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WEAK COMPACTNESS IN ORDERED BANACH SPACES

by

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Abstract. A number of distinguished properties such as the (reciprocal) Dunford-Pettis property, the Grothendieck property, the strict Dieudonné property etc., are discussed in the context of Banach lattices. A characterization in terms of absolute continuity and stability of the weakly compact operators defined on C^*-algebra is also included.

Introduction

In his remarkable paper devoted to the study of the weakly compact operators on C(S) spaces, A. Grothendieck [7] has proved that a bounded subset K of C(S)' is relatively weakly compact if, and only if, $\lim_{n\to\infty} \sup_{\mu\in K} \mu(O_n) = 0$ for all sequences of pairwise disjoint open subsets O_n of S. This was used later by H.P. Rosenthal [19] to formulate the concept of a relatively disjoint family of measures, which is instru-

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mental in his criterion of weak compactness in a space $L_1(\mu)$: A bounded subset K of $L_1(\mu)$ which is not weakly compact contains a basic sequence $(x_n)_n$ which is equivalent to the natural ℓ_1-basis and such that $\overline{\text{Span}}\,(x_n)_n$ is complemented in $L_1(\mu)$.

The main result of the first section extends this criterion in context of weakly sequentially complete Banach lattices and yields also the following dichotomy theorem : An operator T from a Banach space into a weakly sequentially complete Banach lattice is either weakly compact or the restriction of T to a complemented subspace which is isomorphic to ℓ_1 is an isomorphism .

Section 2 is devoted to characterizing the behavior of weakly compact operators defined on Banach lattices in terms of unconditional basic sequences.For example it is proved that every operator T defined on a Banach lattice which contains no complemented isomorph of ℓ_1 is weakly compact provided that T maps weak Cauchy sequences into norm converging sequences (the reciprocal Dunford-Pettis property).

.A special attention is paid to finding the conditions under which the following dichotomy result (a variation of the Dieudonné property) holds : Every operator from E into a Banach space F is either weakly compact or its restriction to a sub - space isomorphic to c_o is an isomorphism .

The paper ends with a non commutative extension of the main result in [13] (see also [15]) concerning the connection between weak compactness, absolute continuity and stability. As an easy consequence we obtain the following noncommutative Dunford-Pettis property : Let T be a weakly compact operator given on a C^*-algebra E. Then :

$$|x_n| \longrightarrow 0 \text{ in the weak topology of E implies } \|Tx_n\| \longrightarrow 0$$

Here the "modulus" of an element $z \in E$ is defined by

$|z| = \left(\frac{z^*z + zz^*}{2}\right)^{1/2}$. In the commutative case $x_n \xrightarrow{w} 0$ allways implies $|x_n| \xrightarrow{w} 0$, so that a weakly compact operator on a C(S) space maps weak converging sequences into norm converging sequences.

Our approach makes use of the classical results due to S. Kakutani, concerning the AM and AL spaces. Particularly we shall need the following construction.

Let E be a Banach lattice. For each $x \in E$, $x > 0$, we can consider the order ideal generated by x :

$$E_x = \{ y \in E ; (\exists)\ \lambda > 0 , |y| \leq \lambda x$$

normed by :

$$\|y\|_x = \inf \{ \lambda ; |y| \leq \lambda x \}$$

Then E_x is an AM space with a strong order unit (which is x) and thus lattice isometric to a space $C(S_x)$. We shall denote by $i_x : E_x \longrightarrow E$ the canonical inclusion.

For each $x' \in E'$, $x' > 0$, we can consider the following relation of equivalence on E

$$x \sim 0 \text{ if, and omly if , } x'(|x|) = 0$$

Then the completion of $E/\sim$ with respect to the norm

$$\|x\|_{x'} = x'(|x|)$$

is an AL space , say $L_1(x')$, and the canonical mapping $E \longrightarrow L_1(x')$ will be denoted by $j_{x'}$. An useful remark is that $(j_{x'})' = i_{x'}$ for each positive $x' \in E'$.

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1. A fundamental dichotomy

The main result of this section (see Theorem 1.4 below) generalizes an important result due to H.P.Rosenthal [18] which asserts that every operator $T \in \mathcal{L}(E, \ell_1(\Gamma))$ is weakly compact or the restriction of T to a complemented subspace isomorphic to $\ell_1(\Gamma)$ is an isomorphism. The essential ingredient in our proof is a weak compactness criterion in terms of sequences of pairwise disjoint elements.

A bounded subset K of a Banach space E is said to be relatively weakly sequentially complete if every weak Cauchy sequence of elements of K is weakly converging (to an element of E). A subset K of a Banach lattice E is said to be solid if $|y| \leqslant |x|$, $x \in K$ implies $y \in K$.

1.1 PROPOSITION. Let K be a bounded convex solid subset of a Banach lattice E. Then K is relatively weakly compact if, and only if , K verifies the following two conditions :

i) K is relatively weakly sequentially complete ;

ii) K contains no sequence of disjoint elements which is equivalent to the natural ℓ_1 - basis.

The proof is a consequence of the main result in [20] (which asserts that every bounded sequence of elements of a Banach space contains a subsequence which is either weak Cauchy or equivalent to the natural ℓ_1-basis) and the following :

1.2 LEMMA. Let K be a relatively weakly sequentially complete convex solid subset of a Banach lattice E. If $(x_n)_n$ is a sequence of elements of K which is equivalent to the natural ℓ_1-basis then there exist an increasing sequence of natural numbers k(n) and a sequence of pairwise disjoint elements $d_n \in K$ which is equivalent to the natural ℓ_1-basis and such that $|d_n| \leq |x_{k(n)}|$, $n \geqslant 1$.

Proof. Without loss of generality we may, and we do assume, that E is separable. Then E' contains strictly positive functionals φ i.e.

$$x \geqslant 0, \quad \varphi(x) = 0 \quad \text{implies} \quad x = 0$$

In this case the canonical mapping j_φ is one-to-one, so we can identify E as a vector subspace of $L_1(\varphi)$.

The remainder of the proof will be covered in four steps.

Step 1. If φ is strictly positive then K is an order ideal of $L_1(\varphi)$ i.e.,

$$x \in L_1(\varphi),\ y \in K \quad \text{and} \quad |x| \leq |y| \quad \text{implies} \quad x \in K$$

Indeed, it suffices to consider the case where $x, y > 0$. Since E is dense in the Banach lattice $L_1(\varphi)$, there exists a sequence of elements $x_n \in E$ such that $\| x_n - x \|_\varphi \to 0$. On the other hand

$$|(x_n \vee 0) \wedge x - x| \leqslant |x_n \vee 0 - x| \leqslant |x_n - x|$$

so that we can assume in addition that $0 < x_n \leqslant x$ and $\| x_n - x \|_\varphi \leqslant 2^{-n}$, $n \geqslant 1$. Put

$$z_n = \dot{\bigvee} \{x_k \ ; \ k \geqslant n\}$$

where the supremum is taken in E''. The sequence $\left(\bigvee_{k=m}^{m+n} x_k \right)_n$, formed by elements of K for all $m \geqslant 1$, is weak converging, so that $z_n \in E$. Since $0 < z_n \leqslant y$ it follows that $z_n \in K$ and in a similar way we can conclude that $z = \bigwedge z_n \in K$. Then :

$$\|x - z\|_\varphi = \lim_{n\to\infty} \|x - \bigvee_{k=n}^{\infty} x_k\|_\varphi \leq$$

$$\leq \lim_{n\to\infty} \|x - x_n\| = 0$$

and thus $x = z \in K$.

Step 2. If $\varphi \leq \psi$ are two strictly positive functionals then the identity of E extends to a mapping $L_1(\psi) \to L_1(\varphi)$ whose restriction to the closure of K in $L_1(\psi)$ is one-to-one.

In fact, we have to prove that

$$x \in \overline{K}, \quad x > 0 \quad \text{implies} \quad \varphi(x) > 0$$

For, choose a sequence of elements $x_n \in K$ with $0 \leq x_n \leq x$ and $\|x_n - x\|_\psi \leq 2^{-n}$ for each $n \in \mathbb{N}$. As above we can conclude that

$$\|x - \bigwedge_{k\geq n} x_k\|_\psi \to 0$$

Because the sequence of elements $z_n = \bigwedge_{k\geq n} x_k \in K$ is non decreasing and $\psi(x) > 0$, it follows that $z_n > 0$ for $n \geq n_0$ and thus $\varphi(x) \geq \varphi(z_{n_0}) > 0$.

Step 3. There exists a strictly positive functional $\varphi_0 \in E'$ and a subsequence $(x_{p(n)})_n$ with no weak converging subsequences in $L_1(\varphi_0)$. In fact, if the contrary is true, then given a strictly positive functional $\varphi \in E'$ there exists a subsequence $(x_{p(n)})_n$ which is weak converging to an $x \in L_1(\varphi)$. For each $x' \in E'$, $x' > 0$, the functional $\psi = \varphi \vee x'$ is also strictly positive so that $(x_{q(p(n))})_n \longrightarrow \tilde{x}$ in the weak topology of $L_1(\psi)$.

A moment's reflection shows that because of the continuity of the canonical mapping $L_1(\psi) \longrightarrow L_1(x')$ and the result in Step 2 the sequence $(x_{p(n)})_n$ must be weak Cauchy in E, in contradiction with the fact that the natural l_1-basis has no weak Cauchy subsequence.

Step 4. Let φ_o be a strictly positive functional as indicated in Step 3 i.e., the sequence $(x_n)_n$ (possibly a sub-sequence of it) has no weak converging subsequence in $L_1(\varphi_o)$. Since E is supposed to be separable, the Banach lattice $L_1(\varphi_o)$ can be viewed as an $L_1(\mu)$-space for μ a suitable positive Radon measure on a compact Hausdorff space S. Then a well known result due to Grothendieck yields the existence of a sequence of pairwise disjoint open subsets $D_n \subset S$ and a subsequence $(x_{k(n)})_n$ of $(x_n)_n$ such that

$$\inf \int_{D_n} |x_{k(n)}(t)| \, d\mu(t) = \delta > 0$$

See [7], Theorem 2, for details.

By Step 1 it follows that $d_n = x_{k(n)} \cdot \chi_{D_n} \in K$. Then $|d_m| \wedge |d_n| = 0 \ (m \neq n)$ and for every finite family of real numbers λ_n we have:

$$\|\sum \lambda_n d_n\| = \|\sum |\lambda_n| \cdot |d_n|\| \geq$$

$$\geq \|\varphi_o\|^{-1} \sum |\lambda_n| \cdot \varphi_o(|d_n|) \geq$$

$$\geq \delta \|\varphi_o\|^{-1} \sum |\lambda_n|$$

so that $(d_n)_n$ is equivalent to the natural l_1-basis, q.e.d.

We need also a combinatorial result due to H.P.Rosenthal [19] .The argument is so elegant and short that we include it here :

1.3 LEMMA.Let $(a_{i,j})_{i,j}$ be an infinite matrix of positive numbers such that

$$\sup_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} a_{i,j} < \infty$$

Then for each $\varepsilon > 0$ there exists an infinite subset $N_\varepsilon \subset \mathbb{N}$ such that

$$\sup_{i \in N_\varepsilon} \sum_{\substack{j \in N_\varepsilon \\ j \neq i}} a_{i,j} < \varepsilon$$

Proof. We can choose by induction , for $i = 1,2, \ldots$ positive integers k_i and infinite subsets N_i of positive integers, such that for all i :

a) If $i > 1$, then $k_i \in N_{i-1}$,$N_{i-1} \supset N_i$ and $k_{i-1} < k_i$;

b) $\sum_{j > \ell} a_{k_i,j} \leqslant 2^{-i}$ for all $\ell \in N_i$;

c) $a_{\ell,k_i} \leqslant 2^{-i}$ for all $\ell \in N_i$.

Then $N_\varepsilon = \{k_p, k_{p+1}, \ldots\}$ for $p = [-\lg_2 \varepsilon] + 1$, q.e.d.

We can now formulate the following criterion of weak compactness which extends previous results due to Grothendieck [7] and Rosenthal [19] :

1.4 THEOREM.Let K be a bounded subset of a Banach lattice E ,whose convex solid hull is relatively weakly sequentially complete.The following assertions are equivalent :

(wC) K is relatively weakly compact ;

(R) There is no sequence of elements $x_n \in K$ which is equivalent to the natural ℓ_1-basis and $\overline{\mathrm{Span}}\,(x_n)_n$ is complemented in E ;

(G') If $(x'_n)_n$ is a weakly summable sequence of pairwise disjoint positive elements of E' then

$$\lim x'_n(x) = 0$$

uniformly for $x \in K$.

Proof. Clearly (wC) $\Longrightarrow$ (R).

(R) $\Rightarrow$ (G'). In fact, if the contrary is true, then there are a weakly summable sequence of pairwise disjoint norm 1 elements $x'_n \in E'$, $x'_n > 0$, and a sequence of elements $x_n \in K$ such that $\inf |x'_n(x_n)| = \delta > 0$. Put

$$\lambda = \sup \{\|x\| \; ; \; x \in K\}$$

$$\mu = \sup \left\{ \sum |x''(x'_n)| \quad ; \quad x'' \in E'', \|x''\| \leq 1 \right\}$$

and

$$\varepsilon = (1/2) \inf \left\{ \delta, \delta^3 \lambda^{-2} \mu^{-1} \right\}$$

By Lemma 1.3 we can assume in addition that

$$\sum_{i \neq n} x'_i(|x_n|) < \varepsilon \quad , n \geq 1$$

Then for every finite family of scalars λ_n and for $\varepsilon_n = \operatorname{sign} \lambda_n x'_n(x_n)$ we obtain that

$$\mu \lambda \sum |\lambda_n| \geq \mu \| \sum \lambda_n x_n \| \geq |< \sum \lambda_n x_n , \sum \varepsilon_i x'_i >| \geq$$

$$\geq \delta \sum |\lambda_n| - \sum_n \sum_{i \neq n} |\lambda_n| \, x'_i (|x_n|) \geq$$

$$\geq \frac{\delta}{2} \sum |\lambda_n|$$

which implies that the sequence $(x_n)_n$ is equivalent to the natural basis of ℓ_1. Let P_n be the band projection generated by x'_n in E'. The functionals $d_n(.) = P_n(.)|x_n|$ are pairwise disjoint, $0 \leq d_n \leq |x_n|$ and in addition

$$d_n(x'_n) \geq \delta \quad \text{and} \quad d_n(x'_m) = 0 \ (m \neq n)$$

which implies that the sequence $(d_n)_n$ is equivalent to the natural basis of ℓ_1. Moreover, there is a positive projection Q of E'' onto $\overline{\text{Span}}\,(d_n)_n$ given by

$$Q(x'') = \sum_{n=1}^{\infty} \frac{x''(x'_n)}{d_n(x'_n)} \cdot d_n$$

The isomorphism $U : \overline{\text{Span}}\,(x_n)_n \longrightarrow \overline{\text{Span}}(d_n)_n$ given by $U(x_n) = \frac{x'_n(x_n)}{d_n(x'_n)}\, d_n$, $n \geq 1$, verifies $\| Ux - Qx \| \leq \frac{1}{2} \| Ux \|$ for all $x \in \text{Span}\,(x_n)_n$. Consequently the operator $P = (Q|\overline{\text{Span}(x_n)_n})^{-1} \circ Q$ provides a projection of E'' onto $\overline{\text{Span}}\,(x_n)_n$.

(G') $\Longrightarrow$ (wC) (The short argument here is due to Ju. Abramovich).

Without loss of generality we can suppose that K is also convex and solid. If K is not relatively weakly compact then by our Proposition 1.1 above there exists a pairwise disjoint sequence of elements $d_n \in K$ which is equivalent to the natural ℓ_1-basis. For the sake of convenience we shall identify $\text{Span}(d_n)$ as ℓ_1. Let $u' \in E'$ a positive extension of $1 = (1,1,\ldots) \in \ell_\infty$ to E. Put $d'_n = u' \circ [d_n] | E$, $n \geq 1$, where $[d_n]$ denotes the band projection generated by d_n in E''. For each $x \in E$,

$x > 0$ we have

$$\sum d'_n(x) \leq u''(x)$$

so that the sequence $(d'_n)_n$ is weakly summable and formed by pairwise disjoint functionals. But $d'_n(d_n) = 1$, $n \geq 1$, in contradiction with (G') , q.e.d.

1.5 COROLLARY. An operator T from a Banach space E into a wekly sequentially complete Banach lattice L is either weakly compact or the restriction of T to a complemented subspace of E which is isomorphic to ℓ_1 is an isomorphism.

The case where L is a space $\ell_1(\Gamma)$ (originally due to H.P.Rosenthal [18]) gives us information on the existence of the weak order units :

1.6 THEOREM. Let E be a Banach space whose dual is isomorphic to a Banach lattice and which is contained in the band generated by a suitable $u'' \in E''$. If E contains no complemented isomorph of ℓ_1 and A is a closed subspace of E' then either

i) A is contained in a weakly compactly generated sub - lattice of E' having a weak order unit ; or ,

ii) A contains an isomorph of a nonseparable $\ell_1(\Gamma)$ space which is complemented in E'.

The interested reader is refered to [16] and [14] for details .

2. The reciprocal Dunford-Pettis property and other distinguished properties

According to [7] a Banach space E is said to have the reciprocal Dunford-Pettis property if every operator T from E into another Banach space is weakly compact provided that T maps weak converging sequences into norm converging sequences. The classical examples of such spaces are C(S) spaces and reflexive spaces. Notice that the reciprocal Dunford-Pettis property is stable under finite products or passing to quotients.

A complete characterization in context of Banach lattices is now given :

2.1 THEOREM. A Banach lattice E has the reciprocal Dunford-Pettis property if, and only if, E contains no complemented isomorph of ℓ_1, if and only if E contains no lattice isomorph of ℓ_1.

Particularly we obtain some few more Banach spaces with the reciprocal Dunford-Pettis property such as

$$\Big(\sum_{n=1}^{\infty}\oplus\, \ell_p(n)\Big)_{c_0}\,, \qquad 1 \le p < \infty$$

An useful remark is that ℓ_1 is complemented in any Banach lattice which contains it as a sublattice. In fact, for each closed sublattice $\overline{\mathrm{Span}}(d_n)_n \subset E$, which is lattice isomorphic to ℓ_1, there is a positive functional $x' \in E'$ with $\inf x'(d_n) > 0$. Then $\overline{\mathrm{Span}}(j_{x'}(d_n))_n$ is a closed sublattice of $L_1(x')$ and thus complemented in $L_1(x')$,If Q denotes a positive projection of $L_1(x')$ onto $\overline{\mathrm{Span}}(j_{x'}(d_n))_n$, then $P = (j_{x'})^{-1} Q \circ j_{x'}$ provides a positive projection of E onto $\overline{\mathrm{Span}}(d_n)_n$.

Then the proof of Theorem 2.1 is an immediate consequence of this fact and the following :

2.2 LEMMA. Let E be Banach lattice which contains no lattice isomorph of ℓ_1 . Then every bounded subset $K \subset E'$ is relatively weakly compact provided that

$$\text{(rDP)} \qquad \lim_{n\to\infty} \sup_{x'\in K} |x'(x_n)| = 0$$

for every weakly converging sequence of pairwise disjoint elements $x_n \in E$.

The proof of Lemma 2.2 makes use of an idea due to P.Mayer-Nieberg [12] .Since E contains no complemented isomorph of ℓ_1 it follows that E' is weakly sequentially complete.See [10] ,[11] or [12] for details .Notice also that (rDP) holds for the convex solid hull $\widetilde{K}$ of K .We shall prove that $\widetilde{K}$ is relatively weakly compact.In fact , if the contrary is true, then by Proposition 1.1 there exists a pairwise disjoint sequence of positive elements $d'_n \in K$, which is equivalent to the unit vector basis of ℓ_1 . Let $(x_n)_n$ be a sequence of norm 1 positive elements of E such that

$$\inf d'_n(x_n) = \alpha > 0$$

Put $x = \sum 2^{-n} x_n$. Then E_x is lattice isometric to a space $C(S_x)$ and the Radon measures $\mu_n = x_n \cdot d'_n \in C(S_x)'$ constitute a basic sequence equivalent to the unit vector basis of ℓ_1. In fact, $\mu_m \wedge \mu_n = 0$ for $m \neq n$ and $\|\mu_n\| = \mu_n(1) = d'_n(x_n) \geq \alpha$. Then it is easy to check the existence of a sequence of pairwise disjoint elements $y_n \in E_x$, $0 < y_n \leq x$, such that (by passing to a subsequence if necessary):

$$\inf \int_{S_x} y_n(t)\, d\mu_n(t) > 0$$

See also [7], Theorem 2. The elements $d_n = x_n \cdot y_n \in E_x \subset E$ are pairwise disjoint, $\|d_n\| \leq 1$ for all n and $\inf d'_n(d_n) > 0$.

To conclude it suffices to recall an argument due to Ju.A. Abramovich which yields that $d_n \longrightarrow 0$ in the weak topology of E. See [1], Lemma 3.1. In fact, if we suppose the contrary, then there exists an $x' \in E'$, $x' > 0$, with $\inf x'(d_n) = c > 0$. Then for every finite family of scalars λ_n we have

$$\sum |\lambda_n| \geq \|\sum \lambda_n d_n\| = \|\sum |\lambda_n|\, d_n\|$$
$$\geq \frac{1}{\|x'\|} \sum |\lambda_n|\, x'(d_n) \geq$$
$$\geq \frac{c}{\|x'\|} \sum |\lambda_n|$$

in contradiction with the fact that E contains no lattice isomorph of ℓ_1, q.e.d.

Another distinguished property introduced in [7] is the so called Dieudonne' property: a Banach space E is said to have the Dieudonné property if every operator T from E into another Banach space F is weakly compact provided that T maps weak Cauchy sequences into weak converging sequences. In the sequel we shall be concerned with a slightly stronger condition of weak compactness related to earlier work of Grothendieck [7] and Pelczyński [17] on C(S) spaces.

2.3 DEFINITION. A Banach space E has the strict Dieudonné property (abbreviated sD) if E verifies the following equivalent conditions :

(P) Every operator from E into a Banach space F is either weakly compact or the restriction of T to a subspace isomorphic to c_o is an isomorphism;

(G) A bounded subset $K \subset E'$ is relatively weakly compact if (and only if) for every weakly summable sequence of elements $x_n \in E$ we have

$$\lim_{n \to \infty} x'(x_n) = 0$$

uniformly for $x' \in K$.

It is clear that a Banach space with sD contains no complemented isomorph of ℓ_1 and we conjecture that sD means precisely this fact.

The strict Dieudonné property is stable under finite products or passing to quotients.

2.4 THEOREM. Let E be a Banach lattice which contains no complemented copy of ℓ_1. Then each of the following conditions implies that E has the strict Dieudonné property :

a) E contains no reflexive sublattice ;

b) For each $x'' \in E''$ there exists an $x \in E$ with $|x''| \leq |x|$;

c) E is σ-complete and with σ-continuous norm (e.g., E has an unconditional basis).

Proof. a) We need a more careful selection of the elements d_n in the proof of Lemma 2.2 above, namely to observe that the following two possibilities occur :

i) $\overline{\text{Span}}(d_n)_n$ contains an isomorphic copy of c_o.

Then $\overline{\text{Span}}(d_n)_n$ contains a block basic sequence ;

$$z_n = \sum_{k=p_n+1}^{p_{n+1}} \lambda_k d_k > 0 , \quad n \in \mathbb{N}$$

which is equivalent to the natural basis of c_o and thus $(z_n)_n$ is weakly summable. See [6], Theorem 2, page 96, for details.

Moreover

$$z'_n = \frac{1}{p_{n+1} - p_n} \sum_{k=p_n+1}^{p_{n+1}} d'_k \in \widetilde{K}$$

and $\qquad \inf z'_n(z_n) = \inf \| z_n \| > 0$

ii) $\overline{\text{Span}}(d_n)_n$ contains no isomorphic copy of c_o and thus by Theorem 4 in [6], page 98, $\overline{\text{Span}}(d_n)_n$ constitutes a reflexive sublattice of E.

b),c). It suffices to prove that a bounded subset $K \subset E'$ is relatively weakly compact provided that

$$\lim_{n\to\infty} \sup_{x' \in K} |x'(x_n)| = 0$$

for every weakly summable sequence of pairwise disjoint elements $x_n \in E$. Without loss of generality we can suppose that K is also solid.

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Let d'_n be as in the proof of Lemma 2.2 above, $n \in \mathbb{N}$. For the sake of convenience we shall identify $\overline{\text{Span}}\,(d'_n)_n$ as ℓ_1. Let $u'' \in E''$ a positive extension of $1 = (1, 1, \dots) \in \ell_\infty$ to E' and put $z_n = u'' \circ [d'_n] \in E''$, $n \geq 1$, where $[d'_n]$ denotes the band projection generated by d'_n in E'. The sequence $(z_n)_n$ is weakly summable and formed by pairwise disjoint elements. Then the proof ends in the case c) by observing that

$$x''(x') = \sup_{\substack{0 < x < x'' \\ x \in E}} x'(x)$$

for all $x' \in E'$ and $x'' \in E''$. An alternative proof is obtained by combining our Lemma 2.2 above with Lemma 2 in [23], which asserts that for every weak Cauchy sequence $(x_n)_n$ in E there exists an weakly summable sequence $(y_n)_n$ in E so that $\left(x_n - \sum_{k=1}^{n} y_k\right)_n$ converges weakly to 0.

In the case b) let us denote by u a positive element of E such that $u \geq u''$. Then we can proceed as in the proof of Lemma 2.2 above by choosing $x_n = u / \|u\|$ for each $n \geq 1$, q.e.d.

2.5 COROLLARY. Let E be a σ-complete Banach lattice such that E' has the Schur property (i.e., every weakly converging sequence of elements of E' is norm converging). Then E has the strict Dieudonné property.

The same is true for E a σ-complete Banach lattice whose dual has the Radon-Nikodym property (i.e., every integral operator with values in E^* is nuclear).

Proof. Since E is σ-complete and contains no isomorphic copy (equivalently, no complemented isomorph) of ℓ_∞ it follows from [10] (see also [11] or [12]) that E has order continuous norm and thus applies Theorem 2.4(c) above, q.e.d.

Particularly from the result above it follows that the Banach space

$$\mathcal{J} = \left(\sum_{n=1}^{\infty} \oplus\, \ell_1(n)\right)_{c_0}$$

has the strict Dieudonné property. The space $\mathcal{J}$ can be isometrically embedded in c_0 e.g., by using the mapping

$$(\alpha_n)_n \longrightarrow (\alpha_1, \alpha_1+\alpha_2, \alpha_1-\alpha_2, \alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2-\alpha_3, \alpha_1-\alpha_2+\alpha_3, -\alpha_1+\alpha_2+\alpha_3, \dots)$$

2.6 THEOREM. Let E be a Banach space isomorphic to a subspace or to a quotient of c_0. Then E has the strict Dieudonné property.

Proof. The nontrivial case is where E is a subspace of c_0. Then E^* is separable and thus every bounded sequence of elements of E contains a weak Cauchy subsequence. If $T \in \mathcal{L}(E,F)$ is not weakly compact then there exists a weak converging sequence $(x_n)_n$ in E such that such that $\|Tx_{n+1} - Tx_n\| \geq \alpha > 0$ for each $n \geq 1$. According to Lemma 9 in [7], the restriction of T to a subspace isomorphic to c_0 is an isomorphism, q.e.d.

A Banach space E is said to have the Grothendieck property if every weak'-converging sequence in E' is weak converging. Grothendieck has noted in [7] that this is the case if E is a space C(S) with S a Stonean compact space.

A complete characterization in context of Banch lattices is now given :

2.7 THEOREM. The following statements are equivalent for E a Banach lattice :

a) E has the Grothendieck property ;

b) Every operator from E into a separable Banach space is weakly compact ;

c) Every operator $T \in \mathcal{L}(E, c_0)$ is weakly compact ;

d) E has the strict Dieudonne property and E contains no complemented isomorph of c_0.

Proof. Clearly , a) $\Longleftrightarrow$ b) $\Longleftrightarrow$ c).

a) $\Longrightarrow$ d). We have only to verify the weak compactness criterion (G') in Definition 2.3 above. For, notice first that because E contains no complemented isomorph of ℓ_1 it follows that E' contains no isomorphic copy of c_0 and thus E' is weakly sequentially complete. If $(d'_n)_n \subset E'$ is a pairwise dijoint sequence which is equivalent to the natural ℓ_1 basis then

$$\overline{\lim_{n\to\infty}}\, d'_n(d) \neq 0$$

for some $d \in E$ (E has the Grothendieck property and $(d'_n)_n$ is not weak Cauchy), so that it remains to repeat the argument of Lemma 2.2 with $x = x_n = d$ for all $n \in \mathbb{N}$.

d) $\Longrightarrow$ c). Let $T \in \mathcal{L}(E, c_o)$. Because E has the strict Dieudonne property it follows that T is either weakly compact or the restriction of T to a subspace F , isomorphic to c_o , is an isomorphism. Because c_o is separably injective (Sobczyk's main result in [21]), there exists a bounded projection P of c_o onto T(F). Then $Q = (T|F)^{-1} \circ P \circ T$ provides a bounded projection of E onto F , in contradiction with the fact that E contains no complemented isomorph of c_o , q.e.d.

From the result above it follows that a space C(S) has the Grothendieck property if, and only if, C(S) contains no complemented isomorph of c_o .

The proof of Theorem 2.7 above suggests also a positive answer to the following extension of Sobczyk's result (which constitutes the case where $E = C[0,1]$:

2.8 PROBLEM. Let E be a Banach lattice which contains no complemented isomorph of ℓ_1 and ℓ_∞ and let F be a subspace isomorphic to c_o . Is F complemented in E ?

3. Absolute continuity and stability

The aim of this section is to precise a result in [13] concerning the connection between weak compactness, absolute continuity and stability.

Recall that a sequence $(x_n)_n$ of elements of a Banach space E is stable (with limit x) if there exists an $x \in E$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} x_{k_i} - x \right\| \longrightarrow 0$$

uniformly in the set of all strictly increasing sequences $(k_n)_n$ of natural numbers. This concept comes from ergodic theory (a measure-preserving point transformation τ is stable if for every $f \in L_2[0,1]$ the sequence $(f \circ \tau^n)_n$ is stable in the Banach space $L_2[0,1]$) and was first studied by Brunel and Sucheston [4]. Stability implies reflexivity but the converse is known to fails. See [3] for details.

Given a positive functional φ on a C^*-algebra E, we can consider the following relation of equivalence

$$x \sim y \quad \text{iff} \quad \varphi[(x-y)^*(x-y)+(x-y)(x-y)^*]=0$$

Then the completion $L_2(\varphi)$ of $E/\sim$ with respect to the norm

$$\|x\|_{L_2(\varphi)} = \varphi^{1/2}\left(\frac{x^*x+xx^*}{2}\right)$$

verifies the parallelogram's law and thus is a Hilbert space.

The following result suggests that every weakly compact operator defined on a C^*-algebra can be factored through a superreflexive space. Compare to the main result in [5].

3.1 THEOREM. Let E be a C^*- algebra and let T be an operator from E into a Banach space F. The following assertions are equivalent :

(wC) T is weakly compact

(AC_2) There exists a positive functional $\varphi \in E'$ such that

$$\|T(.)\| \leq \varepsilon \|\cdot\| + \delta(\varepsilon) \|\cdot\|_{L_2(\varphi)}$$

for each $\varepsilon > 0$;

(st) T maps bounded sequences into sequences with stable subsequences.

Clearly, an appropriate result is true for E a Banach space whose second dual is complemented in a C^*-algebra. Particularly this is the case if E is an $\mathcal{L}_\infty$- space in the sense of Lindenstrauss and Pelczynski [9] and thus the above result extends partially our Theorem 2.3 in [15].

Proof. (wC) $\Longrightarrow$ (AC_2) Indeed, the second dual of a C^* algebra is a von Neumann algebra and a result due to C.A. Akemann [2] yields a normal form φ on E'' such that for each $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ for which

$$x \in E, \ \|x\| \leq 1 \ \text{ and } \ \varphi(x^*x + xx^*) < \delta \ \text{ implies } \|Tx\| < \varepsilon$$

Consequently $\|T(.)\| \leq \varepsilon \|\cdot\| + \frac{\varepsilon}{\sqrt{\delta}} \|\cdot\|_{L_2(\varphi)}$ for all $\varepsilon > 0$.

$(AC_2) \Longrightarrow$ (st). A result due to Brunel and Sucheston [4] asserts that every bounded sequence of elements of a Hilbert space has a stable subsequence and thus thus the proof is an immediate consequence of the following estimate ,guaranted by (AC_2) :

$$\left\| \frac{1}{\operatorname{Card} I} \sum_{i\in I} Tx_i - \frac{1}{\operatorname{Card} J} \sum_{j\in J} Tx_j \right\| \leq$$

$$\leq \varepsilon \sup \|x_n\| + \delta(\varepsilon) \left\| \frac{1}{\operatorname{Card} I} \sum_{i\in I} x_i - \frac{1}{\operatorname{Card} J} \sum_{j\in J} x_j \right\|_{L_2(\varphi)}$$

Here I and J denotes two arbitrary finite subsets of $\mathbb{N}$.

(st) $\Longrightarrow$ (wC).If $T \in \mathcal{L}(E,F)$ is not weakly compact, then a result due to Lindenstrauss and Pelczynski [9] yields the existence of two operators $S_1 \in \mathcal{L}(\ell_1, E)$ and $S_2 \in \mathcal{L}(F, \ell_\infty)$ such that

$$S_2 \circ T \circ S_1 = \sigma$$

where $\sigma : \ell_1 \to \ell_\infty$ is given by

$$\sigma((\alpha_n)_n) = \left(\sum_{k=1}^{n} \alpha_k\right)_n$$

Since σ is not stable , nor T can be stable ,q.e.d.

3.2 COROLLARY.Let T be a weakly compact operator given on a C^*-algebra E , with values in a Banach space F. Then T maps weakly unconditionally convergent series of positive elements into norm unconditionally convergent series .

3.3 COROLLARY (non commutative Dunford-Pettis property)

Let T be a weakly compact operator given on a C^*-algebra E. Then :

$|x_n| \longrightarrow 0$ in the weak topology of E implies $\|Tx_n\| \longrightarrow 0$

Here the nonstandard modulus of an element z E is defined by

$$|z| = \left(\frac{z^*z + zz^*}{2}\right)^{1/2},$$

In the commutative case $x_n \xrightarrow{w} 0$ allways implies $|x_n| \xrightarrow{w} 0$, so that a weakly compact operator given on a C(S) space maps weak converging sequences into norm converging sequences .

For sequences in the unit sphere of a Hilbert space weak and norm convergence agree i.e., if $(x_n)_n$ is a sequence which converges weakly to x and if $\|x_n\| \longrightarrow \|x\|$ then $\|x_n - x\| \longrightarrow 0$. Due to the condition (AC_2) above the following result (of the Radon-Riesz type) is true :

3.5 THEOREM. Let E be a quotient space of a Banach space whose second dual is complemented in a C^*-algebra and let T be a weakly compact operator defined on E .

If $(x_n)_n$ is a sequence which converges weakly to x in E and if $\|x_n\| \longrightarrow \|x\|$ then $\|Tx_n - Tx\| \longrightarrow 0$.

Theorem 3,1 above has a companion in the setting of ordered Banach spaces which was first noted in [15].

3.6 THEOREM. Let E be a Banach lattice whose dual is weakly sequentially complete and such that E is contained in the band generated by a suitable $u'' \in E''$. If T is a weakly compact operator from E into a Banach space F then :

(AC_1) There exists an $u' \in E'$, $u' > 0$ such that

$x \in E$, $|x| \leq x''$, $u'(|x|) \leq \delta(\varepsilon, x'')$ implies $\|T(x)\| \leq \varepsilon$

for every $\varepsilon > 0$ and every $x'' \in E''$, $x'' > 0$; and

(st) If $(x_n)_n$ is a sequence of elements of E which is order bounded in E'' then $(Tx_n)_n$ has a stable subsequence .

We do not know wheather the condition (st) in Theorem 3.6 above implies that T is weakly compact .The answer is affirmative at least for Banach lattices as in Theorem 2.4 .

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