INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

APPROXIMATION PROPERTIES OF THE FORMALLY SMOOTH MORPHISMS

by

Vasile NICA and Dorin POPESCU

PREPRINT SERIES IN MATHEMATICS
No. 46/1979

BUCURESTI

APPROXIMATION PROPERTIES OF THE FORMALLY SMOOTH MORPHISMS

by

Vasile NICA*) and Dorin POPESCU**)

September 1979

*) *Ghirlandei 38, Bl. D 1, ap. 24, 77592 Bucharest, Romania*

**) *Department of Mathematics, Ntional Institute for Scientific and Technical Creation, Bdul. Păcii 220, 77538 Bucharest, Romania*

APPROXIMATION PROPERTIES OF THE FORMALLY SMOOTH MORPHISMS

Vasile Nica and Dorin Popescu

Let L be a system of linear equations with integer coefficients. Then L has solutions in a field $K \supset \mathbb{Z}$ iff it has rational solutions. This is a property of linear saturation for $\mathbb{Q}$. In algebraic case, a system of polynomial equations with rational coefficients has a solution in an extension K of $\mathbb{Q}$ iff it has solutions in the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. This means a property of algebraic saturation for $\mathbb{Q}$. These saturation properties are very strong, because they are reffering to every extension K We can study also weaker saturation properties (linear or algebraic) relative to a given extension K. There are pure extensions in the linear case (see [6]) and algebraic pure extensions in the algebraic case (see [9]).

In the algebraic case it is known (see [1], [5]), the study of the local noetherian rings R which has algebraic saturation property relative to its completion $\hat{R}$, i.e. the morphism $R \longrightarrow \hat{R}$, is algebraically pure.

We say that a ring R has the property of approximation (we write R is an AE-ring) i.e. every "formal" solution (from $\hat{R}$) of an arbitrary finite system of polynomial equations can be well approximate by "algebraic" solutions (from R) in the m-adic topology of $\hat{R}$, m being the maximal ideal in $\hat{R}$ (in fact if every finite system of polynomial equations over R has a solution in R whenever it has one in $\hat{R}$, then R is an AE-ring).

In ch.V [5] (or [8]) it shows the preserving of some properties from R to $\hat{R}$ if R is an AE-ring, properties which can be given by compatibility of some systems of polynomial equations (for example R is reduced (integral domain) iff R is too). Unfortunately, this cannot be done for algebraically pure morphisms (the extension $\mathbb{R} \hookrightarrow \mathbb{R}[X]/(X^2)$ is algebraically pure but does not preserve the property of being reduced). The reason is that the profs from [5] use the possibility to can well approximate the solutions from $\hat{R}$ of an arbitrary system of polynomial equations over R by solutions from R. This is not possible for arbitrary algebraically pure morphisms. However, corollary 2.6, in [9] it shows that if A is a noetherian local complete ring and B is a Cohen A-algebra such that its residue field of B is a ultrapower of the residue field of A, then the solution from B of an arbitrary system of polynomial equations over A can be in a sense "well approximate" by solutions from A. As consequence, it shows that A is reduced (integral domain) iff B is too (proposition 2.10 [9]).

In this work, we give a sense for the above "good approximation" in the case of the formally smooth morphisms ((2.2), (2.3) şi (2.6)). As a consequence, we get the preserving of some properties by algebraically pure formally smooth morphisms (§ 3). Also we extend the remark 2.18 iv [9] showing that if A, B are noetherian local complete rings and B is a formally smooth A-algebra, then B is an algebraically pure A-algebra iff their residue field extension is algebraically pure (2.1).

§ 1. Algebraically and analytically pure morphisms

Algebraically and analytically pure morphisms were introduced in [9] in connection with the study of rings which have the property of approximation (all the rings are here supposed to be commutative and with identity). They generalize the linear case of pure module morphisms [6].

(1.1) DEFINITION. A ring morphism $u:A \longrightarrow B$ is called algebraically pure if every finite system of polynomials $F=(F_1,\dots,F_m)$ with coefficients in A, in an arbitrary number of variables $Y=(Y_1,\dots,Y_N)$ has a solution in A iff it has a solution in B. If so, we say sometimes that B is algebraically pure over A.

(1.2) PROPERTIES AND EXAMPLES. i) If A is a local noetherian ring, then the completion morphisms $A \longrightarrow \hat{A}$ is algebraically pure iff A has the property of approximation. Moreover, if A is an integral domain then $\hat{A}$ is too and the fraction field extension $Q(A) \hookrightarrow Q(\hat{A})$ is algebraically pure.

ii) The class of algebraically pure morphisms is stable under composition and base change. Moreover, if $v \circ u$ is algebraically pure, then u is so.

iii) If $u:A \longrightarrow B$ is a finite presentation morphism, then u is algebraically pure iff it has an A-algebra retraction. In particular if k is a field, and B a finite type k-algebra, the structure morphism $k \longrightarrow B$ is algebraically pure iff Spec B has a closed k-rational point. If B is an integral domain and $k \hookrightarrow B$ is algebraically pure, then k is algebraically closed

iv) More generally, an arbitrary ring morphism $u: A \longrightarrow B$ is algebraically pure iff B is a filtered inductive limit of algebraically pure A-algebras or a filtered inductive limit of A-algebras such that their structure morphisms have retractions.

The above result furnishes some interesting examples of algebraically pure morphisms and also a criterion for recognizing this property (see 1.3 below).

v) If k is an algebraically closed field and B an arbitrary k-algebra then the structure morphism $k \hookrightarrow B$ is algebraically pure.

vi) If k is an infinite field and X a variable, then the morphism $k \hookrightarrow k(X)$ is algebraically pure. More generally, any pure transcedental extension of infinite fields is algebraically pure.

vii) Any algebraically pure field extension of a finite field is trivial.

viii) The algebraically pure morphisms are not in general flat. For instance, $A \hookrightarrow A[[X]]$ can be not flat (nonnoetherian case) but it is algebraically pure (having a retraction).

All these properties and examples are given in [9]. We complete them with the following proposition:

(1.3) <u>PROPOSITION</u>. Let k be a field and B a finite type k-algebra which is an integral domain: $B = k[X_1,\dots,X_n]/p$. Then, the fraction field Q(B) of B is algebraically pure over k iff for every polynomial $F \in k[X_1,\dots,X_n]$, $F \notin p$ which depends at most $s = \mathrm{trdeg}_k B$ variables, we have $Z(p) \not\subset Z(F)$, where Z(p), Z(F) denote the sets of zeros in k^n of p and F respectively.

<u>Proof</u>. Necessity. Let F be a polynomial in $k[X_1,\dots,X_n]$ not in p. From hypothesis, the composition $k \hookrightarrow B_F \hookrightarrow Q(B)$

is algebraically pure and by (1.2.ii) the finite presentation morphism $k \hookrightarrow B_F$ is too. Consequently, $k \hookrightarrow B_F$ has a k-algebra retraction, equivalently the prime ideal p has a zero in k^n, which is not a zero for F.

Sufficiency. Let $s=\mathrm{trdeg}_k B$. We can suppose that images of $X_1,\ldots,X_s$ in B form an algebraically free system over k. The injective morphism $k[X_1,\ldots,X_s] \hookrightarrow B$ is algebraic in the sense that every elemnt of B satisfies a polynomial equation with coefficients in $k[X_1,\ldots,X_s]$. Let consider a $b\neq 0$ in B and let $a_t b^t+\ldots+a_0=0$ be the minimal degree equation over $k[X_1,\ldots,X_s]$ for b. We apply the hypothesis on the polynomial $F=a_o$; we observe that $F\notin p$, otherwise the equation satisfied by b has not minimal degree. There exists a zero $\alpha=\alpha_1,\ldots,\alpha_n)$ for p such that $F(\alpha)\neq 0$. The correspondence $X_i \rightsquigarrow \alpha_i$ produces a well-defined morphism $r:B \longrightarrow k$ with $r(b)\neq 0$, because $F(\alpha)\neq 0$. Then r extends to a k-algebra morphism $B_b \longrightarrow k$. So, the finite presentation morphism $k \hookrightarrow B_b$ is algebraically pure. Since $Q(B)=\bigcup_{b\neq 0} B_b$, we conclude with (1.2 iv).

Q.E.D.

(1.3.1) COROLLARY. If k is infinite and $\mathrm{trdeg}_k B=1$ then $Q(B)$ is algebraically pure over k iff the set $Z(p)\subset k^n$ is infinite.

(1.3.2) EXAMPLES. Let $\mathbb{R}$ the field of real numbers. Then the extension $\mathbb{R} \hookrightarrow Q(\mathbb{R}[X,Y]/(X^2-Y^3))$ is algebraically pure, but $\mathbb{R} \hookrightarrow Q(\mathbb{R}[X,Y]/(X^2+Y^2))$ is not.

(1.3.3) Remark. A field is separable closed iff any its separable extension is algebraically pure (also a field is algebraically closed iff any its extension is algebraically pure). Indeed, let k be a separable closed field and $k'\supset k$ a separable finite generated field extension (by (1.2) iv) this is sufficient).

Then k' has the form $k' \simeq Q(k[X,T]/_{(P)})$, $X=(X_1,\ldots,X_n)$, where P is a (monic in T) separable irreducible polynomial from $k[X,T]$. Let $F \in k[X]$ be a non-zero polynomial. k being infinite, there exists $x \in k^n$ such that $F(x) \neq 0$ and $P'(x,T) \neq 0$. Thus $P(x,T)$ is still separable in $k[T]$ and so it has a solution t in k. We get $Z(P) \not\subset Z(F)$ and by (1.3) the extension $k \subset K$ is algebraically pure. Conversely, let k_s be the separable closure of k. If the extension $k \subset k_s$ is algebraically pure, then $k=k_s$ by (1.2)iii).

In the case of complete rings, the algebraically pure concept is extended in the following manner:

(1.4) DEFINITION. A local morphism of local, noetherian, and complete rings $u:A \longrightarrow B$ is called analytically pure if every system $F=(F_1,\ldots,F_m)$ of polynomials in $A[[Z]][Y]$ - where $Z=(Z_1,\ldots Z_M)$, $Y=(Y_1,\ldots,Y_N)$ are variables has a solution (z,y) in A iff it has a solution (z,y) in B. (Obviously, the components of $z,\bar{z}$ are in the maximal ideals of A and B respectively)

(1.4.1) Note that for M=0, we recover the algebraic case of (1.1). Also in the case of artinian local rings, both definitions coincide.

(1.5) Proposition. Let $u:A \longrightarrow B$ be an algebraically pure morphism of noetherian local rings. Suppose that the maximal ideals $\underline{m}$ of A generates the maximal ideal of B. Then the induced morphism $\hat{u}:\hat{A} \longrightarrow \hat{B}$ is analytically pure. Moreover, if A,B are complete rings then u is algebraically pure iff it is analytically pure.

Proof. Let F be a system of formal power series from $\hat{A}[[Z]][Y]$ which has a solution in $\hat{B}$. By theorem 2.8 [9] (see also theorem 2.5 [8]), there exists $c \in \mathbb{N}$ such that F has a solution in $\hat{A}/_{\underline{m}^c\hat{A}}$ iff it has one in $\hat{A}$. But our system F has a solution in

$B/\underline{m}^c B \simeq \hat{B}/\underline{m}^c \hat{B}$ and thus it has one in $\hat{A}/\underline{m}^c \hat{A} \simeq A/\underline{m}^c$ because the induced morphism $A/\underline{m}^c \longrightarrow B/\underline{m}^c B$ is analytically pure by (1.2)ii) and (1.4.1). Consequently F has solutions in $\hat{A}$.

Q.E.D.

In the context of definition (1.4), we remark that there is no relation between the solution $(\bar{z},\bar{y})$ of F in B and the solution (z,y) in A. Then, it arises the following question: if F has a solution $(\bar{z},\bar{y})$ in B and it is known that it has solutions in A, one can find in A a solution (z,y) which is "near" to $(\bar{z},\bar{y})$ in B? For this, it is necessary a suitable concept of "nearness" between solutions $(\bar{z},\bar{y})$ and (z,y), and one of possible ways to introduce such a concept is the following one:

Suppose that solution $(\bar{z},\bar{y})$ in B satisfies the condition:

$$\text{ord } G_j(\bar{z},\bar{y})=c_j \quad j=1,\ldots p, \quad \text{briefly ord } G(\bar{z},\bar{y})=c$$

where $G=(G_1,\ldots G_p)$ are in $A[[Z]][Y]$, $c=(c_1,\ldots c_p)$ are nonnegative integers and $\text{ord } G_j(\bar{z},\bar{y})=c_j$ means $G_j(\bar{z},\bar{y})\in \underline{n}^{c_j}$, $G_j(\bar{z},\bar{y})\notin \underline{n}^{c_j+1}$, $\underline{n}$ being maximal ideal in B.

In the above notations, we shall say that the solution (z,y) in A is G-near to solution $(\bar{z},\bar{y})$ in B if $\text{ord } G(z,y)= =\text{ord } G(\bar{z},\bar{y})=c$.

So, we can specialize the definition (1.4) looking for the lifting of any solution $(\bar{z},\bar{y})$ of F in B to a solution (z,y) in A G-near to (z,y) in the above sense. We shall see in § 3 that the analytically pure morphisms $A \longrightarrow B$ which lift the solutions from B to near solutions in A, allow to preserve some properties from A to B, properties which can be explicited in terms of compatibility of some polynomial equation systems.

§ 2 Main results

(2.1) THEOREM. Let $u:A \longrightarrow B$ be a local formally smooth morphism between two noetherian local complete rings with residue fields k respectively K. Suppose that K is a separable extension of k. Then the following conditions are equivalent:

i) u is algebraically pure

ii) u is analytically pure

iii) the residue field morphism $k \hookrightarrow K$ induced by u is algebraically pure.

Proof. By hypothesis, B is of the type $B \simeq A'[[X]]$ where A' is a Cohen A-algebra and $X=(X_1,\ldots,X_n)$ are variables (see [3] § 19). Thus u admits the following decomposition

$$A \xrightarrow{u'} A' \xrightarrow{u''} A'[[X]] \simeq B$$

Clearly, u" has a retraction and so it is algebraically pure. It remains to prove the theorem in the case, when B is a Cohen A-algebra (1.2 ii)), which is the subject of the following theorem:

(2.2) THEOREM. Let A be a noetherian, local, complete ring and B a Cohen A-algebra (i.e. a flat, local, complete A-algebra such that $B/_{\underline{m}B}$ is a field separable extension of the residue field $k=A/_{\underline{m}}$ of A). The following conditions are equivalent:

i) The structure morphism $u:A \longrightarrow B$ is algebraically pure

ii) u is analytically pure

iii) For every $F=(F_1,\ldots,F_m)$, $G=(G_1,\ldots,G_p)$ in $A[[Z]][Y]$, $Z=(Z_1,\ldots Z_M)$, $Y=(Y_1,\ldots Y_N)$ being variables, every $c=(c_1,\ldots c_p)$ with c_j nonnegative integers and every $(\bar{z},\bar{y})$ in B such that $F(\bar{z},\bar{y})=0$ and ord $G(\bar{z},\bar{y})=c$, there exist (z,y) in A, such that $F(z,y)=0$ and ord $G(z,y)=c$ (in other words, any solution of F in B lifts to a G-near solution in A).

iv) The residue field morphism $k \hookrightarrow K$ induced by u is algebraically pure.

<u>Proof</u>. Implications iii) $\Rightarrow$ ii) $\Rightarrow$ i) are easy, and i) $\Rightarrow$ iv) comes from the fact that algebraically pure morphisms are stable under base change. Implication iv) $\Rightarrow$ iii) is the object of the following theorem, in which the nearness condition between solutions $(\bar{z},\bar{y})$ and (z,y) is translated in terms of linear systems.

(2.3) <u>THEOREM</u>. Let A be a noetherian, local, complete ring, B a Cohen A-algebra such that the residue field morphism $k \hookrightarrow K$ induced by u is algebraically pure. Then, for every system $F=(F_1,\dots,F_m)$, $G=(G_1,\dots,G_p)$ from $A[[Z]][Y]$, every matrix $\| a_{jk} \|$ $j=1,\dots,p$, $k=1,\dots,q$ with elements in A, and every $(\bar{z},\bar{y})$ in B such that $F(\bar{z},\bar{y})=0$ and the linear system $\sum_{k=1}^{q} a_{jk}T_k=G_j(\bar{z},\bar{y})$ $j=1,\dots p$ is incompatible in B, there exists a solution (z,y) of F in A for which the linear system $\sum_{k=1}^{q} a_{jk}T_k=G_j(z,y)$ is incompatible.

Now we show how we get iv) $\Rightarrow$ iii) (2.2) using (2.3). In the notations of theorem (2.2) let $\{e_{jk}\}$, $\{f_{j\ell}\}$ $j=1,\dots,p$ $k=1,\dots q_j'$, $\ell=1,\dots q_j''$ be systems of generators for ideals $\underline{m}^{c_j}$, $\underline{m}^{c_j+1}$ respectively ($\underline{m}$ being the maximal ideal of A). To say that $G_j(\bar{z},\bar{y})\in \underline{m}^{c_j}B$ means that there exist $\{b_{jk}\}$ in B such that $G_j(\bar{z},\bar{y})=\sum_{k=1}^{q_j'} b_{jk}e_{jk}$, $j=1,\dots p$, in other words the system $G_j(\bar{z},\bar{y})=\sum_{k=1}^{q_j'} e_{jk}U_{jk}$ has a solution in B. These equations will be added to the original system F. To say that $G_j(\bar{z},\bar{y})\notin \underline{m}^{c_j+1}B$ means to say that the linear system

$G_j(\bar{z},\bar{y}) = \sum_{\ell=1}^{2_j''} f_{j\ell} T_{j\ell}$ has no solutions in B.

The proof of (2.3) is difficult enough and it is the subject of § 4.

(2.4) Remark. When B is a Cohen A-algebra such that the residue field morphism $k \longrightarrow K$ is algebraically pure, the theorem (2.2) shows that the morphism $A \longrightarrow B$ is not only analytically pure but has also the property to lift solutions from B of arbitrary finite formal equations to near solutions in A (shortly we say "lifts good"). It arrises the question if this new property is still valid for formally smooth morphisms. The answer is negative in general because the morphism $u'': A' \longrightarrow A'[[X]]$ lifts solutions but not in a "good" way. For instance, the morphism $k \longrightarrow k[[X]]$, where k is an arbitrary field is analytically pure but does not lift good "analytically", because a formal equation with coefficients in k can have nonzero solutions in $k[[X]]$ and only the zero solution in k. If k is finite the above morphism does not lift good also "algebraically", because otherwise the morphism $k \hookrightarrow k((X))$ must be algebraically pure, which is a contradiction (1.2) vii). When k is algebraically closed then clearly the mkorphism $k \longrightarrow k[[X]]$ lifts good "algebraically".

The following theorem precises when the morphism u" from (2.1) lifts good "algebraically".

(2.5) THEOREM. Let K be a field and T a variable. The following affirmations are equivalent:

i) The extension $K \hookrightarrow \Omega(K\langle T\rangle)$ is algebraically pure.

ii) The extension $K \hookrightarrow K((T))$ is algebraically pure.

iii) For every noetherian local complet ring A with residue field K the morphism $A \longrightarrow A[[T]]$ lifts good "algebraically".

Here $K\langle T\rangle$ denotes the ring of algebraic power series with coefficients in K.

Proof. Clearly ii) $\Rightarrow$ i), and i) $\Rightarrow$ ii) is a consequence of (1.2) i) and ii), $K\langle T\rangle$ being an AE-ring (see [1]).

ii) $\Rightarrow$ iii) Let $\underline{m}$ be the maximal ideal of A and denote by B the completion of the ring $A[[T]]_{mA[[T]]}$. Clearly, B is a Cohen A-algebra with residue field K((T)), which is an algebraically pure extension of K(ii)). Applying (2.2), we get that the morphism $A \longrightarrow B$ lifts good algebraically. It remains to show that $A \longrightarrow A[[T]]$ has the same property. Let $a_{jk} \in A$, G_j, $F \in K[Y]$, $Y=(Y_1,\ldots,Y_N)$ and $\bar{y} \in A[[T]]^N$ a solution of F such that the system

$$(+) \sum_k a_{jk} T_k = G_j(\bar{y}) \quad ,$$

has no solutions in $A[[T]]$. Then it is sufficient to show that (+) remains incompatible in B, i.e. there exists j_o such that $G_{j_o}(\bar{y})$ is not contained in the ideal $\underline{a}_{j_o} B$, where $\underline{a}_j$ denote the ideal generated by $(a_{jk})_k$ in A. Let j_o be such that $G_{j_o}(\bar{y}) \notin \underline{a}_{j_o} A[[T]]$ (otherwise (+) is compatible in $A[[X]]$). Then $\underline{a}_{j_o}$ is a proper ideal and we get easy $(\underline{a}_{j_o}.B) \cap A[[T]] = \underline{a}_{j_o}.A[[T]]$. Consequently, if (+) is compatible in B then $G_{j_o}(\bar{y}) \in \underline{a}_j B$ and we get $G_{j_o}(\bar{y}) \in \underline{a}_{j_o} A[[T]]$. Contradiction !

iii) $\Rightarrow$ ii) Using iii) we deduce in particular that the morphism $K \longrightarrow K[[T]]$ lifts good algebraically. Now, let $F \in K[Y]$ be a system of polynomials and $\bar{u}_j$, $\bar{v} \in K[[T]]$, $\bar{v} \neq 0$ such that $\frac{\bar{u}}{\bar{v}}$ is a solution of F in K((T)). Let F′ be the homogeneous polynom associated to F. We get $F'(\bar{v},\bar{u})=0$. Lifting good $(\bar{v},\bar{u})$, we get a solution (v,u) of F′ in K with $v \neq 0$. Clearly, $\frac{u}{v}$ is a solution of F in K.

Q.E.D.

(2.6). Corollary. Let $u:A \longrightarrow B$ be a local formally smooth morphism between two noetherian local complete rings with residue fields k respectively K. Suppose that K is a separable extension of k and $Q(K\langle T\rangle)$ is an algebraically pure extension of K, T being a variable. Then u lifts good algebraically iff K is an algebraically pure extension of k.

For proof is sufficient to apply (2.5) n-times successively in order to get that u" (from (2.1)) lifts good algebraically.

(2.7) Remark. i) Corollary (2.6) is still true if A, B are not complete, but A is an AE-ring. Indeed, by proposition 19.3.6 [3] the induced morphism $\hat{u}:\hat{A} \longrightarrow \hat{B}$ is still formally smooth. If K is an algebraically pure extension of k, then $\hat{u}$ lifts good algebraically (2.6). Let $F, G_j \in K[Y]$, $a_{jk} \in A$, $Y=(Y_1,\dots,Y_N)$ and $\bar{y} \in B^N$ a solution of F such that the system (+) is incompatible in B. The canonical morphism $B \hookrightarrow \hat{B}$ being faithfully flat, we get $(\underline{a}_j\hat{B}) \cap B = \underline{a}_j B$ and so as in the proof of (2.5) we deduce that (+) has no solutions in $\hat{B}$. Now, by (2.6) there exists $\tilde{y} \in \hat{A}^N$ such that $F(\tilde{y})=0$ and the system

$$(\tilde{+}) \quad \sum a_{jk}T_k = G_j(\tilde{y})$$

has no solutions in $\hat{A}$. If $y \in A^N$ is a solution of F such that $y \equiv \tilde{y}$ mod $\underline{m}^c\hat{A}$ for c sufficiently large (A is AE-ring), then the system $\sum a_{jk}T_k = G_j(y)$ is incompatible in A ($\underline{m}$ denotes the maximal ideal of A).

ii) As above, it is enough to have A an AE-ring, instead "A,B complete rings" in the hypothesis of (2.1), (2.2), (2.3).

iii) If we leave the case of formally smooth morphisms, then the results of this section are not in general true. For instance the morphism $\mathbb{C}[[X]] \hookrightarrow \mathbb{C}[[X]][Y]_{(X,Y)}/(Y^2-X)$ is not algebraically pure, but it becomes algebraically pure if we change the base from $\mathbb{C}[[X]]$ to $\mathbb{C}[[X]]/(X) \simeq \mathbb{C}$.

(2.8) Remark. The condition i) from (2.5) is fulfilled for instance when K is separable closed (see (1.3.3)).

§3. Applications

In this section some applications of the theorems announced in § 2 are given.

Let $A \longrightarrow B$ be a formally smooth local morphism between two noetherian local complete rings with residue fields k and K . Suppose that $k \hookrightarrow K$ is a separable and algebraically pure extension, and in the case when B is not a Cohen A-algebra, suppose additionally that

(*) $K \hookrightarrow Q(K\langle T\rangle)$ is an algebraically pure extension,

T being a variable. Using theorems 2.1, 2.2 and corollary 2.6 we shall show that some properties of the ring A which can be formuled in terms of compatibility of some polynomial equation systems are transfered on the ring B.

(3.1) PROPOSITION. Let $\underline{m}$ be the maximal ideal of A, n a non-negative integer, $F=(F_1,\ldots,F_m)$ a finite system of polynomials from $A[Y]=A[Y_1,\ldots,Y_N]$ which has a finite number of solutions (perhaps none) in $\underline{m}^n$. Then, F has not other solutions in $\underline{m}^n B$.

Proof. Let $y^{(1)},\ldots,y^{(s)}$ be the solutions of F in $\underline{m}^n$. Suppose contrary, that F has in $\underline{m}^n B$ a solution $y=(y_1,\ldots,y_N)$ distinct from $y^{(t)}$, t=1,...s. We show that $\overline{y}$ lifts to a solution in $\underline{m}^n$ for F distinct from $y^{(t)}$. Let $\{e_1,\ldots,e_p\}$ be generators of the ideal $\underline{m}^n$. There exist $\{\bar{b}_{ij}\}$ $i=1,\ldots,N$, $j=1,\ldots,p$ in B such that $\bar{y}_i=\sum_{j=1}^{p}\bar{b}_{ij}e_j$, $i=1,\ldots,N$, because $\bar{y}\in\underline{m}^n B$. For every $t=1,\ldots,s$ there exists an index i_t such that $c_i=\text{ord}\,(\bar{y}_{i_t}-y_{i_t}^{(t)})<\infty$. Put $G_{i_t}=Y_{i_t}-y_{i_t}^{(t)}$,

$G=(G_{i_t})$ $t=1,\ldots,s$, $c=(c_1,\ldots,c_s)$, and add the polynomials

$Y_i-\sum_{j=1} e_j Y_{ij}$ $i=1,\ldots,N$ to F. By (2.6), there exists a solution y in A of the extended system F, with the property ord $(G)=c$, i.e, there exists a solution y for F in $\underline{m}^n$ distinct from $y^{(1)},\ldots,y^{(s)}$. Contradiction !

(3.2) PROPOSITION. The following assertions holds:

i) A is a reduced ring iff B is so.

ii) A is integral domain iff B is so, and in this case A is algebraically closed in B.

iii) If p is a prime ideal in A, then pB is a prime ideal in B (in particular, Spec B $\longrightarrow$ Spec A is a closed morphism).

iv) If p is a prime ideal in A, then the canonical morphism $k(p)\hookrightarrow k(pB)$ is algebraically pure ($k(p)$ denotes the residue field of the local ring A_p).

Proof. i) If B is reduced then A is reduced, the morphism $A\hookrightarrow B$ being injective. The necessity of condition follows from (3.1). Indeed, for every nonnegative integer c, equation $Y^c=0$ has in A only the trivial solution. It has not other solutions in B, therefore B is reduced.

ii) Sufficiency being trivial, we prove the necessity. Suppose that B is not an integral domain, and let $\bar{y}_1,\bar{y}_2$ in B such that $\bar{y}_1\cdot\bar{y}_2=0$ but $\bar{y}_1\neq 0$, $\bar{y}_2\neq 0$. Then ord $\bar{y}_1$, ord $\bar{y}_2<\infty$. By theorem (2.1) there exist y_1 , y_2 in A such that $y_1y_2=0$ and ord $y_i=$ord $\bar{y}_i$ $i=1,2$. It follows that y_1,y_2 are different from zero, i.e. y_1,y_2

are zero divisors in A which contradicts the hypothesis.

For the last part of ii) we observe that a polynomial $F \in A[Y]$ has at most deg F solutions in A, therefore it can not have other solutions in B by virtue of (3.1).

iii) follows from ii) by base change $A \longrightarrow A/p$

iv) By base change $A \longrightarrow A/p$ we reduce to prove that Q(B) is an algebraically pure extension of Q(A), in the case when A, B are integral domains. Now the proof is as in Remark 1.9 v) [9]. Using (1.2) iv) it is sufficient to show that the morphism of the type $u:Q(A) \longrightarrow Q(A)[\bar{z}_o, \bar{z}_1, \dots, \bar{z}_s, \bar{z}_o^{-1}]$, $\bar{z}_i \in B$, $\bar{z}_o \neq 0$, $s \in \mathbb{N}$ have retractions. Let $\underline{a} \subset A[Z]$, $Z=(Z_o, \dots, Z_s)$ be the kernel of the map $A[Z] \longrightarrow B$ given by $P \rightsquigarrow P(\bar{z})$ and $c = \text{ord } \bar{z}_o$. By 2.2 or 2.6, there exists a solution z of $\underline{a}$ in A such that ord $z_o = c$. In particular, $z_o \neq 0$ and the map $A[Z] \longrightarrow A$ given by $P \rightsquigarrow P(z)$ induces the retraction $Q(A)[\bar{z}, \bar{z}_o^{-1}] \longrightarrow Q(A)$, we were looking for.

(3.2.1) <u>Corollary</u> i) If q is a p-primary ideal in A then qB is pB-primary ideal in B.

ii) If $\underline{a} = q_1 \cap \dots \cap q_s$ is a reduced primary decomposition of ideal $\underline{a} \subset A$ and p_i are the associated prime ideals of q_i, then $\underline{a}B = q_1 B \cap . \dots \cap q_s B$ is a reduced primary decomposition of ideal $\underline{a}B$ and $p_i B$ are the associated prime ideals of $q_i B$. Moreover, it holds $\sqrt{\underline{a}}B = \sqrt{\underline{a}B}$.

The proof is a consequence of 3.2 iii) and theorem 13 p.60 [7].

(3.2.2) <u>Corollary</u> i) Every saturated prime chain from A remains a saturated prime chain by extension to B.

ii) Every $q \in \text{Spec } A$ holds ht q=ht(qB).

Proof i) If A is an integral domain and $q \subset A$ is a prime ideal with height one then qB is a nonzero ideal with height ≤ 1, using theorem 19 p.79 [7]. B being integral domain (3.2 ii)) we deduce ht(qB)=1. Consequently, the saturated prime chain $(0) \subset q$ remains a saturated prime chain by extension to B. This is sufficient because always we may reduce to this case by a base change.

ii) B being catenar, ii) is a consequence of i).

Q.E.D.

(3.2.3) Corollary i) Let $\underline{a} \subset A$ be an ideal and $q \supset \underline{a}$ a prime ideal of A. Then $\underline{a}A_q$ can be generated in A_q by m-elements iff $\underline{a}B_{qB}$ can be generated in B_{qB} by m-elements.

ii) $p \in \operatorname{Reg} A$ iff $pB \in \operatorname{Reg} B$, where Reg A denote the set of prime ideals of A such that A_p is regular.

Proof. Let $\underline{a}=(f_1,\dots,f_s)$, $\mathfrak{g}=(g_1,\dots,g_t)$. Consider the following system of polynomials

$$F := \begin{cases} Sf_i = \sum_{j=1}^{m} U_{ij} Z_j , & i=1,\dots,s \\ SZ_j = \sum_{k=1}^{s} V_{jk} f_k , & j=1,\dots,m \end{cases}$$

$$G := S - \sum_{k=1}^{t} g_k T_k$$

where S, U_{ij}, V_{jk}, Z_j, T_k are variables. If $\underline{a}B_{qB}$ is generated by m-elements $\bar{z}_1,\dots,\bar{z}_m \in B$ then there exist $\bar{s},\bar{u},\bar{v} \in B$ such that $(\bar{z},\bar{s},\bar{u},\bar{v})$ is a solution of F in B and the equation $G(\bar{s},T)=0$ has no solutions in B. By (2.3) or (2.6), F has a solution (z,s,u,v) in A such taht the equation $G(s,T)=0$ has no solutions in A.

Consequently $z=(z_1,\dots,z_m)$ generates $\underline{a}A_q$.

ii) is a consequence of i) and 3.2.2 ii)

Q.E.D.

(3.2.4) <u>Corollary</u>. If B is a factorial ring then A is too.

<u>Proof</u>. Let $q\subset A$ be a prime ideal with height one. Then qB is still a prime ideal with height one (3.2 ii), (3.2.2) ii)) and thus is principal. Applying (3.2.3) i) it results that q is principal.

Q.E.D.

Now, let $x_1,\dots,x_n\in A$ be a system of parameters of A. The ring A has a big Cohen-Macaulay module if there exists an A-module E such that $x=(x_1,\dots,x_n)$ is a regular sequence on E [4].

(3.3) <u>Proposition</u>. Let B be a Cohen A-algebra. Then A has a big Cohen-Macanlay module iff B has one.

<u>Proof</u>. As M.Hochster showed [4] the non-existence of a big Cohen-Macaulay module over A is equivalent with the compatibility of a system of polynomial equations with integral coefficients in A. Clearly it is sufficient to apply (2.2).

§ 4. <u>Proof of the theorem 2.3</u>

For the beginning, we remaind a result from the theory of ring ultraproducts [2], [9]. Let $\mathbb{N}$ be the set of naturals and D a nonprincipal ultrafilter on $\mathbb{N}$. Let A^* be the ultraproduct of A with respect to D. The ring A^* is local, let A_1 be its separate in the adic topology given by the maximal ideal. Using hypothesis and notations of (2.2), we have the following proposition:

(4.1) PROPOSITION. B_1 is a Cohen A_1-algebra.

Proof. In fact, by structure theorem of noetherian local complete rings we have $A \cong R[[X]]/_{\underline{a}}$, where R is either the field k or a discrete valuation ring of characteristic 0, with residue field k, and $X=(X_1,\ldots,X_n)$ are variables. Let R' be the Cohen-R algebra with residue field K [3]. Then $B=R'[[X]]/_{\underline{a}R'[[X]]}$. It is known [9] that $A_1=R_1[[X]]/_{\underline{a}R_1[[X]]}$, $B_1=R'_1[[X]]/_{\underline{a}R'_1[[X]]}$ and it is sufficient to prove that if $R \longrightarrow R'$ is an unramified extension of discrete valuation rings such that their residue field extension is a separable one, then the morphism $R_1 \longrightarrow R'_1$ has the same property. The extension $R \longrightarrow R'$ being unramified it is sufficient to show that if a field extension $k \longrightarrow K$ is separable then the extension $k^* \longrightarrow K^*$ is too (here k^* is the ultraproduct of the field k with respect to the ultrafilter D). An algebraic closed extension L of K being considered, it is sufficient to show that the fields K^*, $(k^*)^{1/p} \subset L^*$ are linearly disjoint over k^* (Mac-Lane's criterion), where p=char (k)

Let $\alpha_1,\ldots,\alpha_t$ in K^* be linearly independent over k^* with $\sum_{i=1}^{t} a_i\alpha_i=0$, where $a_i \in (k^*)^{1/p}=(k^{1/p})^*$. Let $a_i=[(a_{in})_{n\in\mathbb{N}}]$, $a_{in}\in k^{1/p}$ and $\alpha_i=[(\alpha_{in})_{n\in\mathbb{N}}]$, $\alpha_{in}\in K$, where " [ ] " means the equivalence class modulo the relation given by the ultrafilter D. We obtain that the set $\delta=\{n\in\mathbb{N} \mid \sum_{i=1}^{t} a_{in}\alpha_{in}=0\}\in D$. On the other hand, the set $\delta'=\{n\in\mathbb{N} \mid \alpha_{in},\ldots,\alpha_{t,n}$ are linearly independent over $k\}\in D$. It results that for every $n\in\delta\cap\delta'\in D$ we have $a_{in}=0$ $i=1,\ldots,t$, whence all a_i are zero; therefore $\alpha_1,\ldots,\alpha_t$ are linearly independent over k^*.

Q.E.D.

The proof of the theorem 2.3 will be done in several steps.

Step 1. Reduction to the case $A_1 \longrightarrow B_1$.

Let us consider the commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ A_1 & \xrightarrow{u_1} & B_1 \end{array}$$

and suppose that the theorem 2.3 is true for u_1. Let F, G,... be as in theorem 2.3 and let $(\bar{z},\bar{y})$ be a solution in B of the system F such that the linear system $\sum_k a_{jk}T_k=G_j(\bar{z},\bar{y})$ has no solutions in B. By hypothesis, F has a solution $(\tilde{z},\tilde{y})$ in A_1 such that the system $\sum_k a_{jk}T_k=G_j(\tilde{z},\tilde{y})$ has no solutions in A_1. The ring A_1 being noetherian local complete, it is an AE-ring [8] . Thus the system $\sum_k a_{jk}T_k=G_j(\tilde{z},\tilde{y})$ has no solutions in $A_1/\underline{m}^c A_1$ for a suitable $c \geqslant 1$ ($\underline{m}$ is the maximal ideal of A). By theorem 2.5 [9], there exists a set $\delta_c \in D$ and for every $d \in \delta_c$ a solution $(z^{(d)},y^{(d)})$ for F in A such that $z^{(d)} \equiv \tilde{z}_d$, $y^{(d)} \equiv \tilde{y}_d$ mod $\underline{m}^c$. We claim that for at least $d \in \delta_c$ the system $\sum_k a_{jk}T_k=G_j(z^{(d)},y^{(d)})$ has no solutions in A. Otherwise, let $t^{(d)}$ be a solution in A of the system $\sum_k a_{jk}T_k= =G_j(z^{(d)},y^{(d)})$, $d \in \delta_c$. We consider in A_1 the elements $\tilde{\tilde{z}},\tilde{\tilde{y}},\tilde{\tilde{t}}$ defined by sequences $(z_d)_{d\in \mathbb{N}}$, $(y_d)_{d\in \mathbb{N}}$, $(t_d)_{d\in \mathbb{N}}$ as follows:

$z_d = z^{(d)}$, $y_d = y^{(d)}$, $t_d = t^{(d)}$ if $d \in \delta_c$ and 0 otherwise. Clearly, we have $F(\tilde{\tilde{z}},\tilde{\tilde{y}})=0$ and $\tilde{\tilde{t}}$ is a solution for linear system $\sum_k a_{jk}T_k = G_j(\tilde{\tilde{z}},\tilde{\tilde{y}})$. But $\tilde{\tilde{z}} \equiv \tilde{z}$, $\tilde{\tilde{y}} \equiv \tilde{y}$ mod $\underline{m}^c A_1$, and we get $\sum_k a_{jk}\tilde{\tilde{t}}_k = G_j(\tilde{t},\tilde{y})$ mod $\underline{m}^c A_1$. Contradiction !

Finally, we remark that the residue field extension $k^* \hookrightarrow K^*$ is algebraically pure because the extension $k \hookrightarrow K$ is so.

Step 2. <u>Reduction to the case when A_1 is a formal power series over a field or a discrete valuation ring with characteristic 0.</u>

With notations of the above step, we consider the commutative diagram:

$$\begin{array}{ccc} R_1[[X]] & \longrightarrow & R_1'[[X]] \\ \downarrow & & \downarrow \\ R_1[[X]]/\underline{a}R_1[[X]] \cong A_1 & \longrightarrow & B_1 = R_1'[[X]]/\underline{a}R_1'[[X]] \end{array}$$

We claim that it suffices to prove the theorem 2.3 in case of the morphism $R_1[[X]] \longrightarrow R_1'[[X]]$.

Let $a_1,\dots,a_s \in R_1[[X]]$ be generators for the ideal $\underline{a}$ and consider the systems $F^* = (F_1^*,\dots,F_m^*)$, $F_i^* := \tilde{F}_i - \sum_{j=1}^{s} a_j U_{ij} \in R_1[[X,Z]][Y,U]$ $i=1,\dots,m$ and $\sum_{k=1}^{q} \tilde{a}_{jk}T_k + \sum_{k=1}^{s} a_k V_k = \tilde{G}_j$, $j=1,\dots,p$, where $\tilde{F}_i$, $\tilde{G}_j$, $\tilde{a}_{jk}$ denote some liftings of F_i , G_j , a_{jk} in $R_1[[X]]$. Clearly, F has a solution $(\bar{z},\bar{y})$ in B_1 such that the linear system $\sum_k a_{jk}T_k = G_j(\bar{z},\bar{y})$ is incompatible in B_1 iff F^* has a solution (z',y',u') in $R_1'[[X]]$ such that the linear system

$$\sum_k \tilde{a}_{jk} T_k + \sum a_k V_{jk} = \tilde{G}_j(z', y') \qquad j=1,\dots,p$$

is incompatible in $R_1'[[X]]$. In this way, we may reduce to the case we were looking for.

Step 3. <u>Reduction to the case when $A=R_1$ is a field or a discrete valuation ring with characteristic 0, the system G being finite but F perhaps not</u>.

Now, let $F=(F_1,\dots,F_m)$, $G=(G_1,\dots,G_p)$ be systems from $R_1[[X,Z]][Y]$, where $X=(X_1,\dots,X_n)$, $Z=(Z_1,\dots,Z_M)$, $Y=(Y_1,\dots,Y_N)$. Denote

$$Z_i^* = \sum_{\alpha\in \mathbb{N}^n} Z_{i\alpha} X^\alpha \qquad i=1,\dots,M$$

$$Y_j^* = \sum_{\alpha\in \mathbb{N}^n} Y_{j\alpha} X^\alpha \qquad j=1,\dots,N$$

$$T_k^* = \sum_{\alpha\in \mathbb{N}^n} T_{k,\alpha} X^\alpha \qquad k=1,\dots,q$$

and substitute Z_i, Y_j by Z_i^*, Y_j^* into the systems of formal power series F, G, where $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$, $X^\alpha = X_1^{\alpha_1}\dots X_n^{\alpha_n}$ and $\{Z_{i,\alpha}\}$, $\{Y_{j,\alpha}\}$, $\{T_{k,\alpha}\}$ are countable sets of variables. We get

$$F_i(Z^*,Y^*) = \sum_{\beta\in\mathbb{N}^n} F_{i\beta} X^\beta \qquad i=1,\dots,m$$

$$G_j(Z^*,Y^*) = \sum_{\beta\in\mathbb{N}^n} G_{j\beta} X^\beta \qquad j=1,\dots,p$$

where $F_{i\beta}$, $G_{j\beta}\in R_1[[Z_{io}]][\{Z_{i\alpha}\}_{\alpha\neq 0}, \{Y_{jk}\}]$. We remark that for every $\beta\in\mathbb{N}^n$, the number of variables $Z_{i,\alpha}$, $Y_{j,\alpha}$, $\alpha\in\mathbb{N}^n$

which really appear in $F_{i\beta}$ or in $G_{j\beta}$ is, anyway finite. Also the elements $a_{jk} \in R_1[[X]]$ can be written in the following form:

$$a_{jk} = \sum_{\beta \in \mathbb{N}^n} a_{jk,\beta} \, X^\beta \qquad\qquad a_{jk,\beta} \in R_1$$

Clearly, the compatibility of the original system of equations F=0 is equivalent to the compatibility of the countable system of equations

$$(1) \qquad F_{i\beta} = 0 \qquad\qquad \beta \in \mathbb{N}^n \ .$$

in a countable of variables $Z_{i\alpha}$, $Y_{j\alpha}$. Also the system $\sum a_{jk} T_k = G_j$ is equivalent with the following countable system of equations

$$(2) \qquad \sum_{k=1}^{q} \sum_{\beta+\gamma=\alpha} a_{jk,\beta} \, T_{k,\gamma} = G_{j,\alpha} \qquad \alpha \in \mathbb{N}^n \ , \ j=1,\dots,p$$

which is linearly in $T_{k,\gamma}$.

If $(\bar{z},\bar{y})$ is a solution of F in $B_1 = R_1[[X]]$, and $\bar{z},\bar{y}$ have the form $\bar{z} = \sum_\alpha \bar{z}_\alpha X^\alpha$, respectively $\bar{y} = \sum_\alpha \bar{y}_\alpha X^\alpha$ ($\bar{z}_{i,0}$, $i=1,\dots,M$ are from the maximal ideal of R'_1), then $(\bar{z}_\alpha, \bar{y}_\alpha)$ is a solution of (1) in R'_1 and reciprocally. The system $\sum_k a_{jk} T_k = G_j(\bar{z},\bar{y})$, $j=1,\dots,p$ is incompatible in B_1 iff the system

$$(3) \quad \sum_{k=1}^{q} \sum_{\beta+\gamma=\alpha} a_{jk,\beta} \, T_{k,\gamma} = G_{j,\alpha}(\bar{z}_\alpha, \bar{y}_\alpha), \quad \alpha \in \mathbb{N}^n, \ j=1,\dots,p$$

is incompatible in R'_1 and this holds iff there exists a finite subsystem of (3) which is still incompatible in R'_1, as says the following lemma:

(4.2) Lemma. Let $F=(F_n(Z_i,Y_j)_{n\in\mathbb{N}}$ be a countable system of formal power series from $R_1[[Z]][Y]$, $Z=(Z_i)_{i\in\mathbb{N}}$, $Y=(Y_j)_{j\in\mathbb{N}}$ such that every F_n depends only on finite variables. Then F has solutions in R_1 iff every its finite subsystem has too.

Proof. The system F=0 is equivalent to the countable system of congruences:

$$(4) \quad F_n\equiv 0 \mod p^c R_1 \qquad c,n\in\mathbb{N}$$

p being a local parameter in R_1. Let $F_{n,c}$ be a polynom from $R[Z,Y]$ such that $F_{n,c}\equiv F_n \mod Z^\alpha$ with $|\alpha|\geq c$ ($|\alpha| = \sum \alpha_i$), The system (4) is equivalent with the following system of polynomial congruences:

$$F_{n,c}\equiv 0 \mod p^c R_1 \qquad n,c\in\mathbb{N}$$

$$Z \equiv 0 \mod pR_1$$

Lifting $F_{n,c}$ to R^* we get the system

$$\tilde{F}_{n,c}\equiv 0 \mod p^c R^*$$

$$Z \equiv 0 \mod pR^*$$

which gives a countable system P of polynomial equations over R^*, writing congruences as equalities. BY lemma 2.17 [9], we deduce that the compatibility of F is equivalent to the compatibility of every finite subsystem of P.

Q.E.D. for lemma 4.2.

Step 4. <u>Incompatibility of (2) is equivalent with compatibility of a finite system of polynomial equations, which can be added to F.</u>

The incompatibility of (2) is equivalent to the incompatibility of a finite system of the type

$$(5) \quad \sum a_{jk} T_k = G_j(\bar{z},\bar{y})$$

by step 3. Let r be the rang of $\| a_{jk} \|$ and Δ a nonzero rxr-minor of minimal valuation (if R_1 is not a field). There exists two kinds of incompatibility for (5):

a) (5) is incompatible in the fraction field of R_1'. This happens when there exists a nonzero (r+1) x (r+1)-minor $H(\bar{z},\bar{y})$ of the matrix $\| a_{jk} | G_j(\bar{z},\bar{y}) \|$. Let s=ord $H(\bar{z},\bar{y})$. Adding to F the polynomials $H(Z,Y)-p^s U$, $UU'-1$ (U,U′ are new variables), we settle this case.

b) (5) is compatible in the fraction field of R_1' but incompatible in R_1'. Clearly (5) is equivalent with a system of the following form:

$$(6) \quad \Delta T_i + \sum_{j=r+1}^{\ell} \Delta_{ij} T_j = G_i^*(\bar{z},\bar{y}) \;, \qquad i=1,\ldots,r$$

where $\Delta_{ij} \in R$, $G_{ij}^* \in R[[Z]][Y]$. Remark that $\Delta \mid \Delta_{ij}$ (Δ has the minimal valuation) and thus (6) is incompatible iff there exists $i_o \in \{1,\ldots,r\}$ such that $ord(\Delta) > ord(G_{i_o}^*(\bar{z},\bar{y}))$. Let s=ord $(G_{i_o}^*(\bar{z},\bar{y}))$. Adding to F the polynomials

$$G_{i_o}^*(Z,Y)-p^s V \;, \qquad VV'-1$$

V, V' being variables, we settle also this case.

After Step 4, the proof of (2.3) is a consequence of (4.2) and of the following lemma (the case R_1=field is already over):

(4.3) Lemma. Let $R \hookrightarrow R'$ be an unramified extension of complete discrete valuation rings of characteristic 0 such that the residue field k' of R' is a separable extension of the residue field k of R. Then the following statements are equivalent:

i) the morphism $u: R \longrightarrow R'$ is analytically pure.

ii) u is algebraically pure.

iii) The residue field morphism $k \longrightarrow k'$ is algebraically pure.

Proof. Using (1.5) it is enough to prove iii) $\Rightarrow$ ii). Let $F=(F_1,\dots,F_m)$ be a system of polynomial equations from $R[Y]$, $Y=(Y_1,\dots,Y_N)$ and $\bar{y}$ a solution of F in R'. Let q be the kernel of the map $R[Y] \longrightarrow R'$ given by $P \rightsquigarrow P(\bar{y})$ and denote $r=ht(q)$. Adding some polynomials to F we may suppose that F generates q. R being of characteristic zero, the extension $Q(R) \hookrightarrow Q(R')$ is separable. By jacobian criterion, the matrix $(\frac{\partial F}{\partial Y})$ has a $r \times r$-minor M, which is not in q, let us put $M=\det (\frac{\partial F_i}{\partial Y_i})_{i,j=1,\dots,r}$. Thus

$M(\bar{y}) \neq 0$ and using Neron's p-desingularization we may suppose that $M(\bar{y})$ is invertible in R'. Remark that $\bar{y}$ induces a solution for the system

$$F(Y)=0\ , \qquad M(Y).U=1$$

in k'. Then by iii) it has one in k. Thus there exists $\tilde{y} \in R^N$ such that $F(\tilde{y}) \equiv 0 \bmod p$ and $M(\tilde{y}) \not\equiv 0 \bmod p$. By the Implicit Function Theorem there exists a solution y of $F_1,\dots,F_r$ in R such that $y \equiv \tilde{y} \bmod p$.

It remains to show that y is a solution for the whole system F. Let $\sqrt{(F_1,\ldots,F_r)}=q_1\cap\ldots\cap q_t$ be the reduced primary decomposition of $\sqrt{(F_1,\ldots,F_r)}$, q_i being prime ideals in $R[Y]$. As $q=(F)\supset(F_1,\ldots,F_r)$, q must contain an ideal q_i, let us put $q\supset q_1$. It results $q_1R[Y]_q=qR[Y]_q=(F_1,\ldots,F_r)R[Y]_q$ and we get $q_1=q$. If t=1 then clearly y is a solution for the whole F. If $t>1$ then $M\in\sqrt{q+\underline{a}}$, where $\underline{a}=\bigcap_{i=q}^{t} q_i$. (If $\underline{b}\supset q+\underline{a}$ is a prime ideal which does not contain M, then $C:=(R[Y]/_{(F_1,\ldots,F_r)})_{\underline{b}}$ is not integral domain. This is a contradiction because the morphism $R\to C$ is a smooth one). Thus there exists an integer $d\geq 1$ such that $M^d=M_1+M_2$ with $M_1\in q$ and $M_2\in\underline{a}$. As $q(y)=0$ and $M(y)\neq 0$ it results $M_2(y)\neq 0$. Consequently $F(y)=q(y)=0$ because $M_2q\subset\sqrt{(F_1,\ldots,F_r)}$.

Q.E.D.

(4.5) Remark. We see from the proof of theorem (2.3) that this theorem remains true if we consider more "incompatible" systems of the type $\sum a_{jk}T_k=G_j$. Also the coefficients of the above linear system can be polynomials from $R[[Z]][Y]$ and not only elements from R.

REFERENCES

1. M.Artin, Algebraic approximation of structures over complete local rings, Publ.Math.Inst.Hautes Etudes Sci., 1969, 36, 23-58.
2. J.Becker, J.Denef, L.Lipschitz, and L. van der Dries, Ultraproducts and approximation in local rings I. Inventiones Math., 1979, 51, 189-203.
3. A.Grothendieck and J.Dieudonné, Eléments de Géométrie Algébrique, IV, IHES (1964).
4. M.Hochster, Some applications of the Frobenius in characteristic zero, Bull.Amer.Math.Soc. 84, no.5 (1978), 886-912.
5. H.Kurke, T.Mostowski, G.Pfister, D.Popescu and M.Roczen, Die Approximationseigenschaft lokaler Ringe. Lecture Notes in Math. 634, Springer-Verlag, Berlin, 1978.
6. D.Lazard, Autour de la platitude, Bull.Soc.Math.France, 1969 97, 81-128.
7. H.Matsumura, Commutative Algebra, W.A. Benjamin Co. New York (1970).
8. G.Pfister and D.Popescu, Die strenge Approximationseigenschaft lokaler Ringe, Inventiones Math., 1975, 30, 145-174.
9. D.Popescu, Algebraically pure morphisms, Rev.Roum.Math.Pures et Appl., XXIV, 6 (1979), 947-977.