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STOCHASTIC INTEGRAL EQUATIONS IN THE PLANE

by

Constantin TUDOR

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CONTENTS

Introduction

In the present work we shall consider the following stochastic integral equation:

$$(1) \quad x_{s,t}=x+\int_0^s\int_0^t a(u,v,x_{u,v})dM_{u,v}+\int_0^s\int_0^t b(u,v,x_{u,v})dudv$$

where $(M_{s,t})$ is a two-parameter martingale (a Wiener-Yeh process in particular) and $\int_0^s\int_0^t \phi(u,v)dM_{u,v}$ is the stochastic integral in the Ito sense.

The theory of multiparameter martingales is an field insufficient developed untill now, but enough is known to follow the usual construction of the Ito integral.

The basic references for this martingales are the following papers: CAIROLI [2], [16] (where are investigated the convergence and the Doob decomposition theorems on a product probability space), CAIROLI and WALSH [3] (where is established the Doob decomposition theorem for martingales and strong martingales on a general probability space) and GIHMAN [4] (where are proved the convergence and the Doob decomposition theorems for strong martingales), WONG and ZAKAI [14], [15].

The stochastic integral with respect to the Wiener-Yeh process is defined in CAIROLI [17], PONOMARENCO [5], TARENCO [12], and with a two parameter martingale in general by CAIROLI and WALSH in their fundamental paper [3].

In Section 1 we give, first, the definition for two-parameter martingale, weak, 1 and 2, strong (or)-martingale, then we discus some general properties of them.

Also, in this section are formulated criteria in order to a given strong martingale to have other given process as increasing process (theorem 1.8).

This criteria are particularized to the Wiener-Yeh process (corollary 1.9).

In Section 2 we introduce the stochastic integral with respect to a two-parameter square martingale and we establish some properties of it.

In Section 3 we utilise the technique of optional stopping to prove the existence and the uniqueness of a strong solution of 1) for a,b that satisfy a Lipschitz condition and $(M_{s,t})$ a continuous martingale with the increasing process of the form $[M]_{s,t}=\int_o^s\int_o^t h(u,v)dudv$ where h is continuous and adapted(theorem 3.1).

This result generalises theorem 2 from [5] and theorem 1 from [13]. It is shown that the solution of 1) satisfies a Markov property(theorem 3.6).

Also we consider the question of convergence, allowing the coefficients to converge, the differential to converge.

In all cases the solutions converge in maximal quadratic mean to the solution of the limiting equation(theorems 3.7,3.8,3.9).

In Section 4 we prove the existence of a weak solution of 1) for a,b continuous and bounded and $M_{s,t}$ a continuous martingale with the increasing process of the form $[M]_{s,t}=\int_o^s\int_o^t h(u,v)dudv$ where h is adapted and continuous(theorem 4.3).

In particular when M is a Wiener-Yeh process we improve the previous result by proving the existence of a weak solution of 1) for a,b measurable and bounded and $a(s,t,x)\geq C>o$. for every s,t,x.

The uniqueness in law of the weak solution of 1) with M a Wiener-Yeh process is true if 1) has an unique strong solution.

We formulate the martingale problem for the two-parameter case(in one-dimensional parameter this problem was formulated by Stroock-Varadhan in [7]) and we prove that there exists an equivalence between the existence(and the uniqueness) of a weak solution of 1) and of the solution to the martingale problem(theorem 4.12).

In Section 5 we prove, if the stochastic equation 1) has an unique in law weak solution, the weak convergence of Markov with two-parameters to the law of this solution(theorem 5.1o).

1. Two-Parameter Martingales

Let $(\Omega,\mathcal{F},P)$ be a probability space and $(\mathcal{F}_{s,t})_{s,t\in[0,1]}$ be an increasing family of sub-σ-fields of $\mathcal{F}$ (fixed from now on) satisfying:

i) $\mathcal{F}_{0,0}$ contains all the null sets of $\mathcal{F}$.

ii) $\mathcal{F}_{s,t}=\bigcap\limits_{\substack{u>s\\ v>t}}\mathcal{F}_{u,v}$ for each s,t (i.e $(\mathcal{F}_{s,t})$ is right continuous).

iii) (Cairoli's condition): $\mathcal{F}_{s,1}$ and $\mathcal{F}_{1,t}$ are conditionally independent relative to $\mathcal{F}_{s,t}$ for each s,t or equivalently $M[\cdot/\mathcal{F}_{s,t}/\mathcal{F}_{u,v}]=$ $=M[\cdot/\mathcal{F}_{s\wedge u,t\wedge v}]$ for each s,t,u,v.

As a consequence of iii) we have $\mathcal{F}_{s,t}=\mathcal{F}_{s,1}\cap\mathcal{F}_{1,t}$.

For every s,t we put $\overline{\mathcal{F}}_{s,t}=\mathcal{B}(\mathcal{F}_{s,1}\cup\mathcal{F}_{1,t})$ and if $(x_{u,v})$ is a process and $s<s'$, $t<t'$ then we put $\square_{s,t}x_{s',t'}=x_{s',t'}-x_{s',t}-x_{s,t'}+x_{s,t}$.

Sometimes we shall use the notation $X((s,s']\times(t,t'])$ instead $\square_{s,t}x_{s',t'}$.

DEFINITIONS Let $x=(x_{s,t})_{s,t\in[0,1]}$ be a process.

a) x is a weak martingale if

a_1) $x_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and integrable for every s,t.

a_2) $M[\square_{s,t}x_{s',t'}/\mathcal{F}_{s,t}]=0$ for every $s<s'$, $t<t'$.

b) x is a martingale if

b_1) $x_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and integrable for every s,t.

b_2) $M[x_{s',t'}/\mathcal{F}_{s,t}]=x_{s,t}$ for every $s<s'$, $t<t'$,

c) x is an 1-martingale (2-martingale) if

c_1) $x_{s,t}$ is $\mathcal{F}_{s,1}$-measurable ($\mathcal{F}_{1,t}$-measurable) and integrable for every s,t.

c_2) $M[\square_{s,t}x_{s',t'}/\mathcal{F}_{s,1}]=0$ $(M[\square_{s,t}x_{s',t'}/\mathcal{F}_{1,t}]=0)$ for every $s<s'$, $t<t'$.

d) x is a strong martingale (or $\square$-martingale) if

d_1) $x_{s,t}$ is $\mathcal{F}_{s,t}$-measurable and integrable for every s,t.

d_2) $M[\square_{s,t}x_{s',t'}/\overline{\mathcal{F}}_{s,t}]=0$ for every $s<s'$, $t<t'$.

e) x is an increasing process if

e_1) x vanishes on $I_0=[0,1]\times\{0\}\cup\{0\}\times[0,1]$.

e_2) $x_{s,t}$ is $\mathcal{F}_{s,t}$-measurable for every s,t.

e_3) $\square_{s,t} x_{s',t'} \geq o$ for every $s<s', t<t'$.

REMARK 1.1 Every martingale is an i-martingale.

Moreover if $(x_{s,o}, \mathcal{F}_{s,1}, o\leq s\leq 1)$, $(x_{o,t}, \mathcal{F}_{1,t}, o\leq t\leq 1)$ are martingales, the converse is also true.

For $p\geq 1$ let $\mathcal{M}^p$ be the class of all right-continuous martingales $x=(x_{s,t})_{s,t\in[o,1]}$ such that x vanishes on I_o and $M(|x_{s,t}|^p)<\infty$ for all s,t.

Let $\mathcal{M}^p_c$ ($\mathcal{M}^p_{cs}$) denote the class of continuous (strong) martingales in $\mathcal{M}^p$.

THEOREM 1.2 (see [2], th.1.p.3) If $(x_{s,t})\in\mathcal{M}^p$, $p>1$, then the following inequality holds:

$$M(\sup_{s,t}|x_{s,t}|^p)\leq\left(\frac{p}{p-1}\right)^{2p}M(|x_{11}|^p)$$

The Doob-Meyer martingale decomposition theorem holds in the following weakned form.

THEOREM 1.3 (see [3], th.1.5, p.117) If $x\in\mathcal{M}^2$ then there exists an increasing process $(a_{s,t})$ such that $(x^2_{s,t}-a_{s,t})$ is a weak martingale.

In order to give the form of the above theorem for strong martingales we need the notion of previsibility for two-parameter process.

Let $(\mathcal{R}_{s,t})_{s,t\in[o,1]}$ be a right continuous stochastic basis.

We define the Borel field $\mathcal{P}_{\mathcal{R}}$ of $\mathcal{R}_{s,t}$-previsible sets by

$$\mathcal{P}_{\mathcal{R}} = \mathcal{B}\{(s,s']\times(t,t']\times A \,/\, s<t, s'<t', A\in\mathcal{R}_{s,t}\}.$$

A process $(x_{s,t})$ is previsible if the map $(s,t,\omega)\longmapsto x_{s,t}(\omega)$ is $\mathcal{P}_{\mathcal{R}}$-measurable.

THEOREM 1.4 (see [3], th.1.8, p.121) If $x\in\mathcal{M}^2_s$ then there exists an unique increasing $\mathcal{F}^i_{s,t}$-previsible process $([x]^i_{s,t})$ such that

$(x^2_{s,t}-[x^i]_{s,t})$ be an i-martingale(here $\mathcal{F}^1_{s,t}=\mathcal{F}_{s,1}, \mathcal{F}^2_{s,t}=\mathcal{F}_{1,t}$).

THEOREM 1.5(see [3] ,th,1.9,p.122 and [4] ,prop.5,p.61)

a)If $\mathcal{F}_{s,t}=\mathcal{B}(w_{u,v}/u\le s, v\le t)$, where $(w_{u,v})$ is a Wiener-Yeh process or if $x\in\mathcal{M}^4_s$, and x is continuous, then there exists an unique $\mathcal{F}_{s,t}$-previsible process $([x]_{s,t})$ such that $(x^2_{s,t}-[x]_{s,t})$ be a martingale.

b)If $x\in\mathcal{M}^2_s$, x continuous and the family of random variables

$$\sigma_n^2=\sum_{j,k=0}^{n-1} M[(\square_{\frac{j}{n}s,\frac{k}{n}t}\, x_{\frac{j+1}{n}s,\frac{k+1}{n}t})^2/\mathcal{F}_{\frac{j}{n}s,\frac{k}{n}t}],\ n=1,2,\dots$$

is uniformly integrable, then there exists an unique continuous increasing process $[x]_{s,t}$ such that

$$M[(\square_{s,t}x_{s',t'})^2/\mathcal{F}_{s,t}]=M[\square_{s,t}[x]_{s',t'}/\mathcal{F}_{s,t}]$$

for every $s<s', t<t'$(in this part of the theorem the hypothesis i-iii are not necessary)

Particulary $(x^2_{s,t}-[x]_{s,t})$ is a martingale.

PROPOSITION 1.6 If $(x_{s,t})\in\mathcal{M}^1(\in\mathcal{M}^1_s)$ and T is a $\mathcal{F}_{s,s}$-stopping time then the process $(y_{s,t})=(x_{s\wedge T,t\wedge T})\in\mathcal{M}^1\ (\in\mathcal{M}^1_s)$.

Proof We prove only the first part of the proposition.

It is sufficient to prove that $(y_{s,t})$ is an i-martingale.

First it easy to se that $(y_{s,t})$ is $\mathcal{F}_{s,t}$-measurable.

Now consider the sequence (T_n) of $\mathcal{F}_{s,s}$-stopping times, that decrease to T, defined by $T_n=\frac{k+1}{2^n}$ if $\frac{k}{2^n}\le T\le\frac{k+1}{2^n}$, $k=0,1,\dots,2^n-1$.

Let $y^{(n)}_{s,t}=x_{s\wedge T_n, t\wedge T_n}$, $D_n=\{(\frac{k}{2^n},\frac{l}{2^n})/k,l=0,1,\dots 2^n-1\}$ be.

We claim that:

$$(1)\quad M[\square_{\frac{k}{2^n},\frac{l}{2^n}}\, y^{(n)}_{\frac{k+1}{2^n},\frac{l+1}{2^n}}/\mathcal{F}^i_{\frac{k}{2^n},\frac{l}{2^n}}]=0$$

Let $A\in\mathcal{F}^i_{\frac{k}{2^n},\frac{l}{2^n}}$, $B_{k,l}=\{T>\max(\frac{k}{2^n},\frac{l}{2^n})\}\in\mathcal{F}_{\frac{k}{2^n},\frac{l}{2^n}}$ be.

Then we have:

$$\int_A \square_{\frac{k}{2^n},\frac{l}{2^n}}\, y^{(n)}_{\frac{k+1}{2^n},\frac{l+1}{2^n}}\,dP=\int_{A\cap B_{k,l}} \square_{\frac{k}{2^n},\frac{l}{2^n}}\, x_{\frac{k+1}{2^n},\frac{l+1}{2^n}}\,dP+$$

$$+\int_{A\setminus B_{k,l}} \square_{\frac{k}{2^n},\frac{l}{2^n}}\, y^{(n)}_{\frac{k+1}{2^n},\frac{l+1}{2^n}}\,dP=\int_{A\cap B_{k,l}} \square_{\frac{k}{2^n},\frac{l}{2^n}}\, x_{\frac{k+1}{2^n},\frac{l+1}{2^n}}\,dP=0$$

since $(x_{s,t})$ is an i-martingale and $\Box_{\frac{k}{2^n},\frac{1}{2^n}} y^{(n)}_{\frac{k+1}{2^n},\frac{l+1}{2^n}}=o$ on $A\setminus B_{k,l}$.

If $s<s', t<t'$ and $(s,t)\in D_m$, $(s',t')\in D_m$ then from 1) we have

$M[\Box_{s,t} y^{(n)}_{s,t} / \mathcal{F}^i_{s,t}]=o$ for $n\geq m$ or which is the same

(2) $\int_A \Box_{s,t} y^{(n)}_{s,t}\, dP=o$

for each $A\in\mathcal{F}^i_{s,t}$.

For fixed s,t the process $(z_u)_u=(x_{s\wedge u,t\wedge u})_u$ is a $\mathcal{F}_{s\wedge u,t\wedge u}$-martingale, hence the sequence $(z_{T_n})_n=(y^{(n)}_{s,t})_n$ is uniformly integrable.

Therefore from 2) and from $y^{(n)}_{s,t}\longrightarrow y_{s,t}$ we obtain

$\int_A \Box_{s,t} y_{s',t'}\, dP=o$ for $(s,t),(s',t')$ from D_m, hence from $[o,1]^2$, so that $y_{s,t}$ is $\mathcal{F}^i_{s,t}$-martingale.

Next we are going to give some criteria in order to have, for a given process, other given process as increasing process (in the sense of th.1.5.b).

In this part the hypotesis i)-iii) are not necessary.

<u>PROPOSITION 1.7</u> Let $(x_{s,t})$ be a continuous and $\mathcal{F}_{s,t}$-adapted process that vanishes on I_o.

Let $(a_{s,t})$ be a continuous increasing process such that

$\Box_{s,t} a_{s,t}\leq C(s'-s)(t'-t)$ for every $s<s', t<t'$, where C is a constant.

Assume that:

$$M[\exp(\theta\Box_{s,t} x_{s',t'}-\frac{\theta^2}{2}\Box_{s,t} a_{s',t'} / \overline{\mathcal{F}}_{s,t}]=1$$

for every $s<s', t<t', \theta\in R$.

Then: a) <u>Exponential bound:</u> $P(\sup_{\substack{s\leq u\leq s'\\ t\leq v\leq t'}} |\Box_{s,t} x_{u,v}|\geq 1)\leq 4\exp\left\{-\frac{1^2}{2Ct'(s'-s)}\right\}$

b) $(x_{s,t})$ is a strong martingale with $(a_{s,t})$ as increasing process.

<u>Proofa)</u> It is not difficult to see that for fixed t the process $\{\exp(\theta x_{s,t}-\frac{\theta^2}{2}a_{s,t})\}_{o\leq s\leq 1}$ is $\mathcal{F}_{s,1}$-martingale for every $\theta\in R$.

The exponential bound is now a consequence of the one corresponding to onedimensional time parameter and of the previous remark (see [7]).

b) Let $s<s', t<t', A\in\overline{\mathcal{F}}_{s,t}$ be. From hypothesis we have:

(1) $\int_A \exp(\theta\Box_{s,t} x_{s',t'}-\frac{\theta^2}{2}\Box_{s,t} a_{s',t'})\,dP=P(A)$

for every $\theta\in R$.

By using the exponential bound we get that the family

$$\left\{\frac{d^k}{d\theta^k}\exp(\theta\square_{s,t}x_{s',t'}-\frac{\theta^2}{2}\square_{s,t}a_{s',t'})\right\}_{|\theta|\le\theta_0}$$

is uniformly integrable for every integer k, so that we may take the derivative of every order with respect to θ in 1).

Then b) follows by taking the first and the second derivative with respect to θ in 1) and puting $\theta=o$.

THEOREM 1.8 Let $(x_{s,t})$ be a continuous and $\mathcal{F}_{s,t}$-adapted process that vanishes on I_o.

Let $(a_{s,t})$ be a continuous and $\mathcal{F}_{s,t}$-adapted process such that $o<C'\le a(s,t,\omega)\le C$ for every s,t,ω, where C,C' are constants.

Let $(A_{s,t})$ be the process defined by $A_{s,t}=\int_0^s\int_0^t a(u,v)dudv$.

The following conditions 1)-3) are equivalent to each other.

1) $(x_{s,t})$ is a strong martingale with $(A_{s,t})$ as increasing process.

2) $M[\exp(\theta\square_{s,t}x_{s',t'}-\frac{\theta^2}{2}\square_{s,t}A_{s',t'})/\bar{\mathcal{F}}_{s,t}]=1$ for every $s<s', t<t', \theta\in R$.

3) $M\exp(i\theta\square_{s,t}x_{s',t'}+\frac{\theta^2}{2}\square_{s,t}A_{s',t'}/\bar{\mathcal{F}}_{s,t}]=1$ for every $s<s', t<t', \theta\in R$.

Proof,, 1)$\Longrightarrow$2). Let $s<s', t<t', \theta\in R$ fixed and let $(\bar{x}_u)_{s\le u\le 1}$ be the process defined by $\bar{x}_s=o, \bar{x}_u=\square_{s,t}x_{u,t'}$.

The process $(\bar{x}_u)$ is a $\mathcal{F}_{u,1}$-martingale with $\bar{A}_s=o, \bar{A}_u=\square_{s,t}A_{u,t'}$ as increasing process.

Indeed if $s\le u\le u'$ then we have:

$$M[\bar{x}_{u'}-\bar{x}_u/\mathcal{F}_{u,1}]=M[\square_{u,t}x_{u',t'}/\mathcal{F}_{u,1}]=M[\square_{u,t}x_{u',t'}/\bar{\mathcal{F}}_{u,t}/\mathcal{F}_{u,1}]=o$$

$$M[(x_{u'}-x_u)^2/\mathcal{F}_{u,1}]=M[(\square_{u,t}x_{u',t'})^2/\bar{\mathcal{F}}_{u,t}/\mathcal{F}_{u,1}]=$$

$$=M[\square_{u,t}A_{u',t'}/\bar{\mathcal{F}}_{u,t}/\mathcal{F}_{u,1}]=M[\bar{A}_{u'}-\bar{A}_u/\mathcal{F}_{u,1}]$$

Next by using the Ito formula to the semirmantingale $\theta\bar{x}_u-\frac{\theta^2}{2}\bar{A}_u$ and to the function $f(x)=\exp(\theta x)$ we get:

$$\bar{x}_\theta(s)\overset{def}{=}\exp(\theta\bar{x}_s-\frac{\theta^2}{2}\bar{A}_s)=1+\theta\int_s^{s'}\bar{x}_\theta(u)d\bar{x}_u$$

It remains to show that $M[\int_s^{s'}\bar{x}_\theta(u)d\bar{x}_u/\bar{\mathcal{F}}_{s,t}]=o$.

It is sufficient to prove that:

$$M[\int_s^{s'}\alpha\chi_{(u,u']}(s)d\bar{x}_s/\bar{\mathcal{F}}_{s,t}]=o$$

where α is $\mathcal{F}_{u,1}$-measurable and bounded(the general case follows by passing to the limit).

We have: $M[\int_s^{s'} \alpha \chi_{(u,u']}(s) d\bar{x}_s / \bar{\mathcal{F}}_{s,t}] = M[\alpha(\bar{x}_{u'} - \bar{x}_u) / \bar{\mathcal{F}}_{s,t}] =$

$M[\alpha \Box_{u,t} x_{u',t'} / \bar{\mathcal{F}}_{u,t} / \bar{\mathcal{F}}_{s,t}] = M[\alpha M[\Box_{u,t} x_{u',t'} / \bar{\mathcal{F}}_{u,t} / \bar{\mathcal{F}}_{s,t}] = o$

„$2 \Rightarrow 1$" This was proved in the proposition 1.7.

„$2 \Rightarrow 3$" Since the family $\{\exp(\theta \Box_{s,t} x_{s',t'} - \frac{\theta^2}{2} {}_{s,t} A_{s',t'})\}_{|\theta| \le \theta_0}$ is uniformly integrable (this is a consequence of the exponential bound) it follows that the function $\varphi(\theta) = \int_A \exp(\theta \Box_{s,t} x_{s',t'} - \frac{\theta^2}{2} \Box_{s,t} A_{s',t'}) dP$, $\theta \in C$ is continuous.

According to the Morera theorem φ is holomorfic, so that the equality:

$$\int_A \exp(\theta \Box_{s,t} x_{s',t'} - \frac{\theta^2}{2} \Box_{s,t} A_{s',t'}) dP = P(A)$$

holds for every $\theta \in C$.

„$3 \Rightarrow 2$" Itis sufficient to show that the exponential bound also is true in this case(see the previous implication)

Or this follows from the fact that for fixed t the process $\{\exp(i\theta x_{s,t} + \frac{\theta^2}{2} A_{s,t})\}_s$ is $\mathcal{F}_{s,1}$-martingale for every $\theta \in R$ (see [7], th.lo.1,p.),

COROLLARY 1.9 Let $(x_{s,t})_{s,t \in [o,1]}$ be a continuous and $\mathcal{F}_{s,t} = \mathcal{B}(x_{u,v} / u \le s, v \le t)$-adapted process that vanishes on I_o.

The following assertions are equivalent:

1) $(x_{s,t})$ is a Wiener-Yeh process,i.e $(x_{s,t})$ is gaussian process with $M(x_{s,t}) = o$ and $M(x_{s,t} x_{s',t'}) = \min(s,s') \min(t,t')$.

2) $(x_{s,t})$ is a process with independent increments and for $s < s'$, $t < t'$ the random variables $\Box_{s,t} x_{s',t'}$ is normally distributed with mean o and variance $(t'-t)(s'-s)$.

3) $M[\exp(i\theta \Box_{s,t} x_{s',t'}) / \bar{\mathcal{F}}_{s,t}] = \exp\{-\frac{\theta^2}{2}(s'-s)(t'-t)\}$ for every $s < s'$ $t < t'$, $\theta \in R$.

4) $M[\exp(\theta \Box_{s,t} x_{s',t'}) / \bar{\mathcal{F}}_{s,t}] = \exp\{\frac{\theta^2}{2}(s'-s)(t'-t)\}$ for every $s < s', t < t'$, $\theta \in R$.

$(x_{s,t})$ is a strong martingale with (st) as increasing process.

Proof „$1 \Rightarrow 2$" Let $s < s', t < t'$ be. Since $(x_{s,t})$ is gaussian process all we need to show is that $M[(\Box_{s,t} x_{s,t}) x_{u,v}] = o$ if $u \le s$ or $v \le t$.

We have:

$$M[(\square_{s,t}x_{s',t'})x_{u,v}]=\begin{cases}u(t'\wedge v-t'\wedge v-t\wedge v+t\wedge v)=o & \text{if } u\leq s\\ v(s'\wedge u-s\wedge u-s'\wedge u+s\wedge u)=o & \text{if } v\leq t\end{cases}$$

„$2\Longrightarrow 3$" It is immediate.

The equivalences $3\Leftrightarrow 4\Leftrightarrow 5$ follow from theorem 1.8 where $A_{s,t}\equiv 1$.

„$2\Longrightarrow 1$" Let $o\leq s,s',t,t'\leq 1$. Without loss of generality we may assume that $o<s<s', 0<t<t'$. Then we have:

$$M(x_{s',t'}x_{s,t})=M[(\square_{s,t}x_{s',t'}+\square_{o,t}x_{s,t'}+\square_{s,o}x_{s',t}+x_{s,t})x_{s,t}]=$$

$$M(\square_{s,t}x_{s',t'})M(x_{s,t})+M(\square_{o,t}x_{s,t})M(x_{s,t})+M(\square_{s,o}x_{s',t})M(x_{s,t})$$

$$+M(x^2_{s,t})=M(x^2_{s,t})=st.$$

The theorem is now proved.

2. Stochastic Integrals

Let $x\in\mathcal{M}^2$ and $L^2(x)=\{(\varphi_{s,t})_{s,t\in[o,1]};(\varphi_{s,t})$ is $\mathcal{F}_{s,t}$-previsible with $M(\int_0^1\int_0^1\varphi^2_{s,t}d\langle x\rangle_{s,t})<\infty\}$.

Here $(\langle x\rangle_{s,t})$ is a process as in theorem 1.3 (even if $\langle x\rangle_{s,t})$ is not uniquely determined $M(\int_0^1\int_0^1\varphi^2_{s,t}d\langle x\rangle_{s,t})$ is independent on $\langle x\rangle$; see proposition 2.1, p.124 from [3]).

$L^2(x)$ is a Hilbert space relative to the norm $//\varphi//=\left[M(\int_0^1\int_0^1\varphi^2_{s,t}d\langle x\rangle_{s,t})\right]^{\frac{1}{2}}$

If $\varphi_{s,t}=\alpha\lambda_A(s,t)$ where $A=(s,s']\times(t,t']$ and α is $\mathcal{F}_{s,t}$-measurable and bounded we define <u>the stochastic integral of φ with respect to x by</u>

$$(\varphi.x)_{s,t}=\int_0^s\int_0^t\varphi_{u,v}dx_{u,v}\overset{def}{=}\alpha x(A\cap R_{s,t})$$

where $R_{s,t}=(o,s]\times(o,t]$.

The new process $(\varphi.x)_{s,t}$ posesses the following properties:

a) $\varphi.x\in\mathcal{M}^2$; $\varphi.x\in\mathcal{M}^2_c$ if $x\in\mathcal{M}^2_c$.

b) If $x,x'\in\mathcal{M}^2$ and φ,φ' are as above then

$$\langle\varphi.x,\varphi'.x'\rangle_{s,t}=\int_0^s\int_0^t\varphi_{u,v}\varphi'_{u,v}d\langle x,x'\rangle_{u,v}$$

where we put $\langle x,x'\rangle_{u,v}=\frac{1}{2}(\langle x+x'\rangle_{u,v}-\langle x\rangle_{u,v}-\langle x'\rangle_{u,v})$

c) $M[(\varphi.x)^2_{s,t}]=\int_0^s\int_0^t\varphi^2_{u,v}d\langle x\rangle_{u,v}$

If φ is a simple $\mathcal{F}_{s,t}$-previsible process i.e $\varphi_{s,t}=\sum_{k=1}^{n}\alpha_k \lambda_{A_k}(s,t)$ where $A_k=(s_k,s_k']\times(t_k,t_k']$ and α_k is $\mathcal{F}_{s_k,t_k}$-measurable and bounded then we define:

$$(\varphi.x)_{s,t}=\int_0^s\int_0^t \varphi_{u,v}dx_{u,v}=\sum_{k=1}^{n}\alpha_k x(A_k\cap R_z)$$

The map $\varphi \longrightarrow \varphi.x$ of simple $\mathcal{F}_{s,t}$-previsible process into $\mathcal{M}^2$ is linear and preserves the norm.

Therefore it can be extended by continuity into a linear norm-preserving map of $L^2(x)$ into $\mathcal{M}^2$ which is denoted in the same way. The new process also possesses the properties a)-c).

<u>REMARK 2.1</u>If x is a Wiener-Yeh process then in the definition of $L^2(x)$ we can change the word previsible by measurable and adapted.

<u>PROPOSITION 2.2</u>a) If the hypothesis a) from theorem 1.5 holds and $\varphi\in L^2(x)$ then the process $y_{s,t}=(\varphi.x)^2_{s,t}-\int_0^s\int_0^t\varphi^2_{u,v}d[x]_{u,v}$ is a martingale.

b)If the hypothesis b) of the theorem 1.5 holds and $\varphi\in L^2(x)$ then $\varphi.x\in\mathcal{M}^2_S$ and

$$M[(\square_{s,t}(\varphi.x)_{s,t})^2/\bar{\mathcal{F}}_{s,t}]=M[\int_s^{s'}\int_t^{t'}\varphi^2_{u,v}d[x]_{u,v}/\bar{\mathcal{F}}_{s,t}]$$ for every $s<s'$, $t<t'$(in this point the hypothesis i)-iii) are not necessary).

<u>Proof</u> a)Let $\varphi=\alpha\lambda_A$ be, where $A=(s,s']\times(t,t']$ and α is $\mathcal{F}_{s,t}$-measurable and bounded.

Then: $M[(\varphi.x)^2((s,s']\times(t,t'])/\mathcal{F}^i_{s,t}]=M[\alpha^2x^2(A\cap(o,s]\times(o,t])/\mathcal{F}^i_{s,t}]=$

$=\alpha^2M[x^2(A\cap(o,s]\times(o,t])/\mathcal{F}^i_{s,t}]=\alpha^2M([x](A\cap(o,s]\times(o,t])/\mathcal{F}^i_{s,t}]=$

$=M[\int_0^s\int_0^t\varphi^2_{u,v}d[x]_{u,v}/\mathcal{F}^i_{s,t}]$

The general case follows from L^2-continuity of the stochastic integral.

<u>LEMMA 2.3(Localisation lemma)</u> Let φ, x as in the previous proposition and T be a $\mathcal{F}_{s,s}$-stopping time.Then:

$$(\varphi^T.x^T)_{s\wedge T,t\wedge T}=(\varphi.x^T)_{s\wedge T,t\wedge T}=(\varphi.x)_{s\wedge T,t\wedge T} \quad \text{a.s}$$

for every s,t(here $*^T_{s,t}=*_{s\wedge T,t\wedge T}$)

<u>Proof</u> a) From theorem 1.2 and proposition 1.6 we get that $x^T\in\mathcal{M}^2$.

According to the above proposition we have:

$$(\varphi.x^T)^2_{s\wedge T,t\wedge T}-\int_0^{s\wedge T}\int_0^{t\wedge T}\varphi^2_{u,v}d[x]_{u,v};\quad (\varphi.x)^2_{s\wedge T,t\wedge T}-\int_0^{s\wedge T}\int_0^{t\wedge T}\varphi^2_{u,v}d[x]_{u,v};$$

$$(\varphi^T.x^T)^2_{s\wedge T,t\wedge T}-\int_0^{s\wedge T}\int_0^{t\wedge T}\varphi^2_{u,v}d[x]_{u,v}$$ are martingales.

Then we have :

$$M(|(\varphi.x^T)_{s\wedge T,t\wedge T}-(\varphi.x)_{s\wedge t,t\wedge T}|^2)=M[(\varphi.x^T)^2_{s\wedge T,t\wedge T}]+$$

$$+M[(\varphi.x)^2_{s\wedge T,t\wedge T}]-2M[(\varphi.x^T)_{s\wedge T,t\wedge T}\cdot(\varphi.x)_{s\wedge T,t\wedge T}]=$$

$$=2M\left(\int_0^{s\wedge T}\int_0^{t\wedge T}\varphi^2_{u,v}d[x]_{u,v}\right)-2M\left(\int_0^{s\wedge T}\int_0^{t\wedge T}\varphi^2_{u,v}d[x]_{u,v}\right)=0$$

so that $(\varphi.x^T)_{s\wedge T,t\wedge T}=(\varphi.x)_{s\wedge T,t\wedge T}$ a.s.

The other equality proves in the same manner.

b) Like as in a).

In the end of this section we extend the stochastic integral to a wide class of processes.

Let us assume that x, $(\mathcal{F}_{s,t})$ are as in theorem 1.5 and define

$L^2_{loc}(x)=\{(\varphi_{s,t});\varphi\ \mathcal{F}_{s,t}$-previsible such that there exists a sequence T_n of $\mathcal{F}_{s,s}$-stopping times increasing to 1 such that

$$M\left(\int_0^1\int_0^1\varphi^2_{s\wedge T_n,t\wedge T_n}d[x]^{T_n}_{s,t}\right)<\infty \text{ for each } n\}.$$

Then we can define the stochastic integral $(\varphi^{T_n}.x^{T_n})_{s,t}$ and the localisation lemma tells us that is possible to put

$$(\varphi.x)_{s,t}=\int_0^s\int_0^t\varphi_{u,v}dx_{u,v}\overset{def}{=}(\varphi^{T_n}.x^{T_n})_{s,t} \text{ if } s,t\le T_n.$$

The process $(\varphi.x)_{s,t}$ is called <u>the stochastic integral of with respect to x.</u>

<u>REMARK</u> The localisation lemma holds for $\varphi\in L^2_{loc}(x)$.

3. <u>Strong Solutions for Stochastic Integral Equations in the Plane.</u>

The main result of this section is the following theorem:

<u>THEOREM 3.1</u> Let $(M_{s,t})$ be a continuous process and let $(\alpha_{s,t})_{s,t\in[0,1]}$

be a continuous and adapted process.

Let a b: $[o,1]^2 \times R \longrightarrow R$ be two functions.

Suppose that:

a_1) a b are jointly continuous and satisfy

3.2) $/a(z,x)-a(z,y)/ + /b(z,x)-b(z,y)/ \leq L/x-y/$

for every x,y,z and for some constant $L>o$.

a_2) $(M_{s,t})$ satisfies one of the hypothesis of theorem 1.5 and $[M]_{s,t}=\int_o^s\int_o^t h(u,v)dudv$ where $h(u,v,\omega)$ is continuous as a function of s,t and $\mathcal{F}_{s,t}$-measurable for fixed s,t.

a_3) $M(\sup_{s,t}/\alpha_{s,t}/^2)<\infty$.

Then there exists an unique(in the paths sense) continuous process $(x_{s,t})$ that verifies the following stochastic integral equation:

3.3) $x_{s,t}=\alpha_{s,t}+\int_o^s\int_o^t a(u,v,x_{u,v})dM_{u,v}+\int_o^s\int_o^t b(u,v,x_{u,v})dudv$ a.s

for every $s,t\in[o,1]$.

Proof The existence We shall use the technique of optional stopping.

For every integer k let T_k be the $\mathcal{F}_{s,s}$-stopping time defined by

$T_k=\inf\{s\leq 1;\ \max_{u,v\leq s}/h(u,v)/\geq k\}$ if the set $\{.\}$ is nonempty and $T_k=1$ in opposite case.

Clearly T_k increase to 1.

Let $M^{(k)}_{s,t}=M_{s\wedge T_k,t\wedge T_k}$ be; we have that $M^{(k)}\in\mathcal{M}^2_c$ and $(M^{(k)}_{s,t})^2-[M]_{s\wedge T_k,t\wedge T_k}$ is a martingale (see theorem 1.5 and proposition 1.6) .

Inductively we construct the following sequences of processes

$$x^{o,k}_{s,t}=\alpha_{s,t};\quad x^{n+1,k}_{s,t}=x^{o,k}_{s,t}+\int_o^s\int_o^t a(u,v,x^{n,k}_{u,v})dM^{(k)}_{u,v}+\int_o^s\int_o^t b(u,v,x^{n,k}_{u,v})dudv$$

We claim that $\sup_{s,t}M[/x^{n,k}_{s,t}/^2]<\infty$ for each k,n.

For n=o the assertion result from a_3).Suppose the assertion holds for n.

Since a,b are continuous and satisfy 3.2) it follows that there exists L_1 o such that $a^2(z,x)\leq L_1(1+x^2)$, $b^2(z,x)\leq L_1(1+x^2)$ for all z,x.

By the Schwartz inequality and the inequality $(a+b+c)^2\leq 3(a^2+b^2+c^2)$ we get:

$$(x^{n+1,k}_{s,t})^2 \le 3\Big[(x^{o,k}_{s,t})^2 + \Big(\int_0^s\!\!\int_0^t a(u,v,x^{n,k}_{u,v})dM^{(k)}_{u,v}\Big)^2 + \int_0^s\!\!\int_0^t b^2(u,v,x^{n,k}_{u,v})dudv\Big]$$

hence:

$$\sup_{s,t}(x^{n+1,k}_{s,t})^2 \le 3\sup_{s,t}/\alpha_{s,t}/^2 + 3\sup_{s,t}\Big[\int_0^s\!\!\int_0^t a(u,v,x^{n,k}_{u,v})dM^{(k)}_{u,v}\Big]^2 + 3\int_0^1\!\!\int_0^1 b^2(u,v,x^{n,k}_{u,v})dudv$$

where from (see theorem 1.2 and a_3)):

$$M\Big[\sup_{s,t}(x^{n+1,k}_{s,t})^2\Big] \le 3M\Big[\sup_{s,t}/\alpha_{s,t}/^2\Big] + 48M\Big[\int_0^1\!\!\int_0^1 a^2(u,v,x^{n,k}_{u,v})h_{u\wedge T_k,v\wedge T_k}dudv\Big]$$

$$+3M\Big[\int_0^1\!\!\int_0^1 b^2(u,v,x^{n,k}_{u,v})dudv\Big] \le 3M\Big[\sup_{s,t}/\alpha_{s,t}/^2\Big] + 48KL_1\Big\{1+\int_0^1\!\!\int_0^1 M\big[(x^{n,k}_{s,t})^2\big]dsdt\Big\} +$$

$$+3L_1\Big\{1+\int_0^1\!\!\int_0^1 M\big[(x^{n,k}_{s,t})^2\big]dsdt\Big\} \le 3\,M\Big[\sup_{s,t}(\alpha_{s,t})^2\Big] +$$

$$+48KL_1\Big\{1+M\Big[\sup_{s,t}(x^{n,k}_{s,t})^2\Big]\Big\} + 3L_1\Big\{1+M\Big[\sup_{s,t}(x^{n,k}_{s,t})^2\Big]\Big\} < \infty .$$

Similarly as above (using 3.2) we obtain :

$$M\Big[\sup_{s,t}/x^{n+1,k}_{s,t}-x^{n,k}_{s,t}/^2\Big] \le L_2\int_0^1\!\!\int_0^1 M\big[/x^{n,k}_{s,t}-x^{n+1,k}_{s,t}/^2\big]dsdt$$

where $L_2 > o$ is a constant depending only on L.

Iterating the previous inequality we get:

$$M\Big[\sup_{s,t}/x^{n+1,k}_{s,t}-x^{n,k}_{s,t}/^2\Big] \le \frac{L_2^n}{[(n-1)!]^2}\int_0^1\!\!\int_0^1 M\big[/x^{1,k}_{s,t}-x^{o,k}_{s,t}/^2\big]dsdt$$

$$\le \frac{A_k L_2^n}{[(n-1)!]^2}$$

where from we derive the estimate

$$//\sup_{s,t}/x^{n+1,k}_{s,t}-x^{n,k}_{s,t}///_{L^2(\Omega,\mathcal{F},P)} \le \frac{(A_k L_2^n)^{\frac12}}{(n-1)!}$$

This estimate shows that $\sum_n \sup_{s,t}/x^{n+1,k}_{s,t}-x^{n,k}_{s,t}/<\infty$ a.s so that $x^{n,k}_{s,t}$ converge uniformly a.s for $o\le s,t\le 1$ to a continuous limit $x^{\infty,k}_{s,t}$.

On the other hand the inequality:

$$//\sup_{s,t}/x^{p,k}_{s,t}-x^{\infty,k}_{s,t}///_2 \le \sum_{n\ge p}//\sup_{s,t}/x^{n+1,k}_{s,t}-x^{n,k}_{s,t}///_2$$

implies that $\sup_{s,t}/x^{p,k}_{s,t}-x^{\infty,k}_{s,t}/ \xrightarrow{L^2} o$ as $n \longrightarrow \infty$.

This fact ,3.2 and he bound $h_{u\wedge T_k v\wedge T_k} \le K$ for each u,v tell us that

$$\int_0^s\!\!\int_0^t a(u,v,x^{p,k}_{u,v})dM^{(k)}_{u,v} \xrightarrow{L^2} \int_0^s\!\!\int_0^t a(u,v,x^{\infty,k}_{u,v})dM^{(k)}_{u,v} \text{ and}$$

$$\int_0\!\!\int_0 b(u,v,x^{p,k}_{u,v})\,du\,dv \xrightarrow{L^2} \int_0\!\!\int_0 b(u,v,x^{\infty,k}_{u,v})\,du\,dv$$

(the Schwartz inequality is used also in the last convergence).

Therefore we can pass to limit in the equality that defines $x^{n+1,k}_{s,t}$ and we shall obtain that :

$$x^{\infty,k}_{s,t}=x^{o,k}_{s,t}+\int_0^s\!\!\int_0^t a(u,v,x^{\infty,k}_{u,v})dM^{(k)}_{u,v}+\int_0^s\!\!\int_0^t b(u,v,x^{\infty,k}_{u,v})dudv$$

For $l\geq k$ the last equality implies:

$$x^{\infty,l}_{s\wedge T_k,t\wedge T_k}=x^{o,k}_{s\wedge T_k,t\wedge T_k}+\int_0^{s\wedge T_k}\int_0^{t\wedge T_k} a(u\wedge T_k,v\wedge T_k,x^{\infty,k}_{u\wedge T_k,v\wedge T_k})dM^{(k)}_{u,v}+$$
$$\int_0^{s\wedge T_k}\int_0^{t\wedge T_k} b(u\wedge T_k,t\wedge T_k,x^{\infty,k}_{u\wedge T_k,v\wedge T_k})dudv$$

where from:

$$M\Big[/x^{\infty,k+1}_{s\wedge T_k,t\wedge T_k}-x^{\infty,k}_{s\wedge T_k,t\wedge T_k}/^2\Big]\leq L(k+1)\int_0^s\!\!\int_0^t M\Big[/x^{\infty,k+1}_{u\wedge T_k,t\wedge T_k}-x^{\infty,k}_{u\wedge T_k,v\wedge T_k}/^2\Big]dudv$$

Iterating the previous inequality we get:

$$M\Big[/x^{\infty,k+1}_{s\wedge T_k,t\wedge T_k}-x^{\infty,k}_{s\wedge T_k,t\wedge T_k}/^2\Big]\leq\frac{L^2(K+1)^n}{\big[(n-1)!\big]^2}L_3(k)$$

where from:

$$x^{\infty,k+1}_{s\wedge T_k,t\wedge T_k}=x^{\infty,k}_{s\wedge T_k,t\wedge T_k}\quad \text{a.s for each } s,t.$$

Therefore we can define the process $(x_{s,t})$ by $x_{s,t}=x^{\infty,k}_{s,t}$ if $s,t\leq T_k$.

The process $(x_{s,t})$ satisfies the requirements of the theorem.

The uniqueness Let $(y_{s,t})$ be another process as in theorem. Without loss of generality we can suppose that h is buonded (eventually we pass to stopped processes).

Using 3.2, theorem 1.2 and a_3) we get:

$$M(/y_{s,t}-x_{s,t}/^2)\leq cts.\int_0^s\!\!\int_0^t M(/y_{u,v}-x_{u,v}/^2)dudv$$

Where from $M(/y_{s,t}-x_{s,t}/^2)=o$ hence $y_{s,t}=x_{s,t}$ a.s and this completes the proof of the theorem.

REMARK 3.4 a) The case of the Wiener.Yeh process is covered by the previous theorem (in this case h=1) so that our theorem generalises theorem 2 from [5] or the theorem 1 from [13] .

b)The process $(x_{s,t})$ satisfies : $M(\sup_{s,t} x^2_{s,t})<\infty$.

c)If $(\varphi_{s,t})_{s,t\in I_o}$ is a continuous process such that $M(\sup_{s,t}\varphi^2_{s,t})<\infty$

then in the hypothesis of the theorem there exists an unique continuous process $x^{\varphi}=(x^{\varphi}_{s,t})$ such that x^{φ} agrees with φ on I_o and

3.5) $\Box_{o,o}x_{s,t}=\int_o^s\int_o^t a(u,v,x^{\varphi}_{u,v})dM_{u,v}+\int_o^s\int_o^t b(u,v,x_{u,v})dudv$

(indeed we take $\alpha_{s,t}=\varphi_{s,o}+\varphi_{o,t}-\varphi_{o,o}$).

DEFINITION 3.5 A continuous process $(x_{s,t})$ that satisfies 3.3) is called strong sulution of the equation 3.3).

The Process from 3.4.c) is called the initial process of the solution $(x_{s,t})$.

THEOREM 3.6 The process$(x_{s,t})$ from 3.4.c, where $(M_{s,t})$ is taken the Wiener-Yeh proces and φ deterministe, is a Markov process in the following sense:

$$P(x^{\varphi}_{s+h,t+k}\in B/\overline{\mathcal{F}}_{s,t})=P(x_{s+h,t+k}\in B/\mathcal{F}^{o}_{s,t})$$

for each $s,t\geq o,h,k>o,B\in\mathcal{B}_R$, where $\overline{\mathcal{F}}_{s,t}=\mathcal{B}(M_{u,v}/u\leq s,o\leq v\leq 1$ or $s\leq u\leq 1, v\leq t)$, $\mathcal{F}^{o}_{s,t}=\mathcal{B}(x^{\varphi}_{s,u},x^{\varphi}_{u,t}/t\leq v\leq 1,s\leq u\leq 1)$.

Proof The conclusion follows easily from the following three facts:

i_1) $x^{\varphi}_{s+h,t+k}=x^{s,t,\varphi'}_{s+h,t+k}$ where φ' agrees with x^{φ} on $I_{s,t}=\{s\}\times[t,1]\cup[s,1]\times\{t\}$ and $x^{s,t,\varphi'}_{s+h,t+k}$ is the solution of the equation:

$$\Box_{s,t}x_{s',t'}=\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})dM_{u,v}+\int_s^{s'}\int_t^{t'}b(u,v,x_{u,v})dudv, s\leq s'\leq s+h,$$

$t\leq t'\leq t+k$, that agrees with φ' on $I_{s,t}$ (consequence of uniqueness).

i_2) $x^{s,t,\varphi}_{s+h,t+k}$, φ deterministe, depends only on $\Box_{s,t}M_{s+h,t+k}$ (by construction).

i_3) $(x^{\varphi}_{s,t})$, with φ deterministe, is measurable in φ (even continuous) with respect to the uniform convergence (this is a consequence of thw following inequality:

$$M(\sup_{s,t}/x^{\varphi}_{s,t}-x^{\psi}_{s,t}/^2)\leq cts.\sup_{s,t}/\varphi_{s,t}-\psi_{s,t}/^2$$

that is easy to prove).

The next theorems show that the solution of the stochastic equation 3.3) depends continuosly on initial data.

THEOREM 3.7 Let $a,b,M^{(n)},\alpha^{(n)}$, $n=o,1,\dots$, be so that the system $(a,b,M^{(n)},\alpha^{(n)})$ satisfy the hypothesis of theorem 3.1 for each $n=o,1,\dots$

Moreover suppose that:

a_1) a is bounded.

a_2) $\lim_{n\to\infty} M[\sup_{s,t}/M^{(n)}_{s,t}-M^{(n)}_{s,t}/^2]=o$.

a_3) $\sup_{n\geq 1,s,t} h^{(n)}(s,t)\leq cts.$ a.s.

a_4) $\lim\limits_{n\to\infty} M[\sup\limits_{s,t}|\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}|^2]=o$.

Let $(x^{(n)}_{s,t})$ be the strong solution of the equation:

$$x^{(n)}_{s,t}= {}^{(n)}_{s,t}+\int_o^s\int_o^t a(u,v,x^{(n)}_{u,v})dM^{(n)}_{u,v}+\int_o^s\int_o^t b(u,v,x^{(n)}_{u,v})dudv\ ,\ n=o,1,..$$

Then :

$$\lim_{n\to\infty} M[\sup_{s,t}|x^{(n)}_{s,t}-x^{(o)}_{s,t}|^2]=o.$$

Proof We have:

$$M[\sup_{s,t}|x^{(n)}_{s,t}-x^{(o)}_{s,t}|^2]\leq 6M[\sup_{s,t}|\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}|^2]+6M[\sup_{s,t}|\int_o^s\int_o^t a(u,v,x^{(n)}_{u,v})dM^{(n)}_{u,v}-$$

$$-\int_o^s\int_o^t a(u,v,x^{(o)}_{u,v})dM^{(o)}_{u,v}|^2]+3M[\sup_{s,t}|\int_o^s\int_o^t(a(u,v,x^{(n)}_{u,v})-a(u,v,x^{(o)}_{u,v}))dM^{(n)}_{u,v}|^2]$$

$$+3M[\int_o^1\int_o^1|b(u,v,x^{(n)}_{u,v})-b(u,v,x^{(o)}_{u,v})|^2dudv]\leq 6M[\sup_{s,t}|\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}|^2]+$$

$$+96M[\int_o^1\int_o^1 a^2(u,v,x^{(o)}_{u,v})d[M^{(n)}-M^{(o)}]_{u,v}\quad +96M[\int_o^1\int_o^1|a(u,v,x^{(n)}_{u,v})-$$

$$-a(u,v,x^{(o)}_{u,v})|^2h^{(n)}(u,v)dudv]+3M[\int_o^1\int_o^1|b(u,v,x^{(n)}_{u,v})-b(u,v,x^{(o)}_{u,v})|^2dudv]\leq$$

$$\leq 6M[\sup_{s,t}|\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}|^2]+96\sup_{u,x}a^2(u,x)M[|M^{(n)}_{11}-M^{(o)}_{11}|^2]+L_1\int_o^1\int_o^1 M[|x^{(n)}_{u,v}-$$

$$-x^{(o)}_{u,v}|^2]dudv=\beta_n+L_1\int_o^1\int_o^1 M[|x^{(n)}_{u,v}-x^{(o)}_{u,v}|^2]dudv$$ where $\lim\beta_n=o$, L_1 is a constant independent on n.

Retaining the first and the last member from above inequality and iterating the obtained inequality we get :

$$M[\sup_{s,t}|x^{(n)}_{s,t}-x^{(o)}_{s,t}|^2]\leq\beta_n.cts.\longrightarrow o.$$

THEOREM 3.8 Let $a,b,M^{(n)},\alpha^{(n)}$, $n=o,1,\dots$, be such that the system $(a,b,M^{(n)},\alpha^{(n)})$ satisfies the hypothesis of theorem 3.1 for each n=o, 1,.....

Suppose that there exists a sequence (T_m) of $\mathcal{F}_{s,s}$-stopping times increasing to 1 such that:

b_1) $\lim\limits_{n\to\infty} M[\sup\limits_{s,t\leq T_m}|M^{(n)}_{s,t}-M^{(o)}_{s,t}|^2]=o$ for each m.

b_2) $\sup\limits_{\substack{s,t\leq T_m\\ n\geq 1}}|h^{(n)}(u\wedge T_m, v\wedge T_m)|\leq C_m$ a.s for each m.

b_3) $\lim\limits_{n\to\infty} M[\sup\limits_{s,t\leq T_m}|\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}|^2]=o$ for each m.

Then there exists a sequence (T'_m) of $\mathcal{F}_{s,s}$-stopping times increasing to 1 such that:

$$\lim_{n\to\infty} M[\sup_{s,t\leq T'_m}|x^{(n)}_{s,t}-x^{(o)}_{s,t}|^2]=o$$

for every m(here $x^{(n)}$ is the process from the above theorem).

Proof Let $\overline{T}_m$ be the $\mathcal{F}_{s,s}$-stopping time defined by:

$\overline{T}_m=\inf\{s\leq 1;\ \max\limits_{\substack{u\leq s\\ v\leq s}} /a(u,v,x^{(o)}_{u,v}/\geq m\}$ if the set $\{\cdot\}$ is nonempty

and $\overline{T}_m=1$ in opposite case.

Then $T'_m=\min(T_m,\overline{T}_m)$ are $\mathcal{F}_{s,s}$-stopping times increasing to 1.

Taking into account the stochastic integral equation verified by the process $x^{(n)}_{s\wedge T'_m,t\wedge T'_m}$ we show as in the proof of the previous theorem that:

$$\lim_{n\to\infty} M[\sup_{s,t\leq T'_m} /x^{(n)}_{s,t}-x^{(o)}_{s,t}/^2]=0$$

for every m.

The techniques used in the proof of theorem 3.8 may be utilised to prove the following theorem:

THEOREM 3.9 Let $a_n,b_n,M,\alpha^{(n)}$, $n=0,1,\ldots$, be such that the system $(a_n,b_n,M,\alpha^{(n)})$ satisfies for each n the hypothesis of theorem 3.1 with the same constant Lipschitz.

Moreover suppose that:

c_1) a_n,b_n are uniformly bounded and $a_n\xrightarrow{a.p.t}a$, $b_n\xrightarrow{a.p.t}b$.

c_2) h is a.s bounded.

c_3) $\lim\limits_{n\to\infty} M[\sup\limits_{s,t} /\alpha^{(n)}_{s,t}-\alpha^{(o)}_{s,t}/^2]=0$.

Then:

$$\lim_{n\to\infty} M[\sup_{s,t} /x^{(n)}_{s,t}-x^{(o)}_{s,t}/^2]=0$$

Where $(x^{(n)}_{s,t})$, $n=0,1,\ldots$, is the solution of the following stochastic integral equation:

$$x^{(n)}_{s,t}=\alpha^{(n)}_{s,t}+\int_0^s\int_0^t a_n(u,v,x^{(n)}_{u,v})dM_{u,v}+\int_0^s\int_0^t b_n(u,v,x^{(n)}_{u,v})dudv$$

REMARK 3.10 A similar result as that of theorem 3.8 is holds if in the previous theorem we impose the hypothesis c_2), c_3) only locally.

4. Weak Solutions for Stochastic Integral Equations in the Plane and the Martingale Problem

We consider the following stochastic integral equation:

4.1) $$x_{s,t}=x+\int_0^s\int_0^t a(u,v,x_{u,v})dM_{u,v}+\int_0^s\int_0^t b(u,v,x_{u,v})dudv$$

where $x\in R$, a,b: $[0,1]^2\times R\longrightarrow R$ and $(M_{s,t})\in\mathcal{M}^2_c$.

DEFINITION 4.2 By weak solution of the equation 4.1) we mean a system $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P},\tilde{\mathcal{F}}_{s,t},\tilde{M}_{s,t},\tilde{x}_{s,t};0\le s,t\le 1)$ where:

-$(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ is a probability space and $(\tilde{\mathcal{F}}_{s,t})$ is a increasing family of sub-σ-fields of $\tilde{\mathcal{F}}$.

-$(\tilde{M}_{s,t})\in\mathcal{M}_c^2(\tilde{\mathcal{F}})$ and $(\tilde{M}_{s,t})$ is equivalent with $(M_{s,t})$.

-$(\tilde{x}_{s,t})$ is continuous and $\tilde{\mathcal{F}}_{s,t}$-adapted such that:

$$\tilde{x}_{s,t}=x+\int_o^s\int_o^t a(u,v,\tilde{x}_{u,v})d\tilde{M}_{u,v}+\int_o^s\int_o^t b(u,v,\tilde{x}_{u,v})dudv \qquad \text{a.s}$$

for every $s,t\in[o,1]$.

We have the following result:

THEOREM 4.3 If the following conditions hold:

1) a,b are continuous and bounded.

2) $(M_{s,t})$ satisfies one of the hypothesis of theorem 1.5 with $[M]_{s,t}=\int_o^s\int_o^t h(u,v)dudv$ where h is continuous and adapted.

3) $/[M]_{s,t}-[M]_{s',t'}/\le K//(s,t)-(s',t')//$ for each s,t,s',t' where $//(s,t)//=$ $=\max(/s/,/t/)$, then the equation 4.1) has a weak solution.

Proof Let a_n ,b_n be uniformly and Lipschitz functions such that a_n, b_n converge uniformly on every compact to a,b.

The theorem 3.1 tells us that there exists a strong solution $(x^{(n)}_{s,t})$ of the stochastic integral equation:

$$x^{(n)}_{s,t}=x+\int_o^s\int_o^t a_n(u,v,x^{(n)}_{u,v})dM_{u,v}+\int_o^s\int_o^t b_n(u,v,x^{(n)}_{u,v})dudv$$

It is not difficult to see that:

i_1) $\lim_{\delta\to o}\sup_n P(\sup_{//(s,t)-(s',t')//\le\delta}/x_{s,t}-x_{s',t'}/\ge\varepsilon)=o.$

i_2) $\lim_{l\to\infty}\sup_n P(\sup_{s,t}/x_{s,t}/\ge l)=o.$

for every $\varepsilon>o$, where $x=x^{(n)}$ or M.

According to the corollary 2, p.13 from [6] there exists a sequence of processes $(\tilde{x}^{(n)}_{s,t},\tilde{M}^{(n)}_{s,t})$ such that $(x^{(n)}_{s,t},M_{s,t})_{s,t}$ has the same law with $(\tilde{x}^{(n)}_{s,t},\tilde{M}^{(n)}_{s,t})_{s,t}$ for every n and $\tilde{x}^{(n)}_{s,t}\xrightarrow{p}\tilde{x}_{s,t}$, $\tilde{M}^{(n)}_{s,t}\xrightarrow{p}\tilde{M}_{s,t}$ for every s,t.

The uniqueness of repartition implies that:

$$\tilde{x}^{(n)}_{s,t}=x+\int_o^s\int_o^t a_n(u,v,\tilde{x}^{(n)}_{u,v})d\tilde{M}^{(n)}_{u,v}+\int_o^s\int_o^t b_n(u,v,\tilde{x}^{(n)}_{u,v})dudv$$

We shall show that:

$$\int_o^s\int_o^t a_n(u,v,\tilde{x}^{(n)}_{u,v})d\tilde{M}^{(n)}_{u,v}\xrightarrow{L^2}\int_o^s\int_o^t a(u,v,\tilde{x}^{(n)}_{u,v})d\tilde{M}_{u,v}$$

$$\int_0^s\int_0^t b_n(u,v,x^{(n)}_{u,v})\,du\,dv \xrightarrow{L^2} \int_0^s\int_0^t b(u,v,x_{u,v})\,du\,dv$$

(this ends the proof of the theorem)

We have:

$$M\Big[\Big/\int_0^s\int_0^t a_n(u,v,\tilde{x}^{(n)}_{u,v})\,d\tilde{M}^{(n)}_{u,v} - \int_0^s\int_0^t a(u,v,\tilde{x}_{u,v})\,d\tilde{M}_{u,v}\Big/^2\Big] \le$$

$$\le 3M\Big[\Big/\int_0^s\int_0^t (a_n(u,v,\tilde{x}^{(n)}_{u,v}) - a(u,v,\tilde{x}^{(n)}_{u,v}))\,d\tilde{M}^{(n)}_{u,v}\Big/^2\Big] + 3M\Big[\Big/\int_0^s\int_0^t (a(u,v,\tilde{x}^{(n)}_{u,v}) -$$

$$- a(u,v,\tilde{x}_{u,v}))\,d\tilde{M}^{(n)}_{u,v}\Big/^2\Big] + 3M\Big[\Big/\int_0^s\int_0^t a(u,v,\tilde{x}_{u,v})\,d\tilde{M}^{(n)}_{u,v} - \int_0^s\int_0^t a(u,v,\tilde{x}_{u,v})\,d\tilde{M}_{u,v}\Big/^2\Big]$$

$$= 3M\Big(\int_0^s\int_0^t /a_n(u,v,\tilde{x}^{(n)}_{u,v}) - a(u,v,\tilde{x}^{(n)}_{u,v})/^2\,d[M^{(n)}]_{u,v}\Big) + 3M\Big(\int_0^s\int_0^t /a(u,v,\tilde{x}^{(n)}_{u,v}) -$$

$$- a(u,v,\tilde{x}_{u,v})/^2\,d[M^{(n)}_{u,v}]\Big) + 3\,M\Big(\Big/\int_0^s\int_0^t a(u,v,\tilde{x}_{u,v})\,d\tilde{M}^{(n)}_{u,v} -$$

$$- \int_0^s\int_0^t a(u,v,\tilde{x}_{u,v})\,d\tilde{M}_{u,v}\Big/^2\Big) = 3(I_1+I_2+I_3)$$

$$I_1 \le \sup_{z,/x/\le l} /a_n(z,x)-a(z,x)/^2 M(M^2_{11}) + P(\sup_z/\tilde{x}^{(n)}_z/\ge l)$$

$$I_2 \le \sup_{z,/x/\le l} /a_n(z,x)-a(z,x)/^2 M(M^2_{1,1}) + P(\sup_z/\tilde{x}^{(n)}_z/\ge l) +$$

$$P(\sup_z/\tilde{x}_z/\ge l) \le \sup_{z,/x/\le l} /a_n(z,x)-a(z,x)/^2 M(M^2_{1,1}) +$$

$$P(\sup_z/\tilde{x}_z/\ge l) + P(\sup_z/\tilde{x}^{(n)}_z-\tilde{x}_z/\ge \tfrac{l}{2}) + P(\sup_z/\tilde{x}^{(n)}_z/\ge \tfrac{l}{2})$$

Obviously $I_1+I_2 \le \frac{\varepsilon}{2}$ for large l and n.

Furthemore $I_3 \le \sup_{z,x}/a(z,x)/M(\langle\tilde{M}^{(n)}-\tilde{M}\rangle_{11}) \le \sup_{z,x}/a(z,x)/M[(\tilde{M}^{(n)}_{11}-\tilde{M}_{11})^2] \to 0$.

The other convergence proves similarly.

<u>REMARK 4.4</u> The case of the Wiener-Yeh process is covered by the previous theorem.

Moreover in this case we can weaken the hypothesis on the coefficients.

More precise we have the following result:

<u>THEOREM 4.5</u> If a,b are measurable and bounded and $a(z,x) \ge C > 0$ for every z,x and for some constant C, then the stochastic integral equation:

$$(4.6)\quad x_{s,t} = x + \int_0^s\int_0^t a(u,v,x_{u,v})\,dw_{u,v} + \int_0^s\int_0^t b(u,v,x_{u,v})\,du\,dv$$

has a weak solution(here $(w_{s,t})$ is a Wiener-Yeh process).

<u>Proof</u> Let a_n, b_n be uniformly bounded and Lipschitz functions, $a_n(z,x) \ge C$

for every z,x and such that $//(a_n-a)^2//_{3,[o,1]^2xS(n)}\leq\frac{1}{2^n}$,

$//b_n-b//_{3,[o,1]^2xS(n)}\leq\frac{1}{2^n}$.

Let $(x_{s,t}^{(n)})$ be the solution of the equation:

$$x_{s,t}^{(n)}=x+\int_o^s\int_o^t a_n(u,v,x_{u,v}^{(n)})dw_{u,v}+\int_o^s\int_o^t b_n(u,v,x_{u,v}^{(n)})dudv$$

Let $(\tilde{x}_{s,t}^{(n)})$, $(\tilde{w}_{s,t}^{(n)})$ be equi valent processes with $(x_{s,t}^{(n)})$, $(w_{s,t})$ such that $\tilde{x}_{s,t}^{(n)}\xrightarrow{p}\tilde{x}_{s,t}$, $\tilde{w}_{s,t}^{(n)}\xrightarrow{p}\tilde{w}_{s,t}$.

Without loss of generality we can suppose that $(\tilde{x}_{s,t}^{(n)})$, $(\tilde{w}_{s,t}^{(n)})$,are continuous.

Clearly $(\tilde{w}_{s,t}^{(n)})$, $(\tilde{w}_{s,t})$ are Wiener-Yeh processes and:

$$\tilde{x}_{s,t}^{(n)}=x+\int_o^s\int_o^t a_n(u,v,\tilde{x}_{u,v}^{(n)})d\tilde{w}_{u,v}^{(n)}+\int_o^s\int_o^t b_n(u,v,\tilde{x}_{u,v}^{(n)})dudv$$

Now we are going to prove that:

$$\tilde{x}_{s,t}^{(n)}\xrightarrow{p} x+\int_o^s\int_o^t a(u,v,\tilde{x}_{u,v})d\tilde{w}_{u,v}+\int_o^s\int_o^t b(u,v,\tilde{x}_{u,v})dudv$$

As in onedimensional time parameter it is sufficient to show that:

$$(4.7)\quad M[\chi_{\{\sup_{s,t}/x_{s,t}^{(n)}/\leq l\}}\int_o^1\int_o^1(a_n-a_N)^2(u,v,\tilde{x}_{u,v}^{(n)})dudv]\longrightarrow o$$

$$(4.8)\quad M[\chi_{\{\sup_{s,t}/x_{s,t}/\leq l\}}\int_o^1\int_o^1(a_N-a)^2(u,v,\tilde{x}_{u,v})dudv]\longrightarrow o$$

as $n,N\longrightarrow\infty$.

We prove only 4.7).

We have:

$$M[\chi_{\{\sup_{s,t}/\tilde{x}_{s,t}^{(n)}/\leq l\}}\int_o^1\int_o^1(a_n-a_N)^2(s,t,\tilde{x}_{s,t}^{(n)})dsdt]\leq$$

$$\leq M[\chi_{\{\sup_{s,t}/x_{s,t}^{(n)}/\leq l\}}\int_o^1\int_o^1(a_n-a_N)^2(s,t,\tilde{x}_{s,t}^{(n)})dsdt]+cts.\alpha$$

The process $\exp\{\theta(\tilde{x}_{s,t}^{(n)}-x)-\frac{\theta^2}{2}\int_o^s\int_o^t a_n(u,v,\tilde{x}_{u,v}^{(n)})dudv+\theta\int_o^s\int_o^t b_n(u,v,\tilde{x}_{u,v}^{(n)})dudv$

is a martingale, hence for fixed t is a martingale with onedimensional time parameter.

Then it follows that:

$$\tilde{x}_{s,t}^{(n)}=x+\int_o^s\bar{a}_n(u,t)d\bar{w}_u^{(n)}+\int_o^s\bar{b}_n(u,t)du\quad,o\leq s\leq 1$$

where $(\bar{w}_s^{(n)})_{o\leq s\leq 1}$ is a brownian motion(see [7],corollary 3.2,p.366).

According to the inequality from [1],p.9, there exists a positive and continuous function N(t) such that $\lim_{t\to o}N(t)=\infty$ and:

$$M[\lambda_{\{\sup_{s,t}/\tilde{x}^{(n)}_{s,t}/\leq 1\}}\int_o^1 (a_n-a_N)^2(s,t,\tilde{x}^{(n)}_{s,t})ds]\leq N(t)\ //(a_n-a_N)^2(.,t,.)//_{3,[o,1]\times S(1)}$$

Then:

$$M[\lambda_{\{\sup_{s,t}/x^{(n)}_{s,t}/\leq 1\}}\int_\alpha^1\int_o^1 (a_n-a_N)^2(s,t,\tilde{x}^{(n)}_{s,t})dsdt \leq$$

$$\leq\int_\alpha^1 N(t)//(a_n-a_N)^2(.,t,.)//_{3,[o,1]\times S(1)}dt\leq$$

$$\leq\int_\alpha^1 N^{\frac{3}{2}}(t)^{\frac{2}{3}}//(a_n-a_N)^2//_{3,[o,1]^2\times S(1)}$$

where from:

$$M[\lambda_{\{\sup_{s,t}/x^{(n)}_{s,t}/\leq 1\}}\int_o^1\int_o^1 (a_n-a_N)^2(s,t,\tilde{x}^{(n)}_{s,t})dsdt]\leq$$

$$\leq cts.\alpha + C(\alpha)//(a_n-a_N)^2//_{3,[o,1]^2\times S(1)}$$

if α is small enough and N,n are large enough.

Concerning the uniqueness in law of the weak solution of 4.6) we have the following partial result:

<u>THEOREM 4.9</u> Let a,b: $[o,1]^2\times R\longrightarrow R$ be as in theorem 3.1.

Then there exists an unique in law a weak solution of 4.6).

<u>Proof</u> From theorem 3.1 it follows that 4.6) has an unique(in the paths sense) strong solution.

Then as in the onedimensional time parameter we can prove that the uniqueness in the paths sense implies the uniqueness in law.

Next we are going to give an alternative way to prove the existence and the uniqueness of weak solution of 4.6).

Consider $\Omega=C([o,1]^2,R)$ that is separable and complete metric space relative to the uniform metric.

Let $(x_{s,t})$ be the process $x_{s,t}(\omega)=\omega(s,t)$ and let $\mathcal{F}_{s,t}=\mathcal{B}(x_{u,v}/u\leq s, v\leq t)$ be.

Let a,b: $[o,1]^2\times R\longrightarrow R$ be measurable and bounded functions and moreover let us suppose that $a\geq o$.

<u>DEFINITION 4.1o</u> A probability measure P on $\mathcal{B}_\Omega$ is a solution of the martingale problem with respect to (a,b,x) if:

1) $(x_{s,t})$ agrees with x on I_o a.s.

2) $M[\exp(\theta\square_{s,t}x_{s',t'}-\frac{\theta^2}{2}\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})dudv-\theta\int_s^{s'}\int_t^{t'}b(u,v,x_{u,v})dudv/\mathcal{F}_{s,t}]=1$

for every $s<s', t<t'$ and $\theta\in R$.

<u>PROPOSITION 4.11</u> Let P be a solution of the martingale problem with respect to (a,b,x) and let $\theta(u,v,\omega)$ be a $\mathcal{F}_{s,t}$-previsible and

bounded process. Then:

$$M[\exp(\int_s^{s'}\int_t^{t'}\theta(u,v)dy_{u,v}-\frac{1}{2}\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})\theta^2(u,v)du\,dv/\overline{\mathcal{F}}_{s,t}]=1$$

for every $s<s',t<t'$ where $y_{s,t}=x_{s,t}-x-\int_o^s\int_o^t b(u,v,x_{u,v})du\,dv$.

Proof It follows from theorem 1.8 and proposition 2.2.b).

THEOREM 4.12 If $o<C' \leq a(s,t,x)\leq C$ for each s,t,x (C,C' are constants) then there exists an equivalence between the existence (and the uniqueness) of solution of the martingale problem and the existence (and the uniqueness in law) of the weak solution of equation 4.6).

Proof Let P be a solution of the martingale problem with respect to (a,b,x) and let $y_{s,t}=x_{s,t}-x-\int_o^s\int_o^t b(u,v,x_{u,v})du\,dv$, $W_{s,t}=\int_o^s\int_o^t a^{-\frac{1}{2}}(u,v,x_{u,v})\,dy_{u,v}$ be.

The process $(y_{s,t})$ is a strong martingale (see theorem 1.8) and from previous proposition we have that:

$$1=M[\exp(\theta\int_s^{s'}\int_t^{t'}a^{-\frac{1}{2}}(u,v,x_{u,v})dy_{u,v}-\frac{\theta^2}{2}\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})(a^{-\frac{1}{2}})^2(u,v,x_{u,v})du\,dv$$

$$/\overline{\mathcal{F}}_{s,t}]=M[\exp(\theta\,\square_{s,t}w_{s',t'}-\frac{\theta^2}{2}(s'-s)(t'-t)/\overline{\mathcal{F}}_{s,t}]$$

for every $s<s',t<t',\theta\in R$.

Retaining the first and the last member of the above equalities we obtain that $(w_{s,t})$ is a Wiener-Yeh process (see corollary 1.9).

Obviously 4.6) is satisfied.

Also the law of $(x_{s,t})$ is P, hence the uniqueness in law of the weak solution of 4.6) implies the uniqueness of solution of the martingale problem.

Now, let $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P},\tilde{\mathcal{F}}_{s,t},\tilde{x}_{s,t},\tilde{w}_{s,t})$ be a weak solution of 4.6) and P be the range on Ω of $\tilde{P}$ by function $s,t\longmapsto\tilde{x}_{s,t}(\omega)$.

We have:

$$M_P[\exp(\theta\,\square_{s,t}x_{s',t'}-\frac{\theta^2}{2}\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})du\,dv-\theta\int_s^{s'}\int_t^{t'}b(u,v,x_{u,v})du\,dv)/\overline{\mathcal{F}}_{s,t}]=$$

$$=M_{\tilde{P}}[\exp(\theta\int_s^{s'}\int_t^{t'}a^{\frac{1}{2}}(u,v,\tilde{x}_{u,v})d\tilde{w}_{u,v}-\frac{\theta^2}{2}\int_s^{s'}\int_t^{t'}a(u,v,\tilde{x}_{u,v})du\,dv/\tilde{\overline{\mathcal{F}}}_{s,t}]=1$$

if we keep mind the proposition 4.11 and that $(\tilde{w}_{s,t})$ is a Wiener-Yeh process.

Therefore P is a solution of the martingale problem.

Also it follows that the uniqueness of solution of the martingale problem implies the uniqueness in law of solution of 4.6).

5. An Invariance Principle for Two-Parameter Markov Processes.

Let $(E, \mathcal{E})$ be a measurable space and for integers $o \le k < k'$, $o \le l < l'$ let $P(k,l,k',l',x,y,z,A)$ be a transition probability with respect to $(E^3, \mathcal{E})$.

DEFINITION 5.1 We say that the family $\{P(k,l,k',l',.)\}_{k,k',l,l'}$ is a transition function if hold the following Chapman-Kolmogorov relations:

(5.2) $P(k_1,l_1,k_3,l_2,x,y,z,A)=\int P(k_1,l_1,k_2,l_2,x,y,\xi,d\eta)P(k_2,l_1,k_3,l_2,\xi,\eta,z,A)$

(5.3) $P(k_1,l_1,k_2,l_3,x,y,z,A)=\int P(k_1,l_1,k_2,l_2,x,\xi,z,d\eta)P(k_1,l_2,k_2,l_3,\xi,y,\eta,A)$

for every x,y,z,ξ and $A\in\mathcal{E}$.

THEOREM 5.4 Let E be a polish space, $P(.)$ be a transition function and μ be a law on $\mathcal{B}_E$.
Then there exists an unique probability measure P_μ on $(\prod_{k,l} E_{k,l}, \bigotimes_{k,l} \mathcal{B}_{E_{k,l}})$, where $E_{k,l}=E$ for every k,l, such that:

(5.5) $P_\mu(\prod_{(k,l)\in\Delta} B_{k,l})=\mu(\bigcap_{(k,l)\in\Delta} B_{k,l})$ for every $\Delta\subset\{o\}\times\{o,1,\dots\}\cup\{o,1,\dots\}\times\{o\}$ countable, $B_{k,l}\in\mathcal{B}_E$.

(5.6) $P_\mu(\prod_{(k,l)\in\Gamma\setminus\Gamma_o} B_{k,l}\times\prod_{k_i,l_j} B_{k_i,l_j})=\int d\mu(x)\int P(o,o,k_1,l_1,x,x,x,dx_{11})\dots$
$\dots\int P(o,l_{n-1},k_1,l_1,x,x,x_{1,n-1},dx_{1,n})\int P(k_1,o,k_2,l_1,x,x_{1,1},x,dx_{2,1})\dots$
$\dots\int P(k_1,l_{n-1},k_2,l_n,x_{1,n-1},x_{1,n},x_{2,n-1},dx_{2,n})\dots\int P(k_{m-1},o,k_m,l_1,$
$x,x_{m-1},x,dx_{m,n})\dots\int P(k_{m-1},l_{n-1},k_m,l_n,x_{m-1,n-1},x_{m-1,n},x_{m,n-1},dx_{m,n})$
$\prod_{(u,v)\in\Gamma_o}\lambda_{B_{u,v}}(x)\prod_{k_i,l_j\in\Gamma}\lambda_{B_{k_i,l_j}}(x_{i,j})$

where $k_1<\dots<k_m$, $l_1<\dots<l_n$, $\Gamma_o=\{(k_i,o)/o\le i\le m\}\cup\{(o,l_j)/1\le j\le n\}$, $\Gamma=$
$=\{(k_i l_j)/1\le i\le m, 1\le j\le n\}$, $B_{k,l}\in\mathcal{B}_E$.
Moreover if $x_{k,l}:\prod_{i,j=o}^{\infty}E_{i,j}\longrightarrow E$ are the projections and $\mathcal{F}_{k,l}=$
$=\mathcal{B}(x_{k,u}, u\le l \text{ or } x_{v,l}, v\le k)$ then it is holds the following Markov property:

(5.7) $P(x_{k',l'}\in B/\mathcal{F}_{k,l})=P(k,l,k',l',x_{k,l},x_{k,l'},x_{k',l},B)$ P_μ-a.s for $k<k'$, $l<l'$ and $B\in\mathcal{B}_E$.

Proof For the existence of P_μ one use the projective system of measures given by 5.5),5.6).

The uniqueness of P_μ is obvious and the Markov property follows from 5.2),5.3),5.5),5.6).

LEMMA 5.8) Let $P(k,l,k'',l',x,y,z,A)$ and $(x_{k,l})$ as above for E=R and let ν_x be the probability measure P_{ε_x}.

Consider the process $(y_{k,l})$ defined by:

$$y_{k,o}=y_{o,l}=x$$

$$y_{k+1,l+1}=x_{k+1,l+1}-\sum_{i=o}^{k}\sum_{j=o}^{l}(y-x_{i+1,l}-x_{i,j+1}+x_{i,j})P_{i,j}(x_{i,j},x_{i,j+1},$$

$,x_{i+1,j},dy)$, where $P_{i,j}(x,y,z,A)=P(i,j,i+1,j+1,x,y,z,A)$.

Then :

$$M[\Box_{k,l}y_{k',l'}/\mathcal{F}_{k,l}]=o$$

i.e $(y_{s,t})$ is a strong martingale.

Particulary holds the following inequality:

$$(5.9)\quad P(\max_{\substack{o\le i\le k\\ o\le j\le l}}/y_{i,j+l_o}-y_{i,l_o}/\ge\varepsilon)\le(\tfrac{4}{3})^8\frac{1}{\varepsilon^4}M(/y_{k,l+l_o}-y_{k,l_o}/^4)$$

Proof From Markov property we have:

$$M[x_{k+1,l+1}/\mathcal{F}_{k,l}]=\int yP_{k,l}(x_{k,l},x_{k,l+1},x_{k+1,l},dy)$$

where from:

$$M[x_{k+1,l+1}-x_{k+1,l}-x_{k,l+1}+x_{k,l}/\mathcal{F}_{k,l}]=\int(y-x_{k+1,l}-x_{k,l+1}+x_{k,l})P_{k,l}(x_{k,l},x_{k,l+1},x_{k+1,l},dy)$$

Now the conclusion follows easily.

The inequality 5.9) is a consequence of the Doob inequality from two-parameter case(see [4],proposition 3).

Now for every integer $n\ge 1$ let $P^{(n)}(.)$ be a transition function.

Let $\nu_x^{(n)}$ be the probability measure on $\prod_{k,l=o}\mathcal{B}_{R_{k,l}}$ ($R_{k,l}=R$ for every k,l) built with $P^{(n)}(.)$ and $\mu=\varepsilon_x$.

Let $\Phi_n:\prod_{k,l=o}R_{k,l}\longrightarrow C([0,1]^2,R)$ be the function defined by:

$$\Phi_n(\omega)(s,t)=X_{[ns],[nt]}(\omega)+(ns-[ns])X_{[ns]+1,[nt]}(\omega)+$$

$$(nt-[nt])X_{[ns],[nt]+1}(\omega)+(ns-[ns])(nt-[nt])X_{[ns]+1,[nt]+1}$$

and let $P_x^{(n)}=\nu_x^{(n)}\circ\phi_n^{-1}$, that is a probability measure on $C([0,1]^2,R)$.
Next M_n denote the expectation with resoect to $\nu_x^{(n)}$.
The main result of this section is the following general invariance principle:

THEOREM 5.10 Let a,b: $[0,1]^2\times R\longrightarrow R$ be continuous and bounded functions such that $0<C_1\leq a(s,t,x)$ for every s,t,x.
For every integer $n\geq 0$ let $P^{(n)}(.)$ be a transition function and let $P_{k,l}^{(n)}(.)=P^{(n)}(k,l,k+1,l+1,.)$.
Suppose that:

a_1) $\lim_{n\to\infty}\sup_{k,l,x_i} n^2\int|y-x_2-x_3+x_1|^{4+\eta}P_{k,l}^{(n)}(x_1,x_2,x_3,dy)=0$ for some $\eta>0$.

a_2) $$\int_{|y-x_2-x_3+x_1|\leq 1}(y-x_2-x_3+x_1)P_{k,l}^{(n)}(x_1,x_2,x_3,dy)=$$

$$=\frac{1}{n^2}b\left(\frac{k}{n},\frac{l}{n},x_1\right)+\beta(n,k,l,x_1,x_2,x_3)$$ where

$$\sup_{k,l,x_i}|\beta(n,k,l,x_1,x_2,x_3)|\leq\frac{C_1}{n^2};\ \lim_{n\to\infty}n^2\sup_{k,l,x_i}n^2|\beta(n,k,l,x_1,x_2,x_3)|=0.$$

a_3) $$\int_{|y-x_2-x_3+x_1|\leq 1}(y-x_2-x_3+x_1)^2P_{k,l}^{(n)}(x_1,x_2,x_3,dy)=$$

$$=\frac{1}{n^2}a\left(\frac{k}{n},\frac{l}{n},x_1\right)+\alpha(n,k,l,x_1,x_2,x_3)$$ where

$$\sup_{k,l,x_i}|\alpha(n,k,l,x_1,x_2,x_3)|\leq\frac{C_2}{n^2};\ \lim_{n\to\infty}\sup_{k,l,x_i}n^2|\alpha(n,k,l,x_1,x_2,x_3)|=0.$$

Furthemore suppose that the stochastic equation:

$$5.11)\quad x_{s,t}=x+\int_0^s\int_0^t a^{\frac{1}{2}}(u,v,x_{u,v})dw_{u,v}+\int_0^s\int_0^t b(u,v,x_{u,v})dudv$$

has an unique solution in law.
Let P_x be the law of solution of 5.11) and let $P_x^{(n)}$ be the probability measure on $C([0,1]^2,R)$ built as above and let $x_n\longrightarrow x$.
Then $P_{x_n}^{(n)}$ converge weakly to P_x.
The proof follows closely the one from the onedimensional time parameter given in [7].
We need the following lemmas.

LEMMA 5.12 Let $P(.)$ be a transition function such that:

$$\sup_{k,l,x_i}\left|1-\int\exp\{i\theta(y-x_2-x_3+x_1)\}P_{k,l}(x_1,x_2,x_3,dy)\right|<1$$

for some $\theta\in R$.

Let $\phi_{k,l}^{(\theta)}(x_1,x_2,x_3)=\log\int\exp\{i\theta(y-x_2-x_3+x_1)\}P_{k,l}(x_1,x_2,x_3,dy)$ be.

Then:

$$M[\exp\{i\theta\Box_{k,l}x_{k',l'}-\sum_{r=k}^{k'-1}\sum_{s=l}^{l'-1}\phi^{(\theta)}_{k,l}(x_{r,s},x_{r,s+1},x_{r+1,s})\}/\mathcal{F}_{k,l}]=1$$

for every $k<k', l<l'$.

Proof From the Markov property we have:

$$M[\exp(i\theta\Box_{k,l}x_{k+1,l+1})/\mathcal{F}_{k,l}]=\int\exp\{i\theta(y-x_{k+1,l}-x_{k,l+1}+x_{k,l})\}\cdot$$

$$\cdot P_{k,l}(x_{k,l},x_{k,l+1},x_{k+1,l},dy)$$

From here the conclusion follows easily.

LEMMA 5.13 For every integer $n\geq 1$, let $P^{(n)}(.)$ be a transition function and let $P^{(n)}_{k,l}(.)=P^{(n)}(k,l,k+1,l+1,.)$ be. Suppose that:

a$_1$) $\lim_{n\to\infty} n^2\sup_{k,l,x_i}\int/y-x_2-x_3+x_1/^{4+\eta}P^{(n)}_{k,l}(x_1,x_2,x_3,dy)=0$ for some $\eta>0$.

a$_2$) $\sup_{k,l,x_i}/\int(y-x_2-x_3+x_1)P^{(n)}_{k,l}(x_1,x_2,x_3,dy)/\leq\frac{C_1}{n^2}$

a$_3$) $\sup_{k,l,x_i}\int/y-x_2-x_3+x_1/^2P^{(n)}_{k,l}(x_1,x_2,x_3,dy)\leq\frac{C_2}{n^2}$

Then the family $(P^{(n)}_x)_{n,/x/\leq L}$ is relatively compact in the weak topology, for each $L<\infty$.

Proof We must to show that:

1) $\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{/x/\leq L}P^{(n)}_x(\sup_{/s_i-t_i/\leq\delta}/x_{s_1,s_2}-x_{t_1,t_2}/>\varepsilon)=0$ for every $\varepsilon>0$.

2) $\lim_{a\to\infty}\sup_n\sup_{/x/\leq L}P^{(n)}_x(\sup_{s,t}/x_{s,t}/>a)=0$

or equivalently:

1') $\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{/x/\leq L}\nu^{(n)}_x(\sup_{/s_i-t_i/\leq\delta}/\phi_n(s_1,s_2)-\phi_n(t_1,t_2)/>\varepsilon)=0$

for every $\varepsilon>0$.

2') $\lim_{a\to\infty}\sup_n\sup_{/x/\leq L}\nu^{(n)}_x(\sup_{s,t}/\phi_n(s,t)/>a)=0$.

1') We have:

$$\sup_{/s_i-t_i/<\delta}/\phi_n(s_1,s_2)-\phi_n(t_1,t_2)/\leq 8\max\{\sup_{\substack{0\leq t_1\leq 1\\ k\delta<1}}\sup_{k\delta\leq t_2\leq(k+1)\delta}/\phi_n(t_1,s_2)-\phi_n(t_1,k\delta)/,\ \sup_{0\leq t_2\leq 1,k\delta<1}\sup_{k\delta\leq t_1\leq(k+1)\delta}/\phi_n(t_1,t_2)-\phi_n(k\delta,t_2)/\}$$

$$\sup_{0\leq t_1\leq 1,}\sup_{k\delta\leq t_2\leq(k+1)\delta}/\phi_n(t_1,t_2)-\phi_n(t_1,k\delta)/=$$

$$\max_{0\leq i\leq n}\sup_{k\delta\leq t_2\leq(k+1)\delta}/\phi_n(\tfrac{i}{n},t_2)-\phi_n(\tfrac{i}{n},k\delta)/\leq$$

$$\leq 2\max_{0\leq i\leq n}\max_{0\leq j\leq[n\delta]}/x_{i,[nk\delta]+j}-x_{i,[nk\delta]}/$$

Therefore it is sufficient to show that:

$$\lim_{\delta\to o}\ \lim_{n\to\infty}\ \sup_{/x/\leq L}\ \sum_{k\delta<1} \nu_x^{(n)}\big(\max_{\substack{o\leq i\leq n\\ o\leq j\leq n}} /x_{i,[nk\delta]+j}-x_{i,[nk\delta]}/>\varepsilon\big)=o.$$

From lemma 5.8 we have:

$$\nu_x^{(n)}\big(\max_{\substack{o\leq i\leq n\\ o\leq j\leq n}} /y^{(n)}_{i,[nk\delta]+j}-y^{(n)}_{i,[nk\delta]}/>\varepsilon\big)\leq\left(\tfrac{4}{3}\right)^{8}\tfrac{1}{\varepsilon^4}M_n(/y^{(n)}_{n,[nk\delta]+[n\delta]}-y^{(n)}_{n,[nk\delta]}/^4)$$

Define $T^{(n)}_{j_o}(i,j)=y^{(n)}_{i,j+j_o}-y^{(n)}_{i,j_o}$ and $\Delta^{(n)}_{j_o}(i,j)=y^{(n)}_{i,j+j_o}-y^{(n)}_{i,j+j_o-1}$.

We have:

$$M_n[(T^{(n)}_{j_o}(i,j+1))^4]=M_n[(T^{(n)}_{j_o}(i,j))^4]+6M_n[(T^{(n)}_{j_o}(i,j))^2(\Delta^{(n)}_{j_o}(i,j+1))^2]$$

$$+4M_n[T^{(n)}_{j_o}(i,j)(\Delta^{(n)}_{j_o}(i,j+1))^3]+M_n[(\Delta^{(n)}_{j_o}(i,j+1))^4]$$

$$M_n[(\Delta^{(n)}_{j_o}(i,j+1))^2]=M_n[(\sum_{k=o}^{i-1}\Box_{k,j}\Delta^{(n)}_{j_o}(k+1,j+1))^2]^2=$$

$$=\sum_{k=o}^{i-1}M_n[(\Box_{k,j}\Delta^{(n)}_{j_o}(k+1,j+1))^2]\leq 2\sum_{k=o}^{i-1}M_n[(\Box_{k,j+j_o}x_{k+1,j+j_o+1})^2]+$$

$$+2\sum_{k=o}^{i-1}M_n[(\int(y-x_{k,j+j_o+1}-x_{k+1,j+j_o}+x_{k,j+j_o})P^{(n)}_{k,j+j_o}(x_{k,j+j_o},$$

$$,x_{k,j+j_o+1},x_{k+1,j+j_o+1},dy))^2]\leq 2i\sup_{k,l,x_i}\int(y-x_2-x_3+x_1)^2P^{(n)}_{k,l}(x_1,$$

$$x_2,x_3,dy)+2i\sup_{k,l,x_i}/\int(y-x_2-x_3+x_1)P^{(n)}_{k,l}(x_1,x_2,x_3,dy)/^2\leq$$

$$\leq 2i\frac{C_2}{n^2}+2i\frac{C_1}{n^4}\leq\frac{C_3}{n}\quad\text{for } o\leq i\leq n.$$

Then $M_n[(T^{(n)}_{j_o}(i,j))^2]=\sum_{k=o}^{j-1}M_n[(\Delta^{(n)}_{j_o}(i,l))^2]\leq j\frac{C_3}{n}$

Also: $M_n[(\Delta^{(n)}_{j_o}(i+1,j+1))^3]=M_n[(\Delta^{(n)}_{j_o}(i,j+1))^3]+3M_n[\Delta^{(n)}_{j_o}(i,j+1)\cdot$

$$\cdot\Box_{i,j+j_o}y^{(n)}_{i+1,j+j_o+1})^2]+M_n[(\Box_{i,j+j_o}y^{(n)}_{i+1,j+j_o+1})^3]$$

But $M_n[(\Box_{i,j+j_o}y^{(n)}_{i+1,j+j_o+1})^2]\leq\frac{C_1}{n^2}+\frac{C_1^2}{n^4}\leq\frac{C}{n^2}$; $M_n[(\Delta^{(n)}_{j_o}(i,j+1))^2]\leq\frac{iC}{n^2}$

$$M_n[(\Box_{i,j+j_o}y^{(n)}_{i+1,j+j_o+1})^3]\leq \text{cts.}\big(\alpha\frac{C_2}{n^2}+\frac{1}{\alpha^{\frac{4+\eta}{3}}}\sup_{k,l,x_i}\int/y-x_2-x_3+x_1/^{4+\eta}P^{(n)}_{k,l}($$

$$.(x_1,x_2,x_3,dy)+\frac{C_1^3}{n^6}\big)\leq\frac{f(\alpha,n)}{n^2}\quad\text{where }\lim_{\alpha\to o}\lim_{n\to\infty}f(\alpha,n)=o.$$

Then:

$$M_n[(\Delta^{(n)}_{j_o}(i+1,j+1))^3] \le M_n[(\Delta^{(n)}_{j_o}(i,j+1))^2] + \frac{3C'' i^{\frac{1}{2}}}{n^3} + \frac{f(\alpha,n)}{n^2} \le$$

$$\le A' \frac{i^{\frac{3}{2}}}{n^3} + \frac{i f(\alpha,n)}{n^2} \le \frac{f'(\alpha,n)}{n}$$

It follows that:

$$M_n[(T^{(n)}_{j_o}(i,j+1))^4] \le M_n[(T^{(n)}_{j_o}(i,j))^4] + 6j\frac{C_3^2}{n^2} + 4\frac{f(\alpha,n)}{n^{\frac{3}{2}}} j^{\frac{1}{2}} +$$

$$M_n[(\Delta^{(n)}_{j_o}(i,j+1))^4] \le A(\frac{j}{n})^2 + B(\frac{j}{n})^{\frac{3}{2}} f'(\alpha,n) + \sum_{k=o}^{i} M_n[(\Delta^{(n)}_{j_o}(i,j+1))^2].$$

Also:

$$M_n[(\Delta^{(n)}_{j_o}(i+1,j+1))^4] = M_n[(\Delta^{(n)}_{j_o}(i,j+1))^4] + 6M_n[(\Delta^{(n)}_{j_o}(i,j+1))^2 .$$

$$.(\Box_{i,j+j_o} y^{(n)}_{i+1,j+j_o+1})^2] + 4M_n[\Delta^{(n)}_{j_o}(i,j+1)(\Box_{i,j+j_o} y^{(n)}_{i+1,j+j_o+1})^3] +$$

$$+M_n[(\Box_{i,j+j_o} y^{(n)}_{i+1,j+j_o+1})^4]$$

But: $M_n[(\Box_{i,j+j_o} y^{(n)}_{i+1,j+j_o+1})^4] \le 8(\alpha^2 \frac{C_2^2}{n^2} + \frac{1}{\alpha^\delta} \sup_{k,l,x_i} \int |y-x_2-x_3+x_1|^{4+\eta} P^{(n)}_{k,l}(x_1, x_2, x_3, dy) + \frac{C_1^4}{n^8}) \le \frac{f''(\alpha,n)}{n^2}$ where $\lim_{\alpha\to o}\lim_{n\to\infty} f''(\alpha,n)=o.$

Then $M_n[(\Delta^{(n)}_{j_o}(i+1,j+1))^4] \le M_n[(\Delta^{(n)}_{j_o}(i,j+1))^4] + \frac{6iC''}{n^4} + \frac{4f(\alpha,n)i^{\frac{1}{2}}}{n^3} +$

$$\frac{f''(\alpha,n)}{n^2} \le A'\frac{i^2}{n^4} + B' f(\alpha,n)\frac{i^{\frac{3}{2}}}{n^3} + \frac{i f''(\alpha,n)}{n^2} \le \frac{f'''(\alpha,n)}{n}$$

Where $\lim_{\alpha\to o}\lim_{n\to\infty} f'''(\alpha,n)=o.$

Therefore $M_n[(T^{(n)}_{j_o}(i,j+1))^4] \le A(\frac{j}{n})^2 + B(\frac{j}{n})^{\frac{3}{2}} f'(\alpha,n) + j\frac{f'''(\alpha,n)}{n}$

for $o \le i \le n$, so that:

$$M_n[(T^{(n)}_{[nk\delta]}(n,[n\delta]))^4] \le A(\frac{[n\delta]}{n})^2 + B(\frac{[n\delta]}{n})^{\frac{3}{2}} + \frac{[n\delta]}{n} f'''(\alpha,n).$$

Consequently we have:

$$\overline{\lim_{n\to\infty}} \sup_x \sum_{k\delta<1} \nu^{(n)}_x(\max_{\substack{o\le i\le n\\ o\le j\le[n\delta]}} |x_{i,[nk\delta]+j} - x_{i,[nk\delta]}| > \varepsilon) \le$$

$$\le \lim_{\alpha\to o}\lim_{n\to\infty} \sup_x \sum_{k\delta<1} (\frac{4}{3})^8 \frac{1}{\varepsilon^4}[A(\frac{[n\delta]}{n})^2 + B(\frac{[n\delta]}{n})^{\frac{3}{2}} + \frac{[n\delta]}{n} f'''(\alpha,n) =$$

$$= (\frac{4}{3})^8 \frac{1}{\varepsilon^4}\frac{1}{\delta}(A\delta^2 + B\delta^{\frac{3}{2}}) \longrightarrow o \text{ when } \delta \longrightarrow o.$$

We have: $\nu_x^{(n)}\big(\sup_{s,t}|\phi_n(s,t)|>a\big)=\nu_x^{(n)}\big(\max_{\substack{0\le k\le n\\0\le l\le n}}|x_{k,l}|>a\big)\le$

$$\le\nu_x^{(n)}\big(\max_{0\le k,l\le n}|y^{(n)}_{k,l}|\ge a-L-C_1\big)\le\frac{2}{(a-L-C_1)^2}M_n\big[(y^{(n)}_{n,n})^2\big]\le\frac{K}{(a-L-C_1)^2}\longrightarrow 0$$

as $a\longrightarrow\infty$ uniformly in n and $|x|\le L$.

Proof of theorem 5.1o. From lemma 5.13 it follows that the family $(P_{x_n})_n$ is relatively compact.

Therefore we may suppose that $P_{x_n}\Longrightarrow P$.

We must to show that P is a solution of the martingale problem with respect to (x,a,b) (see theorem 4.12).

As in onedimensional time parameter we have:

$$(5.14)\quad \lim_{n\to\infty}\sup_{k,l,x_r}\Big|\phi^{(n)}_{k,l,\theta}(x_1,x_2,x_3)-\frac{i}{n^2}b\Big(\frac{k}{n},\frac{l}{n},x_1\Big)+\frac{\theta^2}{2n^2}a\Big(\frac{k}{n},\frac{l}{n},x_1\Big)\Big|=0.$$

In wiew of lemma 5.12) we have:

$$M_n\Big[\exp\Big\{i\theta\Box_{k,l}x_{k',l'}-\sum_{r=k}^{k'}\sum_{s=l}^{l'}\phi^{(n)}_{r,s}(x_{r,s},x_{r,s+1},x_{r+1,s})\Big\}\Big/\mathcal{F}_{k,l}\Big]=1$$

for every $k<k'$, $l<l'$.

This implies that:

$$M_{P^{(n)}_{x_n}}\Big[\exp\Big\{i\theta\Box_{\frac{k}{n},\frac{l}{n}}x_{\frac{k'}{n},\frac{l'}{n}}-n^2\int_{[nk]}^{[nk']}\int_{[nl]}^{[nl']}\phi^{(n)}_{[nu],[nv]}\Big(x_{\frac{[nu]}{n},\frac{[nv]}{n}},x_{\frac{[nu]}{n},\frac{[nv]+1}{n}},x_{\frac{[nu]+1}{n},\frac{[nv]}{n}}\Big)du\,dv\Big/\mathcal{F}_{\frac{k}{n},\frac{l}{n}}\Big]=1.$$

Teh theorem is proved if we show that:

$$\lim_{n\to\infty}M_{P^{(n)}_{x_n}}\Big[\xi\exp\Big\{i\theta\Box_{\frac{[ns]}{n},\frac{[nt]}{n}}x_{\frac{[ns']}{n},\frac{[nt']}{n}}-n^2\int_{\frac{[ns]}{n}}^{\frac{[ns']}{n}}\int_{\frac{[nt]}{n}}^{\frac{[nt']}{n}}\phi^{(n)}_{\frac{[nu]}{n},\frac{[nv]}{n}}\Big(x_{\frac{[nu]}{n},\frac{[nv]}{n}},x_{\frac{[nu]}{n},\frac{[nv]+1}{n}},x_{\frac{[nu]+1}{n},\frac{[nv]}{n}}\Big)du\,dv\Big\}\Big]=M_{P_x}\Big[\xi\exp\Big\{i\theta\Box_{s,t}x_{s',t'}-\theta\int_s^{s'}\int_t^{t'}b(u,v,x_{u,v})du\,dv+$$

$$+\frac{\theta^2}{2}\int_s^{s'}\int_t^{t'}a(u,v,x_{u,v})du\,dv\Big\}\Big]$$

for each ξ bounded and continuous.

From 5.14) it follows:

$$\lim_{n\to\infty,\,\omega\in K}\sup\Big|\exp\Big\{i\theta\Box_{\frac{[ns]}{n},\frac{[nt]}{n}}x_{\frac{[ns']}{n},\frac{[nt']}{n}}\int_{\frac{[ns]}{n}}^{\frac{[ns']}{n}}\int_{\frac{[nt]}{n}}^{\frac{[nt']}{n}}\phi^{(n)}_{\frac{[nu]}{n},\frac{[nv]}{n}}\Big(x_{\frac{[nu]}{n},\frac{[nv]}{n}},$$

$$, x_{\frac{[nu]}{n}, \frac{[nv]+1}{n}}, x_{\frac{[nu]+1}{n}, \frac{[nv]}{n}})dudv\} - \exp\{i\theta \square_{s,t} x_{s',t'} - \theta \int_s^{s'}\int_t^{t'} b(u,$$

$$, v, x_{u,v})dudv + \frac{\theta^2}{2}\int_s^{s'}\int_t^{t'} a(u,v,x_{u,v})dudv\}/ = 0$$

for each compact set K.

But $(P_{x_n}^{(n)})_n$ is relatively compact, so that there exists a compact set K such that $P_{x_n}^{(n)}(CK) \leq \frac{\varepsilon}{\Delta}$ for every n, where $\Delta = \sup_n /\!/\xi(\exp\{.\} - \exp\{.\})/\!/$.

Consequently:

$$/M_{P_{x_n}^{(n)}}[\xi \exp\{.\}] - M_P[\xi \exp\{.\}]/ \leq \sup_{\omega \in K} /\exp\{.\} - \exp\{.\}/ +$$

$$+ \Delta P_{x_n}(CK) + /M_{P_{x_n}^{(n)}}[\xi \exp\{.\}] - M_P[\xi \exp\{.\}]/ \leq \varepsilon$$

for n large enough.

Theorem is now proved.

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