal 1-equicodimensional. Then dim $(A_S/\ker \varphi_S) = 1$. Therefore A is universally 1-equicodimensional.

Example 5. Every Jacobson k-algebra essentially of finite type over a field k is universally 1-equicodimensional.

Indeed, a such k-algebra is universally catenarian; then the assertion follows from Corollary 10.

Proposition 5. Let X be an integral universally 1-equicodimensional scheme. If there is a nonempty open subset $U \subset X$ such that U is a catenarian scheme and for every closed point $x \in U$, dim $\mathcal{O}_{U,x}$ = dim U, then X is a catenarian scheme and for every closed point $x \in X$ dim $\mathcal{O}_{X,x}$ = dim X = dim U.

Proof. The proof is like that of Corollary 7.

Remark 5. Let us call a scheme X universally equicodimensional if for every integral scheme Y of finite type over X and every closed point $y \in Y$, dim $\mathcal{O}_{Y,y}$ = dim Y. In Proposition 5, if $U$ is catenarian and universally equicodimensional, then $X$ is catenarian and universally equicodimensional.

The proof is left to the reader.

Corollary 13. Let $f:X \to Y$ be a separated morphism of finite type. Suppose that $Y$ is an universally 1-equicodimensional scheme and $X$ is an integral catenarian scheme with the property that for every closed point $x \in X$, $\dim \mathcal{O}_{X,x} = \dim X$. Then, for every dense compactification $\bar{f}:\bar{X} \to Y$ of $f$, $\bar{X}$ has the above properties of $X$.

Proposition 6. a) If $f : X \to Y$ is a faithful flat morphism of schemes, $X$ is universally 1-equicodimensional and $Y$ Jacobson, then $Y$ is universally 1-equicodimensional.

b) Let $f : X \to Y$ be a surjective separated morphism of schemes over an universally 1-equicodimensional ring $k$, such that $X$ is of finite type over $k$ and $Y$ is a noetherian Jacobson scheme. Suppose that for every closed subscheme $X' \subset X$ of dimension $> 1$ and for every closed point $y \in Y$, the closed subset $X' \cap f^{-1}(y)$ is either empty or of dimension $> 0$. Then $Y$ is universally 1-equicodimensional.

Proof. a) We may suppose that $Y$ is an affine scheme. Let $Y'$ be an integral finite scheme over $Y$, which has a closed point $y \in Y'$, such that $\dim \mathcal{O}_{Y',y} = 1$; $Y'$ being a closed subscheme of $Y \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T_1,\dots,T_n]$ for any $n \geq 0$ and $f_n : X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T_1,\dots,T_n] \to Y \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{Z}[T_1,\dots,T_n]$ being faithful flat, we may suppose that $Y'$ is an integral closed subscheme of $Y$. We must prove that $\dim Y' = 1$.

Because $f$ is a flat surjective morphism, there is an irreducible component $X'$ of $f^{-1}(Y')$ such that the morphism $f|_{X'} : X' \to Y'$ is dominant and $y \in \mathrm{Im}(f|_{X'})$. (We consider $X'$ with the structure of reduced subscheme). We may choose a closed point $x \in X'$, such that $(f|_{X'})(x) = y$ and $\mathfrak{p} \subset \mathcal{O}_{X',x}$ a prime ideal of coheight 1, such that

$(f|_{X'})^*(\underline{m}_x) \notin \underline{p}$ (here, $(f|_{X'})^*$ is the canonical injective homomorphism $\mathcal{O}_{Y',y} \to \mathcal{O}_{X',x}$ and $\underline{m}_x$ is the maximal ideal of $\mathcal{O}_{X',x}$; then the integral closed subscheme $C \subset X'$ passing through x and corresponding to $\underline{p} \subset \mathcal{O}_{X',x}$, is of dimension 1 and $f(C) \supsetneq \{y\}$. Because y is a closed point of codimension 1 in Y', we have $\overline{f(C)} = Y'$. Then dim Y'=1, because, if $g: \overline{C} \to Y'$ is a dense compactification of $f|_C$, by Lemma B, we have $\dim C = \dim \overline{C} = 1$ and g is a finite morphism.

b) As in a), we must prove that if $Y' \subset Y$ is an integral closed subscheme, which has a closed point $y \in Y'$ of codimension 1, then dim Y' = 1. We shall consider two cases:

1) There is an irreducible component X' of $f^{-1}(Y')$ of dimension > 0, such that $y \in f(X')$ and X' is not contracted by f to y. Then the proof is similar to that of a).

2) If every irreducible component X' of $f^{-1}(Y')$, such that $y \in f(X')$, is contracted by f to y, we shall prove that dim Y'=1 in the following manner: let $\overline{f}: \overline{X} \to Y$ be a dense compactification of f with $\overline{X}$ an integral scheme; there is an integral closed subscheme, $\overline{X}' \subset \overline{X}$ such that $\overline{f}(\overline{X}') = \overline{Y}'$; obviously, if $x \in \overline{X}'$ is a closed point such that $\overline{f}(x) = y$, then $x \in \overline{X}' - X'$ (here $X' = \overline{X}' \cap X \neq \emptyset$). Suppose that $\overline{X}'$ is a minimal integral closed subscheme such that $X' = \overline{X}' \cap X \neq \emptyset$ and $f(\overline{X}') = Y'$.

If $\dim \overline{X}' = 1$, then $\overline{X}'$ is finite over Y'; hence dim Y' = 1.

If $\dim \overline{X}' > 1$, let $Z = (f|_{\overline{X}'})^{-1}(y) \subset \overline{X}' - X' = \widetilde{X}'$; by the minimality of $\overline{X}'$, every integral closed subscheme $\overline{V}' \subsetneq \overline{X}'$ such that $\overline{V}' \cap Z \neq \emptyset$ is contained in Z. If $x \in Z$ is a closed point and $\underline{p}_1, \ldots, \underline{p}_n \subset \mathcal{O}_{\overline{X}',x}$ are the prime ideals corresponding to the irreducible components of Y, which contain the point x, it results that every prime ideal $\underline{q} \subset \mathcal{O}_{\overline{X},x}$, $\underline{q} \neq 0$, contains an ideal $\underline{p}_i$. Then $\dim \mathcal{O}_{\overline{X}',x} = 1$ and Z is a finite set of closed points of codimension

1 in $\overline{X}'$. Let $\bar{f}|_{\overline{X}'} : \overline{X}' \longrightarrow Y'$ be the restriction morphism and $U = \{x \in \overline{X}' | x \text{ is isolated in } \bar{f}^{-1}\bar{f}(x)\}$. Then $Z \subset U$ and by Zariski Main Theorem, U is an open subset and $\bar{f}|_{\overline{X}'}$ is generically finite. Thus, there is a closed point $y \in Y'$ such that $X' \cap f^{-1}(y)$ is a 0-dimensional scheme. By the condition given in Corollary, it follows that dim $X' \leqslant 1$, which contradicts the assumption that dim $X' > 1$ ( by Lemma B , since Y is a Jacobson scheme).

Corollary 14. a) If A is a Jacobson noetherian k-subalgebra of an algebra of finite type over an universally 1-equicodimensional ring k, then A is generically universally 1-equicodimensional.

b) If X is a scheme of finite type over an universally 1-equicodimensional ring k, such that $\Gamma(X)$ is Jacobson noetherian, then $\Gamma(X)$ is a generically universally 1-equicodimensional ring.

Proof. These are particular cases of the following:

(***) If $f : X \longrightarrow Y$ is a dominant morphism of finite type of schemes, such that X is universally 1-equidimensional and Y is Jacobson noetherian, then Y is generically universally 1-equicodimensional.

Indeed, $f_{red}$ is a generically faithful flat morphism.

Proposition 7. Let $f:X \longrightarrow Y$ be a separated dominant morphism of schemes over a field k. If X is an algebraic scheme over k and Y is noetherian, then Y is universally 1-equicodimensional.

Proof. We may suppose that X and Y are integral schemes. If $\bar{f}:\overline{X} \rightarrow Y$ is a dense compactification of f, then by Prop.2,d), it suffices to prove that $\overline{X}$ is universally 1-equicodimensional. We have to prove that for $n \geqslant 0$, every closed integral subscheme

$Z \subset \overline{X} \times_{\mathrm{Spec}\,k} \mathbb{A}^n_k$, which contains a closed point $z$ of codimension 1 in $Z$, is of dimension 1. Indeed, otherwise, we have $\dim Z > 1$; let $z' \in Z$ be a closed point such that $\operatorname{codim}_Z \{z'\} > 1$ and let $\{z'\} = Z'_0 \subsetneqq Z'_1 \subsetneqq \dots \subsetneqq Z'_\ell = Z$ be a saturated chain of integral closed subschemes; we have $\ell > 1$. If $Z \subsetneqq Z'_{\ell+1} \subsetneqq \dots \subsetneqq Z'_m = \overline{X} \times \mathbb{A}^n_k$ is a saturated chain of integral closed subschemes, we have two maximal saturated chain of different lengths:

$$\{z'\} = Z'_0 \subsetneqq Z'_1 \subsetneqq \dots \subsetneqq Z'_\ell = Z \subsetneqq Z'_{\ell+1} \subsetneqq \dots \subsetneqq Z'_m = \overline{X} \times \mathbb{A}^n_k$$

and $\{z\} \subsetneqq Z \subsetneqq Z'_{\ell+1} \subsetneqq \dots \subsetneqq Z'_m = \overline{X} \times \mathbb{A}^n_k$. Then Corollary 8, applied to the open immersion $X \times \mathbb{A}^n_k \hookrightarrow \overline{X} \times \mathbb{A}^n_k$, implies that the lengths of these two chains are equal, which is a contradiction. Hence $\dim Z = 1$.

Remark 6. Proposition 7 is a strong generalisation of Corollary 3 and thus generalizes Goodman-Landman result (see Remark 3), because, by Proposition 3, for every morphism $f: Z \to W$ of schemes of finite type over $Y$ ($Y$ as in Theorem 2) $f$ is proper iff every closed integral subscheme $C \subset Z$ of dimension 1 is proper over $W$.

As an application of the theory of universally 1-equicodimensional schemes, we shall prove the following:

Corollary 15. Let $f: X \to Y$ be a separated dominant morphism of schemes over a field $k$. Suppose that $X$ is an integral algebraic scheme over $k$ and $Y$ is noetherian. Then $Y$ is catenary and for every closed point $y \in Y$, $\dim \mathcal{O}_{Y,y} = \dim Y = \dim.\mathrm{al}._k K(Y)$. More, every integral scheme $Z$ of finite type over $Y$ has these properties.

Proof. Let $U \subset X$ be the open subset of regular points of U. Because U is dense in X, f(U) is a constructible dense subset of Y; hence there is a nonempty open subset $V \subset f(U)$. Because, for every closed point $y \in V$ $f^{-1}(y) \cap U \neq \phi$, by EGA IV, we have that V is a regular scheme, if we choose $V \subset f(U)$ such that $f|_{f^{-1}(V)}: f^{-1}(V) \longrightarrow V$ is flat; then, for every $x \in X$, we have $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{Y,y} + \dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)$ with $y=f(x)$. By EGA IV, Prop.13.2.3 $f|_{f^{-1}(V)}$ is an equicodimensional morphism; then for every $x \in X$ $\dim \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y) = \dim_x f^{-1}f(x) = \dim f^{-1}f(x) = \dim f^{-1}(\eta)$ with $\eta$ the generic point of Y. It follows that for every closed point $y \in V$, $\dim \mathcal{O}_{Y,y} = \dim X - \dim f^{-1}(\eta)$; hence for every closed point $y \in V$ $\dim \mathcal{O}_{Y,y} = \dim V$. By Prop.7 and Prop.5, it follows that Y is catenarian and for every closed point $y \in Y$, $\dim \mathcal{O}_{Y,y} = \dim Y$.

Then $\dim Y = \dim X - \dim f^{-1}(\eta) = \dim.\mathrm{al}._k K(X) - \dim.\mathrm{al}._{K(Y)} K(X) = \dim \mathrm{al}._k K(Y)$.

If Z is an integral scheme of finite type over Y, then locally it is a closed subscheme of $U \times \mathbb{A}^n_k$, with $U \subset Y$ an open affine subset. We may suppose Y and Z affine schemes and $Z \subset Y \times \mathbb{A}^n_k$ an integral closed subscheme. Because $f_n : X \times \mathbb{A}^n_k \longrightarrow Y \times \mathbb{A}^n_k$ is a dominant morphism and $X \times \mathbb{A}^n_k$ is algebraic over k, it suffices to prove that every closed integral subscheme $Z \subset Y$ has the properties given in the corollary. Indeed, because Y is a catenarian scheme and every maximal saturated chain of closed integral subscheme has the same length, it follows that Z is catenarian and for every closed point $z \in Z$, $\dim \mathcal{O}_{Z,z} = \dim Z$

We must prove that $\dim Z = \dim \mathrm{al}._k K(Z)$. If we choose a maximal saturated chain of closed integral subschemes

$$\{z\} = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_l = Z \subsetneqq Z_{l+1} \subsetneqq \cdots \subsetneqq Z_n = Y$$

because $\dim Y = \dim \mathrm{al}._k K(Y)$, it suffices to prove that

$$\dim.\mathrm{al}._k K(Z_i) < \dim.\mathrm{al}._k K(Z_{i+1})$$

Indeed, if $Z_{i+1} = \operatorname{Spec} A$, then $Z_i = \operatorname{Spec} A/\mathfrak{p}$, $\mathfrak{p}$ being a non-zero prime ideal of A; if $f \in \mathfrak{p}$ is a non-zero element and $\{f_1, \dots, f_m\}$ is a subset of elements of A such that $\{f_1|_{Z_i}, \dots, f_m|_{Z_i}\}$ is a transcendental basis of $K(Z_i)$, then $\{f, f_1, \dots, f_m\}$ is algebrically free over k.

Corollary 15 is proved.

We can write Corollaries 4,5 and 15 in the following equivalent pure algebraic form, which seems to be an extension of the classical dimension theory of k-algebras of finite type:

Theorem 2. Let A be a noetherian k-subalgebra of an algebra of finite type over a field k. Then every integral A-algebra B of finite type has the following properties:

a) B is an universally catenarian Jacobson ring.

b) for every maximal ideal $\underline{m} \subset B$, $B/\underline{m}$ is a finite extension of $k$.

c) $\dim B < \infty$ and for every maximal ideal $\underline{m} \subset B$, $\dim B_{\underline{m}} = \dim B$.

d) if K(B) is the field of quotients of B, then $\dim \text{al.}_k K(B) < \infty$ and $\dim.\text{al.}_k K(B) = \dim B$.

Indeed, we have a separated dominant morphism of finite type $f: X \rightarrow \operatorname{Spec} A$, where X is an algebraic scheme over k. Then a), c) and d) follow from Corollary 5 and Corollary 15 and b) from Corollary 4 and from the fact that Spec A is a Jacobson scheme.

Remark 7. a) For every algebraic scheme X over an algebraically closed field k such that $\Gamma(X)$ is noetherian, $\Gamma(X)$ has the properties of A from Theorem 2.

b) From a (more general) theorem, due to M.Nagata and K.Otsuka (see [14] ,Theorem 3), it follows the following: if k is a field, A is a k-subalgebra of a k-algebra B of finite type and $\mathfrak{p} \subset A$ is a prime ideal such that there is a prime ideal $\mathfrak{q} \subset B$ lying over $\mathfrak{p}$, then $\mathrm{ht}\,\mathfrak{p} + \dim.\mathrm{al}._k K(A/\mathfrak{p}) = \dim\,\mathrm{al}._k K(A)$.

c) Theorem 2 can be generalized in the following form: let k be a noetherian ring such that every integral k-algebra of finite type is generically regular and equicodimensional. Then, for every noetherian k-subalgebra A of a k-algebra of finite type, the following are equivalent:

(i) A is universally 1-equicodimensional.

(ii) A is a Jacobson universally catenarian ring

In these conditions we have that A is universally equicodimensional and for every prime ideal $\mathfrak{p} \subset A$ we have the relation: $\mathrm{ht}\,\mathfrak{p} + \dim.\mathrm{al}._{K(k/k\cap\mathfrak{p})} K(A/\mathfrak{p}) = \mathrm{ht}\,(k \cap \mathfrak{p}) + \dim.\mathrm{al}._{K(k)} K(A)$ (if k and A are integral rings) (see also Nagata - Otsuka Theorem).

The proof is like that of Corollary 15 and Theorem 2.

The Theorem 2, Proposition 1, Corollary 8, Lemma 2 and other assertions of this paper, lead to the following

PROBLEM 1. A noetherian integral k-subalgebra of a k-algebra of finite type over a field k is of finite type over k?

In geometrical language: If $f: X \to Y$ is a dominant morphism of schemes over a field k, Y is integral noetherian and X is a separated algebraic scheme over k, is Y an algebraic scheme over k?

The author was asked by professor A.Landman a question which amounts to a particular case of Problem 1: if X is an algebraic variety with $\Gamma(X)$ noetherian, is $\Gamma(X)$ a finite generated k - algebra?

An important result is the following Goodman-Landman Theorem ( see [6] ) :

Let X be an integral algebraic variety over an algebraically closed field k and $f: X \to Y$ a dominant morphism of schemes over k (Y is not necessary noetherian!). If f is proper, then Y is an algebraic variety over k.

In connection with Problem 1, one may further ask also the following question:

PROBLEM 2. Let $f: X \to Y$ be a proper surjective morphism of integral schemes. If X is noetherian , is Y noetherian? (in some " reasonable" conditions )

A particular case of this problem is the Nagata-Eakin Theorem (the case when the morphism f, in Problem 2, is affine). Via Problem 1, the above Theorem of J.E.Goodman and A.Landman might turn out to be a particular case of Problem 2.

The Problem 2 seems to be an important step for generalisations of Theorem 1, when Y is not a noetherian scheme.

Proposition 9. Let $\mathcal{C}$ be a category of integral schemes of finite type over an universally 1-equicodimensional ring k, such that:

i) if $X \in \mathcal{C}$ , then $\Gamma(X)$ is a noetherian Jacobson ring

ii) if $X \in \mathcal{C}$ , then for every $n>0$, $X \times_k \mathbb{A}^n_k \in \mathcal{C}$

iii) for every $X \in \mathcal{C}$ and every prime ideal $\mathfrak{p} \subset \Gamma(X)$ there exists $Y \in \mathcal{C}$ such that $\Gamma(Y)$ contains $\Gamma(X)/\mathfrak{p}$ and it is integral over $\Gamma(X)/\mathfrak{p}$ .

Then for every $X \in \mathcal{C}$, the ring $\Gamma(X)$ is universally 1-equicodimensional.

Proof. Let $X \in \mathcal{C}$ and A an integral $\Gamma(X)$ - algebra of finite type which contains a maximal ideal $\underline{m}$ of height 1. Because Spec A is an integral closed subscheme of Spec $\Gamma(X)[T_1, ..., T_n]$, for some $n \geq 0$ and $\Gamma(X)[T_1, ..., T_n] = \Gamma(X \times \mathbb{A}^n_k)$, it results, by i) and ii) that it is sufficient for us to prove the following: if $X \in \mathcal{C}$ and if $Y \subset \mathrm{Spec}\,\Gamma(X)$ is an integral closed subscheme which contains a closed point of codimension 1, then dim Y=1. By iii) there is a scheme $Z \in \mathcal{C}$, such that there exists an integer surjective morphism Spec $\Gamma(Z) \longrightarrow Y$; then $\Gamma(Z)$ has a maximal ideal of height 1 and, if we prove that dim $\Gamma(Z) = 1$, it follows that dim Y = 1.

Therefore, it suffices to prove that if $X \in \mathcal{C}$ and $\Gamma(X)$ has a maximal ideal $\underline{m}$ height 1, then dim $\Gamma(X) = 1$.

Let $I = \{\underline{n} \mid \underline{n} \subset \Gamma(X),\ \underline{n}$ maximal ideal, ht $\underline{n}=1\}$. Then I is either a finite set or a dense subset of Spec $\Gamma(X)$; indeed, let $\overline{I}_1, ... \overline{I}_\ell$ be the irreducible components of the closure $\overline{I}$ of I in Spec $\Gamma(X)$; for every j, $\overline{I}_j$ contains an ideal $\underline{n}_j \in I$ and then, because ht $\underline{n}_j=1$, either $\overline{I}_j = \{\underline{n}_j\}$ or $\overline{I}_j$=Spec $\Gamma(X)$; if, for every j, $\overline{I}_j = \{\underline{n}_j\}$, then I is finite set; otherwise, I is dense in Spec $\Gamma(X)$.

We shall prove that U=Spec $\Gamma(X)$-I is an affine scheme, if I is finite; indeed, for any coherent $\mathcal{O}_U$- module $\mathcal{F}$ and every extension $\overline{\mathcal{F}}$ of $\mathcal{F}$ to Spec $\Gamma(X)$, in the exact sequence: $H^1(\mathrm{Spec}\,\Gamma(X), \overline{\mathcal{F}}) \to H^1(U, \mathcal{F}) \to H^2_I(\mathrm{Spec}\,\Gamma(X), \overline{\mathcal{F}})$, we have $H^2_I(\mathrm{Spec}\,\Gamma(X), \overline{\mathcal{F}}) = \bigoplus_{x \in I} H^2_{\{x\}}(\mathrm{Spec}\,\mathcal{O}_x, \mathcal{F}_x)$; because dim $\mathcal{O}_x = 1$, it results that, for every $x \in I$, $H^2_{\{x\}}(\mathrm{Spec}\,\mathcal{O}_x, \mathcal{F}_x) = 0$ and then $H^2_I(\mathrm{Spec}\,\Gamma(X), \overline{\mathcal{F}}) = 0$; thus $H^1(U, \mathcal{F}) = 0$, because $H^1(\mathrm{Spec}\,\Gamma(X), \overline{\mathcal{F}}) = 0$; by Serre's Criterion, it follows that U is an affine open subset.

Let $\pi: X \to \mathrm{Spec}\,\Gamma(X)$ be the canonical morphism. If I is a finite set, then $I \cap \pi(X) \neq \emptyset$; otherwise, because U=Spec $\Gamma(X)$-I is an affine set, if $A = \Gamma(U, \mathcal{O}_{\mathrm{Spec}\,\Gamma(X)})$, it follows that $\pi$

factors through U: $X \xrightarrow{\pi'} \operatorname{Spec} A \overset{i}{\hookrightarrow} \operatorname{Spec}\Gamma(X)$ and hence the homomorphism $\pi'^{*}\circ i^{*}: \Gamma(X) \overset{i^*}{\hookrightarrow} A \overset{\pi'^*}{\hookrightarrow} \Gamma(X)$ is $1_{\Gamma(X)}$; it results from here that $i^*$ is an isomorphism, which contradicts the fact $I \neq \emptyset$. If I is a dense subset then $I \cap \pi(X) \neq \emptyset$, because $\pi(X)$ contains a non-empty open subset of Spec $\Gamma(X)$.

Therefore, there is a maximal ideal $\underline{n} \in \pi(X)$ such that ht $\underline{n}=1$. Let $x \in X$ be a closed point such that $\pi(x)=\underline{n}$ and $C \subset X$ a closed integral subscheme of dimension 1 passing through x and which is not contracted to $\underline{n}$ by $\pi$. Then the closure $\overline{\pi(C)} \supsetneq \underline{n}$; hence $\overline{\pi(C)} = \operatorname{Spec}\Gamma(X)$, because ht $\underline{n} = 1$. If $\tilde{\pi}: \tilde{C} \to \operatorname{Spec} \Gamma(X)$ is a dense compactification of $\pi|_C : C \to \operatorname{Spec}\Gamma(X)$, it follows that $\tilde{\pi}$ is a finite morphism; then $\dim \Gamma(X) = \dim \operatorname{Spec}\Gamma(X) = \dim \tilde{C}=1$ (see Lemma B).

Proposition 9 is proved.

In the proof of Proposition 9 we have used and we have proved the following:

<u>Lemma 3. Let $\overset{\circ}{X}$ be an integral scheme of finite type over an universally 1-equicodimensional ring k, such that $\Gamma(X)$ is a noetherian Jacobson ring. If $\dim \Gamma(X) \geq 2$, then for every maximal ideal $\underline{m} \subset \Gamma(X)$, ht $\underline{m} \geq 2$.</u>

We finish this Section with the following example, which generalizes Example 4,2.

<u>Example 6.</u> Every Jacobson noetherian 2 - dimensional ring is universally 1 - equicodimensional.

In fact, we may suppose that the ring is integral. Then its integral closure is noetherian Cohen - MaCaulay, hence it is universally catenarian and so, universally 1 - equicodimensional, by Corollary 10. Then the assersion follows from Prop. 2, d).

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