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STOCHASTIC STABILITY OF DISCRETE - TIME SYSTEMS WITH JUMP MARKOV PERTURBATIONS

by

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1. Introduction

The problem of stochastic stability and control for continuous or discrete-time systems with jump Markov disturbances has been developed in many papers. See, e.g. references [1] - [14].

In this paper the method of Liapunov functions is used to discuss the stability in probability and mean square exponential stability of some discrete-time systems with jump Markov perturbations.

In Section 2 we get some notations which will be used in sequel. Section 3 contains results concerning the stability in probability.

In Section 4 necessary and sufficient conditions for mean square exponential stability of nonlinear systems are given. In the last section the general results in the preceding section are applied to the linear case. It is also shown that in the linear case the uniform asymptotic stability in mean square implies the exponential stability in mean square.

2. Preliminaries

2.1. The following notations will be used

$\mathbb{R}^l$ is the real l - dimensional Euclidean space,

$$|x|^2 = \sum_{i=1}^{l} x_i^2 , \quad x \in \mathbb{R}^l$$

If A is a matrix (or a vector) A^* means the transposed and $|A| = \sup_{|x|\leq 1} |Ax|$, $\quad |A|_0 = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}$;

$G > 0$, means that G is a positive definite matrix.

We say that $G_n \gg 0$, $n \geq 0$ if $G_n^* = G_n$ and there exist $\alpha, \beta > 0$ such that $\alpha|x|^2 \leq x^* G_n x \leq \beta |x|^2$ for all n and x.

I is the identity matrix.

In this paper $\{\Omega, \mathcal{K}, P\}$ will be a probability field.

If x is a random variable and $\mathcal{F} \subset \mathcal{K}$ is a σ- algebra of subsets of Ω we denote by Ex the mean value of x and by $E[x|\mathcal{F}](\cdot)$ the conditional mean of x with respect to $\mathcal{F}$.

If x,y are random variables, $y_0 \in \mathbb{R}$ and $P\{y(\omega)=y_0\} > 0$ then by definition

$$E[x|y(\omega)=y_0] = \frac{1}{P\{y(\omega)=y_0\}} \int_{\{y(\omega)=y_0\}} x(\omega)\, P(d\omega)$$

If $A \in \mathcal{K}$, φ_A means the characteristic function of the set A.

2.2. Throughout this paper $\eta_n(\omega)$, $n \geq 0$ is a homogeneous Markov chain with state space the set of integers $D = \{1, 2, \ldots, s\}$ and transition matrix $p_{ij} = P\{\eta_{n+1}(\omega)=j | \eta_n(\omega)=i\}$, $i,j \in D$, $n \geq 0$; $\mathcal{F}_n$ will be the σ - algebra generated by the random variables $\eta_0, \ldots, \eta_n$

We assume $P\{\eta_n(\omega)=i\} > 0$ for all $n \geq 0$, $i \in D$;

We remark that this condition is not essential; all results in this paper can be adapted to the case when the above condition is not satisfied.

3. Stability in probability

3.1. Throughout this section a(r) and b(r) will be continuous and increasing functions defined for $r \geq 0$ and $a(0)=b(0)=0$, $\Psi : \mathbb{R}^l \to [0, \infty)$ is a continous function with $\Psi(0)=0$, $\Psi(x) > 0$, $x \neq 0$

$\beta : \{x \in \mathbb{R}^{\ell}, |x| \leq q\} \to [0,\infty)$ is a continuous function with $\beta(0)=0$, $\beta(x)>0$, $x \neq 0$ (q is a given positive number); r_n, $n \geq 0$ will be a sequence of positive numbers with $\sum_{n=0}^{\infty} r_n = \infty$

3.2. Let us consider the system

$$x_{n+1}(\omega) = f_n(x_n(\omega), \eta_n(\omega), \eta_{n-1}(\omega)), \quad n \geq 1, \omega \in \Omega \tag{1}$$

where $f_n : \mathbb{R}^{\ell} \times D^2 \to \mathbb{R}^{\ell}$ are Borel functions.

Such systems are related to the design of a certain feedback law in some stochastic control problems with jump Markov perturbations (see [12]).

If k is a natural number and $x \in \mathbb{R}^{\ell}$ by $x_n(k,x,\omega)$ we denote the solution of the system (1), defined for $n \geq k$ and which verifies $x_k(k,x,\omega) - x$ for all ω

Suppose that $f_n(0,j,i) = 0$ for all $n \geq 1$, j, $i \in D$.

Definition 1.

(1) The trivial solution of the system (1) is strongly stable in probability if for all $\varepsilon > 0$, $\eta > 0$ there exists $\delta(\varepsilon, \eta) > 0$ such that if $|x| < \delta(\varepsilon, \eta)$ then

$$P\{\sup_{n \geq k} |x_n(k,x,\omega)| > \eta\} < \varepsilon \quad \text{for all } k \geq 1$$

(2) The trivial solution of the system (1) is strongly asymptotically stable in probability if it is strongly stable in probability and $\lim_{|x| \to 0} P\{\lim_{n\to\infty} |x_n(k,x,\omega)| = 0\} = 1$ for all $k \geq 1$

(3) The trivial solution of the system (1) is globally strongly asymptotically stable in probability if it is strongly stable in probability and $P\{\lim_{n\to\infty} |x_n(k,x,\omega)| = 0\} = 1$ for all $k \geq 1$ and $x \in \mathbb{R}^{\ell}$

Definition 2.

The trivial solution of the system (1) is strongly exponentially stable in mean square if there exist $\alpha\in(0,1)$ and $\delta>0$ such that

$$E\left[|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right] \le \delta\,\alpha^{n-k}|x|^2$$

for all k, all $n\geq k\geq 1$ and $x\in\mathbb{R}^\ell$, and all $i\in D$.

Remark 1. Since

$$E|x_n(k,x,\omega)|^2 = \sum_{j=1}^{s} P\{\eta_{k-1}(\omega)=j\}\,E[|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=j] \le$$
$$\le \sum_{j=1}^{s} E[|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=j]$$

it follows that if the trivial solution of the system (1) is strongly exponentially stable in mean square then there exist $\alpha\in(0,1)$ and $\delta>0$ such that

$$E|x_n(k,x,\omega)|^2 \le \delta\,\alpha^{n-k}|x|^2$$

for all $n\geq k\geq 1$ and $x\in\mathbb{R}^\ell$.

Conversely, if the last property is true and if in addition $\inf_{n\geq 0} P\{\eta_n(\omega)=i\}>0$, $i\in D$ then the trivial solution of the system (1) is strongly exponentially stable in mean square.

In the following we shall prove some theorems for the stability in probability.

Theorem 1. Suppose that there exist Borel functions $V_n:\mathbb{R}^\ell\times D\to[0,\infty)$, $n\geq 1$, with the following properties:

(1) $a(|x|)\le V_n(x,i)\le b(|x|)$ for all $n\geq 1$, $x\in\mathbb{R}^\ell$, $i\in D$.

(2) $\sum_{j=1}^{s} V_{n+1}(f_n(x,j,i),j)\,p_{ij}\le V_n(x,i)$ for all $n\geq 1$, $i\in D$ and $|x|\le q$

Then the trivial solution of the system (1) is strongly stable in probability.

Proof.

Since $x_n(\kappa, x, \omega)$ is measurable with respect to $\mathcal{F}_{n-1}$ and $\eta_n(\omega)$, $n \geq 0$ is a homogeneous Markov chain, we can write [15]

$$E[\, V_{n+1}(x_{n+1}(\kappa, x), \eta_n) \mid \mathcal{F}_{n-1}](\omega) =$$

$$= E[\, V_{n+1}(f_n(x_n(\kappa, x), \eta_n, \eta_{n-1}), \eta_n) \mid \mathcal{F}_{n-1}](\omega) =$$

$$= \sum_{j=1}^{s} E[\, V_{n+1}(f_n(x_n(\kappa, x), j, \eta_{n-1}), j)\, \varphi_{\eta_n = j} \mid \mathcal{F}_{n-1}](\omega) =$$

$$= \sum_{j=1}^{s} V_{n+1}(f_n(x_n(\kappa, x), j, \eta_{n-1}), j)\, E[\varphi_{\eta_n = j} \mid \mathcal{F}_{n-1}](\omega) =$$

$$= \sum_{j=1}^{s} V_{n+1}(f_n(x_n(\kappa, x, \omega), j, \eta_{n-1}(\omega)), j)\, E[\varphi_{\eta_n = j} \mid \eta_{n-1}](\omega) =$$

$$= \sum_{j=1}^{s} V_{n+1}(f_n(x_n(\kappa, x, \omega), j, \eta_{n-1}(\omega)), j)\, p_{\eta_{n-1}(\omega) j}$$

Thus, we have proved.

$$(2) \quad E[\, V_{n+1}(x_{n+1}(\kappa, x), \eta_n) \mid \mathcal{F}_{n-1}](\omega) =$$

$$= \sum_{j=1}^{s} V_{n+1}(f_n(x_n(\kappa, x, \omega), j, \eta_{n-1}(\omega)), j)\, p_{\eta_{n-1}(\omega) j} \quad , \text{ a.e.}, n \geq \kappa \geq 1$$

Let

$$\gamma = \inf_{r \geq q} a(r)$$

Obviously $\gamma > 0$.

Let $k \geq 1$, and $|x| \leq q$.

We consider

$$W_n(\omega) = \min\{V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)), \gamma\}, \quad n \geq k;$$

$W_n(\omega)$ is measurable with respect to $\mathcal{F}_{n-1}$.

From Condition (1) it follows

$$\{W_n(\omega) < \gamma\} \subset \{|x_n(k,x,\omega)| \leq q\}$$

Thus, by virtue of the equalities (2) and Condition (2) of the theorem, we can write, for all $A \in \mathcal{F}_{n-1}$

$$\int_{A\cap\{W_n(\omega)<\gamma\}} W_{n+1}(\omega)\, P(d\omega) \leq \int_{A\cap\{W_n(\omega)<\gamma\}} V_{n+1}(x_{n+1}(k,x,\omega), \eta_n(\omega))\, P(d\omega) =$$

$$= \int_{A\cap\{W_n(\omega)<\gamma\}} E[V_{n+1}(x_{n+1}(k,\cancel{x}), \eta_n) \mid \mathcal{F}_{n-1}](\omega)\, P(d\omega) =$$

$$= \int_{A\cap\{W_n(\omega)<\gamma\}} \sum_{j=1}^{s} V_{n+1}(f_n(x_n(k,x,\omega), j, \eta_{n-1}(\omega)), j)\, p_{\eta_{n-1}(\omega) j}\, P(d\omega) \leq$$

$$\leq \int_{A\cap\{W_n(\omega)<\gamma\}} V_n(x_n(k,x,\omega), \eta_{n-1}(\omega))\, P(d\omega) =$$

$$= \int_{A\cap\{W_n(\omega)<\gamma\}} W_n(\omega)\, P(d\omega)$$

Therefore

$$\int_A W_{n+1}(\omega)P(d\omega) = \int_{A\cap\{W_n(\omega)=\gamma\}} W_{n+1}(\omega)P(d\omega) + \int_{A\cap\{W_n(\omega)<\gamma\}} W_{n+1}(\omega)P(d\omega) \leq$$

$$\leq \int_{A\cap\{W_n(\omega)=\gamma\}} \gamma P(d\omega) + \int_{A\cap\{W_n(\omega)<\gamma\}} W_n(\omega)P(d\omega) = \int_A W_n(\omega)P(d\omega)$$

Hence

$$E[W_{n+1}|\mathcal{G}_n](\omega) \leq W_n(\omega) \quad , \text{a.e.}$$

where $\mathcal{G}_n = \mathcal{F}_{n-1}$

So that $(W_n, \mathcal{G}_n)$, $n \geq k$ is a supermartingale.

Hence [15]

$$a(\delta)\, P\{\sup_{n\geq k} W_n(\omega) > a(\delta)\} \leq E\, W_k(\omega) \leq E\, V_k(x, \eta_{k-1}(\omega)) \leq$$

$$\leq b(|x|) \quad , \quad \delta > 0$$

Therefore, by virtue of Condition (1), we have for all $\delta > 0$ with $a(\delta) < \gamma$

$$a(\delta)\, P\{\sup_{n\geq k} |x_n(k,x,\omega)| > \delta\} = a(\delta)\, P\{\sup_{n\geq k} a(|x_n(k,x,\omega)|) > a(\delta)\}$$

$$\leq a(\delta)\, P\{\sup_{n\geq k} V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)) > a(\delta)\} \leq$$

$$\leq a(\delta)\, P\{\sup_{n\geq k} W_n(\omega) > a(\delta)\} \leq b(|x|)$$

and, since $\lim_{|x|\to 0} b(|x|) = 0$, the theorem is proved.

<u>Theorem 2. Suppose that there exist Borel functions</u> $V_n : \mathbb{R}^{\ell} \times D \to [0,\infty)$ <u>with the following properties</u>:

(1) $a(|x|) \le V_n(x,i) \le b(|x|)$ for all $n \geq 1$, $x \in \mathbb{R}^{\ell}$, $i \in D$

(2) $\sum_{j=1}^{s} V_{n+1}(f_n(x,j,i),j)\, p_{ij} \le V_n(x,i) - r_n \beta(x)$ <u>for</u> all $n \geq 1$, $i \in D$ and $|x| \le q$.

<u>Then the trivial solution of the system</u> (1) <u>is strongly asymptotically stable in probability.</u>

<u>Proof.</u>

From Theorem 1 it follows the trivial solution of the system (1) is strongly stable in probability.

We have to prove that

$$\lim_{|x|\to 0} P\{ \lim_{n\to\infty} |x_n(k,x,\omega)| = 0\} = 1$$

We shall use the same notations as in the proof of Theorem 1.

Using the reasoning in the proof of the preceding theorem and taking account Condition (2) and the equalities (2) we get

$$E[W_{n+1} | \mathcal{G}_n](\omega) \le W_n(\omega) - r_n \alpha_n(\omega), \quad n \geq k, \quad \text{a.e.}$$

where $\mathcal{G}_n = \mathcal{F}_{n-1}$ and

$$\alpha_n(\omega) = \begin{cases} \beta(x_n(k,x,\omega)) & \text{if } W_n(\omega) < \gamma \\ 0 & \text{if } W_n(\omega) = \gamma \end{cases}$$

Further we shall use the same reasoning as in the proof of Theorem 1 in [16].

From the above inequality it follows that

$$\sum_{n=k}^{\infty} r_n E\alpha_n(\omega) < \infty$$

Since $\sum_{n=k}^{\infty} r_n = \infty$ there exists a sequence n_i with $\lim_{i\to\infty} n_i = \infty$ and $\lim_{i\to\infty} E\alpha_{n_i}(\omega) = 0$

Thus there exists a sequence k_i with $\lim_{i\to\infty} k_i = \infty$ and $\lim_{i\to\infty} \alpha_{k_i}(\omega) = 0$ with probability one.

Since $W_n(\omega)$ is a supermartingale there exists a positive random variable $\kappa(\omega)$ such that $P\{\lim_{n\to\infty} W_n(\omega) = \kappa(\omega)\} = 1$

But $b(r)$ is a continuous function and $b(0)=0$.

Thus there exists $0 < \delta_0 < q$ such that

$$\{x \in R^l, |x| \le \delta_0\} \subset \{x \in R^l, b(|x|) < \gamma\}$$

γ being the number in the proof of the theorem 1.

Let

$$A_x = \{\omega, \lim_{n\to\infty} W_n(\omega) = \kappa(\omega), \lim_{i\to\infty} \alpha_{k_i}(\omega) = 0, \sup_{n \ge k} |x_n(k,x,\omega)| \le \delta_0\}$$

We shall prove that

$$\{\omega, \lim_{n\to\infty} |x_n(k,x,\omega)| = 0\} \supset A_x$$

Indeed, let $\omega \in A_x$.

From Condition (1) it follows that for $n \geq k$ we have

$$|x_n(k,x,\omega)| \le \delta_0, \quad V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)) < \gamma,$$
$$W_n(\omega) = V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)) \text{ and } \beta(x_n(k,x,\omega)) = \alpha_n(\omega)$$

Since $\lim_{i\to\infty} \alpha_{k_i}(\omega) = 0$, from the above relations, using Condition (1) of the Theorem and the properties of the function β we conclude that the sequence $x_{k_i}(k,x,\omega)$. has a subsequence $x_{n_p}(k,x,\omega)$ which converges to zero.

But $W_{n_p}(\omega) \leq b(|x_{n_p}(k,x,\omega)|)$

Hence $c(\omega) = 0$ and thus by Condition 1 $\{\omega, \lim_{n\to\infty} |x_n(k,x,\omega)|=0\} \supset A_x$

But $P(A_x) = 1 - P\{\sup_{n\geq k} |x_n(k,x,\omega)| > \delta_0\}$

By virtue of Theorem 1 $\lim_{|x|\to 0} P\{\sup_{n\geq k} |x_n(k,x,\omega)|>\delta_0\} = 0$

Thus $\lim_{|x|\to 0} P\{\lim_{n\to\infty} |x_n(k,x,\omega)| = 0\} = \lim_{|x|\to 0} P(A_x) = 1$

and Theorem is proved.

Theorem 3. Suppose that there exist Borel functions $V_n : \mathbb{R}^l \times D \to [0,\infty)$ with the following properties:

(1) $a(|x|) \leq V_n(x,i) \leq b(|x|)$, $n \geq 1$, $x \in \mathbb{R}^l$, $i \in D$

(2) $\lim_{r\to\infty} a(r) = \infty$

(3) $\sum_{j=1}^{s} V_{n+1}(f_n(x,j,i),j)\, p_{ij} \leq V_n(x,i) - r_n \psi(x)$

for all $n \geq 1$, $x \in \mathbb{R}^l$, $i \in D$,

Then the trivial solution of the system (1) is globally strongly asymptotically stable in probability.

Proof.

From Theorem 1 it follows the trivial solution of the system (1) is strongly stable in probability.

We shall prove $P\{\lim_{n\to\infty} x_n(k,x,\omega) = 0\} = 1$ for all $k \geq 1$, $x \in \mathbb{R}^l$

Let $k \geq 1$, $x \in \mathbb{R}^l$ and $s_n(\omega) = V_n(x_n(k,x,\omega), \eta_{n-1}(\omega))$

From Condition (3) and the equalities (2), it follows:

(i) $E\, s_{n+1}(\omega) \leq E\, s_n(\omega)$ for all $n \geq k$

(ii) $E\, s_n(\omega) < \infty$

(iii) $r_n\, E\, \psi(x_n(k,x,\omega)) \leq E\, s_n(\omega) - E\, s_{n+1}(\omega) < \infty$

(iv) $E[s_{n+1} | \mathcal{G}_n](\omega) \leq s_n(\omega) - r_n \psi(x_n(k,x,\omega))$, a.e., $n \geq k$

where $\mathcal{G}_n = \mathcal{F}_{n-1}$

From (iv) and (ii) it follows that $(s_n, \mathcal{G}_n)$ is a supermartingale.

Hence [15], there is an integrable function $c(\omega)$ such that $\lim_{n\to\infty} s_n(\omega) = c(\omega)$ a.e.

From (iii) we get

$$\sum_{n=k}^{\infty} r_n E\,\Psi(x_n(k,x,\omega)) < \infty$$

Since $\sum_{n=k}^{\infty} r_n = \infty$, from the above relation it follows that there exists a sequence n_i with $\lim_{i\to\infty} n_i = \infty$ and $\lim_{i\to\infty} \Psi(x_{n_i}(k,x,\omega)) = 0$ a.e.

Let

$$A = \{\omega,\ c(\omega) < \infty,\ \lim_{n\to\infty} s_n(\omega) = c(\omega),\ \lim_{i\to\infty} \Psi(x_{n_i}(k,x,\omega)) = 0\}$$

Let $\omega \in A$.

Since $s_n(\omega)$ is a bounded sequence, from Conditions (1) and (2) it follows that the sequence $|x_n(k,x,\omega)|,\ n \geq k$ is a bounded.

Thus the sequence $x_{n_i}(k,x,\omega),\ i \geq 1$ has a convergent subsequence.

Let $x_{n_p}(k,x,\omega)$ be this subsequence and $\tilde{x}(\omega)$ its limit.

We have $\Psi(\tilde{x}(\omega)) = 0$. Thus $\tilde{x}(\omega) = 0$, $\lim_{p\to\infty} b(|x_{n_p}(k,x,\omega)|) = 0$, $\lim_{p\to\infty} s_{n_p}(\omega) = 0$. Therefore $c(\omega) = 0$ and from Condition (1) $\lim_{n\to\infty} |x_n(k,x,\omega)| = 0$. Since $P(A)=1$, the theorem is proved.

3.3. Let us consider the system

$$(3) \quad z_{n+1}(\omega) = F_n(z_n(\omega), \eta_n(\omega)), \quad n \geq 0, \omega \in \Omega$$

where $F_n : \mathbb{R}^\ell \times D \to \mathbb{R}^\ell$ are Borel functions.

By $z_n(k,x,\omega)$, $n \geq k \geq 0$ we denote the solution of the systems (3) with $z_k(k,x,\omega) \equiv x$

Suppose that $F_n(0,i)=0$, $n \geq 0$, $i \in D$

Let $A_i = \{\omega, \eta_0(\omega) = i\}$

Remark 2. Since $\bigcup_{i=1}^{s} A_i = \Omega$ and $z_n(0,x,\omega) = z_n(1, F_0(x,i), \omega)$ for all $n \geq 1$, $\omega \in A_i$, $i \in D$ and all $x \in \mathbb{R}^\ell$ then it is easy to prove that if the functions $F_0(\cdot,i)$ are continuous in $x=0$ then the following assertions hold:

(i) if the trivial solution ~~solution~~ of the system (3) is strongly stable in probability then for all $\varepsilon > 0$, $\eta > 0$ there exists $\delta(\varepsilon,\eta) > 0$ such that if $|x| < \delta(\varepsilon,\eta)$ then $P\{\sup_{n \geq k} |z_n(k,x,\omega)| > \eta\} < \varepsilon$ for all $k \geq 0$

(ii) if the trivial solution of the system (3) is strongly asymptotically stable in probability then $\lim_{|x| \to 0} P\{\lim_{n \to \infty} |x_n(k,x,\omega)| = 0\} = 1$ for all $k \geq 0$

(iii) if the trivial solution of the system (3) is globally strongly asymptotically stable in probability then

$$P\{\lim_{n \to \infty} |x_n(k,x,\omega)| = 0\} = 1 \qquad \text{for all } k \geq 0,\ x \in \mathbb{R}^\ell.$$

4. MEAN SQUARE EXPONENTIAL STABILITY OF NONLINEAR SYSTEMS

We shall give some necessary and sufficient conditions for strong mean square exponential stability of the systems (1) and (3).

4.1. First, we prove the next lemma.

Lemma 1. *If* $x_n(k,x,\omega)$ *is the solution of the system (1) then*

$$\sum_{j=1}^{1} p_{ij} E[\,|x_n(k+1, f_k(x,j,i),\omega)|^2 \mid \eta_k(\omega)=j] =$$
$$= E[\,|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i]\,, \quad k\geq 1,\ n\geq k+1,\ i\in D,\ x\in\mathbb{R}^l$$

Proof.

We have, for $n \geq k+1$

$$E[\,|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i] = E[\,|x_n(k+1, x_{k+1}(k,x,\omega),\omega)|^2 \mid$$
$$\mid \eta_{k-1}(\omega)=i] =$$
$$= E[\,|x_n(k+1, f_k(x,\eta_k(\omega),i),\omega)|^2 \mid \eta_{k-1}(\omega)=i]$$

Since $\eta_n(\omega),\ n\geq 0$ is a Markov process and $x_n(k+1, f_k(x,\eta_k(\omega),i),\omega)$ is measurable with respect to the σ-algebra generated by $\{\eta_m(\omega),\ m\geq k\}$ we can write

$$E[\,|x_n(k+1, f_k(x,\eta_k,i))|^2 \mid \mathcal{F}_k](\omega) =$$
$$= E[\,|x_n(k+1, f_k(x,\eta_k,i))|^2 \mid \eta_k](\omega) =$$
$$= \sum_{j=1}^{1} E[\,|x_n(k+1, f_k(x,\eta_k(\omega),i),\omega)|^2 \mid \eta_k(\omega)=j]\, \varphi_{\eta_k=j}(\omega)$$

Hence

$$E[|x_n(k+1, \xi_k(x, \eta_k(\omega), i), \omega)|^2 \mid \eta_{k-1}(\omega)=i] =$$

$$= E[(E[|x_n(k+1, f_k(x, \eta_k, i))|^2 \mid \mathcal{F}_k](\omega)) \mid \eta_{k-1}(\omega)=i] =$$

$$= \sum_{j=1}^{1} E[|x_n(k+1, \xi_k(x, j, i), \omega)|^2 \mid \eta_k(\omega)=j]\, p_{ij}$$

Thus Lemma 1 is proved.

Theorem 4. <u>The trivial solution of the system (1) is strongly exponentially stable in mean square if and only if there exist the Borel functions</u> $V_n : \mathbb{R}^l \times D \to [0,\infty)$, $n \geq 1$, <u>with the following properties:</u>

(i) $\alpha_1 |x|^2 \leq V_n(x,i) \leq \quad |x|^2$, $n \geq 1$, $x \in \mathbb{R}^l$, $i \in D$

(ii) $\sum_{j=1}^{1} V_{n+1}(f_n(x,j,i),j)\, p_{ij} \leq V_n(x,i) - \alpha_3 |x|^2$, $n \geq 1$, $x \in \mathbb{R}^l$, $i \in D$

<u>where</u> $\alpha_1, \alpha_2, \alpha_3$ <u>are positive numbers.</u>

<u>Proof.</u>

Suppose that there exist Borel function V_n with properties (i) and (ii). From the relations (2) and Conditions (ii), (i) we get

$$E[V_{n+1}(x_{n+1}(k,x), \eta_n) \mid \mathcal{F}_{n-1}](\omega) \leq V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)) -$$

$$- \alpha_3 |x_n(k,x,\omega)|^2 \leq \left(1 - \frac{\alpha_3}{\alpha_2}\right) V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)), \text{ a.e.}, n \geq k \geq 1$$

Hence

$$E[V_{n+1}(x_{n+1}(k,x,\omega), \eta_n(\omega)) \mid \eta_{k-1}(\omega)=i] \leq$$

$$\leq \left(1 - \frac{\alpha_3}{\alpha_2}\right) E[V_n(x_n(k,x,\omega), \eta_{n-1}(\omega)) \mid \eta_{k-1}(\omega)=i]$$

for all $n \geq k \geq 1$, $i \in D$

Therefore

$$E[\ V_n(x_n(k,x,\omega),\eta_{n-1}(\omega))\ |\ \eta_{k-1}(\omega)=i] \leq$$

$$\leq \left(1-\frac{\alpha_3}{\alpha_2}\right)^{n-k} E[\ V_k(x, \eta_{k-1}(\omega))\ |\ \eta_{k-1}(\omega)=i] \leq$$

$$\leq \left(1-\frac{\alpha_3}{\alpha_2}\right)^{n-k} \alpha_2 |x|^2$$

Hence

$$\alpha_1 E[\,|x_n(k,x,\omega)|^2\ |\ \eta_{k-1}(\omega)=i] \leq \left(1-\frac{\alpha_3}{\alpha_2}\right)^{n-k} \alpha_2 |x|^2$$

and thus the trivial solution of the system (1) is strongly exponentially stable in mean square.

Suppose now the trivial solution of the system (1) is strongly exponentially stable in mean square.

Let us consider the functions

$$V_k(x,i)=\sum_{n=k}^{\infty} E[\,|x_n(k,x,\omega)|^2\ |\ \eta_{k-1}(\omega)=i],\quad k\geq 1,\ x\in R^l,\ i\in D$$

Obviously V_k are Borel functions and

$$|x|^2 \leq V_k(x,i) \leq \sum_{n=k}^{\infty} \delta\alpha^{n-k}|x|^2 \leq \delta\frac{1}{1-\alpha}|x|^2$$

By virtue of Lemma 1, we can write

$$\sum_{j=1}^{d} V_{k+1}(f_k(x,j,i),j)p_{ij} =$$

$$= \sum_{n=k+1}^{\infty} \sum_{j=1}^{d} p_{ij} E[|x_n(k+1, f_k(x,j,i),\omega)|^2 \,|\, \eta_k(\omega)=j] =$$

$$= \sum_{n=k+1}^{\infty} E[|x_n(k,x,\omega)|^2 \,|\, \eta_{k-1}(\omega)=i] = V_k(x,i) - |x|^2$$

for all $k \geq 1$, $i \in D$, $x \in \mathbb{R}^l$.

Thus, the theorem is proved.

If $f_n(x,j,i)$ are linear functions in x, then the functions $V_k(x,i)$ in the proof of Theorem 7 have the form $V_k(x,i) = x^* H_k(i) x$ (see also [14]).

From Theorem 3 and 4 it follows.

<u>Corollary 1. If the trivial solution of the system</u> (1) <u>is strongly exponentially stable in mean square then it is globally strongly asymptotically stable in probability.</u>

4.2. Let us consider the system

$$(4) \qquad x_{n+1}(\omega) = f(x_n(\omega), \eta_n(\omega), \eta_{n-1}(\omega)) \quad , \ n \geq 1, \omega \in \Omega$$

where $f: \mathbb{R}^l \times D^2 \to \mathbb{R}^l$ is a Borel function.

<u>Lemma 2. If</u> $x_n(k,x,\omega)$ <u>is the solutions of</u> (4) <u>then</u>

$$E[|x_n(k,x,\omega)|^2 \,|\, \eta_{k-1}(\omega)=i] =$$

$$= E[|x_{n-k+1}(1,x,\omega)|^2 \,|\, \eta_0(\omega)=i] \ , \quad n \geq k \geq 1, \ x \in \mathbb{R}^l, \ i \in D$$

<u>Proof.</u>

Obviously the equality is true for $n = k$.

Let $n \geqslant k+1$. We define Borel functions $M_n: \mathbb{R}^\ell \times D^n \to \mathbb{R}^\ell$ $n \geqslant 2$ by the following recurrence formula

$$M_n(x, i_{n-1}, \ldots, i_1, i_0) = f(M_{n-1}(x, i_{n-2}, \ldots, i_1, i_0), i_{n-1}, i_{n-2}),$$

if $n \geqslant 3$

and

$$M_2(x, i_1, i_0) = f(x, i_1, i_0) \quad , \quad x \in \mathbb{R}^\ell$$

It is easy to prove by induction that

$$x_n(k, x, \omega) = M_{n-k+1}(x, \eta_{n-1}(\omega), \ldots, \eta_k(\omega), \eta_{k-1}(\omega)),$$

$$x_{n-k+1}(1; x, \omega) = M_{n-k+1}(x, \eta_{n-k}(\omega), \ldots, \eta_1(\omega), \eta_0(\omega)),$$

$n \geqslant k+1$, $x \in \mathbb{R}^\ell$, $\omega \in \Omega$

Hence

$$E[|x_n(k, x, \omega)|^2 \mid \eta_{k-1}(\omega) = i] =$$

$$= E[|M_{n-k+1}(x, \eta_{n-1}(\omega), \ldots, \eta_{k-1}(\omega))|^2 \mid \eta_{k-1}(\omega) = i] =$$

$$= \sum_{i_1, \ldots, i_{n-k}} |M_{n-k+1}(x, i_{n-k}, \ldots, i_1, i)|^2 p_{i i_1} p_{i_1 i_2} \cdots p_{i_{n-k-1} i_{n-k}} =$$

$$= E[|M_{n-k+1}(x, \eta_{n-k}(\omega), \ldots, \eta_1(\omega), \eta_0(\omega))|^2 \mid \eta_0(\omega) = i] =$$

$$= E[|x_{n-k+1}(1, x, \omega)|^2 \mid \eta_0(\omega) = i]$$

Lemma 2 is thus proved.

Suppose that $f(0, j, i) = 0$, $j, i \in D$

From Lemma 2 it follows that the functions $V_k(x, i)$ in the proof of Theorem 4 do not depend on k, i.e.

$$V_k(x, i) = V_1(x, i) = V(x, i)$$

Thus, from Theorem 4 and Lemma 2, we conclude that the following theorem holds.

Theorem 5. The following assertions are equivalent:

(i) The trivial solution of (4) is strongly exponentially stable in mean square

(ii) There exists a Borel function $V: \mathbb{R}^{\ell} \times D \to [0,\infty)$ with the following properties: $\alpha_1 |x|^2 \le V(x,i) \le \alpha_2 |x|^2$,

$$\sum_{j=1}^{s} V(f(x,j,i),j) p_{ij} \le V(x,i) - \alpha_3 |x|^2 , \quad x \in \mathbb{R}^{\ell}, i \in D$$

(iii) There exist $\delta > 0$, $\alpha \in (0,1)$ such that

$$E |x_n(1,x,\omega)|^2 \le \delta \alpha^n |x|^2 , \quad n \ge 1, x \in \mathbb{R}^{\ell}$$

In that follows we shall discuss the strong mean square exponential stability of the system (3).

4.3. Let us consider the system (3) and the following system

$$\hat{x}_{n+1}(\omega) = \hat{f}_n(\hat{x}_n(\omega), \eta_n(\omega), \eta_{n-1}(\omega)) , \quad n \ge 1, \omega \in D \tag{5}$$

where $\hat{f}_n(x,j,i) = F_{n-1}(x,i) , \quad n \ge 1, i,j \in D, x \in \mathbb{R}^{\ell}$

By induction it is easy to prove

$$\hat{x}_n(k,x,\omega) = z_{n-1}(k-1,x,\omega) , \quad n \ge k \ge 1, x \in \mathbb{R}^{\ell}, \omega \in \Omega \tag{6}$$

From the above relations and Theorem 4 it follows.

Theorem 6. The following assertions are equivalent:

(i) There exist $\delta > 0$, $\alpha \in (0,1)$ such that

$$E[|z_n(k,x,\omega)|^2 \mid \eta_k(\omega)=i] \le \delta \alpha^{n-k} |x|^2 \text{ for all } k \ge 0,\ n \ge k,\ x \in \mathbb{R}^l,\ i \in D$$

(ii) The trivial solution of the system (5) is strongly exponentially stable in mean square

(iii) There exist the Borel functions $W_n : \mathbb{R}^l \times D \to [0,\infty)$, $n \geq 0$, with the following properties:

$$\alpha_1 |x|^2 \le W_n(x,i) \le \alpha_2 |x|^2,$$

$$\sum_{j=1}^{s} W_{n+1}(F_n(x,i),j) p_{ij} \le W_n(x,i) - \alpha_3 |x|^2,$$

$$n \ge 0,\ i \in D,\ x \in \mathbb{R}^l$$

The above theorem is a discrete-time analogue of a result of Katz and Krasovskii [3].

We shall prove that if there exists $\gamma_0 > 0$ such that $|F_0(x,i)| \le \gamma_0 |x|^2$, $x \in \mathbb{R}^l$, $i \in D$, then the trivial solution of the system (3) is strongly exponentially stable in mean square iff the trivial solution of the system (5) has the same property.

First we prove the next lemma.

Lemma 3. We have

$$\sum_{j=1}^{s} p_{ij} E[|z_n(k,x,\omega)|^2 \mid \eta_k(\omega)=j] =$$

$$= E[|z_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i],\quad k \ge 1,\ n \ge k,\ x \in \mathbb{R}^l,\ i \in D$$

Proof.

Since $\eta_n(\omega), n \geq 0$ is a Markov process and $z_n(k,x,\omega)$ is measurable with respect to the σ - algebra generated by $\{\eta_m(\omega), m \geq k\}$ we can write

$$E[\,|z_n(k,x)|^2 / \mathcal{F}_k](\omega) = E[\,|z_n(k,x)|^2 \,|\, \eta_k](\omega) =$$

$$= \sum_{j=1}^{1} E[\,|z_n(k,x,\omega)|^2 \,|\, \eta_k(\omega) = j]\, \varphi_{\eta_k = j}(\omega) \quad , \quad a.e.$$

Hence

$$E[\,|z_n(k,x)|^2 \,|\, \eta_{k-1}](\omega) = \sum_{j=1}^{1} E[\,|z_n(k,x,\omega)|^2 \,|\, \eta_k(\omega) = j]\, E[\varphi_{\eta_k = j} \,|\, \eta_{k-1}](\omega)$$

Therefore

$$E[\,|z_n(k,x,\omega)|^2 \,|\, \eta_{k-1}(\omega) = i] = \sum_{j=1}^{1} E[\,|z_n(k,x,\omega)|^2 \,|\, \eta_k(\omega) = j]\, p_{ij}$$

Theorem 7. Assume there exists $\gamma_0 > 0$ such that

$$|F_0(x,i)| \leq \gamma_0 |x| \quad , \quad x \in \mathbb{R}^\ell, \ i \in D$$

Then the trivial solution of the system (3) is strongly exponentially stable in mean square iff there exist $\delta > 0$, $\alpha \in (0,1)$ such that

$$E[\,|z_n(k,x,\omega)|^2 \,|\, \eta_k(\omega) = i] \leq \delta\, \alpha^{n-k} |x|^2 \quad , \quad k \geq 0, n \geq k, x \in \mathbb{R}^\ell, i \in D$$

Proof.

Suppose there exist $\delta > 0$, $\alpha \in (0,1)$ such that

$$E[\,|z_n(k,x,\omega)|^2 \mid \eta_k(\omega)=i\,] \le \delta\alpha^{n-k}|x|^2, \quad n\ge k\ge 0,\, x\in R^{\ell},\, i\in D$$

From Lemma 3 it follows

$$E[\,|z_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i\,] \le \sum_{j=1}^{s} E[\,|z_n(k,x,\omega)|^2 \mid \eta_k(\omega)=j\,] \le$$

$$\le s\,\delta\alpha^{n-k}|x|^2$$

and thus the trivial solution of the system (3) is strongly exponentially stable n mean square.

Suppose now there exist $\delta > 0$, $\alpha \in (0,1)$ such that

$$E[\,|z_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i\,] \le \delta\alpha^{n-k}|x|^2, \quad n\ge k\ge 1,\, x\in R^{\ell},$$

$$i\in D$$

But

$$P\{\eta_k(\omega)=u\} = \sum_{j=1}^{s} P\{\eta_{k-1}(\omega)=j\}\,p_{ju}, \quad k\ge 1,\, u\in D$$

Hence for every $u\in D$ there exists $i_u \in D$ such that

$$p_{i_u u} > 0$$

Let

$$D_0 = \{(i,j)\in D^2 \mid p_{ij} > 0\} \text{ and } \delta_0 = \min_{(i,j)\in D_0} p_{ij}$$

By virtue of Lemma 3 , we can write

$$\delta_0 E[|z_n(k,x,\omega)|^2 | \eta_k(\omega)=u] \le p_{i_u u} E[|z_n(k,x,\omega)|^2 | \eta_k(\omega)=u] \le$$

$$\le \sum_{j=1}^{s} p_{i_u j} E[|z_n(k,x,\omega)|^2 | \eta_k(\omega)=j] =$$

$$= E[|z_n(k,x,\omega)|^2 | \eta_{k-1}(\omega)=i_u] \le \delta\alpha^{n-k}|x|^2, \quad k\ge 1, n\ge k,$$

$$, \quad x\in\mathbb{R}^\ell, u\in D$$

Also, we have

$$E[|z_n(0,x,\omega)|^2 | \eta_0(\omega)=i] =$$

$$= E[|z_n(1,F_0(x,i),\omega)|^2 | \eta_0(\omega)=i] \le \delta\alpha^{n-k}|F_0(x,i)|^2 \le$$

$$\le \delta\alpha^{n-k}\gamma_0^2|x|^2 \qquad , \quad n\ge 1$$

Thus, the theorem is proved.

Theorem 7 generalizes the first assertion of Theorem 6 in [11]. From Theorems 5 and 7 it follows

Corollary 2. Under the assumption of Theorem 7, the trivial solution of the system (3) is strongly exponentially stable in mean square iff there exist Borel functions $W_n : \mathbb{R}^\ell \times D \to [0,\infty)$ with the properties in Theorem 6.

4.4. Let us consider the systems

(6) $$z_{n+1}(\omega) = F(z_n(\omega), \eta_n(\omega)), \quad n\ge 0, \omega\in\Omega$$

where $F : \mathbb{R}^\ell \times D \to \mathbb{R}^\ell$ is a Borel function

(7) $$\hat{x}_{n+1}(\omega) = f(\hat{x}_n(\omega), \eta_n(\omega), \eta_{n-1}(\omega)) \quad , \quad n\ge 1, \omega\in\Omega$$

where

$$\hat{f}(x,j,i) = F(x,i) \quad , \quad x \in \mathbb{R}^{\ell}, \; j, i \in D$$

Since $\hat{x}_n(k,x,\omega) = z_{n-1}(k-1,x,\omega)$, $n \geq k \geq 1, x \in \mathbb{R}^{\ell}, \omega \in \Omega$

from Theorem 5 it follows

Theorem 8. The following assertion are equivalent:

(i) There exist $\delta > 0, \alpha \in (0,1)$ such that

$$E[\,|z_n(k,x,\omega)|^2 \mid \eta_k(\omega) = i] \leq \delta \alpha^{n-k} |x|^2, \quad k \geq 0, n \geq k, x \in \mathbb{R}^{\ell},$$

$$i \in D$$

(ii) the trivial solution of the system (7) is strongly exponentially stable in mean square

(iii) there exists a Borel function $W : \mathbb{R}^{\ell} \times D \to [0,\infty)$ with the following properties: $\alpha_1 |x|^2 \leq W(x,i) \leq \alpha_2 |x|^2$,

$$\sum_{j=1}^{d} W(F(x,i),j)\, p_{ij} \leq W(x,i) - \alpha_3 |x|^2, \quad x \in \mathbb{R}^{\ell}, i \in D$$

(iv) There exist $\delta > 0, \alpha \in (0,1)$ such that

$$E(|z_n(0,x,\omega)|^2) \leq \delta \alpha^n |x|^2 \; , \quad n \geq 0, x \in \mathbb{R}^{\ell}$$

From Corollary 2 and Theorem 8 it follows

Theorem 9. Assume there exist $\delta_0 > 0$ such that $|F(x,i)| \leq \delta_0 |x|^2$. Then the trivial solution of the system (6) is strongly exponentially stable in mean square iff there exists a Borel function $W : \mathbb{R}^{\ell} \times D \to [0,\infty)$ with the properties in Theorem 8.

5. Mean square exponential stability of linear systems

In this section we shall apply the results in the preceding section to the linear case.

5.1. Let us consider the linear system

$$(8)\qquad x_{n+1}(\omega) = A_n(\eta_n(\omega), \eta_{n-1}(\omega))\, x_n(\omega), \quad n \geq 1, \omega \in \Omega$$

where $A_n(j,i)$, $n \geq 1$, $j,i \in D$ are $l \times l$ matrices.

We define the random matrices

$$X_n(k,\omega) = A_{n-1}(\eta_{n-1}(\omega), \eta_{n-2}(\omega)) \cdots A_k(\eta_k(\omega), \eta_{k-1}(\omega))$$

if $n \geq k+1$

and

$$X_k(k,\omega) = I$$

Obviously $x_n(k,x,\omega) = X_n(k,\omega)x$

Theorem 10. *The trivial solution of the system (8) is strongly exponentially stable in mean square iff there exist* $H_n(i) \gg 0$, $n \geq 1$, $i \in D$ *such that*

$$\sum_{j=1}^{s} A_n^*(j,i)\, H_{n+1}(j)\, A_n(j,i)\, p_{ij} - H_n(i) = -I, \quad n \geq 1, i \in D$$

Proof.

Suppose that there exist $H_n(i) \gg 0$, $n \geq 1$, $i \in D$ such that the equalities in the theorem hold.

Let

$$W_n(x,i) = x^* H_n(i)x, \quad x \in \mathbb{R}^l, \; i \in D, \; n \geq 1$$

The function $W_n(x,i)$ satisfy the conditions of Theorem 4 and thus the trivial solution of the system (8) is strongly exponentially stable in mean square.

Suppose now the trivial solution of the system (8) is strongly exponentially stable in mean square.

Let us consider the functions

$$V_k(x,i) = \sum_{n=k}^{\infty} E\left[|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right], \quad k\geq 1, x\in \mathbb{R}^{\ell}, i\in D$$

and the matrices

$$H_k(i) = \sum_{n=k}^{\infty} E\left[X_n^*(k,\omega) X_n(k,\omega) \mid \eta_{k-1}(\omega)=i\right]$$

Obviously $x^* H_n(i) x = V_n(x,i)$

From the proof of Theorem 4 it follows that $H_n(i) \gg 0$, $n\geq 1$, $i\in D$ and

$$\sum_{j=1}^{s} A_n^*(j,i) H_{n+1}(j) A_n(j,i) p_{ij} - H_n(i) = -I, \quad n\geq 1, i\in D$$

Theorem is thus proved.

For the system (8) we can define the following concept of the stability.

Definition 3.

The trivial solution of the system (8) is strongly uniformly asymptotically stable in mean square if the following two assertions hold:

(i) there exists $\delta_0 > 0$ such that $E\left[|X_n(k,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right] < \delta_0$ for all $n\geq k\geq 1$, $i\in D$

(ii) for all $\varepsilon > 0$ there exists a natural number $n(\varepsilon)$ such that $E\left[\,|X_n(k,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right] < \varepsilon$ for all $n \geq n(\varepsilon) + k$, $k \geq 1$, $i \in D$.

Theorem 11. The trivial solution of the system (8) is strongly exponentially stable in mean square iff it is strongly uniformly asymptotically stable in mean square.

Proof.

It is obvious that the strong exponential stability in mean square implies the strong uniform asymptotic stability in mean square.

Suppose the trivial solution of (8) is strongly uniformly asymptotically stable in mean square. Hence there exists a sequence of positive numbers a_n such that $\lim_{n\to\infty} a_n = 0$ and

$$E\left[\,|X_n(k,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right] \leq a_{n-k} \quad \text{for all } n \geq k \geq 1,\ i \in D$$

Let N_0 be a natural number such that $a_{N_0+1} < \frac{1}{2}$

Let us consider

$$H_k(i) = \sum_{n=k}^{k+N_0} E\left[X_n^*(k,\omega)\, X_n(k,\omega) \mid \eta_{k-1}(\omega)=i\right],$$

$$V_k(x,i) = x^* H_k(i)\, x$$

We have $|x|^2 \leq V_k(x,i) \leq |x|^2 \sum_{n=k}^{k+N_0} a_{n-k} \leq \delta |x|^2$

By virtue of Lemma 1 we can write

$$\sum_{j=1}^{s} V_{k+1}(A_k(j,i)x, j)\, p_{ij} = \sum_{n=k+1}^{k+1+N_0} \sum_{j=1}^{s} p_{ij}\, E\left[\,|x_n(k+1, A_k(j,i)x,\omega)|^2 \mid \eta_k(\omega)=j\right] = \sum_{n=k+1}^{k+1+N_0} E\left[\,|x_n(k,x,\omega)|^2 \mid \eta_{k-1}(\omega)=i\right] \leq V_k(x,i) - |x|^2 +$$

$$+\, a_{N_0+1}|x|^2 \leq V_k(x,i) - \frac{1}{2}|x|^2$$

Thus by virtue of Theorem 4, the theorem is proved.

5.2. Let us consider the system

$$x_{n+1}(\omega) = A(\eta_n(\omega), \eta_{n-1}(\omega))\, x_n(\omega), \quad n \geq 1,\ \omega \in \Omega \tag{9}$$

where $A(j,i)$, $j, i \in D$ are $l \times l$ matrices

From Lemma 2 (for the system (9)) it follows the matrices $H_k(i)$ in the proof of Theorem 10 do not depend on k, and thus we can conclude that the next theorem holds

Theorem 12. The trivial solution of the system (9) is strongly exponentially stable in mean square iff there exist

$$H(i) > 0,\ i \in D \text{ such that } \sum_{j=1}^{s} A^*(j,i)H(j)A(j,i)p_{ij} - H(i) = -I,\ i \in D$$

Corollary 3. If the trivial solution of the system (9) is strongly exponentially stable in mean square then

$$\sqrt{p_{ii}}\,\rho(A(i,i)) < 1 \quad \text{for all } i \in D$$

(where $\rho(A)$ is the spectral radius of A).

Proof.

From Theorem 12 it follows that there exist $H(i) > 0$, $i \in D$ such that

$$\left(\sqrt{p_{ii}}\,A(i,i)\right)^* H(i)\left(\sqrt{p_{ii}}\,A(i,i)\right) - H(i) \le \sum_{j=1}^{s} A^*(j,i)H(j)A(j,i)p_{ij} - H(i) < 0$$

Thus $\rho\left(\sqrt{p_{ii}}\,A(i,i)\right) < 1$

Theorem 13. The following assertions are equivalent:

(i) The trivial solution of the system (9) is strongly exponentially stable in mean square

(ii) The trivial solution of the system (9) is strongly uniformly asymptotically stable in mean square

(iii) $\lim_{n\to\infty} E|x_n(1,x,\omega)|^2 = 0$ for all $x \in \mathbb{R}^l$

Proof.

Obviously (i) $\Rightarrow$ (iii). By virtue of Theorem 11 we have (i) $\Leftrightarrow$ (ii).

Suppose (iii) holds. Hence $\lim_{n\to\infty} E|X_n(1,\omega)e_r|^2 = 0$ where $e_r^* = (0\ldots 010\ldots 0)$, $r = 1,2,\ldots,l$

Since $(|X_n(1,\omega)|_o)^2 = \sum_{r=1}^{\ell} |X_n(1,\omega)e_r|^2, \quad |X_n(1,\omega)| \le |X_n(1,\omega)|_o$ we get $\lim_{n\to\infty} \gamma_n = 0$, where $\gamma_n = E(|X_n(1,\omega)|_o)^2$

Thus, from Lemma 2 it follows

$$E[\,|X_n(k,\omega)|^2 \mid \eta_{k-1}(\omega)=i] \le \sum_{r=1}^{\ell} E[\,|X_{n-k+1}(1,\omega)e_r|^2 \mid \eta_o(\omega)=i] \le$$

$$\le \frac{1}{P\{\eta_o(\omega)=i\}}\,\gamma_{n-k+1}$$

Therefore (iii) $\Rightarrow$ (ii) and the theorem is proved.

5.3. Let us consider the systems

(10) $$z_{n+1}(\omega) = B_n(\eta_n(\omega))\, z_n(\omega) \quad , n \ge 0, \ \omega \in \Omega$$

where $B_n(i)$, $i \in D$ are l x l matrices

(11) $$z_{n+1}(\omega) = B(\eta_n(\omega))\, z_n(\omega) \ , \ n \ge 0, \ \omega \in \Omega$$

where B(i) are l x l matrices

From Theorem 10 and Corollary 2 it follows

<u>Theorem 14. The trivial solution of the system</u> (10) <u>is strongly exponentially stable in mean square iff there exist</u> $\hat{H}_n(i) \gg 0$, $n \ge 0$, $i \in D$ <u>such that</u> $\sum_{j=1}^{s} B_n^*(i)\hat{H}_{n+1}(j)B_n(i)p_{ij} - \hat{H}_n(i) = -I$, $n \ge 0, i \in D$

From Theorems 7,8,13 and 14 it follows

<u>Theorem 15. The following assertions are equivalent:</u>

(i) <u>The trivial solution of the system</u> (11) <u>is strongly exponentially stable in mean square</u>

(ii) <u>There exist</u> $\hat{H}(i) > 0$, $i \in D$ <u>such that</u> $\sum_{j=1}^{s} B^*(i)\hat{H}(j)B(i)p_{ij} - \hat{H}(i) = -I$

(iii) $\lim_{n\to\infty} E|z_n(0,x,\omega)|^2 = 0$ <u>for all</u> $x \in R^{\ell}$

($z_n(k,x,\omega)$ being the solution of (11)).

Theorem 15 can be used to discuss the problem of the existence for some linear discrete - time systems with jump Markov parameters of a stabilizing linear feedback control which is optimal with respect to a quadratic cost (see [11]).

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