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by

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A note on quasitriangularity and trace-class selfcommutators

by Dan Voiculescu

In [3] C.A.Berger and B.I.Shaw proved that for a hypernormal operator T the following inequality holds:

$$\mathrm{Tr}\,[T^*, T] \leq \frac{1}{\pi} m(T)\, \omega(\sigma(T))$$

where $\sigma(T)$ is the spectrum of T, ω is planar Lebesgue measure and $m(T) \in N \cup \{\infty\}$ denotes the multicycilcity of T. The aim of present note is to give a new proof and an extension of the result of Berger and Shaw by connecting it with quasitriangularity relative to the Hilbert-Schmidt class. Thus we shall prove that the hyponormality condition can be replaced by the condition that the negative part $([T^*, T])_-$ of $[T^*, T]$ be trace class. Even more, for such T we shall prove that

$$\mathrm{Tr}\,[T^*, T] \leq \frac{1}{\pi} m(T+X)\, \omega(\sigma(T+X))$$

where X is any Hilbert-Schmidt operator. In particular if

$$\pi\, \mathrm{Tr}\,[T^*, T] > \omega(\sigma(T))$$

then every Hilbert-Schmidt perturbation of T has a non-trivial invariant subspace.

Quasitriangular operators were introduced by P.R.Halmos [6] and it was shown by Apostol, Foiaş and Voiculescu [2] that there is a spectral characterization of these operators. A refinement of the notion of quasitriangular operator relative to a norm-ideal was considered in [14].

Throughout $\mathcal{H}$ will denote a complex separable Hilbert space of infinite dimension. By $\mathcal{L}(\mathcal{H})$ we denote the bounded operators on $\mathcal{H}$ and by $\mathcal{P}(\mathcal{H})$ the set of finite-rank orthogonal projections on $\mathcal{H}$ with its natural order. Then the analogue of Apostol's modulus of quasitriangularity relative to a Schatten-von Neumann class is:

$$q_p(T) = \liminf_{P \in \mathcal{P}(\mathcal{H})} |(I-P)TP|_p$$

where $T \in \mathcal{L}(\mathcal{H})$ and $|X|_p = \mathrm{Tr}((X^*X)^{p/2})$. $(1 \leq p < \infty)$

Then if $P_n \in \mathcal{P}(\mathcal{H})$ and $P_n \xrightarrow{w} I$ we have

$$\liminf_{n \to \infty} |(I-P_n)TP_n|_p \geq q_p(T)$$

Moreover one can find $P_n \in \mathcal{P}(\mathcal{H})$ such that $P_n \uparrow I$ and

$$\lim_{n \to \infty} |(I-P_n)TP_n|_p = q_p(T)$$

For $T \in \mathcal{L}(\mathcal{H})$ we shall denote by Rat(T) the algebra of operators of the form f(T) where f is a rational function with poles off the spectrum $\sigma(T)$ of T. The multicyclicity $m(T) \in \mathbb{N} \cup \{\infty\}$ is the least cardinal of a set $\Xi \subset \mathcal{H}$ such that the closed linear span of $\mathrm{Rat}(T)\,\Xi$ is $\mathcal{H}$.

<u>Proposition</u> 1. <u>For</u> $T \in \mathcal{L}(H)$ <u>and</u> $1 \leq p < \infty$ we have

$$q_p(T) \leq (m(T))^{1/p} \, \|T\|$$

<u>Proof</u>

If $m(T) = \infty$ there is nothing to prove. So assume $m(T) = n < \infty$ and consider $\{\xi_1, \ldots, \xi_n\}$ a multicyclic set for T_j. Consider

$$\mathcal{H}_j = \bigvee_{k=1}^{j} \mathrm{Rat}(T)\,\xi_k, \quad \mathcal{H}_0 = 0$$

$$\mathcal{K}_j = \mathcal{H}_j \ominus \mathcal{H}_{j-1}, \quad T_j = P_{\mathcal{K}_j} T | \mathcal{K}_j$$

$$\eta_j = P_{\mathcal{K}_j} \xi_j$$

Then, using Proposition 2.1 of [11] we have: $q_p(T) \leq (\sum_{k=1}^{n} (q_p(T_k))^p)^{1/p}$. Now, it is easily seen that $\sigma(T_k) \subset \sigma(T)$ and η_k is a multicyclic vector for T_k. This reduces the proof of the proposition to the case n=1.

Consider a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of points contained and dense in the union of the bounded components of $\mathbb{C}\setminus\sigma(T)$. Since ξ_1 is multicyclic for T, it is easily seen that denoting by P_m the proposition onto the finite-dimensional subspace of H sanned by the vectors $T^k(T-\lambda_1)^{-1}\cdot\ldots\cdot(T-\lambda_m)^{-1}\xi_1$ where $0 \leq k \leq 2m$, we have $P_m \leq P_{m+1}$, $P_m \uparrow I$ and rank $((I-P_m)TP_m)=1$. It follows that $|((I-P_m)TP_m)|_p \leq \|T\|$ and hence $q_p(T) \leq \|T\|$

Q.E.D.

For a hermitian operaror $A \in \mathcal{L}(\mathcal{H})$ such that the negative part A_- of A is trace-class, we shall denote by Tr A the trace of A, in case A is trace-class and ∞ in case A is not trace-class.

<u>Proposition 2. Let $T \in L(H)$ be an operator such that the negative part $([T^*,T])_-$ of $[T^*,T]$ is trace-class. Then we have:</u>

$$\mathrm{Tr}[T^*,T] \leq (q_2(T))^2$$

<u>Proof</u>. Let $P_m \in \mathcal{P}(\mathcal{H})$, $P_m \uparrow I$ be such that

$$\lim_{m\to\infty} |(I-P_m)TP_m|_2 = q_2(T)$$

We have

$$(q_2(T))^2 = \lim_{m\to\infty} |(I-P_m)TP_m|_2^2 =$$

$$= \lim_{m\to\infty} \mathrm{Tr}(P_mT^*TP_m - P_mT^*P_mTP_m) =$$

$$= \lim_{m\to\infty} \mathrm{Tr}(P_m[T^*,T]P_m + P_mT(I-P_m)T^*P_m) \geq$$

$$\geqslant \limsup_{m\to\infty} \operatorname{Tr}(P_m [T^*,T] P_m) = \operatorname{Tr}[T^*, T]$$

Q.E.D.

Proposition 3. Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that the negative part $([T^*,T])_-$ of $[T^*, T]$ be trace-class and let $X \in \mathcal{L}(\mathcal{H})$ be a Hilbert-Schmidt operator. Then we have

$$\operatorname{Tr}[T^*, T] \leq \frac{1}{\pi} m(T+X)\, \omega(\sigma(T+X))$$

where ω denotes planar Lebesgue-measure.

Proof. It is clearly sufficient to consider the case when $m(T+X)=n < \infty$.

Using the "computational lemma" of the paper of Berger and Shaw [3] it is easily seen that given $\varepsilon > 0$ and denoting by Ω the open set

$$\Omega = \{ z \in \mathbb{C} \,|\, |z| < \|T+X\| + \varepsilon \} \setminus \sigma(T+X)$$

we can find a hyponormal operator D such that:

$$\sigma(D) \subset \Omega, \quad m(D)=n, \quad \|D\| \leq \|T+X\| + \varepsilon$$

$$[D^*, D] \geqslant 0, \quad \operatorname{Tr}[D^*,D] \geqslant \frac{n}{\pi}(\omega(\Omega) - \varepsilon)$$

Using Proposition 1 we have

$$\operatorname{Tr}[(T \oplus D)^*, (T \oplus D)] \leq (q_2(T \oplus D))^2$$

and hence

$$\operatorname{Tr}[T^*,T] + \frac{n}{\pi}(\omega(\Omega) - \varepsilon) \leq (q_2(T \oplus D))^2$$

But $q_2(T \oplus D) = q_2((T+X) \oplus D)$ since X is Hilbert-Schmidt. Moreover $m((T+X) \oplus D) = n = m(T+X)$ and hence using Proposition 2 we have

$$(q_2((T+X) \oplus D))^2 \leq (m(T+X))(\|T+X\| + \varepsilon)^2 =$$

$$= \frac{1}{\pi} m(T+X)(\omega(\Omega) + \omega(\sigma(T+X)))$$

It follows that

$$\operatorname{Tr}[T^*,T] \leq \frac{1}{\pi} m(T+X)(\omega(\sigma(T+X)) + \varepsilon)$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\operatorname{Tr}[T^*,T] \leq \frac{1}{\pi} m(T+X)\, \omega(\sigma(T+X))$$

which is the desired result.

Q.E.D.

Consider $\sigma_{le}(T)$, $\sigma_{re}(T)$ the left-essential and the right-essential spectra of T and remark that if $\sigma(T+X)$ in the proposition above is bigger than $\sigma_{le}(T) \cap \sigma_{re}(T)$ then T+X has a non-trivial invariant subspace. This together with Proposition 3 gives the following corollary.

Corollary. *If T is an operator with $([T^*, T])_-$ of trace class and*

$$\mathrm{Tr}\,[T^*,T] > \frac{1}{\pi}\omega(\sigma_{le}(T) \cap \sigma_{re}(T))$$

then every operator T+X with X Hilbert-Schmidt has a non-trivial invariant subspace.

Consider also $E(\sigma(T))$ the polynomially convex hull of $\sigma(T)$, i.e., the complement of the unbounded component of $\mathbb{C} \setminus \sigma(T)$ and remark that for X a compact operator $\sigma(T+X) \cap (\mathbb{C} \setminus E(\sigma(T))$ is an at most countable set and hence $\omega(\sigma(T+X)) \le \omega(E(\sigma(T)))$. This together with Propositions 3 gives the following corollary

Corollary : *If T is an operator with $([T^*, T])_-$ of trace-class and if*

$$\mathrm{Tr}\,[T^*,T] > \frac{1}{\pi}\omega(E(\sigma(T)))$$

then $m(T+X) > 1$ for every Hilbert-Schmidt operator X.

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