INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

OPEN EMBEDDINGS OF ALGEBRAIC VARIETIES IN SCHEMES AND APPLICATIONS

(Revised version)

by

Adrian CONSTANTINESCU

PREPRINT SERIES IN MATHEMATICS
No. 30/1979

BUCURESTI

OPEN EMBEDDINGS OF ALGEBRAIC VARIETIES IN SCHEMES AND APPLICATIONS

(Revised version)

by

Adrian CONSTANTINESCU*)

June 1979

*) *Department of Mathematics, National Institute for Scientific and Technical Creation, Bdul. Păcii 220, 77538 Bucharest, Romania.*

This preprint is a revised version of our previous preprint with the same title (INCREST Preprint no.5/1979). The revision was necessary because of a wrong application in the proof of Lemma 2, in the first version, of a Lemma of Nagata. Because of this, the results we were able to recuparate are weaker.

The author thanks professor D.Mumford for pointing out this error.

This preprint is a revised version of our previous preprint with the same title (INCREST Preprint no.5/1979). The revision was necessary because of a wrong application in the proof of Lemma 2, in the first version, of a lemma of Kasato. Because of this, the results we were able to recuperate are weaker.

The author thanks professor D.Mumford for pointing out this error.

C O N T E N T S

Introduction 0

1. Open embeddings of algebraic varieties in schemes 3

2. Two Theorems about the schemes dominated by algebraic varieties 18

3. Some consequences and applications 21

4. Universally 1-equicodimensional rings and finite generatedness of subalgebras 31

References 34

Appendix: Chain Conjectures and finite generatedness of subalgebras 36

C O N T E N T S

Introduction 3

1. Open embeddings of algebraic varieties in schemes 4

2. Two theorems about the schemes dominated by algebraic varieties 10

3. Some consequences and applications 21

4. Universally 1-equidimensional rings and finite extensions of subrings 31

References 34

Appendix. Chain conjectures and finite generatedness of subalgebras 35

Introduction

In §1 we discuss the following problem:

Let $i:X \hookrightarrow X^*$ be an open immersion of the algebraic variety X into an integral scheme and let $x \in X^*$ be an arbitrary point. Under which local conditions on X at x, does the point x have an open neighbourhood which is an algebraic variety over k ?

Theorem 1 provides the following answer to this problem: x has an open algebraic neighbourhood iff the local ring $\mathcal{O}_{X^*,x}$ is noetherian universally catenary and $\dim \mathcal{O}_{X^*,x} + \dim.al._k k(x) = \dim X^*$ (where $k(x)$ is the residue field of $\mathcal{O}_{X^*,x}$).

In Lemmas 1-3 and Remark 1, we prove that $x \in X^*$ has an open algebraic neighbourhood if one of the following conditions is fulfilled:

1) $\dim \mathcal{O}_{X^*,x} = 1$ and $\dim \mathcal{O}_{X^*,x} + \dim.al_k k(x) = \dim X^*$

2) $\dim \mathcal{O}_{X^*,x} = 2$, $\mathcal{O}_{X^*,x}$ is a Krull ring and $\dim \mathcal{O}_{X^*,x} + \dim.al._k k(x) = \dim X^*$

3) $\dim \mathcal{O}_{X^*,x} \geq 3$, $\mathcal{O}_{X^*,x}$ is a noetherian normal ring and $\dim \mathcal{O}_{X^*,x} + \dim.al._k k(x) = \dim X^*$

In §2 we give sufficient conditions under which a dominant morphism $f:X \longrightarrow Y$ from an algebraic variety to an integral scheme, has the property that the subset of all points $y \in Y$, for which $\mathcal{O}_{Y,y}$ is noetherian (resp. Krull, if dim Y=2), is an open subset,

Introduction

In §1 we discuss the following problem:

Let $X^* \supset X$ be an open immersion of the algebraic variety X into an integral scheme and let $x \in X^*$ be an arbitrary point. Under which hypotheses on X^* at x, does the point x have an open neighbourhood which is an algebraic variety over k?

Theorem 1 provides the following answer to this problem: *x has an open algebraic neighbourhood iff the local ring $\mathcal{O}_{X^*,x}$ is noetherian, universally catenary and $\dim \mathcal{O}_{X^*,x} + \dim.cl_k \, k(x) =$ $= \dim X^*$* (where $k(x)$ is the residue field of $\mathcal{O}_{X^*,x}$).

In Lemmas 1-4 and Remark 1, we prove that $x \in X^*$ has an open algebraic neighbourhood if one of the following conditions is fulfilled.

1) $\dim \mathcal{O}_{X^*,x} = 1$ and $\dim \mathcal{O}_{X^*,x} + \dim.cl_k \, k(x) =$ $= \dim X^*$

2) $\dim \mathcal{O}_{X^*,x} = 2$, $\mathcal{O}_{X^*,x}$ is a Krull ring and $\dim \mathcal{O}_{X^*,x} + \dim.cl_k \, k(x) = \dim X^*$

3) $\dim \mathcal{O}_{X^*,x} \geq 3$, $\mathcal{O}_{X^*,x}$ is a noetherian normal ring and $\dim \mathcal{O}_{X^*,x} + \dim.cl_k \, k(x) = \dim X^*$

In §2 we give sufficient conditions under which a dominant morphism $f: X \to Y$ from an algebraic variety to an integral scheme, has the property that the subset of all points $y \in Y$, for which $\mathcal{O}_{Y,y}$ is noetherian, (resp. if dim Y=2), is an open subset, algebraic over k. The sufficient conditions we find are: f is surjective and Y is normal or Y is normal and has the property that all maximal chains of closed integral subschemes have the same length (cf. Theorems 2 and 3).

In § 3, applications of the results obtained in § 1 and § 2 are given.

Thus Proposition 1 and Corollaries 1-3 give necessary and sufficient conditions for finite generatedness of k-subalgebras of finite k-algebras. We prove the following result (cf. Proposition 1 and Corollary 1):

Let A be an integral k-subalgebra of a finite type algebra. The following assertions are equivalent:

i) A is finitely generated

ii) for every maximal ideal $\underline{m} \subset A$, the ring $A_{\underline{m}}$ is noetherian universally catenary and $\dim A_{\underline{m}} = \dim A$

iii) If A′ is the integral closure of A in its field of quotients, every maximal ideal $\underline{m} \subset A'$ is such that $A'_{\underline{m}}$ is noetherian and $\dim A'_{\underline{m}} = \dim A$.

Also in § 3 conditions are given for the finite generatedness of the ring of global functions of a normal algebraic variety (Corollary 4 and Proposition 2). We recover the known affirmative cases of Zariski's form of Hilbert's 14th Problem (Proposition 3) and we give a new proof for a Theorem of Goodman-Landman (Proposition 4).

In § 4 a connection between finite generatedness of subalgebras and a class of rings, we did consider in [4], is exhibited.

The author thanks professor D.Mumford for pointing out an important error in the first version of this paper.

Conventions. Throughout we shall use the definitions and notations of EGA, except the term of "preschemes" which is replaced by "scheme"

Therefore for a scheme X and a point $x \in X$ we denote by $\mathcal{O}_{X,x}$ and by $k(x)$ the local ring, resp. the residue field of X at x. If X is an integral scheme, then $K(X)$ denotes the field of rational functions on X.

§1. Open embeddings of algebraic varieties in schemes

We shall give some properties of the open immersions of algebraic schemes into arbitrary schemes.

Lemma 1. Let $i : X \hookrightarrow X^*$ be an open immersion of integral k-schemes over a field k, where X is an algebraic k-scheme. Then:

a) $\dim X = \dim X^* = \dim.al._k K(X^*)$

b) for every $x \in X^*$, $\dim \mathcal{O}_{X^*,x} + \dim.al._k k(x) \leq \dim X^*$

c) If $x \in X^*$ and $\mathcal{O}_{X^*,x}$ is noetherian such that $\dim \mathcal{O}_{X^*,x} + \dim.al_k k(x) = \dim X^*$, then $k(x)$ is a finitely generated extension of k.

Proof. a) If $X_0^* \subset X_1^* \subset \ldots \subset X_n^*$ is a saturated chain of integral closed subschemes of X^*, it is easy to see that $\dim.al._k K(X_i^*) < \dim.al._k K(X_{i+1}^*)$ for every i, $0 \leq i \leq n-1$. Hence $n \leq \dim.al._k K(X^*)$ and then $\dim X^* \leq \dim.al._k K(X^*) = \dim.al._k K(X) = \dim X \leq \dim X^*$.

b) If X_0^* is the topological closure of x in X^*, then for every saturated chain $X_0^* \subset X_1^* \subset \ldots \subset X_n^* = X^*$ of integral closed subschemes of X^*, by the same argument as in a), it follows that $\dim.al._k k(x) + n = \dim.al._k K(X_0^*) + n \leq \dim.al._k K(X^*) = \dim X^*$. From here, b) follows.

c) If $x \in X$, all is clear. Let $x \in X^* - X$ and $Z \subseteq X^* - X$ the topological closure of x in X^*. We shall proceed by induction over $\dim \mathcal{O}_{X^*,x}$.

If $\dim \mathcal{O}_{X^*,x} = 1$, then the integral closure $\mathcal{O}'_{X^*,x}$ of

$\mathcal{O}_{X^*,x}$ is its field of quotients is noetherian, by Krull-Akizuki Theorem. If $\mathcal{O}$ is the localisation of the ring $\mathcal{O}'_{X^*,x}$ with respect to some maximal ideal, then $\mathcal{O}$ is a discrete valuation ring. Because its field of quotients $K(X^*)$ is a finitely generated extension of k and the residue field k' of $\mathcal{O}$ is an algebraic extension of the residue field k(x) of $\mathcal{O}_{X^*,x}$, it follows that $\dim.al._k k' = \dim.al._k k(x) = \dim.al._k K(X^*)-1$ and it results that k' is a finitely generated extension of k, by [21], Ch.VI, Theorem 31. Since $k(x) \subset k'$, we have that k(x) is a finitely generated extension of k.

Suppose that $\dim \mathcal{O}_{X^*,x} > 1$. Let $\underline{a} \subset \mathcal{O}_{X^*,x}$ be the ideal corresponding to the closed subset $X^* - X$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ the minimal prime ideals of $\mathcal{O}_{X^*,x}$ containg $\underline{a}$. If $\mathfrak{q}_0 = 0 \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m$ is a saturated chain of prime ideals of $\mathcal{O}_{X^*,x}$ of length equal to $m = \dim \mathcal{O}_{X^*,x}$, replacing $\mathfrak{q}_1$ by a prime ideal $\mathfrak{q}_1'$, $0 \subset \mathfrak{q}_1' \subset \mathfrak{q}_2$, we may assume that $\mathfrak{q}_1 \neq \mathfrak{p}_i$, for every i, $1 \leq i \leq n$. Then ht $\mathfrak{q}_1 = 1$ and $\dim \mathcal{O}_{X^*,x}/\mathfrak{q}_1 = \dim \mathcal{O}_{X^*,x} - 1$. Let $X^{*\prime}$ be the integral closed subscheme of X^* passing through x and corresponding to $\mathfrak{q}_1$. Then $X' = X^{*\prime} \cap X \neq \phi$; otherwise, $X^{*\prime}$ is contained in some irreducible component of $X^* - X$ passing through x and this fact implies that $\mathfrak{q}_1$ is equal to some of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, which is a contradiction. $X^{*\prime}$ being 1-codimensional in X^*, we have that X' is 1-codimensional in X. Therefore $\dim X^{*\prime} = \dim X' = \dim X^* - 1$, by a). Since $\mathcal{O}_{X^{*\prime},x} = \mathcal{O}_{X^*,x}/\mathfrak{q}_1$ we have that $\dim \mathcal{O}_{X^{*\prime},x} + \dim.al._k k(x) = \dim X^{*\prime}$. Applying the induction hypothesis to the open immersion $i: X' \hookrightarrow X^{*\prime}$ and to the point $x \in X^{*\prime}$, it follows that k(x) is a finitely generated extension of k.

<u>Lemma 2</u>. <u>Let</u> $i: X \hookrightarrow X^*$ <u>be an open immersion of normal</u>

schemes over a field k, where X is an algebraic k-scheme and Z an integral component of X^*-X. Suppose that the local ring $\mathcal{O}_{X^*,Z}$ of Z is noetherian and $\dim \mathcal{O}_{X^*,Z} + \dim.\mathrm{al.}_k K(Z) = \dim X^*$. Then there exists an open subset $X' \subseteq X^*$ with the following properties:

a) X' is an algebraic scheme over k

b) $X' \supset X$

c) $X' \cap Z \neq \phi$

Proof. Replacing X^* by an open neighbourhood of the generic point of Z, it is obvious that we may suppose that

1) X^* is an affine scheme

By Lemma 1 c), the field K(Z) of the rational functions of Z is a finitely generated extension of k. Let $\{\tilde{\beta}_1, \dots, \tilde{\beta}_m\}$ be a finite set of generators of the field K(Z) over k. By restricting X^* to an affine open subset which meets Z, we may assume that there are $\beta_1, \dots, \beta_m \in \Gamma(X^*, \mathcal{O}_{X^*})$ such that for every j, $1 \leq j \leq m$, $\beta_j|_Z = \tilde{\beta}_j$.

Let $\{\alpha_1, \dots, \alpha_\ell\}$ be a set of generators of the maximal ideal of $\mathcal{O}_{X^*,Z}$. By restricting X^* to an affine open subset meeting Z, we may assume that for every i, $1 \leq i \leq \ell$, $\alpha_i \in \Gamma(X^*, \mathcal{O}_{X^*})$. Let $\underline{a} \subset \Gamma(X^*, \mathcal{O}_{X^*})$ be the nilideal of the closed subset X^*-X and $\underline{p} \supseteq \underline{a}$ the prime ideal of $\Gamma(X^*, \mathcal{O}_{X^*})$ corresponding to the irreducible component Z of X^*-X. The ideal $\underline{p}$ being minimal among those containing $\underline{a}$, it is clear that there exists $s \in \Gamma(X^*, \mathcal{O}_{X^*}) - \underline{p}$ such that for every i, $1 \leq i \leq \ell$, $s\alpha_i \in \underline{a}$. Replacing $\alpha_1, \dots, \alpha_n$ by $s\alpha_1, \dots, s\alpha_n$, we may assume that $\alpha_i \in \underline{a}$ for every i, $1 \leq i \leq \ell$, and $\alpha_1, \dots, \alpha_\ell$ is a set of generators for the maximal ideal of $\mathcal{O}_{X^*,Z}$. Then $X^*_{\alpha_i} = \{x \mid x \in X^*_{\alpha_i}, \alpha_i(x) \neq 0\}$ is an open subset of X for every i. X being a quasicompact scheme, we may find a finite set $\alpha_{\ell+1}, \dots, \alpha_n \in \underline{a}$ such that $X = \bigcup_{i=l+1}^{n} X^*_{\alpha_i}$. Then $\alpha_1, \dots, \alpha_n$ also generate the maximal ideal of $\mathcal{O}_{X^*,Z}$; for every i. $\alpha_i \in \Gamma(X^*, \mathcal{O}_{X^*})$ and $X = \bigcup_{i=1}^{n} X^*_{\alpha_i}$.

For every i, $1\leq i\leq n$, the ring of quotients $\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_i}=\Gamma(X^*_{\alpha_i},\mathcal{O}_{X^*})$ is a finite type k-algebra. Let $\{\gamma_{ij}/\alpha_i^{n_j}\}_{1\leq j\leq m_i}$ be a finite system of generators of $\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_i}$ over k, such that $\gamma_{ij}\in\Gamma(X^*,\mathcal{O}_{X^*})$, and let $A=k[\ldots,\alpha_i,\ldots,\beta_j,\ldots,\gamma_{rs},\ldots]$ be the k-subalgebra of $\Gamma(X^*,\mathcal{O}_{X^*})$ generated over k by all the elements $\{\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m,\{\gamma_{ij}\}_{i,j}\}$. For every t, $1\leq t\leq n$, we have:

$$A_{\alpha_t}=k[\ldots,\alpha_i,\ldots,\beta_j,\ldots,\gamma_{rs},\ldots,1/\alpha_t]\supseteq k[\ldots,\gamma_{ts}/\alpha_t^{n_s},\ldots]_{1\leq s\leq m_t}=$$
$$=\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_t}.$$

Therefore $A_{\alpha_t}=\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_t}$.

If A' is the integral closure of A in its field of quotients, A' is also a finite type k-algebra. Since X^* is a normal scheme, $A'\subseteq\Gamma(X^*,\mathcal{O}_{X^*})$ and for every i, $1\leq i\leq n$, $A'_{\alpha_i}=\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_i}$.

Let $f: X^*\longrightarrow Y^*$ be the morphism of affine k-schemes corresponding to the inclusion of k-algebras $A'\subseteq\Gamma(X^*,\mathcal{O}_{X^*})$. Clearly Y^* = Spec A' is a normal algebraic k-scheme. In 2)-4) below, we shall point out certain properties of the morphism f:

2) $Y=f(X)$ <u>is an open subset of</u> Y^* <u>and</u> $f|_X:X\longrightarrow Y$ <u>is an isomorphism of k-schemes.</u>

Indeed, from $A'_{\alpha_i}=\Gamma(X^*,\mathcal{O}_{X^*})_{\alpha_i}$, it follows that $f(X^*_{\alpha_i})=\operatorname{Spec} A'_{\alpha_i}$ and $f(X^*_{\alpha_i})$ is an open subset of Y^*. Hence $Y=f(X)=f(\bigcup_{i=1}^n X^*_{\alpha_i})=\bigcup_{i=1}^n f(X^*_{\alpha_i})$ is open and $f|_X: X\longrightarrow Y$ is surjective. For any points $x,x'\in X^*$ such that $f(x)=f(x')=y$, we hawe that $\mathcal{O}_{X^*,x}=\mathcal{O}_{Y^*,y}=\mathcal{O}_{X^*,x'}$, since $f|_{X^*_{\alpha_i}}: X^*_{\alpha_i}\longrightarrow\operatorname{Spec} A'_{\alpha_i}$ is an isomorphism, for every i. The scheme X^* being affine, it follows that $x=x'$ and so $f|_X: X\longrightarrow Y$ is an isomorphism.

3) <u>If</u> $W\subset Y^*$ <u>is the closure of</u> $f(Z)$ <u>in</u> Y^* <u>with the reduced closed subscheme structure, then</u> $f|_Z: Z\longrightarrow W$ <u>is a birational</u>

morphism of scheme.

In fact, let $(f|_Z)^*: K(W) \longrightarrow K(Z)$ be the canonical homomorphism of the fields of rational functions. For every j, $1 \leq j \leq m$, $\beta_j \in \Gamma(Y^*, \mathcal{O}_{Y^*})$ and if $\beta_j|_W$ is the restriction of β_j on the closed subscheme $W \subset Y^*$ and $\{\tilde{\beta}_1, \ldots, \tilde{\beta}_m\}$ is the set of generators of the field $K(Z)$ over k chosen above, we have $\tilde{\beta}_j = (f|_Z)^*(\beta_j|_W)$ for every j. Therefore $(f|_Z)^*$ is an isomorphism of fields.

Let $\varphi \in \Gamma(Y^*, \mathcal{O}_{Y^*})$ be such that $\varphi|_W \neq 0$ and such that for every irreducible component $Y^{*\prime}$ of the closed subset $Y^* - Y$ not containing W, $\varphi|_{Y^{*\prime}} = 0$. Then $X^*_\varphi = \{x \mid x \in X^*,\ f^*(\varphi)(x) \neq 0\}$ is an affine open subset of X^* which meets Z and $Y^*_\varphi = \{y \mid y \in Y^*, \varphi(y) \neq 0\}$ is an affine open subset of Y^* such that every irreducible component of $Y^*_\varphi - (Y^*_\varphi \cap Y)$ contains $W \cap Y^*_\varphi$.

Therefore, by restricting X^* and Y^* to open affine subsets meeting Z, respectively W, we may assume that $f: X^* \longrightarrow Y^*$ has the following property:

4) every irreducible component of Y^*-Y contains W.

Let $f^*: \mathcal{O}_{Y^*,W} \rightarrow \mathcal{O}_{X^*,Z}$ be the canonical homomorphism of local rings. We have $\dim \mathcal{O}_{X^*,Z} = \dim X^* - \text{dim.al.}_k K(Z) = \dim Y^* - \text{dim.al.}_k K(W)$. The last is equal to $\dim \mathcal{O}_{Y^*,W}$, since Y^* is an algebraic k-scheme.

Therefore $\dim \mathcal{O}_{X^*,Z} = \dim \mathcal{O}_{Y^*,W}$.

For every i, $1 \leq i \leq n$, $\alpha_i \in \Gamma(Y^*, \mathcal{O}_{Y^*})$. Since $f|_{X^*_{\alpha_i}}: X^*_{\alpha_i} \longrightarrow Y^*_{\alpha_i}$ is an isomorphism, it follows that $\alpha_i|_{Y^*-Y} = 0$ for every i and so $\alpha_1, \ldots, \alpha_n$ are in the maximal ideal $\underline{m}_W$ of the local ring $\mathcal{O}_{Y^*,W}$. It follows that $\underline{m}_Z = \underline{m}_W \mathcal{O}_{X^*,Z}$. This fact and 3) imply that the graduation homomorphism $\text{gr}\, f^*: \text{gr}\mathcal{O}_{Y^*,W} \longrightarrow \text{gr}\mathcal{O}_{X^*,Z}$ is surjective. Then it is known (cf. [2] or [20]) that the canonical homomorphism $\hat{f}^*: \hat{\mathcal{O}}_{Y^*,W} \longrightarrow$

$\longrightarrow \hat{\mathcal{O}}_{X^*,Z}$ between the completions of $\mathcal{O}_{Y^*,W}$ and $\mathcal{O}_{X^*,Z}$ in the radical topologies must be surjective.

Since $\mathcal{O}_{Y^*,W}$ is a k-algebra essentially of finite type, it is a normal excellent ring. Then, by EGA IV 7.8.3 (vii), (or by [10] , 2.10.1 and 2.10.5) $\hat{\mathcal{O}}_{Y^*,W}$ is an integral ring.

It follows that f^*, being a surjective homomorphism from an integral ring onto a ring of the same dimension, is an isomorphism. If $K = Q(\mathcal{O}_{X^*,Z}) = Q(\mathcal{O}_{Y^*,W})$ and $\hat{K} = Q(\hat{\mathcal{O}}_{X^*,Z}) = Q(\hat{\mathcal{O}}_{Y^*,W})$ are the fields of quotients of these integral rings, then, by [1](Ch.III, § 3.5), we have the following equalities among subrings of $\hat{K}$:

$$\mathcal{O}_{X^*,Z} = \hat{\mathcal{O}}_{X^*,Z} \cap K = \hat{\mathcal{O}}_{Y^*,W} \cap K = \mathcal{O}_{Y^*,W}$$

Hence $f^*: \mathcal{O}_{Y^*,W} \longrightarrow \mathcal{O}_{X^*,Z}$ is an isomorphism of local rings and so, via the morphism f, the integral closed subschemes $X^{*'}$ of X^* containing Z are in one-to-one correspondence with the integral closed subscheme $Y^{*'}$ of Y^* containing W.

Now $Y^* - Y = W$. Indeed, via the morphism f, to a closed integral subscheme $X^{*'}$ of X^* meeting X there corresponds a closed integral subscheme $Y^{*'}$ of Y meeting $Y = f(X)$. Since there exists a unique closed integral subscheme of X^* containing Z which does not meet X (namely Z itself), it follows that there exists an unique closed integral subscheme of Y^* containing W and not meeting Y; this subscheme must be W.

Let $f^* : \Gamma(Y^*,\mathcal{O}_{Y^*}) \rightarrow \Gamma(X^*,\mathcal{O}_{X^*})$ be the homomorphism induced by f. Then f^* is bijective. In fact, if $\alpha \in \Gamma(X^*,\mathcal{O}_{X^*})$ then $\alpha \in K(Y^*) = K(X^*)$ is defined on $Y \simeq X$. Since $\alpha \in \mathcal{O}_{Y^*,W} = \mathcal{O}_{X^*,Z}$ and $Y^* = Y \cup W$, it follows that α is defined on an open subset $V \subset Y^*$, such that $\operatorname{codim}_{Y^*}(Y^*-V) \geqslant 2$. But Y^* is a normal k-algebraic scheme; hence $\alpha \in \Gamma(Y^*,\mathcal{O}_{Y^*})$ and so f^* is bijective. The morphism f being dominant, it follows that f^* is bijective.

Then $f : X^* \longrightarrow Y^*$ is an isomorphism of affine schemes and so X^* is an algebraic k-scheme. This completes the proof.

Lemma 3. Let $i : X \hookrightarrow X^*$ be an open immersion of normal schemes over a field k, where X is an algebraic k-scheme and let $x \in X^*$

a) If $\dim \mathcal{O}_{X^*,x} = 1$, then x has an open algebraic neighbourhood iff $\dim.al._k k(x) = \dim X^* - 1$

b) If $\dim \mathcal{O}_{X^*,x} = 2$, then x has an open algebraic neighbourhood iff $\dim.al._k k(x) = \dim X^* - 2$ and $\mathcal{O}_{X^*,x}$ is a Krull ring.

Proof. If x has an open algebraic neighbourdhood, then $\mathcal{O}_{X^*,x}$ is a Krull ring, and $\dim.al._k k(x) = \dim X^* - \dim \mathcal{O}_{X^*,x}$

Suppose that $\mathcal{O}_{X^*,x}$ is either 1 - dimensional such that $\dim.al._k k(x) = \dim X^* - 1$, or 2 - dimensional Krull ring such that $\dim.al._k k(x) = \dim X^* - 2$.

Clearly we may assume that $x \in X^* - X$ and let $Z \subseteq X^* - X$ be the closure of x in X^*. We may suppose that:

1) X^* is an affine scheme

In the case when $\dim \mathcal{O}_{X^*,x} = 1$, then Z is an integral component of $X^* - X$.

In the case when $\dim \mathcal{O}_{X^*,x} = 2$, let $\underline{a} \neq 0$ be the nilideal of the closed subscheme $X^* - X$ in $\mathcal{O}_{X^*,Z}$. There exist finitely many prime ideals $p \subset \mathcal{O}_{X^*,Z}$ such that $\underline{p} \supset \underline{a}$ and ht $\underline{p} = 1$, since $\mathcal{O}_{X^*,Z}$ is Krull. For every such $\underline{p}$, the quotient ring $(\mathcal{O}_{X^*,Z})_{\underline{p}}$ is noetherian. Therefore the set $\mathcal{M} = \{ X^{*\prime} \subseteq X^* - X \mid X^{*\prime}$ integral closed subscheme such that $\operatorname{codim}_{X^*} X^{*\prime} = 1$, and $X^{*\prime} \supseteq Z \}$ is finite and via Lemma 2, for every $X^{*\prime} \in \mathcal{M}$ we can find an open subset $U_{X^{*\prime}} \subseteq X^*$ which is algebraic over k and $\bigcup_{X^{*\prime}} \cap X^{*\prime} \neq \phi$. Then replacing X by the algebraic k-scheme $X \bigcup (\bigcup_{X^{*\prime}\in \mathcal{M}} U_{X^{*\prime}})$ we shall suppose that

2) Z is an integral component of $X^* - X$.

Let $\{\tilde{\beta}_1,\ldots,\tilde{\beta}_m\}$ be an algebraic basis of the field K(Z) over k. By restricting X^* to an open affine subset meeting Z, we shall assume that there exist $\beta_1,\ldots,\beta_m \in \Gamma(X^*,\mathcal{O}_{X^*})$ such that for every i, $1\leq i\leq m$, $\beta_i\Big|_Z = \tilde{\beta}_i$.

Let $\alpha_1,\ldots,\alpha_n$ be elements of $\Gamma(X^*,\mathcal{O}_{X^*})$ such that the open subset $X^*_{\alpha_i} = \{x\in X^* \mid \alpha_i(x)\neq 0\}$ cover X.

As in the proof of Lemma 2, using the elements $\{\alpha_i\}_{1\leq i\leq n}$ and $\{\beta_j\}_{1\leq j\leq m}$, we can construct in the same way an affine normal algebraic scheme Y^* over k and a dominant morphism $f^* : X^* \longrightarrow Y^*$ of k-schemes such that:

3) $Y = f(X)$ is an open subset of Y^* and $f\big|_X : X \longrightarrow Y$ is an isomorphism of schemes.

4) if $W \subseteq Y^*$ is the closure of f(Z) in Y^* with reduced closed subscheme structure, then $\dim.al._k K(Z) = \dim.al._k K(W)$.

5) every irreducible component of $Y^* - Y$ contains W.

Let $f^* : \mathcal{O}_{Y^*,W} \longrightarrow \mathcal{O}_{X^*,Z}$ be the canonical homomorphism of local rings. By 3) and 4), we have:
$\dim.al._k K(W) = \dim.al._k K(Z) = \dim.al._k K(X^*) - \dim \mathcal{O}_{X^*,Z} = \dim.al._k K(Y^*) - \dim \mathcal{O}_{X^*,Z} = \dim Y - \dim \mathcal{O}_{X^*,Z}$. Y^* being an algebraic k-scheme, it follows that $\dim \mathcal{O}_{Y^*,W} = \dim \mathcal{O}_{X^*,Z} = 2$.

If $\dim \mathcal{O}_{X^*,Z} = 1$, then $\mathcal{O}_{X^*,Z} \supseteq \mathcal{O}_{Y^*,W}$ and $\mathcal{O}_{Y^*,W}$ is a noetherian 1-dimensional ring. By the Krull-Akizuki Theorem, $\mathcal{O}_{X^*,Z}$ is noetherian and a) of Lemma 3 is proved, by Lemma 2.

If $\dim \mathcal{O}_{X^*,Z} = 2$, let us denote $\mathfrak{X} = \operatorname{Spec} \mathcal{O}_{X^*,Z}$, $\mathfrak{Y} = \operatorname{Spec} \mathcal{O}_{Y^*,W}$, $\underline{m}_x$ and $\underline{m}_y$ the closed points of $\mathfrak{X}$ and $\mathfrak{Y}$ and $\varphi : \mathfrak{X} \to \mathfrak{Y}$ the morphism of affine schemes corresponding to f^*.

The restriction homomorphism $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow$ $\longrightarrow \Gamma(\mathfrak{X} - \underline{m}_x, \mathcal{O}_{\mathfrak{X}})$ is an isomorphism, since $\mathcal{O}_{X^*,Z}$ is a Krull ring.

We shall prove that there is no 1-codimensional closed irreducible subset $Y^{*'} \subseteq Y^* - Y$. Indeed, otherwise we have, by 5), that $Y^{*'} \supseteq W$ and so $f^{-1}(Y^{*'}) = Z$, by virtue of 3). If $p \subseteq \mathcal{O}_{Y^*,W}$ is the prime ideal corresponding to $Y^{*'}$, then $\operatorname{ht} \underline{p} = 1$ and $\underline{p}\mathcal{O}_{X^*,Z}$ is $\underline{m}$-primary. Then it is known (cf. SGA II, Example III.a) that $\mathfrak{Y} - \{\underline{m}_y, \underline{p}\}$ is affine, since $\dim \mathfrak{Y} = 2$ and $\mathfrak{Y}$ is normal. But φ being an affine morphism of scheme and $\varphi^{-1}(\mathfrak{Y} - \{\underline{m}_y, \underline{p}\}) = \mathfrak{X} - \underline{m}_x$, it follows that $\mathfrak{X} - \underline{m}_x$ is an affine scheme. Since $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \longrightarrow$ $\to \Gamma(\mathfrak{X} - \underline{m}_x, \mathcal{O}_{\mathfrak{X}})$ is an isomorphism, the open immersion of affine scheme $\mathfrak{X} - \underline{m}_x \hookrightarrow \mathfrak{X}$ is an isomorphism, which is not possible.

Therefore, $Y^* - Y = W$, since $\operatorname{codim}_{Y^*} W = 2$ and by virtue of 5).

Then $f^*: \Gamma(Y^*, \mathcal{O}_{Y^*}) \longrightarrow \Gamma(X^*, \mathcal{O}_{X^*})$ is an isomorphism. In fact, every $\alpha \in \Gamma(X^*, \mathcal{O}_{X^*}) \subset K(X^*) = K(Y^*)$ is defined on every 1-codimensional closed integral subscheme $Y^{*'} \subset Y^*$, since every such $Y^{*'}$ meets $Y \simeq X$. Hence $\alpha \in \Gamma(Y^*, \mathcal{O}_{Y^*})$, since Y^* is an algebraic normal scheme.

It follows that $f : X^* \longrightarrow Y^*$ is an isomorphism and this completes the proof of Lemma 3 b).

Remark 1. In Lemma 3 a) is not necessary to assume that

X^* is normal. More precisely:

Let $i: X \hookrightarrow X^*$ be an open immersion of an algebraic scheme in an integral scheme over a field k and let $x \in X^*$ such that $\dim \mathcal{O}_{X^*,x} = 1$. Then x has an open algebraic neighbourhood iff $\dim.\mathrm{al}._k k(x) = \dim X^* - 1$.

In fact, if $p : X^{*'} \longrightarrow X^*$ is the normalization morphism of X^*, then for every point $x' \in X^{*'}$ being over x we have $\dim \mathcal{O}_{X^{*'},x'} = 1$ and $\dim.\mathrm{al}._k k(x') = \dim X^{*'} - 1$. Then every such point $x' \in X^{*'}$ has an open algebraic neighbourhood. If U is the union of all these neighbourhoods, then $V = X^* - p(X^* - U)$ is an open neighbourhood of x and $p^{-1}(V) \subset U$. Since p is integral, it follows that V is locally algebraic over k.

Lemma 4. Let $i: X \hookrightarrow X^*$ be an open immersion of normal schemes over a field k, where X is an algebraic k-scheme and let $x \in X^*$. Then x has an open algebraic neighbourhood iff the following conditions are satisfied:

i) $\mathcal{O}_{X^*,x}$ is noetherian

ii) $\dim \mathcal{O}_{X^*,x} + \dim.\mathrm{al}._k k(x) = \dim X^*$.

Proof. First we shall prove that for every point $y \in X^*$ satisfying the conditions i) and ii) of Lemma 4, $\mathcal{O}_{X^*,y}$ is essentially of finite type over k.

Let $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_p\}$ be a set of generators of the function field $K(X^*)$ over k such that:

a) $\alpha_r, \beta_s, \gamma_t \in \mathcal{O}_{X^*,y}$ for every r,s,t

b) the k-algebra $A = k[\ldots, \alpha_r, \ldots, \beta_s, \ldots, \gamma_t, \ldots]$ is normal

c) $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for the maximal ideal $\underline{m}$ of $\mathcal{O}_{X^*,y}$

d) $\{\beta_1,\ldots,\beta_m\}$ gives a set of generators for the residue field $k(y)$ of $\mathcal{O}_{X^*,y}$, which is finitely generated by virtue of Lemma 1 c).

Let $\underline{n} = \underline{m} \cap A$ and $f: A_{\underline{n}} \hookrightarrow \mathcal{O}_{X^*,y}$ be the canonical inclusion of local rings. From a), c), d), it follows that $grf : gr A_{\underline{n}} \to gr\mathcal{O}_{X^*,y}$ is surjective and so the canonical homomorphism between the completions: $\hat{f}: \hat{A}_{\underline{n}} \to \hat{\mathcal{O}}_{X^*,y}$ is surjective. But $\hat{A}_{\underline{n}}$ is integral, since $A_{\underline{n}}$ is essentially of finite type over k and normal. Since $\dim A_{\underline{n}} + \dim.al._k A_{\underline{n}}/\underline{n}A_{\underline{n}} = \dim A = \dim.al._k K(X^*) = \dim \mathcal{O}_{X^*,y} + \dim.al._k k(y)$, it follows that $\dim A_{\underline{n}} = \dim \mathcal{O}_{X^*,y}$ because $k(y)$ is isomorphic with $A_{\underline{n}}/\underline{n}A_{\underline{n}}$. Hence $\hat{f}$ is an isomorphism and then it results that f is an isomprhism, as in the proof of Lemma 2. Hence $\mathcal{O}_{X^*,y}$ is essentially of finite type over k.

Therefore for every prime ideal $\underline{p} \subset \mathcal{O}_{X^*,y}$ we have that $\dim (\mathcal{O}_{X^*,y})_{\underline{p}} + \dim.al._k k(\underline{p}) = \dim X^*$, if $y \in X^*$ verifies the conditions i) and ii) of Lemma 4.

By induction over n, we shall prove that we may find a chain of open subsets of X^*: $V^o \subseteq V^1 \subseteq \ldots \subseteq V^n$ such that for every i, $0 \leq i \leq n$, V^i is locally an algebraic k-scheme containing all the points $y \in X^*$ such that $\mathcal{O}_{X^*,y}$ is a noetherian i-dimensional ring and such that $\dim \mathcal{O}_{X^*,y} + \dim.al._k k(y) = \dim X^*$.

If $i=0$, we may take $V^o = X$. Suppose that we have found a chain $V^o \subseteq V^1 \subseteq \ldots \subseteq V^n$ with the above properties. We shall define V^{n+1}.

If there exists no point $y \in X^* - V^n$ such that $\mathcal{O}_{X^*,y}$ is noetherian (n+1) - dimensional ring and $\dim \mathcal{O}_{X^*,y} + \dim.al._k k(y) = \dim X^*$, then we put $V^{n+1} = V^n$. If there exists such a point y, then it is easy to see that the closure Z_y of y in X^* is an irreducible component of $X^* - V^n$. Let $i_y: \mathrm{Spec}\,\mathcal{O}_{X^*,y} \hookrightarrow X^*$ be the canonical morphism corresponding to the localisation of X^* in y. Because $i_y^{-1}(V^n) =$

$= \operatorname{Spec} \mathcal{O}_{X^*,y} - \underline{m}_y$ is a noetherian scheme, ($\underline{m}_y$ being the maximal ideal of $\mathcal{O}_{X^*,y}$), it follows that we can find finitely many open affine subsets $U_1,\ldots,U_{\ell_y}$ of V^n such that $\operatorname{Spec} \mathcal{O}_{X^*,y} - \underline{m}_y = i_y^{-1}(\bigcup_{j=1}^{\ell_y} U_j)$. It is clear that Z_y is an irreducible component of $X^* - \bigcup_{j=1}^{\ell_y} U_j$ and applying Lemma 2 to the open immersion $i: \bigcup_{j=1}^{\ell_y} U_j \hookrightarrow X^*$ and to Z_y, we find an open affine neighbourhood V_y of y which is an algebraic k-scheme. If $\mathcal{M}_{n+1} = \{y \in X^* - V^n \mid \mathcal{O}_{X^*,y}$ is a noetherian $(n+1)$ - dimensional ring and $\dim \mathcal{O}_{X^*,y} + \dim.\mathrm{al.}_k k(y) = \dim X^*\}$, then we take $V^{n+1} = V^n \cup (\bigcup_{y\in\mathcal{M}_{n+1}} V_y)$.

The open set $V = \bigcup_n V^n$ is locally an algebraic k-scheme and it is the subset of all points $y \in X^*$ such that $\mathcal{O}_{X^*,y}$ is noetherian and $\dim \mathcal{O}_{X^*,y} + \dim.\mathrm{al.}_k k(y) = \dim X^*$.

Recall the following

Marot Lemma ([11] , Lemma 2). If A is a noetherian integral ring and for every prime ideal $\mathfrak{p} \subset A$, $\mathfrak{p} \neq 0$, the ring $A/\mathfrak{p}$ is japanese, then the integral closure of A in every finite extension of its field of quotients is noetherian.

In the case of the open immersions into arbitrary schemes we can prove:

Theorem 1. Let $i : X \hookrightarrow X^*$ be an open immersion of integral k-schemes over a field k, where X is algebraic over k and let $x \in X^*$. Then x has an open algebraic neighbourhood iff the following conditions are satisfied:

i) $\mathcal{O}_{X^*,x}$ is a noetherian universally catenary ring

ii) $\dim \mathcal{O}_{X^*,x} + \dim.\mathrm{al.}_k k(x) = \dim X^*$.

Proof. Suppose that i) and ii) are satisfied. We shall prove that x has an open algebraic neighbourhood by induction over $\dim X^*$.

If $\dim X^*=0$, the assertion is trivial, since $X=X^*$.

Suppose that $\dim X^*>0$.

We claim that every integral closed subscheme $X^{*'}$ of X^* passing through x is generically algebraic over k. In fact, let $X^*=X_0^*\supset X_1^*\supset\ldots\supset X_n^*=X^{*'}=X_{n+1}^*\supset\ldots\supset X_m^*=\overline{\{x\}}$ be a saturated chain of integral closed subschemes of X^*, which contains $X^{*'}$. Since $\mathcal{O}_{X^*,x}$ is catenary, we have $m=\dim\mathcal{O}_{X^*,x}$. Since, by condition ii),

$m+\dim.\text{al.}_kK(X_m^*)=\dim.\text{al.}_kK(X^*)$ and for every i, $0\leq i\leq m-1$

$\dim.\text{al.}_kK(X_i^*)>\dim.\text{al.}_kK(X_{i+1}^*)$, we have

$\dim.\text{al.}_kK(X_i^*)=\dim.\text{al.}_kK(X_{i+1}^*)+1$ for every i, $0\leq i\leq n-1$.

We shall prove by induction over i that for every i, $0\leq i\leq n$, X_i^* is generically algebraic over k. If $i=0$, the assertion is clear. Suppose that X_i^* is generically algebraic over k, and let x_{i+1} be the generic point of X_{i+1}^*. Then $\mathcal{O}_{X_i^*,x_{i+1}}=1$ and $\dim.\text{al.}_kk(x_{i+1})=$ $=\dim.\text{al.}_kK(X_{i+1}^*)=\dim.\text{al.}_kK(X_i^*)-1$.

By Remark 1, it follows that x_{i+1} has an open algebraic neighbourhood in X_i^*. Hence X_{i+1}^* is generically algebraic over k.

For every integral closed subscheme $X^{*'}$ of X^* passing through x, we have $\dim\mathcal{O}_{X^{*'},x}+\dim.\text{al.}_kk(x)=\dim X^{*'}$. Indeed if $X^*=X_0^*\supset X_1^*\supset\ldots\supset X_n^*=X^{*'}\supset X_{n+1}^*\ \ldots\supset X_m^*=\overline{\{x\}}$ is a saturated chain of integral closed subschemes of X^*, which contains $X^{*'}$, we can prove, by induction over i, that $\dim\mathcal{O}_{X_i^*,x}+\dim.\text{al.}_kk(x)=$ $=\dim X_i^*$: for $i=0$, it is the condition ii); we have $\dim\mathcal{O}_{X_{i+1}^*,x}=\dim\mathcal{O}_{X_i^*,x}-1$, since $\mathcal{O}_{X^*,x}$ is catenary, and $\dim X_{i+1}^*=$ $\dim X_i^*-1$ since $\dim.\text{al.}_kK(X_{i+1}^*)=\dim.\text{al.}_kK(X_i^*)-1$.

Therefore we may apply the induction hypothesis to every closed integral subscheme $X^{*'}\neq X^*$ passing through x, and to the point $x\in X^{*'}$. Then for every prime ideal $\mathfrak{p}\subset\mathcal{O}_{X^*,x}$, $\mathfrak{p}\neq 0$, the ring

$\mathcal{O}_{X^*,x}/\mathfrak{p}$ is essentially of finite type over k. By Marot Lemma, it follows that the integral closure $\mathcal{O}'_{X^*,x}$ of $\mathcal{O}_{X^*,x}$ in its field of quotients is noetherian.

Every maximal chain of prime ideals $0=\mathfrak{p}_0\subset\mathfrak{p}_1\subset\ldots\subset\mathfrak{p}_n$ in $\mathcal{O}'_{X^*,x}$ has the length $n=\dim\mathcal{O}_{X^*,x}$. Indeed, since $\mathcal{O}_{X^*,x}$ is noetherian there exist finitely many prime ideals of $\mathcal{O}'_{X^*,x}$ lying over $\mathfrak{p}_i$. Hence we can find a finite $\mathcal{O}_{X^*,x}$ - subalgebra A of $\mathcal{O}'_{X^*,x}$ such that for every i, $0\leq i\leq n$, $\mathfrak{p}_i$ is the unique prime ideal of $\mathcal{O}'_{X^*,x}$ lying over $\mathfrak{p}_i\cap A$. It is clear that $0=\mathfrak{p}_0\cap A\subset\mathfrak{p}_1\cap A\subset\ldots\subset\mathfrak{p}_n\cap A$ is a maximal chain of prime ideals in A. By EGA IV, Proposition 5.6.10, $\dim A_{\mathfrak{p}_n\cap A}=$ $=\dim\mathcal{O}_{X^*,x}$; since $A_{\mathfrak{p}_n\cap A}$ is catenary , it follows that $n=$ $=\dim\mathcal{O}_{X^*,x}$.

Let $p: X^{*\prime}\longrightarrow X^*$ be the normalisation morphism of X^*. By the above, for every point $x'\in X^{*\prime}$ lying over x, we have that $\mathcal{O}_{X^{*\prime},x'}$ is a noetherian ring of dimension equal to $\dim\mathcal{O}_{X^*,x}$. Thus $\dim\mathcal{O}_{X^{*\prime},x'}+\dim.\mathrm{al}._k k(x')=\dim X^{*\prime}$. Via Lemma 4, it follows that every point of $X^{*\prime}$ lying over x has an open algebraic neighbourhood. Let U be the union of these neighbourhoods; then $V=X^*-p(X^*-U)$ is open in X^*, $p^{-1}(V)\subset U$ and $x\in V$. Since p is integral, it follows that V is an open algebraic neighbourhood of x, which ends the proof of Theorem 1.

Remark 2 a). The condition of universally catenarity for $\mathcal{O}_{X^*,x}$ in Theorem 1 is not a consequence of the other conditions. The following Example, which draws upon Example 2 of Appendix to [13], shows this fact:

Example - An open embedding $i: X\hookrightarrow X^*$ of an algebraic k-scheme into an integral k-scheme such that:

1) <u>$\dim X^* = 2$</u>

2) <u>X^*-X is a closed point x and $\mathcal{O}_{X^*,x}$ is a noetherian 2-dimensional ring which is not universally catenary.</u>

Then it is clear that $\mathcal{O}_{X^*,x}$ is catenary.

In fact, let x be an indeterminate over k and $f(x) = \sum_{i=1}^{\infty} a_i x^i$ a formal power series which is transcendental over the field $k(x)$. Let $A' = k[x, f(x), f(x)-a_1x/x, \ldots, f(x)-a_1x-\ldots-a_nx^n/x^n, \ldots]$ and $A = k[x^2-x, f(x)-a_1x/x, \ldots, f(x)-a_1x-\ldots a_nx^n/x^n, \ldots]$ be the k-subalgebra of $k[[x]]$ generated by the indicated elements.

If $\underline{m}_1$ is the ideal of A' generated by x and $\underline{m}_2$ the ideal generated by $x-1$ and $y = f(x)$, we have:

Spec A'=Spec $A'[1/x] \cup \{\underline{m}_1\}$, $A'[1/x] = k[x, y, 1/x]$,

$\dim A'_{\underline{m}_1} = 1$, $\dim A'_{\underline{m}_2} = 2$ and $A'_{\underline{m}_1}$ is a discrete valuation ring.

If $\underline{m} = (x^2-x, y) \subset A$ is the maximal ideal of A generated by x^2-x and y, we have $\underline{m}_1 \cap A = \underline{m}_2 \cap A = \underline{m}$. The ring A' is the integral closure of A in its field of quotients and A' is finite over A. Since A' is noetherian, it follows, by <u>Eakin-Nagata</u> Theorem ([6] or [16]), that A is noetherian. Therefore $A_{\underline{m}}$ is a noetherian 2-dimensional ring which is not universally catenary since $\dim A'_{\underline{m}_1} = 1$ (cf. EGA IV, Proposition 5.6.10).

Since Spec A' is generically algebraic over k, it follows that Spec A is so. If X^*=Spec A, we have an open immersion $i: X \hookrightarrow X^*$ of an algebraic k-scheme X in X^*. Let $x \in X^*$ be the closed point corresponding to $\underline{m} \subset A$. For every closed integral 1-dimensional subscheme $X^{*\prime}$ of X^* passing through x, we have that $\dim \mathcal{O}_{X^*,X^{*\prime}} = 1$ and dim.al.$_k K(X^{*\prime}) = 1$. By Remark 1, it results that the generic point of $X^{*\prime}$ has an open algebraic neighbourhood. Replacing X by the union of X with all these algebraic neighbourhoods, it follows that $\{x\}$ is a component of X^*-X. By restricting X^* to an open neighbourhood of x, we obtain the desired Example.

Remark 2 b) - In Lemma 4, the condition of universally catenarity for $\mathcal{O}_{X^*,x}$, follows from the fact that $\mathcal{O}_{X^*,x}$ is a normal noetherian ring and $\dim \mathcal{O}_{X^*,x} + \text{dim.al.}_k k(x) = \dim X^*$ (in the proof, we have shown that $\mathcal{O}_{X^*,x}$ follows essentially of finite type over k).

From a more general point of view, there is an open problem, called the Chain Conjecture, (cf. [26], p.1071) which comes from Nagata ([23] ,[24]) and Grothendieck (EGA IV, 5.6) and whose affirmative answer implies that every normal local noetherian ring is universally catenary.

§2. Two Theorems about the schemes dominated by algebraic varieties.

The next Lemma, gives the possibility to deduce some properties of the schemes dominated by algebraic varieties, using the previously established properties of the open immersions of algebraic varieties in schemes.

Lemma 5. Let $f:X \longrightarrow Y$ be a dominant morphism of integral schemes over a field k. Suppose that X is an algebraic k-scheme. Then Y is generically an algebraic k-scheme.

Proof. It is sufficient to prove that if A is a k-subalgebra of an integral k-algebra of finite type B, then there exists a non-zero element $\alpha \in A$, such that the ring of quotients A_α is still of finite type.

Let $\{x_1,\dots,x_n\}$ be an algebraic basis of the field of quotients Q(B) of B over the field of quotients Q(A) of A, so that

$x_i \in B$. Then B is algebraic and of finite type over the subring $A[x_1,\dots,x_n]$.

Let $\{y_1,\dots,y_m\}$ be a finite set of generators of B over $A[x_1,\dots,x_n]$ and for every j, $1\leq j\leq m$, an algebraic equation of y_j over $A[x_1,\dots,x_n]$:

$$\varphi^{(j)}_{n_j}\, y_j^{n_j} + \dots + \varphi^{(j)}_1\, y_j + \varphi^{(j)}_0 = 0$$

where $\varphi^{(j)}_{n_j} \neq 0$. If we denote $\varphi = \varphi^{(1)}_{n_1} \cdot \varphi^{(2)}_{n_2} \cdot \ldots \cdot \varphi^{(m)}_{n_m}$, then $\varphi \neq 0$ and B_φ is finite over $A[x_1,\dots,x_n]_\varphi$. Hence $A[x_1,\dots,x_n]_\varphi$ is an algebra of finite type over k.

We shall consider two cases:

a) A is a finite ring. Then A is a finite type k-algebra and this completes the proof.

b) A is an infinite ring. We may assume that $\varphi \in A[x_1,\dots,x_n]$ is a polynomial in the indeterminates $x_1,\dots,x_n$; then there exists $(\alpha_1,\dots,\alpha_n)$ such that $\varphi(\alpha_1,\dots,\alpha_n) \neq 0$. If $\underline{a} \subset A[x_1,\dots,x_n]$ is the ideal generated by the set $\{x_1-\alpha_1,\dots,x_n-\alpha_n\}$, we have the isomorphisms:

$$A[x_1,\dots,x_n]_\varphi/\underline{a}_\varphi \simeq (A[x_1,\dots,x_n]/\underline{a})_\varphi \simeq A_{\varphi(\alpha_1,\dots,\alpha_n)}$$

Therefore $A_{\varphi(\alpha_1,\dots,\alpha_n)}$ is a k-algebra of finite type and Lemma 5 is proved.

Let us recall in a particular case the following

<u>Nagata-Otsuka Theorem</u> (cf. [15], Theorem 3) - <u>let A be a k-subalgebra of an integral algebra B of finite type over a field k and $\mathfrak{p} \subset A$ a prime ideal such that there exists a prime ideal of B lying over $\mathfrak{p}$. Then $\dim A_{\mathfrak{p}} + \text{dim.al.}_k k(\mathfrak{p}) = \dim A$.</u>

Recall that a scheme X (resp. a ring A) is called catenary and equicodimensional if the following condition holds:

(C1) all the maximal chains of closed integral subschemes of X (resp. all the maximal chains of prime ideals of A) have the same length.

It is obvious that a ring A has the property (C1) iff the affine scheme Spec A has the property (C1).

Now we shall give the two Theorems:

Theorem 2. Let $f:X \to Y$ be a morphism of integral k-schemes over a field k, where X is an algebraic k-scheme and Y is normal. Suppose that one of the following conditions is satisfied:

a) f is dominant and Y has the property (C1).

b) f is surjective.

Then the subset of all points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is noetherian is open and locally an algebraic k-scheme.

Proof. By Lemma 5, Y is generically an algebraic k-scheme. If the condition a) is satisfied then it is easy to see that for every point $z \in Y$ we have $\dim \mathcal{O}_{Y,z} + \text{dim.al.}_k k(z) = \dim Y$. If the condition b) is satisfied, then the above equality holds for every point $z \in Y$, by Nagata-Otsuka Theorem. Therefore for every point $y \in Y$ such that $\mathcal{O}_{Y,y}$ is noetherian, the conditions of Lemma 4 are verified and y has an open neighbourhood which is algebraic over k. Hence the subset of all points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is noetherian is an open locally algebraic subset of Y.

Theorem 3. Let $f:X \to Y$ be a morphism of integral k-schemes over a field k such that X is an algebraic k-scheme

1) If dim $Y \leq 1$, then Y is locally an algebraic k-scheme

2) If dim $Y=2$ and if one of the following conditions is satisfied:

a) f is dominant and Y is normal with property (C1)

b) f is surjective and Y is normal

then the subset of all points $y \in Y$ such that $\mathcal{O}_{Y,y}$ is a Krull ring is open and is locally an algebraic k-scheme.

Proof. 1) If dim $Y=0$, the assertion is obvious.

If dim $Y=1$, by Lemma 5, it follows that Y is generically algebraic over k. For every closed point $y \in Y$, we have that $\dim \mathcal{O}_{Y,y}=1$ and then $\dim.\mathrm{al.}_k k(y)=0$, by Lemma 1 b). Via Remark 1, it follows that every closed point of Y has an open algebraic neighbourhood.

2) From a) or b) it follows that $\dim \mathcal{O}_{Y,y} + \dim.\mathrm{al.}_k k(y) = \dim Y$ for every point $y \in Y$. Via Lemma 3, every 1-codimensional point of Y and every 2-codimensional point $y \in Y$, such that $\mathcal{O}_{Y,y}$ is a Krull ring, have an open algebraic neighbourhood. The union of these neighbourhoods is the set of all "Krull points" of Y, which ends the proof.

§3. Some consequences and applications

In the following Proposition is given a characterization of the finite generatedness of a k-subalgebra of a finite type algebra:

Proposition 1. Let A be a k-subalgebra of an integral algebra of finite type over a field k. Then the following assertions are equivalent:

i) A is a finite type k-algebra

ii) for every maximal ideal $\underline{m}\subset A$, $A_{\underline{m}}$ is an universally catenary noetherian ring and $\dim A_{\underline{m}} = \dim A$.

iii) for every maximal ideal $\underline{m}\subset A$, $A_{\underline{m}}$ is noetherian and every integral A-algebra B, which is finite over A and has a maximal ideal of height 1, is 1-dimensional.

Proof. (i)$\Leftrightarrow$(ii) follows from Lemma 5 and Th.1: Spec A is generically an algebraic scheme over k and if (ii) is satisfied then every closed point of Spec A has an open algebraic neighbourhood; then A is of finite type, since Spec A is quasicompact.

iii) $\Rightarrow$ i) Let $X^* = \operatorname{Spec} A$ be the affine scheme corresponding to A and $X\subset X^*$ an open non-empty subset, which is algebraic over k (cf.Lemma 5).

From (iii) every integral scheme, which is finite over X^* and has a closed 1-codimensional point is 1-dimensional. It is obvious that for every $x\in X^*$, $\mathcal{O}_{X^*,x}$ is a noetherian ring.

We shall prove that X^* is algebraic over k by induction over $\dim X^*$.

If $\dim X^*\leq 1$, then X^* is algebraic over k, by Theorem 3, because X^* is quasicompact.

Suppose that $\dim X^* > 1$.

We claim that X^* satisfies the condition (C1). In fact let $X_o\subset X_1\subset\ldots\subset X_{n-1}\subsetneq X_n = X^*$ be a maximal chain of closed integral subschemes of X^*. If n=1, then by (iii) it follows that $\dim X^*=1$, since X^* has a closed 1-codimensional point; this fact contradicts the assumption that $\dim X^*>1$. Therefore $n\geq 2$. Since $\mathcal{O}_{X^*,X_{n-2}}$ is noetherian and $\mathcal{O}_{X^*,X_{n-2}}$ has a maximal chain of prime ideals of length 2, by a Theorem of McAdam (cf.[1]), there exist infinitely many maximal chains of prime ideals of length 2 in $\mathcal{O}_{X^*,X_{n-2}}$. If $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ are the prime ideals of $\mathcal{O}_{X^*,X_{n-2}}$, corresponding to the irreducible components of X^*-X containing X_{n-2}, we may choose a maximal chain $0\subset\mathfrak{p}\subset\mathfrak{m}$ in $\mathcal{O}_{X^*,X_{n-2}}$ of length 2 such that $\mathfrak{p}\neq\mathfrak{p}_i$, for every i, $1\leq i\leq n$. If X'_{n-1} is the closed integral subscheme of X^* corresponding to $\mathfrak{p}$, we have that $X_0\subset X_1\subset\dots\subset X_{n-2}\subset X'_{n-1}\subset X_n=X^*$ is a maximal chain and $X'_{n-1}\cap X\neq\emptyset$. X'_{n-1} being generically algebraic over k and $\dim X'_{n-1}<\dim X^*$, by induction hypothesis we have that X'_{n-1} is algebraic over k. Hence $n-1=\dim X'_{n-1}$. Since $X'_{n-1}\cap X$ is of codimension 1 in X, we have $\dim X'_{n-1}=\dim X-1=\dim X^*-1$. Therefore $n=\dim X^*$ and X^* satisfies the condition (C1).

Every integral closed subscheme X' of X^*, $X'\neq X^*$, is algebraic over k. Indeed, it is sufficient to prove this for $X'\subset X^*$, such that codim X'=1. If x is the generic point of such a subscheme X', then $\mathcal{O}_{X^*,x}$ is an 1-dimensional ring and $\dim\mathcal{O}_{X^*,x}+\text{dim.al.}_k k(x)=\dim\mathcal{O}_{X^*,x}+\text{dim.al.}_k K(X')=\dim X^*$, because X^* has the property (C1). By Remark 1, x has an open algebraic neighbourhood and then X' is generically algebraic over k. Because dim X' < dim X, by induction hypothesis X' is an algebraic k-scheme.

Then for every point $x\in X^*$ and for every non-zero prime ideal $\mathfrak{p}\subset\mathcal{O}_{X^*,x}$, the ring $\mathcal{O}_{X^*,x}/\mathfrak{p}$ is essentially of finite type over k. By Marot Lemma, the integral closure $\mathcal{O}'_{X^*,x}$ of $\mathcal{O}_{X^*,x}$ in its field of quotients is noetherian. Therefore if X^{*N} is the normalization of X^*, for every point $x\in X^{*N}$, the local ring $\mathcal{O}_{X^{*N},x}$ is noetherian.

For every closed integral subscheme $Z\subset X^{*N}$, $Z\neq X^{*N}$, if $Y\subset X^*$, $Y\neq X^*$, is its image in X, we have that K(Z) is a finite extension of K(Y), by Mori-Nagata Theorem (cf [13], 33.10) applied to the noetherian ring $\mathcal{O}_{X^*,Y}$. Since Y is algebraic over k, it is easy to see that Z is algebraic over k.

X^{*N} has not closed 1-codimensional points. Indeed,

$\Gamma(X^{*N}, \mathcal{O}_{X^{*N}})=A'$ is integral over A and if $\underline{m}_1 \subset A'$ is a maximal ideal of height 1, let $\underline{n} = \underline{m}_1 \cap A$ and let $\underline{m}_2, \ldots, \underline{m}_r$ be the other prime ideals of A' lying over $\underline{n}$. If $f_i \in \underline{m}_i - \bigcup_{j=1} \underline{m}_j$ and $B=A[f_1, \ldots, f_r] \subset A'$ is the A-algebra generated by $f_1, \ldots, f_r$ we have that for every i,j $i \neq j$, $\underline{m}_i \cap B \neq \underline{m}_j \cap B$. Then $\underline{m}_1$ is the unique prime ideal of A' lying over $m_1 \cap B$. By Cohen-Seidenberg Theorem, it follows that $ht(\underline{m}_1 \cap B)=1$ since $ht\, \underline{m}_1=1$. By (iii) we have dim B=1 and then dim A=1, which contradicts the fact that $\dim X^* > 1$.

Hence every maximal chain of closed integral subschemes of X^{*N} has the length ≥ 2. Let $Z_0 \subset Z_1 \subset \ldots \subset Z_{m-1} \subset Z_m = X^{*N}$ be a maximal chain of integral closed subschemes. Then as above for X^*, we may replace Z_{m-1} by Z'_{m-1} such that $Z'_{m-1} \cap X^N \neq \phi$, where $X^N \subset X^{*N}$ is the normalization of $X \subset X^*$, and $Z_{m-2} \subset Z'_{m-1} \subset Z_m = X^{*N}$ is a saturated chain Since Z'_{m-1} is algebraic over k, it follows that $m = \dim X^{*N}$, in the same way as for X^*.

Therefore X^{*N} is a normal scheme which is generically algebraic over k and satisfies the condition (C1). By Theorem 2, it follows that X^{*N} is algebraic over k, since for every point $x \in X^{*N}$ the ring $\mathcal{O}_{X^{*N}, x}$ is noetherian. Then X^* is algebraic over k.

Proposition 1 is proved.

For normal subalgebras we have the following characterization of the finite generatedness in terms of chains of ideals:

Corollary 1. Let A be a normal k-subalgebra of an integral algebra of finite type over a field k. The following assertions are equivalent:

i) A is of finite type over k.

ii) A is noetherian and for every maximal ideal $\underline{m} \subset A$, $ht\, \underline{m} = \dim A$.

iii) for every maximal ideal $\underline{m} \subset A$, $A_{\underline{m}}$ is a noetherian ring and $\dim A = \dim A_{\underline{m}}$.

Indeed, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. (iii) $\Rightarrow$ (i) it follows from Lemma 3, applied to $X^* = \operatorname{Spec} A$ and to every closed point of X^*: X^* is generically algebraic over k, by Lemma 5, and if $x \in X^*$ is a closed point then $\dim \mathcal{O}_{X^*,x} + \dim k(y) = \dim X^*$, by Lemma 1 b) and by (iii). Therefore every closed point of X^* has an algebraic neihbourhood, and then (i) follows, since X^* is quasicompact.

For the normal subalgebras of small dimensions, we have weaker conditions of finite generatedness:

Corollary 2 - a) Every 1-dimensional k-subalgebra of an integral algebra of finite type over a field k is still of finite type.

b) Let A be a normal 2-dimensional k-subalgebra of an integral algebra of finite type over a field k. The following assertions are equivalent:

i) A is finitely generated

ii) A is a Krull ring and for every maximal ideal $\underline{m} \subset A$, ht $\underline{m} = 2$

iii) for every maximal ideal $\underline{m} \subset A$, $A_{\underline{m}}$ is a 2-dimensional Krull ring.

We shall point out the following:

Corollary 3. Let A be a normal k-subalgebra of an integral algebra B of finite type over a field k. Suppose that for every prime ideal of A there exists a prime ideal of B lying over it. Then:

a) A is finitely generated over k iff for every maximal ideal $\underline{m} \subset A$ the ring $A_{\underline{m}}$ is noetherian

If dim A=2, A is finite generated over k iff for every maximal ideal $\underline{m} \subset A$, the ring $A_{\underline{m}}$ is Krull.

Corolaries 2 and 3 follow from Theorem 2 and 3 applied to the morphism Spec B $\longrightarrow$ Spec A, where B is the finite type k-algebra containing A, and using the fact that Spec A is quasicompact.

Corollary 4. Let X be a normal algebraic variety over a field k, such that the weak "Nullstellensatz" holds for X (cf.[7], Proposition 3.2). If $\Gamma(X,\mathcal{O}_X)$ is noetherian, then it is a k-algebra of finite type.

Indeed, since the morphism $\pi : X \longrightarrow \mathrm{Spec}\,\Gamma(X)$ has the property that Spec.max.$\Gamma(X) \subseteq \pi(X)$, by Nagata - Otsuka Theorem it follows that for every closed point $\underline{m} \in \mathrm{Spec}\,\Gamma(X)$ we have dim $\Gamma(X)_{\underline{m}}$ + dim.al.$_k k(\underline{m})$=dim $\Gamma(X)$. Since $\Gamma(X)$ is normal, by Lemma 3 every closed point $\underline{m} \in \mathrm{Spec}\,\Gamma(X)$ has an open algebraic neighbourhood.

Proposition 2. Let X be a normal algebraic variety over a field k. If dim $\Gamma(X,\mathcal{O}_X) \leq 2$, then $\Gamma(X,\mathcal{O}_X)$ is a k-algebra of finite type.

Proof. The ring $\Gamma(X,\mathcal{O}_X)$ is a Krull ring. In fact, if $(U_i)_{i\in I}$ is a finite covering of X with open affine subsets, then $\Gamma(X,\mathcal{O}_X) = \bigcap_{i\in I} \Gamma(U_i,\mathcal{O}_X)$, where $\Gamma(U_i,\mathcal{O}_X)$ are Krull rings having the same field of quotients.

Let $\pi : X \longrightarrow \mathrm{Spec}\,\Gamma(X,\mathcal{O}_X)$ be the canonical morphism. Then Proposition 2 follows from Theorem 2 if we prove that in the case when dim$\Gamma(X,\mathcal{O}_X)$=2, Spec$\Gamma(X,\mathcal{O}_X)$ has not closed 1-codimensional points.

Let $y \in Y = \mathrm{Spec}\,\Gamma(X,\mathcal{O}_X)$ be a closed 1-codimensional point. By Lemma 6 below, $y \notin \pi(X)$ and Y-$\{y\}$ is an affine scheme. The canonical homomorphism $\pi^*: \Gamma(Y,\mathcal{O}_Y) \longrightarrow \Gamma(X,\mathcal{O}_X)$ factors in the following way:

$$\Gamma(Y,\mathcal{O}_Y) \longrightarrow \Gamma(Y-\{y\},\mathcal{O}_Y) \longrightarrow \Gamma(X,\mathcal{O}_X)$$

where the first homomorphism is the restriction of the sections. Since π^* is an isomorphism, it follows that this restriction is an isomorphism. Then $Y = Y - \{y\}$, which is not possible.

Lemma 6. Let $f:X \longrightarrow Y$ be a dominant morphism of integral k-schemes such that X is algebraic over k and Y is a 2-dimensional Krull scheme. If y is a closed 1-codimensional point of Y, then $y \notin f(X)$ and $i:Y - \{y\} \hookrightarrow Y$ is an affine morphism.

Proof. In fact, if there exists a closed 1-codimensional point $y \in Y$ such that $y \in f(X)$, then $\dim \mathcal{O}_{Y,y} + \mathrm{dim.al.}_k k(y) = \dim Y$; by Lemmas 5 and 3, it follows that y has a 2-dimensional open algebraic neighbourhood which is not possible.

For the second part of Lemma 6, we may assume that Y is an affine scheme. Let $V \subset Y$ be an open subset not containing the closed 1-codimensional point $y \in Y$. Then $\{y\}$ is an irreducible component of $Y-V$ and thus $V \cup \{y\}$ is an open subset of Y. Using this fact it is easy to see that $Y - \{y\}$ is quasicompact.

Therefore $i:Y - \{y\} \hookrightarrow Y$ is a quasicompact morphism. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_{Y-\{y\}}$-module on $Y - \{y\}$. By EGA I, 9.2.2., $\mathcal{F}' = i_* \mathcal{F}$ is a quasicoherent $\mathcal{O}_Y$- module on Y extending $\mathcal{F}$. In the exact sequence of $\Gamma(Y,\mathcal{O}_Y)$ -modules:

$$H^1(Y,\mathcal{F}') \longrightarrow H^1(Y-\{y\},\mathcal{F}) \longrightarrow H^2_{\{y\}}(Y,\mathcal{F}')$$

the first term is null and for the last therm we have $H^2_{\{y\}}(Y,\mathcal{F}') =$

$= H^2_{\{y\}}(\mathrm{Spec}\,\mathcal{O}_{Y,y}, \mathcal{F}'_y) = 0$, since $\mathcal{O}_{Y,y}$ is a discrete valuation ring. Therefore $H^1(Y-\{y\}, \mathcal{F}) = 0$ and so, by Serre Criterion (cf. EGA II, 5.2.1), it follows that $Y - \{y\}$ is affine. Hence i is an affine morphism.

The next Corollary was proved by Zariski in [22], §7.

Corollary 5. Let X be a normal algebraic surface over a field k. Then $\Gamma(X, \mathcal{O}_X)$ is a k-algebra of finite type.

Proof. By Lemmas 5 and 1, $\dim \Gamma(X, \mathcal{O}_X) = \mathrm{dim.al.}_k Q(\Gamma(X, \mathcal{O}_X)) \leq \mathrm{dim.al.}_k K(X) \leq 2$, Hence Corollary 5 follows from Proposition 2.

With the same proof as for Proposition 2, we recover the affirmative cases of Zariski's form of Hilbert's 14th Problem (cf. [22], [12]).

Proposition 3. Let A be a normal algebra of finite type over a field k and L a subfield of the field of quotients of A containing k. If $\mathrm{dim.al.}_k L \leq 2$, then $L \cap A$ is a finite type k-algebra.

Proof. It is obvious that $L \cap A$ is a Krull ring. Let $f: X = \mathrm{Spec}\, A \to Y = \mathrm{Spec}\, L \cap A$ be the canonical morphism. Since $\dim L \cap A \leq 2$, Proposition 3 follows from Theorem 3 if we prove that in the case when $\dim Y = 2$, Y has not 1-codimensional closed points. By above Lemma 6, if there exists a closed 1-codimensional point $y \in Y$, then $y \notin f(X)$ and $Y - \{y\}$ is affine. The canonical homomorphism $f^*: \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ factors in the following way:

$$\Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y - \{y\}, \mathcal{O}_Y) \longrightarrow \Gamma(X, \mathcal{O}_X)$$

If $\alpha\in\Gamma(Y-\{y\},\mathcal{O}_Y)$, then $\alpha\in\Gamma(X,\mathcal{O}_X)\cap K(Y)\subseteq A\cap L=\Gamma(Y,\mathcal{O}_Y)$. Therefore the restriction homomorphism $\Gamma(Y,\mathcal{O}_Y)\to\Gamma(Y-\{y\},\mathcal{O}_Y)$ is an isomorphism and then $Y=Y-\{y\}$, which is not possible.

We shall give in the following Proposition an alternative proof for a Theorem of Goodman-Landman (cf[7], Corollary 3.9), even for arbitrary fields.

First, recall the following

Mori-Nishimura Theorem ([17], p.397). *If A is a Krull ring such that for every prime ideal $\mathfrak{p}\subset A$, $\mathfrak{p}\neq 0$, $A/\mathfrak{p}$ is noetherian, then A is noetherian.*

Proposition 4. *Let $f:X\to Y$ be a surjective proper morphism of integral schemes over a field k. If X is algebraic over k, then Y is also algebraic.*

Proof. Let $f':X'\to Y'$ be the morphism between the normalizations of X and Y induced by f and $f'=p\circ\varphi$ the Stein factorisation of f', where $\varphi: X\to Z=\operatorname{Spec} f'_*\mathcal{O}_{X'}$ and $p: Z\to Y'$. We may assume that Y is affine. Then Y' and Z are affine schemes. Moreover Z is a Krull scheme. Indeed, if $(U_i)_{i\in I}$ is a finite covering of X' with affine open subsets, then $\Gamma(Z,\mathcal{O}_Z)=\Gamma(X',\mathcal{O}_{X'})=\bigcap_{i\in I}\Gamma(U_i,\mathcal{O}_{X'})$, where $\Gamma(U_i,X_i)$ are Krull rings with the same field of quotients.

We shall proceed by induction over $\dim Y$.

If $\dim Y=0$ then Y is algebraic over k.

Suppose $\dim Y>0$; then $\dim Z>0$. The morphism φ being surjective and proper, for every integral closed subscheme $Z'\subset Z$, there exists a closed integral subscheme $W'\subset\varphi^{-1}(Z')$ such that $\varphi|_{W'}: W'\to Z'$ is surjective and proper. By the induction hypothesis, every such subscheme $Z'\neq Z$ is an algebraic k-scheme. Hence, by

Mori-Nishimura Theorem, it follows that Z is noetherian. By Theorem 2, we have that Z is algebraic over k. Hence Y is an algebraic k-scheme, since Z is integral over Y.

For universally open morphisms, we shall prove the following consequence of Proposition 1, iii) $\Rightarrow$ i):

Corollary 6. *Let $f:X\to Y$ be a surjective universally open morphism of integral schemes over a field k, where X is an algebraic k-scheme. Then*

a) *Y has the property (C1)*

b) *Y is an algebraic k-scheme iff $\mathcal{O}_{Y,y}$ is noetherian for every $y\in Y$.*

Proof. a) Let $Y=Y_0\supset Y_1\supset\ldots\supset Y_n$ be a maximal chain of integral closed subschemes. For every i, $0\leq i\leq n$ and for every $y\in Y_i$, there exists a component X_{ij} of $f^{-1}(Y_i)$ dominating Y_i, such that $y\in f(X_{ij})$, since $f|_{f^{-1}(Y_i)}: f^{-1}(Y_i)\longrightarrow Y_i$ is an open morphism. Therefore, by Nagata-Otsuka Theorem, for every $y\in Y_i$, we have $\dim \mathcal{O}_{Y_i,y} + \dim.\mathrm{al.}_k k(y)=\dim Y_i$.

We shall prove, by induction over i, that Y_i is generically an algebraic k-scheme over k and $\dim Y_i=\dim Y-i$. Then it follows that $n=\dim Y$, which completes the proof of Corollary 6 a).

If $i=0$, the assertion follows from Lemma 5.

Suppose $i>0$ and assume that Y_{i-1} is generically algebraic over k of dimension $\dim Y-i+1$. Let y_i be the generic point of Y_i. Since $\dim \mathcal{O}_{Y_{i-1},y_i}=1$, then it follows $\dim.\mathrm{al.}_k k(y_i)=\dim Y_{i-1}-1$. By virtue of Remark 1, y_i has an open algebraic neighbourhood V in Y_{i-1}.

Therefore Y_i is generically algebraic over k and $\dim Y_i = \dim Y_i\cap V=\dim V-1=\dim Y_{i-1}-1=\dim Y-i$.

b) Y being quasicompact, we may assume that Y is an affine scheme. By Proposition 1, iii) $\Rightarrow$ i), it suffices to prove that every integral scheme Y', which is finite over Y and has a closed 1-codimensional point, is 1-dimensional. There exists $n \geq 0$ and a closed immersion $i: Y' \hookrightarrow Y \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$. Since $f \times 1_{\mathbb{A}^n_{\mathbb{Z}}} : X \times \mathbb{A}^n_{\mathbb{Z}} \longrightarrow$ $\longrightarrow Y \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ is a surjective universally open morphism, it follows that $Y \times_{\mathbb{Z}} \mathbb{A}^n_{\mathbb{Z}}$ has the property (C1), by a). Then Y' has this property and so dim Y'=1.

Remark 3. With the same proof as for Corollary 6 a), it follows that for every closed surjective morphism $f: X \to Y$ of integral k-schemes, where X is an algebraic k-scheme, Y has the property (C1).

Corollary 7. Let $f: X \to Y$ be a faithfully flat morphism of integral k-schemes, where k is a field. If X is an algebraic k-schemes, then Y is also algebraic over k.

Corollary 7 follows from Corollary 6 b), since Y is noetherian.

§4. Universally 1-equicodimensional rings and the finite generatedness of subalgebras.

In [4] we have introduced the following

Definition. A ring A is called universally 1-equicodimensional if it is noetherian and every integral A-algebra B of finite type which has a maximal ideal of height 1 is 1-dimensional.

A scheme X is called universally 1-equicodimensional if there exists a finite covering $(U_i)_{i \in I}$ of X with affine open subsets such that for every $i \in I$, $\Gamma(U_i, \mathcal{O}_X)$ is an universally 1-equicodimensional ring

We have proved in [4], that if Z is a scheme, then the following assertions are equivalent:

i) Z is universally 1-equicodimensional

ii) Z is noetherian and every separated morphism $f: X \to Y$ of integral schemes of finite type over Z is proper iff every integral closed 1-dimensional subscheme of X is proper over Y.

iii) Z is noetherian and for every integral scheme X of finite type over Z, and for every closed point $x \in X$ the subset of all closed points $x' \in X$, such that there exists an integral (resp. connected) closed 1-dimensional subscheme passing through x and x', is dense in X.

iv) Z is a noetherian Jacobson scheme and every integral scheme X, which is finite over Z and has a closed 1-codimensional point, is 1-dimensional.

We shall prove that i) is equivalent to:

v) Z is a noetherian Jacobson scheme and if Z' is an integral closed subscheme of Z, such that its normalization has a closed 1-codimensional point, then $\dim Z' = 1$.

In fact, iv) $\Rightarrow$ v): if Z'^N is the normalization of a closed integral subscheme Z' of Z and $z \in Z'^N$ is a closed 1-codimensional point, then there exists an integral scheme Z'' finite over Z' such that Z'^N is a dominating scheme over Z'' and such that $\{z\}$ is a fiber of the morphism $Z'^N \to Z''$. Then Z'' has a closed 1-codimensional point and so $\dim Z'' = 1$. Therefore, $\dim Z' = 1$.

v) $\Rightarrow$ iv). Indeed, if Z'' is an integral finite scheme over Z and Z' is the (closed integral) image of Z'' in Z, we have a commutative diagram:

$$\begin{array}{ccc} Z''^N & \longrightarrow & Z'' \\ \downarrow & & \downarrow \\ Z'^N & \longrightarrow & Z' \end{array}$$

where Z'^N and Z''^N are the normalization schemes of Z' and Z''. If Z'' has a closed 1-codimensional point, then Z''^N and Z'^N have such points; then dim $Z'^N = 1$, by (v). Hence dim $Z'=1$.

In [4], are shown some general properties for the universally 1-equicodimensional schemes.

Clearly, we may complete Proposition 1 with the following:

Proposition 1'. Let A be a subalgebra of an integral algebra of finite type over a field k. Then the following assertions are equivalent:

i) A is a finite type algebra over k

iv) A is an universally 1-equicodimensional ring.

Remark 4. In [25], L.J.Ratliff Jr., proves the following

Theorem (Theorem 3.1, loc. cit.)

Let A be a noetherian local ring. Then the following are equivalent:

i) A is universally catenary (i.e. A satisfies the altitude formula, loc. cit.)

ii) the completion $\hat{A}$ of A is equidimensional (i.e A is quasi-unmixed, loc. cit.)

Following the proof of i) ⇒ ii) of this Theorem in [25], it is easy to see that we may add the following equivalent property:

iii) A is catenary and every integral A-algebra B, which is finite over A and has a maximal 1-height ideal, is 1-dimensional.

This remark allows an alternate proof for Propositions 1': iv) of Proposition 1' ⇒ ii) of Proposition 1.

REFERENCES

1. St. McAdam, Saturated chains in noetherian rings, Indiana Univ.Math.J.23(1973/1974), 719-728.
2. N.Bourbaki, Algèbre commutative Herman , Paris
3. A.Brezuleanu, N.Radu, Lectures on algebra, Tip.Univ.Bucharest (1977).
4. A.Constantinescu, Schemes with closed 1-codimensional points Preprint Series in Mathematics, No.48/1978, INCREST, Bucharest.
5. A.Constantinescu, Open embeddings of algebraic varieties in schemes and applications, Preprint Series in Mathematics, No.5/1979, INCREST, Bucharest.
6. P.M.Eakin Jr., The converse to a well known theorem on noetherian rings, Math.Ann.177(1968), 278-282.
7. J.E.Goodmann, A. Landman, Varieties Proper over Affine Schemes, Inv.Math., vol.20, Fasc.4(1973), 267-312.
8. A.Grothendieck, Cohomologie locale des faisceaux cohérents et thèoremes de Lefschetz locaux et globaux, I.H.E.S. (1962).
9. A.Grothendieck, J.Dieudonné, Éléments de Géometrie Algébrique, Publ.Math.de l'I.H.E.S., 4,8,11,32 (1960-1967).
10. H.Kurke, G.Pfister, M.Roczen, Henselsche Ringe und algebraische Geometrie, VEB, Deut.Ver.Wiss., Berlin (1975).
11. J.Marot, Sur les anneaux universellement japonais C.R.Acad. Paris, T.277 (1973), Serie A, 1029-1031; T.278 (1974), Serie A, 1169-1172.
12. M.Nagata, On the 14 th problem of Hilbert, Amer.J.Math.,81

(1959), 766-772.

13. M.Nagata, Local rings, John Wiley, New York (1962).
14. M.Nagata, Imbedding of an abstract variety in a complete variety, J.Math.Kyoto Univ. 2(1962), 1-10.
15. M.Nagata, K.Otsuka, Some remarks on the 14-th problem of Hilbert, J.Math.Kyoto Univ.(1965), 61-66.
16. M.Nagata, A type of subrings of a noetherian ring, J.Math. Kyoto Univ.8(1968), 465-467.
17. J.Nishimura, Note on Krull domains, J.Math.Kyoto Univ.15 (1975), 397-400.
18. J.Nishimura, Note on integral closure of a noetherian integral domain, Kyoto Univ.16(1976),117-122.
19. N.Radu, Inele locale, vol.I-II, Ed.Acad.R.S.R.,(1968,1970).
20. J.P.Serre, Algèbre locale, Multiplicites, Lect.notes, Vol.11, 2-ième ed.(1965).
21. O.Zariski, P.Samuel, Commutative Algebra, vol.I-II, New York (1958, 1960).
22. O.Zariski, Interpretationes algébrico-géometriques du quatorzième problème de Hilbert , Bull.Sci.Math.(1954), 155-168.
23. M.Nagata, On the chain problem of prime ideals, Nagoya Math. J.10(1956), 51-64.
24. M.Nagata, Note on a chain condition for prime ideals, Mem. Col.Sci.Kyoto Univ., 32 (1959), 85-90.
25. L.J.Ratliff Jr., On quasi-unmixed local rings, the altitude formula and the chain condition for prime ideals (I) Am. J.Math. 91 (1969), 509-528.
26. L.J.Ratliff Jr., Characterizations of Catenary rings, Amer. J.Math.93(1971),1070-1108.

27. L.J.Ratliff Jr., Chain Conjectures in Ring Theory. An Exposition of Conjectures on Catenary Chains. Lect.Notes, Vol.647 (1978).

APPENDIX

Chain Conjectures and finite generatedness of subalgebras

Recall the following two properties for a ring A, called "the second chain condition", resp. "the chain condition":

(C2) for every minimal prime ideal $\mathfrak{p} \subset A$, every integral extension domain of $A/\mathfrak{p}$ satisfies (C1).

(C) for every pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in A, $(A/\mathfrak{p})_{\mathfrak{q}/\mathfrak{p}}$ satisfies (C2).

Via Proposition 1, an affirmative answer to each of the following two open problems allows some new characterizations of the finite generatedness of the subalgebras of a finite type k-algebra:

The Chain Conjecture: the integral closure of an integral noetherian local ring satisfies (C).

The Normal Chain Conjecture: if the integral closure of an integral noetherian local ring A satisfies (C1), then A satisfies (C2).

Some equivalent satements of each above problems are discussed in [27], Chapters 3, 4 and 12.

In [27], Ch.3, Theorem (3.3), it is shown that the Normal Chain Conjecture follows from the Chain Conjecture.

An affirmative answer to the Chain Conjecture allows the following completion of Proposition 1:

Proposition 1"- Let A be a k-subalgebra of an integral algebra of finite type over a field k. Then the following assertions equivalent:

(i) A is finitely generated.

(v) A is noetherian and all the maximal ideals of the integral closure A′ of A have the same height.

In fact, if (v) is satisfied then for every maximal ideal $\underline{m} \subset A$ the local ring $A_{\underline{m}}$ is noetherian and all the maximal ideals of the integral closure $A'_{\underline{m}}$ of $A_{\underline{m}}$ have the same height. Via the Chain Conjecture, $A'_{\underline{m}}$ satisfies (C1) and then $A'_{\underline{m}}$ verifies (C2) (by the Normal Chain Conjecture). By Theorem 3.1. of [25], it follows that $A_{\underline{m}}$ is universally catenary. By (v), dim $A_{\underline{m}}$=dim A. Then (i) follows from Proposition 1.

It is clear that Corollary 1 is then a direct consequence of above Proposition 1".

An affirmative answer to the Normal Chain Conjecture allows the following weaker completion of Proposition 1:

Proposition 1‴ - If A is a k-subalgebre of an integral algebra of finite type over a field k, the following statements are equivalent:

(i) A is finitely generated

(vi) A is noetherian and all the maximal chains of prime ideals in the integral closure A′ of A have the same length.

Indeed, for every maximal ideal $\underline{m} \subset A$, the integral closure $A'_{\underline{m}}$ of $A_{\underline{m}}$ has the property (C1). By the Normal Chain Conjecture $A'_{\underline{m}}$ verifies (C2) and by Theorem 3.1 of [25], $A_{\underline{m}}$ is universally catenary. Since dim $A_{\underline{m}}$=dim A, Proposition 1‴ follows from Proposition 1.

It is clear that the above Propositions are proved if the Chain Conjecture or the Normal Chain Conjecture have an affirmative answer for noetherian local k-subalgebras A of a function field K over k, such that dim A = dim.al.$_k$ K.