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GORENSTEINNESS OF SEGRE - VERONESE GRADED ALGEBRAS

by

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0. Abstract

The characterization of the Gorenstein property for the Veronese algebras was done by V.Brînzănescu [5] and independently by A.Matsuoka [7], in a pure algebraic manner.
The general case of the Segre-Veronese singularities was treated by L.Bădescu [1] and by L.Bădescu and N.Manolache [2], the characterization of the Gorenstein property of these singularities, in terms of their numerical character, being obtained geometrically, using Serre's duality, in the natural context of the Segre-Veronese embeddings.
The present paper contains an alternate proof of the same result, based upon two facts: the characterization of the Gorenstein property for G-algebras in terms of their Hilbert functions, given in [8] by R.Stanley and the analytic expression of the Hadamard product of two rational functions of one complex variable, given in [4] by L.Bieberbach.

1. G-Algebras

All the rings involved are commutative, with unit element.

Let k be a field. By a "G-algebra" over k we mean a noetherian, $\mathbb{N}$-graded ring; $R = \bigoplus_{n\geq 0} R_n$, with $R_o = k$.
Such a structure satisfies:

$$(\forall)\ n\geq o\ ,\ \dim_k R_n \text{ is finite.}$$

The numerical function: $\chi_R : n \mapsto \dim_k R_n$, called " the Hilbert function " of R, is polynomial i.e. there is a polynomial $Q(t) \in \mathbb{Z}[t]$

and there is integer $n_o > 0$, such that:

$$n \geq n_o \Rightarrow \chi_R(n) = Q(n).$$

The least of such integers n_o is called " the regularity index " of R (cf. [9]) and is denoted by $r(R)$.

The dimension of R is uniquely determined by χ_R, namely:

$$\dim R = 1 + \deg \chi_R = 1 + \deg Q(t).$$

The generating series of the sequence $(\chi_R(n))$ i.e. the formal power series: $H_R(t) = \sum_{n \geq 0} \chi_R(n).t^n \in \mathbb{Z}[[t]]$ is called " the Hilbert-Poincaré series " of R.

As χ_R is polynomial , it results that H_R is rational, i.e. defines an element of $\mathbb{Q}(t)$.

If $\{x_1,\ldots,x_d\}$ is a set of homogeneous generators for the k-algebra structure of R, of degrees respectively $m_1,\ldots,m_d$, then there is a canonical surjective k-homomorphism:

$$k[X_1,\ldots,X_d] \longrightarrow R$$

sending X_j into x_j $(j=1,\ldots,d)$. Hilbert's theorem on syzygies says that R has finite homological dimension over $k[X_1,\ldots,X_d]$, i.e. R has a finite free resolution over $k[X_1,\ldots,X_d]$ (of lenght $\leq d$), showing that:

$$H_R(t) = P_R(t) . \prod_{j=1}^{d} (1-t^{m_j})^{-1},$$ where $P_R(t)$ is a polynomial with integral coefficients.

When $m_1=m_2=\ldots=m_d=1$ i.e. when $R = k[R_1]$ the G-algebra R is called "standard" and, in this case,

$$H_R(t)=P_R(t)/(1-t)^d, \text{ where } P_R(1) \neq 0 \text{ and}$$

d = Krull dim R, as the dimension of R is the order of the pole : t=1 of the rational function H_R. We shall constantly make use of this normalized form of the Hilbert function of a standard G-algebra.

The Gorensteinness of G-algebras is characterized in the following:

1.1. Theorem (R.Stanley, [8])

Let R be a Cohen-Macaulay domain and a G-algebra. Then R is Corenstein if and only if its Hilbert-Poincaré series, considered as a rational function, satisfies the following functional relation:

$$H_R(1/t)=(-1)^{\dim R}.t^{q(R)}.H_R(t)$$

where q(R) is an integer uniquely determined by R.

1.2. Remark

If, in the Theorem 1.1, the G-algebra R is standard, then, looking at the normalized form of H_R, we see that Gorensteinness in this case means that the polynomial P_R is reciprocical, i.e. satisfies

$$P_R(1/t) = t^{-\deg P_R} . P_R(t).$$

Moreover, in this case one obtains:

$q(R) = \dim R - \deg P_R$ (see also [8]).

2. Veronese G- algebras

Let k be a field and r,s be two positive integers.

2.1. Definition

The "Veronese k-algebra of type (r;s)" is the k-subalgebra of the polynomial algebra $k[T_1,\dots,T_r]$, generated by all the monomials of degree s, i.e. by:

$$\left\{ T_1^{i_1} \cdot T_2^{i_2} \cdot \ldots \cdot T_r^{i_r} \;/\; i_1+i_2\ldots+i_r = s \text{ in } \mathbb{Z}_+ \right\}.$$

This algebra will be denoted by V_{rs}, when the field k is fixed. According to this definition, it is easy to see that V_{rs} is a standard G-algebra and a domain of Krull dimension r. The graded structure of $V_{rs} = \bigoplus_{n\geq 0} V_{rs}(n)$ is given by

$$(\forall)\ n\geq 0,\ \dim_k V_{rs}(n) = \binom{r+ns-1}{r-1}.$$

V_{rs} is always Cohen-Macaulay: indeed, $\{T_1^s,\dots,T_r^s\}$ is a system of parameters and a regular sequence in V_{rs} (another argument for the Cohen-Macaulayness of V_{rs} is based on a theorem of Hochster [6], asserting that a monoïdal algebra $k[M]$ over a field k is Cohen-Macaulay if the monoid M is normal; in our case the monoid of all the monomials contained in V_{rs} is obviously normal).

Let $H_{rs}(t) = P_{rs}(t)/(1-t)^r$ be the (normalized) Hilbert-Poincaré series of V_{rs}, written in its rational form, and let $H_r(t) = (1-t)^{-r}$ be the Hilbert-Poincaré series of the ambient polynomial ring $k[T_1,\dots,T_r]$. Directly from 2.1 we see that H_{rs} is obtained from H_r" by selecting the terms from s to s ".

The following lemma yields a general procedure for such a selection

2.2. Lemma

Let $f(t)$ be a rational function from $\mathbb{C}(t)$ and let:

$$f(t) = \sum_{n\geq 0} a_n \cdot t^n$$

be its development near the origin in $\mathbb{C}$.

Then, if $f_s(t) =: \sum_{n\geq 0} a_{ns} \cdot t^n$, then the following relation holds in the Puiseaux power series field over $\mathbb{C}$:

$$1/s\left[\sum_{j=1}^{s} f(z_j t^{1/s})\right] = f_s(t)$$

where $z_1, \ldots, z_s$ are the s-roots of the unity in $\mathbb{C}$.

Proof.

For any s indeterminates $X_1, \ldots, X_s$ we denote by $\sigma_k(X_1, \ldots, X_s)$ the k-th elementary symmetric function, on th X_i's, i.e.:

$$\sigma_k(X_1, \ldots, X_s) = \sum_{1\leq i_1 < i_2 \ldots < i_k \leq s} X_{i_1} X_{i_2} \ldots X_{i_k} .$$

For every positive integer m, let $p_m(X_1, \ldots, X_s)$ be the sum of the m-th powers of the X_i's , i.e.:

$$p_m(X_1, \ldots, X_s) = X_1^m + \ldots + X_s^m .$$

Then the p_m's may be determined through the σ_k's ,because of Newton's formulas (holding in $\mathbb{Z}[X_1, \ldots, X_s]$ (cf. [3])):

(a) $1 \leq k \leq s$:

$$p_k - p_{k-1}\sigma_1 + p_{k-2}\sigma_2 - \dots + (-1)^{k-1} p_1 \sigma_{k-1} + (-1)^k . k . \sigma_k = 0$$

(b) $s < k$

$$p_k - p_{k-1}\sigma_1 + p_{k-2}\sigma_2 - \dots + (-1)^{s-1} p_{k-s+1}\sigma_{s-1} + (-1)^s p_{k-s}\sigma_s = 0$$

Specializing:

$X_1 = z_1, \dots, X_s = z_s$ (z_j being the s-roots of the unity in $\mathbb{C}$), and making use of the Vieté's relations for $Z^s - 1 = 0$, we obtain

$$\sigma_k(z_1, \dots, z_s) = 0, \quad 1 \leq k \leq s-1$$
$$\sigma_s(z_1, \dots, z_s) = (-1)^{s+1}$$

and thus:

$$p_m(z_1, \dots, z_s) = \begin{cases} 0, & \text{if } m \not\equiv 0 \pmod{s} \\ s, & \text{if } m \equiv 0 \pmod{s}. \end{cases}$$

Then, in the conditions of the enounce:

$$\sum_{j=1}^{s} f(z_j t) = \sum_{n \geq 0} a_n p_n(z_1, \dots, z_s) t^n,$$

which immediately gives the desired relation, if we look at the above values of the p_n's.

q.e.d.

Now we are in position to prove the following:

2.3.Theorem

The Veronese algebra V_{rs} of type $(r;s)$ over a field k is Gorenstein

<u>if and only if</u> $r \equiv 0$ <u>(mod s)</u>.

<u>Proof</u>.

According to 1.1 and 1.2 we need only to show that (in the above notations): P_{rs} is reciprocical iff $r \equiv 0$ (mod s).

Or, applying lemma 2.2 to $H_r(t)=(1-t)^{-r}=\sum_{n\geq 0}\binom{r+n-1}{r-1}.t^n$, one obtains:

$$(*)\ P_{rs}(t)=1/s.\sum_{j=1}^{s}\left(\sum_{i=0}^{s-1} z_j^i .t^{i/s}\right)^r$$

This relation actually holds in $\mathbb{Z}[t]$, i.e. the right-side member in (*) is a polynomial with integral coefficients. To see this we use the multinomial formula, expanding every term in the right-side member of (*) and making use of the possible values of $p_n(z_1,..,z_s)$ we obtain:

$$(**)\ P_{rs}(t)=\sum_{a=0}^{b} C_a.t^a,\text{ where } b=\left[r(s-1)/s\right] \text{ (integer part) and}$$

the coefficients C_a are, for every a:

$$C_a = \sum_{(i_1,\dots,i_s)\in S_a} r!/i_1!\dots i_s!$$

the index sets S_a being:

$$S_a=\left\{(i_1,\dots,i_s)\in \mathbb{Z}_+^s \ / \ i_1+\dots+i_s=r \text{ and } i_2+2i_3+\dots+(s-1)i_s=a.s\right\}$$

Suppose P_{rs} is reciprocical as polynomial from $\mathbb{Z}[t]$.

This means that the coefficients C_a in (**) are two by two equal when going from the extremities to the middle of P_{rs}.

In particular:

$$C_0 = C_b.$$

Or, it is immediate that $C_0 = 1$. Then, looking at (**) and taking into account all the possible values for the p_n's, we see that $C_b=1$ iff $r(s-1)\equiv 0 \pmod s \Leftrightarrow r\equiv 0 \pmod s$, obtaining the neccesity of the condition in theorem 2.3.

For the sufficiency, let us suppose that $r=m.s$, with $m\in\mathbb{N}$.

Then, in the above notations, $b=\left[r(s-1)/s\right]= m.(s-1)$.

Let $S = \dot{\bigcup}_{a=0}^{b} S_a$ be the total index set in (**). In every coefficient C_a, every term: $r!/i_1!\dots i_s!$ is a positive integer.

Bearing this in mind, let us look at the function:

$g:S \longrightarrow S$, given by:

$(\forall)(i_1,\dots,i_s)\in S, g((i_1,i_2,\dots,i_{s-1},i_s))=(i_s,i_{s-1},\dots,i_2,i_1)$

So defined, g establishes a bijection between S_a and S_{b-a}, for any $a\in\{0,1,\dots,b\}$, and immediately given:

$$C_a = C_{b-a}$$

because every term: $r!/i_1!\dots i_s!$ of C_a differs from the correspon-ding one in C_{b-a} only by the permutation g on the indices $(i_1,\dots,i_s)$.

Or, this shows that P_{rs} is reciprocical in this case and ends the proof of the theorem.

q.e.d.

3. Segre-Veronese G-algebras

Let $R = \oplus_{n\geq 0} R_n$ and $S = \oplus_{n\geq 0} S_n$ be two G-algebras over the same field k.

The " Segre product " of R and S is, by definition, the G-algebra over k:

$$R \circ S = \bigoplus_{n \geq 0} R_n \otimes_k S_n ,$$

which is a domain of standard type if R and S are such.

Cohen-Macaulayness is preserved by the Segre product iff ([9]):

$$r(R) \leq \min\{n / S_n \neq 0\} \text{ and } r(S) \leq \min\{n / R_n \neq 0\}.$$

The Hilbert function of RoS is obviously:

$\chi_{RoS} = \chi_R \cdot \chi_S$, showing that:

$\dim RoS = 1 + \deg \chi_{RoS} = 1 + \deg \chi_R . \chi_S = 1 + \deg \chi_R + \deg \chi_S =$
$= 1 + (\dim R - 1) + (\dim S - 1) = \dim R + \dim S - 1.$

Finally, the Hilbert-Poincaré series of RoS is:

$H_{RoS} = \sum_{n \geq 0} \chi_R(n) . \chi_S(n) . t^n$, i.e. H_{RoS} is the Hadamard product of H_R and H_S.

We shall use the following integral representation for the Hadamard product of two rational functions f and g of complex argument t (cf.[4]):

$$(1) \quad (f \circ g)(t) = 1/2\pi i \int_C f(1/z) g(tz) dz/z$$

where, if r_f and r_g are the convergence radii of f and g, then the convergence radius r_{fg} of f o g verifies: $r_{fg} \leq \min\{r_f, r_g\}$

and the integration road in (1) is the circle:

$$C : |z| = \rho , \text{ with } 1/r_f < \rho < r_g/|t| .$$

3.1. Remark

The Hadamard product f o g is a commutative multiplication on $\mathbb{C}[[t]]$. In order to see the commutativity on the integral representation (1), one must change the variable: $z \to 1/u$ and then one must invert the orientation of the integration road C.

3.2. Proposition

Let R and S be two Gorenstein domains and G-algebras over the same field k. Then their Segre product, R o S is Gorenstein, if (in the notations of part 1):

$$q(R) = q(S).$$

Proof.

In the notations of part 1, we have:

$$2\pi i \; (H_R \circ H_S)(t) = \int_C H_R(1/z) H_S(tz)\, dz/z, \text{ where}$$

$C : |z| = \rho$, with:

$1/r_R < \rho < r_S/|t|$, r_R and r_S being the convergence radii of H_R and H_S respectively.

We make use of theorem 1.1.Then:

$$2\pi i (H_R \circ H_S) \; (1/t) = \int_{C'} H_R(1/z) H_S(z/t)\, dz/z$$

where now $C' : |z| = \rho'$, $1/r_R < \rho' < |t| . r_S$.

Because R and S are Gorenstein, we see, from theorem 1.1 , that:

$$2\pi i(H_R \circ H_S)(1/t)=\int_{C'}(-1)^{\dim R}.z^{q(R)}H_R(z).(-1)^{\dim S}.\,.t^{q(S)}z^{-q(S)}H_S(t/z)\,dz/z=$$

$$=(-1)^{\dim R+\dim S}.t^{q(S)}\int_{C'}z^{q(R)-q(S)}.H_R(z)H_S(t/z)dz/z\ .$$

Now, we change the variable: $z \to 1/u$, and, making use of 3.1, we obtain:

$$(2)\quad 2\pi i\ (H_{RoS})(1/t)=(-1)^{\dim RoS}t^{q(S)}\int_{\Gamma}u^m H_R(1/u)H_S(tu)du/u$$

where: $\Gamma : |u|=\lambda$, $1/r_R < \lambda < r_S/|t|$ and $m = q(R)-q(S)$.

The integrand in the right-side member of (2) differs from the suited one, as given in (1), only by the multiplicative factor: u^m. Thus, if m=0, one obtains the conclusion of Proposition 3.2 from the theorem 1.1

q.e.d.

3.3.Corollary

In the above assumptions and notations:

$$q(R \circ S) = q(R) = q(S).$$

Proof.

Obvious, if we look at the proof of 3.2 and at 1.1.

Now, let $\gamma = (r_1,\dots,r_n;s_1,\dots,s_n)$ be a sequence of 2n positive integers.

In the notations of part 2 we give the following:

3.4. Definition

The "Segre-Veronese algebra of numerical character $(r_1,\dots,r_n;$ $;s_1,\dots,s_n)$ " over a field k, is the Segre product of the Veronese algebras over k : $V_{r_1 s_1},\dots,V_{r_n s_n}$.
Keeping fixed the field k, we denote simply by V_ν the Segre-Veronese algebra of numerical character $\nu=(r_1,\dots,r_n;s_1,\dots,s_n)$. It is easy to see that V_ν is a domain of Krull dimension $r_1+\dots+r_n-n+1$, naturally embedded in the polynomial k-algebra in $r_1+\dots+r_n$ variables. V_ν is a standard G-algebra, its graded structure:

$$V_\nu=\bigoplus_{n\geq 0} V_\nu(n)$$

being given by:

$$(\forall)\ n\geq 0,\ \dim_k V_\nu(n)=\prod_{j=1}^{n}\binom{r_j+ns_j-1}{r_j-1}.$$

V is Cohen-Macaulay, as one can see using either the above quoted characterization of the Cohen-Macaulayness of the Segre product of two G-algebras (cf.[9]) or the theorem of Hochster [6], as the normality of the monoides of monomials is preserved by the Segre product.
Now, we shall prove the main result of this paper, namely:

3.5. Theorem

The Segre-Veronese algebra of numerical character $(r_1,\dots,r_n;s_1,\dots,s_n)$ over a field k, is Gorenstein if and only if there is an integer q, such that:

$r_1/s_1=r_2/s_2=\ldots=r_n/s_n=q$

Proof.

First, using 1.2 and 2.3, we see that, for any Veronese algebra V_{rs} over a field k, the Gorensteinness of V_{rs} implies that:

$$q(V_{rs}) = \dim V_{rs} - \deg P_{rs} = r - r(s-1)/s = r/s.$$

Then, using an induction on n, we see that all we have to prove is, because of 3.2 and 3.3, that the sufficient condition in 3.2 is also necessary in order to insure the Gorensteinness of the Segre product R o S, in the case when R and S are two Veronese algebras: $R = V_{rs}$, $S = V_{r's'}$.

Looking at (2) in the proof of 3.2, we must show that (in the notations of part 2):

(3) $\int_\Gamma (u^m-1).H_{rs}(1/u)H_{r's'}(tu)du/u=0$ if and only $m=0$.

(Here $\Gamma : |u| = \rho$, with $1<\rho<1/|t|$, because H_{rs} and $H_{r's'}$ both have the convergence radius equal to 1).

According to the results of part 2, we know that:

$$H_{rs}(t)=P_{rs}(t)/(1-t)^r \text{ and } H_{r's'}(t)=P_{r's'}(t)/(1-t)^{r'}$$

with P_{rs} and $P_{r's'}$ reciprocical polynomials in $\mathbb{Z}[t]$, of degrees respectively: $r(s-1)/s$ and $r'(s'-1)/s'$ (V_{rs} and $V_{r's'}$ being Gorenstein) and satisfying:

$P_{rs}(1)\neq 0$ and $P_{r's'}(1)\neq 0$ (cf.part 1).

Then (3) becomes:

$$(4)\quad \int_{\Gamma} (1-u^m)/(1-u)^r \, u^{r/s-1} . \, P_{rs}(u) . P_{r's'}(tu)/(1-tu)^{r'} \, du=0$$

with the same Γ.

Or, in the domain bounded by the circle Γ, the integrand in (4) has a unique pole, namely u=1. Thus, the integral in (4) equals the residue in u=1 of the rational function:

$$h(u)=(1-u^m)/(1-u)^r \, . \, p(u), \quad \text{where:}$$

$$p(u)=u^{r/s-1} . P_{rs}(u) . P_{r's'}(tu)/(1-tu)^{r'} .$$

Or, one directly sees that p(u) is analytic near u=1 and more, that $p(1)\neq 0$. Then, except for the trivial case r=1, a direct computation shows that the residue in u=1 of h(u) is non zero when $m\neq 0$, implying (3) and ending the proof of the theorem.

q.e.d.

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