INSTITUTUL
DE
MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
STIINȚIFICĂ ȘI TEHNICĂ

SINGULAR SETS OF A MODULE AND LOCAL COHOMOLOGY

by

Constantin Bănică and Manuela Stoia

PREPRINT SERIES IN MATHEMATICS
No. 7/1978

BUCUREȘTI

SINGULAR SETS OF A MODULE AND LOCAL COHOMOLOGY

by

Constantin Bănică*) and Manuela Stoia**)

February 1978

*) Department of Mathematics, National Institute for Scientific and Technical Creation, B-dul Pacii 220, 77538 Bucharest, Romania

**) Institute of Mathematics, Str. Academiei 14, 70109 Bucharest, Romania

Singular sets of a module and local cohomology

by

Constantin Bănică and Manuela Stoia

When X is an analytic complex space and $\mathcal{F}$ an analytic coherent sheaf, one can define singular sets $S_k(\mathcal{F})$ ([8],[11]). These ones have an important place in the study of local cohomology. In particular, they provide enunciations to vanishing and coherence theorems for sheaves $\mathcal{H}^i_d\mathcal{F}$ of local cohomology with supports of dimension $\leq d$ ([11] §3; see also [1] Ch.II, §5).

In the case of algebraic geometry the singular sets $S^*_k(\mathcal{F})$ are introduced in ([2] Ch.IV, §10). In accordance with the analytic case, the sheaves of local cohomology with supports of dimension $\leq$d can be considered and vanishing and coherence theorems can be also proved. The proofs, which would been obtained by this way, were of geometric nature making systematically use of sheaves and for the finitude theorem one need too the difficult theorem of finitude of the cohomology with supports in a closed subset ([3] Exposé VIII; [11]). However, as is pointed out in [6] and [10], it is interesting to define local cohomological invariants in algebraic language and to have for them enunciations and proofs envolving only commutative algebra.

The aim of this paper is to introduce the modules of cohomology with supports of dimension $\leq$d and to prove pure algebraically vanishing and finitude theorems for these modules.

As one knows, the depth of a module behaves badly to localization; the vanishing and finitude theorems we will give, show that this behaviour (under resonable assumptions) is not so bad.

(I) Let A be a commutative noetherian ring (with unit element "1"), M an A-module of finite type. For any integer k, put:

$$S_k^*(M)=\{\mathfrak{p}\in \operatorname{Spec} A \mid \operatorname{depth} M_{\mathfrak{p}} + \dim A/\mathfrak{p} \leq k\}.$$

These are called the singular sets of M and are introduced in [2]. It is true the following:

"If A is quotient of a regular ring biequidimensional and M is an A-module of finite type, then the singular sets are closed in Zariski topology of Spec A and moreover, $\dim S_k^*(M) \leq k$."

The first assertion is proved in ([2] Ch.IV, §). Let us sketch another argument (cf.[12]). We can suppose A regular, biequidimensional and denote $n= \dim A$. Then

$$S_k^*(M)=\bigcup_{i\geq 0} \operatorname{Supp} (\operatorname{Ext}_A^{n-k+i}(M,A)).$$

Indeed, for any $\mathfrak{p}$, $n= \dim A/\mathfrak{p} + \dim A_{\mathfrak{p}}$, $\operatorname{depth} M_{\mathfrak{p}} = \dim A_{\mathfrak{p}} - \operatorname{dh} M_{\mathfrak{p}}$ and one can apply to $A_{\mathfrak{p}}$ the following well-known characterization of the homological dimension:

"If A is a local regular ring, M an A-module of finite type and s an integer, then $\operatorname{dh} M \leq s$ iff $\operatorname{Ext}_A^i(M,A)=0$ for all $i > s$."

The evaluation of the dimension is done in [12] and is inspired by the analytic case. It is obtained by localization from the following result [4]:

"If A is a regular local ring and M is an A-module of finite type, then $\dim (\operatorname{Ext}_A^s(M,A)) \leq \dim A-s$."

(II) Let A be a commutative ring, M an A-module and d an integer ≥ 0. We will write :

$$L_dM = \{m\in M \mid \dim (Am) \leq d\}.$$

Evidently, $L_dM= \{m\in M \mid \exists\, \underline{a},\ \dim A/\underline{a} \leq d,\ \underline{a}m =0\}$.

If $M \xrightarrow{\varphi} N$ is an A-liniar map, there is an induced map $L_d\varphi : L_dM \longrightarrow L_dN$. One gets a left-exact functor, canonically isomorpic to the functor $\varinjlim_{\dim A/\underline{a}\leq d} \mathrm{Hom}(A/\underline{a}, *)$, where $\varinjlim$ is taken over ideals $\underline{a}\subset A$ such that $\dim A/\underline{a} \leq d$.

We denote $(H_d^i)_{i\geq 0}$ the right-derived functor associated to L_d. $(H_d^iM)_{i\geq 0}$ are called the modules of cohomology with supports of dimension $\leq d$, associated to M.

H_d^0M equals L_dM. There are isomorphisms:

$$H_d^iM \simeq \varinjlim_{\dim A/\underline{a}\leq d} \mathrm{Ext}_A^i(A/\underline{a}, M).$$

In [10], one defines for an ideal $\underline{a}$ the functor $L_{\underline{a}}(*)$ and its derived functors $R^iL_{\underline{a}}(*)$- the modules of cohomology with supports in $V(\underline{a})$. One gets easily:

$$H_d^iM \simeq \varinjlim_{\dim A/\underline{a}\leq d} R^iL_{\underline{a}}M;$$

these isomorphisms, as well as those written above, are functorial and agree with short exact sequences.

Lemma Let A be a commutative ring and M an A-module. Then

a) if A is the quotient of a noetherian ring B, under the morphism φ, then

$$(H_d^iM)_{[\varphi]} \simeq H_d^iM_{[\varphi]};$$

b) if A is the quotient of a noetherian, catenary, biequidimensional ring, then for any $\underline{p}\in \mathrm{Spec}\ A$,

$$(H_d^iM)_{\underline{p}} \simeq H^i_{d-\dim A/\underline{p}} M_{\underline{p}}.$$

Proof a) If I is an injective A-module, then $R^iL_{\underline{b}}I_{[\varphi]}=0$ for any ideal $\underline{b}\subset B$ and $i>0$ ([10], Corollary 4.2). Consequently, $H_d^i I_{[\varphi]}=0$ for $i>0$, therefore $I_{[\varphi]}$ is acyclic relatively to L_d. As $(H_d^0I)_{[\varphi]}\simeq H_d^0I_{[\varphi]}$ for any A-module I, the conclusion follows.

b) Using a) one may suppose A itself noetherian, catenary and biequidimensional. Let consider an injective resolution of M; since the injectivity is preserved by localization, it suffices to prove b) when i=0, that is $(L_dM)_{\mathfrak{p}} = L_{d-\dim A/\mathfrak{p}} M_{\mathfrak{p}}$. Both modules are sub-modules of $M_{\mathfrak{p}}$ and for any $m \in M$, the trace of Supp(Am) on Spec $A_{\mathfrak{p}}$ equals the support of the $A_{\mathfrak{p}}$-module $A_{\mathfrak{p}}m$. Accordingly, the inclusion $(L_dM)_{\mathfrak{p}} \subset L_{d-\dim A/\mathfrak{p}} M_{\mathfrak{p}}$ easily follows. Let m/s be a class of $M_{\mathfrak{p}}$ (i.e., $m \in M$, $s \notin \mathfrak{p}$) such that $\dim(A_{\mathfrak{p}}.m/s) \leq d-\dim A/\mathfrak{p}$. Let $\mathfrak{q}$ be a prime minimal in Supp(Am). If $\mathfrak{q} \subset \mathfrak{p}$, then $\dim A/\mathfrak{q} = \dim A/\mathfrak{p} + \dim(A/\mathfrak{q})_{\mathfrak{p}} \leq \dim A/\mathfrak{p} + d - \dim A/\mathfrak{p} = d$. Suppose now, $\mathfrak{q} \not\subset \mathfrak{p}$ and choose $l \in \mathfrak{q} \setminus \mathfrak{p}$. The $A_{\mathfrak{q}}$-module of finite type $(Am)_{\mathfrak{q}}$ is concentrated in $\mathfrak{q}A_{\mathfrak{q}}$, hence killed by a suitable power $\mathfrak{q}^nA_{\mathfrak{q}}$. In particular, $l^n m$ is equal to zero in $M_{\mathfrak{q}}$. $\mathrm{Supp}(Al^n m) \subset \mathrm{Supp}(Am)$ but $\mathfrak{q} \notin \mathrm{Supp}(Al^n m)$. Going on we can build an element $t \notin \mathfrak{p}$ such that $\dim(Atm) \leq d$ and the proof is over.

(III) In ([3], Exposé III) one considers the depth of a module relative to an ideal. It is convenient to consider for an integer $d \geq 0$ and a module M of finite type over a noetherian ring A:

$$\mathrm{depth}_d M = \inf_{\dim A/\mathfrak{a} \leq d} \mathrm{depth}_{\mathfrak{a}} M$$

which will be called d-depth of M.

By ([3] Exposé III Prop.2.9) one gets $\mathrm{depth}_d M = \inf_{\dim A/\mathfrak{p} \leq d} \mathrm{depth}\, M_{\mathfrak{p}}$

<u>Vanishing Theorem</u> Let A be a noetherian ring and M an A-module of finite type, $d \geq 0$ and $n \geq 0$ two integers. Then the following are equivalent:

a) $H^i_d M = 0$ for $i < n$.

b) $\mathrm{depth}_d M \geq n$.

c) (If A is the quotient of a regular biequidimensional ring) $\dim S^*_{k+n} M \leq k$ for all $k < d$.

Proof a) $\Longleftrightarrow$ b) is nothing but a straightforward consequence of ([3] Exposé III,2):

a) $\Longrightarrow$ b) For any $\underline{a}$ such that $\dim A/\underline{a} \leq d$, we need $\mathrm{Ext}^i_A(A/\underline{a}, M)=0$ for all $i < n$. Since $\mathrm{depth}_{\underline{a}} M \geq \mathrm{depth}_d M \geq n$, this follows by ([3] Exposé III, Prop.2.4).

b) $\Longrightarrow$ a) For, let proceed inductively by n. The case n=0 is obvious. Let suppose the implication already proved for n-1 and let show it for n. One has $\mathrm{depth}_d M \geq n-1$ or equivalently $\mathrm{depth}_{\underline{a}} M \geq n-1$ for all ideals $\underline{a}$ such that $\dim A/\underline{a} \leq d$. Consequently, by the same proposition of [3], $\mathrm{Ext}^i_A(N,M)=0$ for $i \leq n-2$ and each A-module N of finite type annihilated by an ideal $\underline{a}$ such that $\dim A/\underline{a} \leq d$, or equivalently for which $\dim N \leq d$. Let $\underline{a} \subset \underline{b}$, $\underline{a}$,$\underline{b}$ ideals such that $\dim A/\underline{a} \leq d$ and $\dim A/\underline{b} \leq d$. From the exactness of

$$\mathrm{Ext}^{n-2}_A(\underline{b}/\underline{a},M) \longrightarrow \mathrm{Ext}^{n-1}_A(A/\underline{b},M) \longrightarrow \mathrm{Ext}^{n-1}_A(A/\underline{a}, M)$$

the mapping $\mathrm{Ext}^{n-1}_A(A/\underline{b} ,M) \longrightarrow \mathrm{Ext}^{n-1}_A(A/\underline{a},M)$ arises injective. The hypothesis $H^{n-1}_d M = \varinjlim_{\dim A/\underline{a} \leq d} \mathrm{Ext}^{n-1}_A(A/\underline{a},M)=0$ implies that $\mathrm{Ext}^{n-1}_A(A/\underline{a},M)=0$, hence $\mathrm{depth}_{\underline{a}} M \geq n$ for $\underline{a}$ such that $\dim A/\underline{a} \leq d$ and therefore $\mathrm{depth}_d M \geq n$.

b) $\Longrightarrow$ c) Let $\mathfrak{p} \in S^*_{k+n}(M)$ $(k < d)$, i.e. $\mathrm{depth}\, M_{\mathfrak{p}} + \dim A/\mathfrak{p} \leq k+n$. In accordance with the hypothesis $\mathrm{depth}_d M = \inf_{\dim A/\mathfrak{p}' \leq k} \mathrm{depth}\, M_{\mathfrak{p}'} \geq n$.

Let suppose $\dim A/\mathfrak{p} \leq d$. Since $\mathrm{depth}\, M_{\mathfrak{p}} \geq \mathrm{depth}_d M \geq n$, $n + \dim A/\mathfrak{p} \leq$ $\leq \mathrm{depth}\, M_{\mathfrak{p}} + \dim A/\mathfrak{p} \leq k+n$, hence $\dim A/\mathfrak{p} \leq k$. It remains only to show that the case $\dim A/\mathfrak{p} = d+r$, $r > 0$ does not occur.

Let $\mathfrak{p}' \supset \mathfrak{p}$ such that $\dim A/\mathfrak{p}' = d$. As $S^*_{k+n}(M)$ is a closed set $\mathfrak{p}'$ must belong too to $S^*_{k+n}(M)$, hence by the same argument we get $d = \dim A/\mathfrak{p}' \leq k$, contradiction!

c) $\Longrightarrow$ b) results from definitions.

This theorem provides characterizations for torsion-free and reflexive modules.

Corollary Let A be an integral domain, quotient of a regular biequidimensional ring and M an A-module of finite type. Then

i) M is torsion-free iff dim $S_k^*(M) \leq k-1$ for any $k < \dim A$.

ii) M is reflexive iff dim $S_k^*(M) \leq k-2$ for any $k < \dim A$.

The proof can be done exactly as in analytic case ([11], §1). We will give only the proof of a). If M is torsion-free, then there is a monomorphism $0 \to M \to A^r$ for a suitable r and let denote $K = \mathrm{coker}(M \to A^r)$. So we get the short exact sequence

$$0 \longrightarrow M \longrightarrow A^r \longrightarrow K \longrightarrow 0.$$

We claim $S_k^*(M) \subset S_{k-1}^*(K)$ for $k < t = \dim A$ and the conclusion follows using the estimation of the dimension of singular sets. Let $\mathfrak{p} \in S_k^*(M)$. Since $k < t$ it results that $M_{\mathfrak{p}}$ is not free. As a consequence, $K_{\mathfrak{p}}$ cannot be free, hence depth $K_{\mathfrak{p}}$ = depth $M_{\mathfrak{p}}-1$ and henceforth $\mathfrak{p} \in S_{k-1}^*(K)$.

For the converse, let consider the canonical morphism $M \to M^{**}$ (here, $*$ means the dual). It suffices to show that $K = \ker(M \to M^{**})$ is null. For any $\mathfrak{p} \notin S_{t-1}^*(M)$, $M_{\mathfrak{p}}$ will be a free module and therefore $K_{\mathfrak{p}}=0$. Accordingly, Supp $K \subset S_{t-1}^*(M)$, hence dim $K \leq t-1$. Therefore $K = H_{t-1}^0 K$. As $H_{t-1}^0 K \subset H_{t-1}^0 M$, if we will prove that $H_{t-1}^0 M = 0$, we get $K = 0$. But this follows by means of the vanishing theorem for M, n = 1 and d = t-1.

(IV) We will make use of the following peculiar case of the finitude theorem of Grothendieck for local cohomology sheaves:

"Let A be a regular local ring, $\mathfrak{m}$ the maximal ideal, M an A-module of finite type and $n \geq 0$ an integer. Then $H_{\mathfrak{m}}^i M$ are A-modules of finite type (that is of finite length, because their supports are $\subseteq \{\mathfrak{m}\}$) for any $i \leq n$ iff depth $M_{\mathfrak{p}} > n - \dim A/\mathfrak{p}$ whenever $\mathfrak{p} \neq \mathfrak{m}$."

This assertion can be proved pure algebraically ([3] , Corollaire 3.6, p.71) : in virtue of the theorem of local duality ([3], Th.2.1 , p.64, or [6] Vortrag 5, Satz 5.2 and Satz 5.9), $H^i_{\mathfrak{m}}(M)$ is of finite type iff $Ext^{t-i}_A(M,A)$ (t = dim A) is of finite length, thus iff $(Ext^{t-i}_A(M,A))_{\mathfrak{p}} \simeq Ext^{t-i}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, A_{\mathfrak{p}})$ are zero provided that $\mathfrak{p} \neq \mathfrak{m}$, and one applies the characterization of the depth in regular rings reminded in (I).

Theorem of finiteness Let A be a ring, quotient of a regular biequidimensional ring, M an A-module of finite type and $d \geq 0$, $n \geq 0$ two integers. Then the following are equivalent

a) $H^i_d M$ is of finite type for any $i \leq n$.

b) $\dim S^*_{n+d}(M) \leq d$.

Proof By the previous lemma, one may suppose A itself regular and biequidimensional and denote t= dim A.

a) $\Longrightarrow$ b) Let $\mathfrak{p} \in S^*_{n+d}(M)$ and suppose contrary to b) that $\dim A/\mathfrak{p} > d$. Choose a prime ideal $\mathfrak{q} \supset \mathfrak{p}$ such that $\dim A/\mathfrak{q} = d$. By the lemma, $(H^i_d M)_{\mathfrak{q}} \simeq H^i_{d-\dim A/\mathfrak{q}} M_{\mathfrak{q}} \simeq H^i_o M_{\mathfrak{q}} \simeq H^i_{\mathfrak{q}A_{\mathfrak{q}}} M_{\mathfrak{q}}$. Therefore $H^i_{\mathfrak{q}A_{\mathfrak{q}}} M_{\mathfrak{q}}$ is of finite type over $A_{\mathfrak{q}}$ for $i \leq n$. Consequently ,

depth $M_{\mathfrak{p}} > n - \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) = n - \dim A/\mathfrak{p} + \dim A/\mathfrak{q} = n - \dim A/\mathfrak{p} + d$, contradiction!

b) $\Longrightarrow$ a) Let study first the case $n \geq t-d$. In this case the hypothesis implies $\dim M \leq d$, but therefore $H^0_d M = M$ and $H^i_d M = 0$ for all $i > 0$ (one uses lemma a) for $A \longrightarrow A/\mathrm{Ann}\,M$!) and the finiteness of the modules $H^i_d M$ is proved.

Now, take $n = t-d-1$. In this case, the hypothesis means that $\dim S^*_{t-1}(M) \leq d$, i.e. the set $\{\mathfrak{p} \mid M_{\mathfrak{p}}$ is not free$\}$ is of dimension smaller than d.

We must actually prove that $H^i_d M$ are of finite type if $i \leq t-d-1$. We will proceed by induction on i, $0 \leq i \leq t-d-1$, to prove that the hypothesis done on $S^*_{t-1}M$ implies the finiteness of $H^i_d M$. If $i = 0$, $H^0_d M$ is always of finite type. Let $i < t-d-1$ and prove the induction step $i \to i+1$. We use an idea of Trautmann [15] to consider a resolution of the dual and dualizing it. Let $0 \to N \to L \to M^* \to 0$ be an exact sequence with L a free module of finite rank. By dualizing we get an injective map $M^{**} \to L^*$ and let $P = \mathrm{coker}(M^{**} \to L^*)$. Therefore we get the exact sequence:

$$0 \longrightarrow M^{**} \longrightarrow L^* \longrightarrow P \longrightarrow 0.$$

We claim $\dim S^*_{t-1}P \leq d$. For, we need that if $\dim A/\mathfrak{p} > d$ then $P_{\mathfrak{p}}$ is free; but $M_{\mathfrak{p}}$ is free, hence $M^*_{\mathfrak{p}}$ is also free and the sequence $0 \to N_{\mathfrak{p}} \to L_{\mathfrak{p}} \to M^*_{\mathfrak{p}} \to 0$ splits; therefore $N_{\mathfrak{p}}$ will be free too and $P_{\mathfrak{p}}$ equals the very dual of $N_{\mathfrak{p}}$, so it will be free. Consequently, $H^i_d P$ arises of finite type in virtue of the induction hypothesis.

Now, consider the exact sequence

$$H^i_d P \longrightarrow H^{i+1}_d M^{**} \longrightarrow H^{i+1}_d L^*.$$

By the vanishing theorem $H^{i+1}_d L^* = 0$ $(i+1 < t-d)$. Therefore $H^{i+1}_d M^{**}$ is of finite type. The natural morphism $M \to M^{**}$ is an isomorphism whenever we will localize at prime ideal of dimension $> d$, hence its kernel and cokernel are of dimension $\leq d$. As for such modules H^i_d vanish for $i > 0$ (and H^0_d is obviously of finite type) hence easily follows that $H^{i+1}_d M$ is of finite type and the step of induction is proved.

The implication b) $\Rightarrow$ a) being proved when $n \geq t-d-1$ it reminds only to consider the case $n < t-d-1$. We will proceed by decreasing induction on n, $0 \leq n \leq t-d-2$. Let consider an exact sequence of the type $0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$ with L free module of finite rank. We claim that N fulfils b) for $n+1$, i.e. $\dim S^{*}_{n+1+d}N \leq d$; from the exact sequence

$$H^i_d L \longrightarrow H^i_d M \longrightarrow H^{i+1}_d N$$

we deduce that the inductuon works and the proof of the theorem will be over.

Choose $\mathfrak{p} \in S^{*}_{n+1+d}N$. $M_{\mathfrak{p}}$ is not free, otherwise $N_{\mathfrak{p}}$ will be free, hence $\operatorname{depth} N_{\mathfrak{p}} + \dim A/\mathfrak{p} = t > n+1+d$ -contradiction! By means of the exact sequence $0 \to N_{\mathfrak{p}} \longrightarrow L_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow 0$ we get $\operatorname{depth} M_{\mathfrak{p}} = \operatorname{depth} N_{\mathfrak{p}} - 1$ and therefore $\mathfrak{p} \in S^{*}_{n+d}M$. Consequently $S^{*}_{n+1+d}N \subseteq S^{*}_{n+d}M$ and the claim is proved.

<u>Remark</u> If the ring A is Jacobson, then one can avoid in the proof of the theorem the use of finiteness theorem concerning the invariants $H^i_{\mathfrak{m}}(M)$. This remark is useful for the geometric case of schemes (in accordance with the next section).

Let retake the proof of a) $\Longrightarrow$ b). First note that if $H^i_d M$ is an A-module of finite type then $\dim(H^i_d M) \leq d$. Indeed, if $s_1,\dots,s_k$ is a system of generators for $H^i_d M$, then $\operatorname{Supp} H^i_d M \subseteq$ $\subseteq (\operatorname{Supp} As_1) \cup \dots \cup (\operatorname{Supp} As_k)$ and moreover $\dim(As_j) \leq d$ for all j. Let $\mathfrak{p} \in S^{*}_{n+d}M$; we will show that $\dim A/\mathfrak{p} > d$ cannot happen. On the contrary, assume $\dim A/\mathfrak{p} > d$. By the previous remark $\mathfrak{p} \notin \bigcup_{i=0}^{n} \operatorname{Supp} H^i_d M$. Since A is Jacobson, we can find a prime ideal $\mathfrak{q}$ which does not lie in the closed set $\bigcup_{i=0}^{n} \operatorname{Supp} H^i_d M$ and moreover $\mathfrak{q} \supset \mathfrak{p}$ and $\dim A/\mathfrak{q} = d$. The modules $(H^i_d M)_{\mathfrak{q}} = H^i_0 M_{\mathfrak{q}}$ are zero for all $i \leq n$. By the vanishing theorem $\operatorname{depth} M_{\mathfrak{q}} > n$, hence $\operatorname{depth} M_{\mathfrak{q}} + \dim A/\mathfrak{q} > n+d$. As $\mathfrak{q} \supset \mathfrak{p}$ and $S^{*}_{n+d}M$ is a closed set, that is a contradiction!

(V) Remarks a) Just as in the analytic case ([14] and [11] Ch.II §2) the connection between $H_d^0 M$ and the primary decomposition is stated as follow:

"Let A be a noetherian ring, M an A-module of finite type. If $0 = \bigcap M_i$ is a primary decomposition of $0 \subseteq M$ and $\mathfrak{p}_i$ are the corresponding prime ideals, then

$$H_d^0 M = \bigcap_i \{ M_i \mid \dim(A/\mathfrak{p}_i) > d \} ."$$

The proof can be easily derived from definitions. Let note the following consequence of this :

"Let A be a noetherian local ring, quotient of a Cohen-Macaulay ring, M an A-module of finite type and $d \geqslant 0$ an integer. Then $H_d^0(M)\hat{} \simeq H_d^0(\hat{M})$ (here ^ means the completion in the radical topology)."

Proof Due to the invariance with respect to surjections one can assume A to be Cohen-Macaulay ring. Let $0 = \bigcap_i M_i$ be a primary decomposition of $0 \subset M$, $\mathfrak{p}_i$ the corresponding prime ideals. For each i, let consider the primary decomposition $\widehat{M}_i = \bigcap P_{j_i}$ of $\widehat{M}_i$ in $\widehat{M}$. If $\mathfrak{p}_{j_i} \in \operatorname{Spec} \widehat{A}$ are the corresponding prime ideals , then $\dim \widehat{A}/\mathfrak{p}_{j_i} = \dim A/\mathfrak{p}_i$ for all j_i ([9] Ch.IV Corollary of th. 7) and the conclusion follows.

Without the additional hypothesis done with respect to A, the statement need not be true, in accordance with the fact that in the primary decompositon of a completion of a prime ideal can arise embedded components.

The comparison of the invariants $H_d^i M$ and $H_d^i \widehat{M}$ seems to be a difficult question. A partial result is the following:

"Let A be a noetherian local ring, quotient of a regular ring, M an A-module of finite type and $d \geqslant 0$, $n \geqslant 0$ two integers.

Assume $H^i_d M$ of finite type for all $i \leq n$. Then $H^i_d \hat{M}$ are of finite type for $i \leq n$ and moreover $(H^i_d M)^{\wedge} \simeq H^i_d \hat{M}$ if $i \leq n$."

<u>Proof</u> We may suppose A regular. The condition that $H^i_d M$ are of finite type is equivalent with $\dim S^*_{n+d} M \leq d$. Take again the argument (as well as the notations) used in the proof of the implication b) $\Longrightarrow$ a) of the finitude theorem. First, remark that there are natural morphisms $\theta^i(M) : H^i_d M \otimes_A \hat{A} \longrightarrow H^i_d \hat{M}$, which are functorial with respect to M and which agree with short exact sequences, because we have natural morphisms $L_d M \otimes_A \hat{A} \longrightarrow L_d (M \otimes_A \hat{A})$ and $\hat{A}$ is A-flat.

Let $t = \dim A$. If $n \geq t-d$ then $\dim M \leq d$, therefore $\dim \hat{M} \leq d$, and anyway $\theta^i(M)$ are evidently isomorphisms.

Actually, let $n = t-d-1$. We proceed by induction on i, $0 \leq i \leq t-d-1$, to prove that θ^i is an isomorphism. If $i = 0$ the statement was already proved above. Let $i < t-d-1$ and show the induction step $i \to i+1$. One has $H^i_d P \simeq H^{i+1}_d M^{**}$, $H^i_d \hat{P} \simeq H^{i+1}_d \hat{M}^{**}$. In virtue of the commutative diagram:

$$\begin{array}{ccc}
(H^i_d P) \otimes_A \hat{A} & \longrightarrow & (H^{i+1}_d M^{**}) \otimes_A \hat{A} \\
\downarrow & & \downarrow \\
H^i_d(\hat{P}) & \longrightarrow & H^i_d(\hat{M}^{**})
\end{array}$$

we get $\theta^{i+1}(M^{**})$ is isomorphism. The kernel and the cokernel of the natural morphism $M \longrightarrow M^{**}$ are both of dimension $\leq d$; the same is still true for $\hat{M} \longrightarrow (\hat{M})^{**} \simeq (M^{**})^{\wedge}$. Now one derives easily that $\theta^{i+1}(M)$ is isomorphism.

It remains only the case $n < t-d-1$. We will do induction (decreasing) on n, $0 \leq n < t-d-1$. If $i < t-d-1$ there are isomorphisms morphisms $H^i_d M \simeq H^{i+1}_d N$, $H^i_d \hat{M} \simeq H^{i+1}_d \hat{N}$ and N fulfils the hypothesis for the integer n+1. So, one sees how the induction works.

b) Again let A be a commutative ring and M an A-module. For an integer $d \geqslant 0$, denote $L_{d/d-1}M = L_dM/L_{d-1}M$. The right-derived functors of the functor $L_{d/d-1}$ will be denoted by $H^i_{d/d-1}$. Then one gets the long sequence of cohomology

$$\cdots \to H^i_{d-1}M \to H^i_dM \longrightarrow H^i_{d/d-1}M \longrightarrow H^{i+1}_{d-1}M \longrightarrow \cdots \quad .$$

Moreover, there are natural isomorphisms

$$H^i_{d/d-1}M \simeq \bigoplus_{\dim A/\mathfrak{p}=d} H^i_{\mathfrak{p}A_{\mathfrak{p}}} M_{\mathfrak{p}} .$$

For this see ([5] ,p.225, Motif F),but a direct algebraic argument is not hard to be done. As a consequence one can prove that if A is Cohen-Macaulay (i.e. all localizations are Cohen-Macaulay) biequidimensional, n-dimensional ring, then $H^i_dA = 0$ whenever $i \neq n-d$. Consequently, when L. is a free resolution of an A-module M, one can compute the invariants H^i_dM taking the homology of the complex $H^{n-d}_d L.$. Using the invariants $H^{\cdot}_{d/d-1}$ one derives easily, by the characterization of the dimension for modules of finite type over local rings ([4] , p.88,Prop.6.4 or [6], Satz 4.12),the following

"Let A be a noetherian ring and M an A-module of finite type, $d \geqslant 0$ and $n \geqslant 0$ two integers. Then $H^i_dM = 0$ for $i > n$ iff $\dim_d M \geqslant n$ (where $\dim_d M =: \sup_{\dim A/\mathfrak{p} \leq d} \dim M_{\mathfrak{p}}$)."

For "if" implication one proceed by increasing induction on d and for the converse, one localizes at prime ideals $\mathfrak{p}$ of dimension = d and uses the equality $\dim_d M = \sup_{\dim A/\mathfrak{p}=d} \dim M_{\mathfrak{p}}$.

c) One can consider invariants which are more general as H^i_d. Let M and N be modules over a commutative noetherian ring A. Denote $Hom_{A,d}(N,M)$ the submodule of $Hom_A(N,M)$ given by the morphisms with supports of dimension $\leqslant d$; that is $Hom_{A,d}(N,M)=$

$=L_d(\mathrm{Hom}_A(N,M))$. When N is fixed one gets a functor and we will denote its right-derived functors by $\mathrm{Ext}^{\bullet}_{A,d}$ and called them "exts" with supports $\leq d$.

For N = A one falls under H^i_d . Remark the following:

"Suppose N of finite type. Then there is a spectral sequence of term $E_2^{p,q} = H^p_d(\mathrm{Ext}^q_A(N,M))$ which converges to $\mathrm{Ext}^{p+q}_{A,d}(N,M)$."

To prove that, we need only to notice that if I is an injective module, then for any module N, $\mathrm{Hom}_A(N,I)$ is L_d-acyclic. For the reader of [5], the sheaf theoretically argument is the following: the sheaf $\tilde{I}$ on Spec A is $\tilde{A}$-injective ([5], Ch.II, Corollary 7.14), $\mathcal{H}om_{\tilde{A}}(\tilde{N}, \tilde{I})$ follows flaby, further the flaby sheaves are acyclic to the local cohomology....

As we pointed out in the introduction, we want to remain with the arguments in the frame of commutative algebra, hence we write an algebraic argument as a straightforward consequence of results due to Matlis [7].

First, we will do the following remark : if $0 \to P' \longrightarrow P \longrightarrow P'' \to 0$ is an exact sequence of A-modules of finte type, then the sequence

$$0 \longrightarrow \mathrm{Hom}_{A,d}(P'',I) \longrightarrow \mathrm{Hom}_{A,d}(P\ ,I) \longrightarrow \mathrm{Hom}_{A,d}(P',I) \longrightarrow 0$$

is still exact. For, let prove that any morphism $\theta':P' \longrightarrow I$ extends to a morphism $\theta: P \longrightarrow I$ such that Supp θ' = Supp θ . We may assume I to be a non-zero indecomposable injective, that is the injective envelope $I(\mathfrak{p})$ of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, where $\mathfrak{p} \in$ Spec A. Supp $I(\mathfrak{p}) = V(\mathfrak{p})$ and moreover all fibers on $V(\mathfrak{p})$ equal $I(\mathfrak{p})$. As a consequence $\theta' = 0$ or $\mathfrak{p} \in$ Supp θ' and then Supp $\theta' = V(\mathfrak{p})$. If $\theta' = 0$ we put $\theta = 0$. If $\theta' \neq 0$, every extension $\theta:P \longrightarrow I$ is good. Let show that $H^i_d \mathrm{Hom}_A(N,I) = 0$ for $i \geq 1$. Let $0 \to P \longrightarrow L \longrightarrow N \longrightarrow 0$ be an exact sequence with L free of finite rank. We get the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{A,d}(N,I) \longrightarrow \mathrm{Hom}_{A,d}(L,I) \longrightarrow \mathrm{Hom}_{A,d}(P,I) \longrightarrow 0$$

Since $\mathrm{Hom}_A(L,I)$ is an injective module, hence $H_d^i\mathrm{Hom}_A(L,I) = 0$ for all $i > 0$, and the conclusion follows standardly.

d) As it is pointed out in the introduction, the aim of this paper is to give in the frame of commutative algebra some results in analogy to those of Thimm, Scheja, Siu and Trautmann in analytic geometry and to obtain on this way informations concerning the behaviour of the depth relative to localizations. Therefore we will do only some remarks about the geometrical case of schemes (on the other hand, one can give proofs closely following the analytic case).

Let X be a locally noetherian scheme, $\mathcal{F}$ an algebraic coherent sheaf on X. For a point $x \in X$ define the "rectified depth" ([2] Ch. IV)

$$\mathrm{depth}_x^* \mathcal{F} = \mathrm{depth}_{\mathcal{O}_x} \mathcal{F}_x + \dim \overline{\{x\}}.$$

This notion is interesting in the geometric case if it has local character. For, X must fulfil in addition topological conditions, which are satisfied when X is Jacobson (for example, of finte type over Spec k (k field) or Spec Z).

One writes $S_k^*(\mathcal{F}) = \{ x \in X \mid \mathrm{depth}_x^* \mathcal{F} \leq k \}$. These sets are closed in X [2], of dimension $\leq k$ [12]. For an integer $d \geq 0$ and an algebraic sheaf $\mathcal{F}$ (not necessarily coherent) one denotes by $\mathcal{H}_d^0 \mathcal{F}$ the subsheaf of $\mathcal{F}$ given by the sections with supports of dimension $\leq d$. One gets a functor and its right-derived functors will be denoted $\mathcal{H}_d^{\cdot}$ and will be called the sheaves of local cohomology with supports of dimension $\leq d$.

If X = Spec A and $\mathcal{F} = \tilde{F}$, then canonically $\mathcal{H}_d^i \mathcal{F} \simeq (H_d^i F)^{\sim}$. This comes out because $\tilde{I}$ is an $\tilde{A}$-injective sheaf when I is an injective module over A [5] or using results from ([3] Expose III).

Actually one can state:

Vanishing Theorem Let X be a locally noetherian scheme, $\mathcal{F} \in \mathrm{Coh}\, X$, n and d two fixed integers. Then the following are equivalent:

1) $\mathcal{H}_d^i \mathcal{F} = 0$ if $i < n$.

2) $\mathrm{depth}_d \mathcal{F} \geq n$ ($\mathrm{depth}_d \mathcal{F} = \inf_{\substack{\dim A \leq d \\ A \subset X \text{ closed}}} \mathrm{depth}_A \mathcal{F}$).

3) (If X is locally of finite type over an Jacobson regular biequidimensional scheme) $\dim S^*_{k+n} \mathcal{F} \leq k$ whenever $k < d$.

Finitude Theorem Let X be locally of finite type over an Jacobson, regular, biequidimensional scheme, $\mathcal{F} \in \mathrm{Coh}\, X$, n and d two fixed integers. Then, there are the equivalent statements

1) $\mathcal{H}_d^i \mathcal{F} \in \mathrm{Coh}\, X$ if $i \leq n$.

2) $\dim S^*_{d+n} \mathcal{F} \leq d$.

Geometric proofs following the analytic ones [11] and making use of the vanishing and finitude theorems for the cohomology with supports in a closed set due to Grothendieck ([3], Expose III, Prop. 3.3 and th. 2.1 of Expose VIII) can be find in [13].

The sheaf $\mathcal{H}_d^0 \mathcal{F}$ is also denoted by $\mathcal{F}_{[d]}$; it is a coherent subsheaf of $\mathcal{F}$ and it was introduced by Thimm in the analytic case. A detailed study can be found in [14] and ([11] §2). Another important sheaf is the so called d-gap-sheaf, denoted by $\mathcal{F}^{[d]}$ and given by the presheaf:

$$U \rightsquigarrow \varinjlim_{\dim A \leq d} \mathcal{F}(U \setminus A).$$

There is a natural morphism $\mathcal{F} \longrightarrow \mathcal{F}^{[d]}$ for which the kernel is $\mathcal{H}_d^0 \mathcal{F} = \mathcal{F}_{[d]}$ and the cokernel $\mathcal{H}_d^1 \mathcal{F}$. In virtue of the theorems one gets: $\mathcal{F}^{[d]} \in \mathrm{Coh}\, X$ iff $\dim S^*_{d+1} \mathcal{F} \leq d$; the canonical morphism $\mathcal{F} \longrightarrow \mathcal{F}^{[d]}$ is injective (resp. bijective) iff dim $S^{*}_{k+1}\mathcal{F} \leq k$ (resp. dim $S^{*}_{k+2}\mathcal{F} \leq k$) for all $k < d$.

When $\mathcal{F} \subset \mathcal{G}$ are algebraic coherent sheaves, exactly as in the analytic case ([11],[14]) one can introduce and study the gap-sheaves of $\mathcal{F}$ relative to $\mathcal{G}$.

In the analytic case the local cohomology is very useful to obtain theorems of extension of coherent sheaves. In the algebraic case, the coherent sheaves defined on an open set extend automatically to whole space ([2] Ch.I, 9.4). However, also in the algebraic case the invariants $\mathcal{H}^{\circ}_{d}$ may be of some use if we have to extend under special conditions, for example: "if the given sheaf has no sections of dimension $\leq d$,(or has no sections and 1-dimensional cohomology classes of dimension $\leq d$), then one requires an extension preserving the property."

REFERENCES

1.C.Bănică et O.Stănăşilă,"Méthodes algébriques dans la théorie globale des espaces complexes," Bucarest 1974 et Gauthier-Villars,Paris,1977.

2.A.Grothendieck et J.Dieudonné,"Élements de géométrie algébrique" (EGA) Ch.I,IV,Publ.IHES,Paris,(1960-..)

3.A.Grothendieck,Séminaire de géométrie algébrique 2 (SGA 2), (Cohomologie locale des faisceaux coherents et théorèmes de Lefschetz locaux et globaux), North-Holland Publ.Company,(1968).

4.A.Grothendieck,"Local cohomology" Lecture Notes in Math.,n.41, Springer Verlag,(1967).

5.R.Hartshorne,"Residues and duality" Lecture Notes in Math.n.20, Springer Verlag,(1966).

6.J.Hertzog und E.Kunz,"Der kanonische Modul eines Cohen-Macaulay Ring",Lecture Notes in Math.n.238,Springer Verlag,(1971).

7.E.Matlis,"Injective modules over Noetherian rings", Pacific J. Math.8(1958),511-28.

8.G.Scheja,Fortsetzungssatze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung,Math.Ann.157 (1964),75-94.

9.J.-P.Serre,"Algebre locale:multiplicites" Lecture Notes in Math. n.11, Springer Verlag,(1965).

10.R.Y.Sharp, Local cohomology theory in commutative algebra,Quart .J.Math.Oxford (2),21 (1970),425-34.

11.Y.T.Siu and G.Trautmann,"Gap-sheaves and extension of coherent analytic subsheaves" Lecture Notes in Math.n.172,Springer Verlag (1971).

12.M.Stoia,Une remarque sur la profondeur,C.R.Acad.Sc.Paris,276, (1973), 929-930.

13.M.Stoia,Mulţimile singulare ale unui fascicol,(to appear).

14.W.Thimm,Luckengarben vonkoharenten analytischen Modulgarben, Math.Ann.,148 (1962), 372-394.

15.G.Trautmann, Coherence de faisceaux analytiques de la cohomologie locale, C.R.Acad.Sc.Paris, 276 (1968), 694-695.

C.Bănică
INCREST
Bd.Păcii 220
Bucharest

M.Stoia
Inst.de Matematică
Str.Academiei 14
Bucharest

ROMANIA

INSTITUTUL
DE MATEMATICĂ

INSTITUTUL NAȚIONAL
PENTRU CREAȚIE
ȘTIINȚIFICĂ ȘI TEHNICĂ

PREPRINT SERIES IN MATHEMATICS
1977 ISSUES

1. H.Bercovici and D.Voiculescu : *TENSOR OPERATIONS ON CHARACTERISTIC FUNCTIONS OF C_0 CONTRACTIONS.*
2. C.Vârsan : *OPTIMAL BANG-BANG FEEDBACK CONTROL FOR LINEAR STOCHASTIC SYSTEMS.*
3. N.Popa : *SUR LES APPLICATIONS DU TYPE $\leqslant p$ et $\geqslant p$.*
4. H.I.Ene and Elena Ungureanu–David : *ON THE ELECTROHYDRODYNAMIC EQUATIONS OF THE PERMEABLE MEDIA.*
5. Gr.Arsene and S.Strătilă : *A BIBLIOGRAPHY ON OPERATOR ALGEBRAS AND RELATED TOPICS. SUPPLEMENT NO. 2.*
6. H.Moscovici and A.Verona : *COCYCLE REPRESENTATIONS OF SOLVABLE LIE GROUPS.*
7. D.Burghelea and R.Lashof : *THE HOMOTOPY STRUCTURE OF THE GROUPS OF AUTOMORPHISMS IN STABLE RANGES AND NEW HOMOTOPY FUNCTORS.*
8. S.Teleman : *ON THE IRREDUCIBLE DISINTEGRATION OF THE REPRESENTATIONS OF C* - ALGEBRAS.*
9. M.Pimsner and S.Popa : *ON THE EXT-GROUPS OF AN AF-ALGEBRA.*
10. Ioana Ciorănescu : *ABSTRACT BEURLING SPACES OF CLASS (M_p) AND ULTRA-DISTRIBUTION SEMI-GROUPS.*
11. F.-H.Vasilescu : *A UNIQUENESS RESULT IN OPERATOR THEORY.*
12. C.Bănică and V.Brînzănescu : *HILBERT-SAMUEL POLYNOMIALS OF PROPER MORPHISM.*
13. M.Pimsner and S.Popa : *EXT-GROUPS OF SOME NON-COMMUTATIVE C* - ALGEBRAS.*
14. N.Boboc, Gh.Bucur, A.Cornea : *HILBERTIAN AND LATTICE THEORETICAL METHODS IN POTENTIAL THEORY.*
15. C.Peligrad : *ON TRANSITIVE AND REDUCTIVE ALGEBRAS.*
16. H.Moscovici and A.Verona : *COHERENT STATES AND SQUARE INTEGRABLE REPRESENTATIONS.*
17. Ioana Ciorănescu et L.Zsido : *LA THEORIE DES ω - ULTRADISTRIBUTIONS.*
18. R.Iordănescu and I.Popovici : *SOME BASIC PROPERTIES OF ROTATION AND LORENTZ GROUPS.*
19. C.Vârsan : *DEGENERATE STOCHASTIC CONTROL SYSTEMS ; OPTIMAL FEEDBACK LAWS.*
20. S.Teleman ; *ON THE CHOQUET AND BISHOP - DE LEEUW THEOREMS.*
21. L.Zsido : *ON THE EQUALITY OF TWO WEIGHTS.*
22. F.-H.Vasilescu : *A MARTINELLI TYPE FORMULA FOR THE ANALYTIC FUNCTIONAL CALCULUS.*
23. S.Strătilă and D.Voiculescu : *ON A CLASS OF KMS STATES FOR THE UNITARY GROUP $U(\infty)$.*
24. Gr.Arsene and S.Strătilă : *A BIBLIOGRAPHY ON OPERATOR ALGEBRAS AND RELATED TOPICS.*
25. Horia I.Ene : *SUR LES EQUATIONS DU MOUVEMENT DANS LES MILIEUX POREUX DEFORMABLES.*
26. Nicolae Popa : *SOME IDEALS OF OPERATORS ON l_p.*
27. Ion Suciu and Ilie Valușescu : *FATOU AND SZEGÖ THEOREMS FOR OPERATOR VALUED FUNCTIONS.*
28. Zoia Ceaușescu and Ciprian Foiaș : *ON INTERTWINING DILATIONS. V.*
29. M.Pimsner, S.Popa and D.Voiculescu : *HOMOGENEOUS C* - EXTENSIONS OF $C(X) \otimes K(H)$. PART I.*
30. Ion Suciu and Ilie Valușescu : *LINEAR PREDICTOR FOR STATIONARY PROCESSES IN COMPLETE CORRELATED ACTIONS.*
31. Zoia Ceaușescu and Ciprian Foiaș : *ON INTERTWINING DILATIONS. VI.*
32. Constantin Niculescu : *ABSOLUTE CONTINUITY IN BANACH SPACE THEORY.*
33. S.Strătilă and D.Voiculescu : *A SURVEY ON REPRESENTATIONS OF THE UNITARY GROUP $U(\infty)$.*
34. Valeriu St.Udrescu : *A STRONGER BERTRAND'S POSTULATE.*
35. Adelina Georgescu : *VARIATIONAL FORMULATION OF SOME NON-SELFADJOINT PROBLEMS OCCURING IN BENARD INSTABILITY THEORY.I.*