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by

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ON THREE DIMENSIONAL LOCAL RINGS WITH THE PROPERTY OF APPROXIMATION

G.Pfister and D.Popescu

Let (A,m) be a local noetherian ring (all the rings are supposed to be commutative with identity). A is called a ring with the property of approximation (shortly $A\in AE$) if the following holds (cf. [4], [10]):

" Let $f=(f_1,\ldots,f_m)$ be an arbitrary system of polynomials in some variables $Y=(Y_1,\ldots,Y_N)$ with coefficients in A. Then every solution $\bar{y}$ of f in $\hat{A}$ ($\hat{A}$ denotes the completion of A) can be well approximated in the $\underline{m}$-adic topology) by a solution of f in A, i.e. for every positive integer c there exists a solution y of f in A such that $y\equiv\bar{y} \mod \underline{m}^c A$". Clearly, the noetherian local complete rings are trivial examples of AE-rings. More general, we call an extension of rings $A\hookrightarrow B$ algebraically pure if every system of polynomials with coefficients in A has a solution in B iff it has one in A (cf [12]). It is easy to see that $A\in AE$ iff the extension $A\hookrightarrow\hat{A}$ is algebraically pure.

Which rings with the property of approximation do we know?

1) Let R be a field or an excellent henselian discrete valuation ring, then the ring of algebraic power series $R\langle T\rangle, T=(T_1,\ldots,T_n)$, i.e the henselization of $R[T]_{(T)}$, is an AE-ring as M.Artin proved (cf. [4]). The case n=0 was already investigated by M.J.Greenberg [5].

2) Let R be a valued field of characteristic 0, or a complete valued field in characteristic p>0, then the ring $R\{T\}$,

$T=(T_1,\dots,T_r)$ of convergent power series with coefficients in R is an AE=ring, as M.Artin, M.André and others proved (see [3], [1], [13], [7], [9]):

3) A one dimensional local, noetherian, reduced ring is an AE-ring iff it is henselian and universally japanese (this is an easy consequence of R.Elkik's theorem [6]).

4) A two dimensional local regular ring is an AE-ring iff it is henselian and universally japanese (cf [9], [11]), (conversely all AE-rings are henselian and universally japanese [4], [8]).

Remark that in all these examples of AE-rings with dimension $\geqslant 3$ the Weierstrass Preparation Theorem holds.

In [2], M.Artin put the following question:

i) Let R be a complete discrete valuation ring and $X=(X_1,\dots,X_n)$, $T=(T_1,\dots,T_s)$ some variables.

Does $A:=R[[X]]\langle T\rangle$ have the property of approximation?

In [8], a positive answer to i) is given, but the proof is wrong. We see that for A the Weierstrass Preparation Theorem does not hold, if $n,s\geqslant 1$. A has also not "enough" automorphisms, i.e. a formal power series $\not\equiv 0$ mod p (p denotes a local parameter in R) cannot be regularized by an automorphism of A (as it happens in $\hat{A}$). These facts make i) difficult. However the case n=0 of i) is already known (see 1)) and clearly it is enough to prove i) for s=1 and all $n\geqslant 1$.

A positive answer of i) would give some interesting examples of AE-rings in dimension 3 but first of all it would yield some nice applications in deformation theory, based on the following consequence of i):

" Let K be a field and $f=(f_1,\dots,f_m)$ an arbitrary system of polynomials with coefficients in $K\langle X\rangle$, $X=(X_1,\dots X_s)$. Suppose f has a formal solution $\bar{y}=(\bar{y}_1,\ldots,\bar{y}_N)$ such that $\bar{y}_i \in K[[X_1,\ldots,X_{n_i}]]$, where the natural numbers n_i satisfy $1 \leq n_1 \leq \ldots \leq n_N \leq n$. Then y can be well approximated by solutions of f in $K\langle X\rangle$ having the same property ".

For K being an algebraically closed field the above result was already obtained by T.Mostowski [9] using some other methods.

The aim of this paper is to show that $R[[X]]\langle T\rangle$, n, s=1 is an AE-ring for an arbitrary complete discrete valuation ring R. This gives us many interesting examples of AE-rings in dimension 3: let A be a two dimensional AE-ring which is supposed to be a domain in unequal characteristic case or arbitrary in equal characteristic case, then $A\langle T\rangle$ is also an AE-ring.

§ 1. Rings with the property of approximation

(1.1) Theorem. Let $(A,\underline{m})$ be an one dimensional local noetherian ring, Then A is an AE-ring iff A is henselian and universally japanese.

The proof is given in section 2.

(1.2) Remark. The equivalence stated by (1.1) does not hold for three dimensional local rings. Indeed, if we consider A to be the henselization of the ring R constructed by C. Rotthaus (see § 1 [14]), then $A/\omega_c A$ is an integral domain but $\hat{A}/\omega_c \hat{A}$ is not (see § 4 [14]). Thus A is not an AE-ring (see [8]).

Let R be a complete discrete valuation ring and X,T some variables.

(1.3) Theorem. $R[[X]]\langle T\rangle$ is an AE-ring.

The proof is given in section 3.

(1.4) <u>Corollary.</u> If A is a two dimensional AE-ring which is supposed to be an integral domain in unequal characteristic case, then $A\langle T\rangle$ is AE-ring too ($A\langle T\rangle$ denotes the henselization of the local ring $A[T]_{(T)}$).

<u>Proof.</u> First we consider the case in which A is a local complete ring. Then, by the Cohen Structure Theorem, A is a finite extension of a local complete regular ring B of dimension two.

By (1.3), B T is an AE-ring and thus $A\langle T\rangle$ is also AE-ring, since it is a finite extension of $B\langle T\rangle$ (apply (1.2) chapter II from [9]).

Now, if A is an AE-ring, then the morphism $A \hookrightarrow \hat{A}$ is algebraically pure. Thus the morphism $A\langle T\rangle \hookrightarrow \hat{A}[[T]]$ is still algebraically pure by corollary 1.12 [12]. As $\hat{A}\langle T\rangle$ is an AE-ring (see above), the morphism $\hat{A}\langle T\rangle \hookrightarrow \hat{A}[[T]]$ is still algebraically pure. Consequently, $A\langle T\rangle \hookrightarrow \hat{A}[[T]]$ is algebraically pure and thus $A\langle T\rangle$ is an AE-ring.

Q.E.D.

(1.5) <u>Corollary.</u> If A is a two dimensional noetherian locally complete domain (or more general an AE-domain), then $A\langle T\rangle$ is factorial iff $A[[T]]$ is also factorial.

<u>Proof.</u> By (1.4) $A\langle T\rangle$ is an AE-ring and it is enough to apply (5.7) chapter V [9].

(1.6) <u>Remark.</u> If A is a two dimensional AE-ring which is supposed to be domain in unequal characteristic case, then the following statements hold:

1) Every prime ideal $q\subset A\langle T\rangle$ extends to a prime ideal $qA[[T]]$.

2) A prime ideal $q\subset A\langle T\rangle$ is regular iff $qA[[T]]$ is a regular prime ideal.

3) Every primary decomposition of an ideal $\underline{a}\subset A\langle T\rangle$, $\underline{a}=q_1\cap\ldots\cap q_s$ having $p_i=\sqrt{q_i}$ as associated prime ideals, extends to a primary decomposition $\underline{a}A[[T]] = q_1 A[[T]]\cap\ldots\cap q_sA[[T]]$ having $p_iA[[T]]=\sqrt{q_iA[[T]]}$ as associated prime ideals.

4) $A\langle T\rangle$ is an universally catenary ring.

For the proof we apply (5.1), (5.2), (5.5), chapter V [9].

(1.7) Remark. With the same methods used in the proof of theorem 1.3 one can also prove that for any one dimensional local AE-ring A, which is supposed to be a domain in unequal characteristic case, $A\langle T\rangle$ has also the property of approximation.

(1.8) Theorem: Let K be a field and X,Z,T variables, then $K\langle T\rangle[[X,Z]]$ is an AE-ring. If K is a valued field of characteristic zero or a complete valued field of characteristic $p>0$, then $K\{T\}[[X,Y]]$ is also an AE-ring.
The proof is similar to the proof of (1.3) (cf. remark (3.3)).

§ 2. Proof of theorem (1.1)

Let $B=A[Y]/_{(f)}$ be an A-algebra of finite type.
The set of prime ideals $q\in\text{Spec } B$ such that the morphism

$A \longrightarrow B_q$ is not smooth form a closed set defined by an ideal H_f. By a result of R.Elkik (see [6]), there exists a function $d: \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{N}$ with the following property:

" For every $y\in A^N$ such that $f(y)\equiv 0 \bmod \underline{m}^{d(s,c)}$ and $H_f(y)\ni \underline{m}^s$, there exists a solution $\tilde{y}$ of f in A such that $\tilde{y}\equiv y \bmod \underline{m}^c$".

Now, let $f=(f_1,\ldots,f_m)$ be an arbitrary system of polynomials from $A[Y]$, $Y=(Y_1,\ldots,Y_N)$ and $\bar{y}\in \hat{A}^N$ a "formal" solution of f. Adding some polynomials to f we may suppose that f generates the kernel of the map $\nabla : A[Y]\longrightarrow \hat{A}$ given by $P \rightsquigarrow P(\bar{y})$. We consider the ideal $H_f(\bar{y})$ generated in $\hat{A}$ by elements of the form $P(\bar{y})$, $P\in H_f$. We have the following cases:

Case 1) ht $H_f(y)=1$ or $H_f(y)=A$.

Case 2) ht $H_f(y)=0$.

Case 1. If $H_f(\bar{y})$ is a $\underline{m}$-primary ideal let us put $H_f(\bar{y})\supset \underline{m}^s\hat{A}$. Denote $t=\max\{d(s,c),2s\}$ and choose $y\in A^N$ such that $y\equiv\bar{y} \bmod \underline{m}^t\hat{A}$. By Taylor's formula, we get $f(y)\equiv 0 \bmod \underline{m}^{d(s,c)}$ and $\underline{m}^s\hat{A}\subset H_f(\bar{y})\subset H_f(y).\hat{A} + \underline{m}^{2s}\hat{A}$. It results $\underline{m}^s\hat{A}\subset H_f(y).\hat{A}$ and thus $\underline{m}^s\subset H_f(y)$. Consequently, there exists a solution $\tilde{y}\in A^N$ of f such that $y\equiv\tilde{y} \bmod \underline{m}^c$. If $H_f(\bar{y})=\hat{A}$ then the morphism $A\longrightarrow A[Y]/_{(f)}$ is smooth and we may apply the Implicit Function Theorem.

Case 2. We shall use the following lemma to reduce this case to the first one (case 2 can only appear if A is not reduced).

(2.1) Lemma. Let $B\subset\hat{A}$ be an A-algebra of finite type. Then there is a B-algebra $B'\subset\hat{A}$ of finite type such that B'_p is a smooth A_p- algebra for all minimal primes $p\in \operatorname{Spec} A$.

Apply (2.1) to our situation $B:=\operatorname{Im}\varphi \simeq A[Y]/_{(f)}$, the isomorphism being induced by φ.
Then there exists a B-algebra $B' \subset \hat{A}$ of finite type; let us put $B' \simeq A[Y,X]/_{(g)}$, $X=(X_1,\dots,X_r)$, $g=(g_1,\dots,g_s)$, the isomorphism being given by $Y \rightsquigarrow \bar{y}$, $X \rightsquigarrow \bar{x}$, $\bar{x}=(\bar{x}_1,\dots,\bar{x}_r)\in \hat{A}^r$, such that $\operatorname{ht} H_g(\bar{y},\bar{x})=1$ or $H_g(\bar{y},\bar{x})=A$.

Now as in first case we get a solution $(y,x)\in A^{N+r}$ of $g=0$. In particular, y is a solution of $f=0$.

<u>Proof of Lemma</u> (2.1). Let M be the set of minimal primes of A and $S=A\setminus\bigcup_{p\in M} p$. The inclusions $S^{-1}A \hookrightarrow S^{-1}B \hookrightarrow S^{-1}\hat{A}$ split into a product induced by the canonical maps

$$\prod_{p\in M} A_p \hookrightarrow \prod_{p\in M} B\otimes_A A_p \hookrightarrow \prod_{p\in M} \hat{A}\otimes_A A_p$$

(Remark that $\prod_{p\in M} \hat{A}\otimes_A A_p \simeq \prod_{p\in M} \hat{A}_{p\hat{A}}$).

Now we see, that it is sufficient to prove that $\prod_{p\in M} \hat{A}_{p\hat{A}}$ is a filtered inductive limit of smooth finite type $\prod_{p\, M} A_p$-algebras or equivalently to prove that $\hat{A}_{p\hat{A}}$ is a filtered inductive limit of smooth finite type A_p-algebras for all $p\in M$. Indeed, then there exists a smooth $S^{-1}A$-algebra $\tilde{B}\subseteq S^{-1}\hat{A}$ of finite type which contains $S^{-1}B$ and we may choose $B'\subset\hat{A}$ to be a B-algebra of finite type such that

$S^{-1}B' \simeq \tilde{B}$.

Finally, it remains to prove that $\hat{A}_{p\hat{A}}$ is a filtered inductive limit of smooth finite type A_p-algebras, $p\in M$. Let $k\hookrightarrow K$ be the residue field extension of $A_p \longrightarrow \hat{A}_{p\hat{A}}$ and $K' \subset K$ a finitely generated k-extension. K' is a separable extension of k (A universally japanese implies K/k separable). Choose $x=(x_1,\dots,x_t) \in K'^t$ algebraically independent over k and $y \in K'$ algebraically separable over k(x) such that $K'=k(x,y)$. Let $X=(X_1,\dots,X_t)$, Y be variables and $F(Y) \in A_p[X,Y]$ be a polynomial, which lifts the irreducible polynomial of y over k(x). Let $(\tilde{x},\tilde{y})$ be a lifting of (x,y) to $\hat{A}_{p\hat{A}}$. We have $F(\tilde{x},\tilde{y}) \in p\hat{A}$ and $\frac{\partial F}{\partial Y}(\tilde{x},\tilde{y}) \notin p\hat{A}$. $\hat{A}_{p\hat{A}}$ being local artinian, there exists $y' \in \hat{A}_{p\hat{A}}$ such that $F(\tilde{x},y')=0$ and $y' \equiv \tilde{y}$ mod $p\hat{A}$. Let q be the kernel of the map $\tau : A_p[X,Y] \longrightarrow \hat{A}_{p\hat{A}}$ given by $P \rightsquigarrow P(\tilde{x},y')$. It results $ht(q) = 1$ because $q \subset (F) + pA_p[X,Y]$ and so $C_{K'} = A_p[X,Y]/_q$ is a smooth A_p-algebra of finite type.

Now we remark that $\hat{A}_{p\hat{A}} = \bigcup_{\substack{K' \subset K \\ K'/k \text{ separable}}} (C_{K'})_{pC_{K'}}$ and so $\hat{A}_{p\hat{A}}$ is a filtered inductive limit of A_p-algebras of the type $(C_{K'})_g$, $g \notin pC_{K'}$.

§ 3. Proof of theorem (1.3)

Let $F=(F_1,\dots,F_m)$ be a system of m-polynomials in variables $Y=(Y_1,\dots,Y_N)$ with coefficients in $A := R[[X]]\langle T\rangle$ such that it has a solution $\bar{y}=(\bar{y}_1,\dots,\bar{y}_N)$ in $\hat{A}=R[[X,T]]$.

Adding (perhaps) some polynomials to F we may suppose that F generates the Kernel q of the morphism ∇: $A[Y] \to \hat{A}$ given by $P \rightsquigarrow P(\bar{y})$. Denote $r=ht(q)$. If $g=(g_1,\dots,g_s)$ is a system of s-polynomials from q, then we consider the ideal $\Delta(g,\bar{y}) \subset \hat{A}$ generated by all r x r - minors of the matrix $(\frac{\partial g}{\partial Y}(\bar{y}))$; sometimes we denote $\Delta(g,\bar{y})$ by Δ_g.

We shall prove in some steps that F has a solution in A.

Step 1. Desingularization step. Reduction to the case $ht(\Delta(F,\bar{y})) \geq 2$

In order to get this reduction it is enough to apply the following lemma, which is in fact proposition 3 from [11].

(3.1) Lemma. Let A, A', $A \subset A'$ be noetherian factorial rings such that every prime element t from A remains prime in A' and the extension $Q(A/_{(t)}) \hookrightarrow Q(A'/_{tA'})$ is separable. Suppose A' to be local complete and the extension $Q(A) \subset Q(A')$ to be separable. Then every morphism $\sigma: A[Y] \longrightarrow A'$, $Y=(Y_1,\dots,Y_N)$ can be extended to a morphism $\sigma': A[Y,U] \longrightarrow A'$, $U=(U_1,\dots,U_{N'})$ such that $ht(\Delta_{\sigma'}) \geq 2$.

Indeed, we may apply (3.1) for $A'=\hat{A}$, the hypothesis of (3.1) being fulfilled because A is an excellent henselian ring.

Step 2. Reduction to the case in which q contains r-polynomials $g=(g_1,\dots,g_r)$ such that $ht(\Delta(g,y)) \geq 2$

Let $U=(U_{ij})_{i=1,\dots,r;\ j=1,\dots,m}$ be some variables and $G_i = \sum_{j=1}^{m} U_{ij} F_j$ polynomials from $A\langle U\rangle[Y]$. As before, let $\Delta(G,\bar{y}) \subset \hat{A}\langle U\rangle$ be the ideal generated by all $r \times r$-minors of the matrix $(\frac{\partial G}{\partial Y}(\bar{y}))$. We maintain $ht(\Delta(G,\bar{y})) \geq 2$. Indeed, we have $(\frac{\partial G}{\partial Y}(\bar{y})) = (U_{ij}) \cdot (\frac{\partial F}{\partial Y}(\bar{y}))$ and thus a $r \times r$-minor of $(\frac{\partial G}{\partial Y}(\bar{y}))$ is a polynomial in U (of degree ≤ 1 in every variable U_{ij}) having minors of $(\frac{\partial F}{\partial Y}(\bar{y}))$ as coefficients. Clearly, there exists no common divisor of the $r \times r$-minors of $(\frac{\partial G}{\partial Y}(\bar{y}))$ because otherwise the $r \times r$-minors of $(\frac{\partial F}{\partial Y}(\bar{y}))$ would have a common divisor in $\hat{A}$ which is a contradiction ($ht(\Delta(F,\bar{y})) \geq 2$). Consequently, $ht(\Delta(G,\bar{y})) \geq 2$. Now, if there exist $u=(u_{ij})$ some non-units in A such that the map $\tau_u: \hat{A}\langle U\rangle \longrightarrow \hat{A}$ given by $P \rightsquigarrow P(u)$ satisfies $ht(\tau_u(\Delta(G,\bar{y})) \geq 2$, then $g_i = G_i(u)$, $i=1,\dots,r$ are from $A[Y]$ ($u \in A^{r \times N}$) in fact from q and $\Delta(g,\bar{y}) = \tau_u(\Delta(G,\bar{y}))$. So it is sufficient to prove the following lemma:

(3.2) Lemma. Let A be a local regular ring of dimension ≥ 2 and $\hat{A}\langle U\rangle$ the ring of algebraic power series in variables $U=(U_1,\ldots,U_n)$ with coefficients in the completion $\hat{A}$ of A. Let $\underline{a}\subset\hat{A}\langle U\rangle$ be an ideal with height ≥ 2. Then there exist some non-unit elements $u=(u_1,\ldots,u_n)$ in A such that the map $\tau_u:\hat{A}\langle U\rangle\longrightarrow\hat{A}$ given by $P\rightsquigarrow P(u)$ yields $ht(\tau_u(\underline{a}))\geq 2$.

Proof. Using proposition 4 from [11] there exist some non-unit elements $\bar{u}=(\bar{u}_1,\ldots,\bar{u}_n)$ in $\hat{A}$ such that the map $\lambda_{\bar{u}}:\hat{A}[[U]]\longrightarrow\hat{A}$ given by $S\rightsquigarrow S(\bar{u})$ yields $ht(\lambda_{\bar{u}}(\underline{a}\,A[[U]]))\geq 2$. Thus, the map $\tau_{\bar{u}}=\lambda_{\bar{u}}|_{\hat{A}\langle U\rangle}$ yields $ht(\tau_u(\underline{a}))\geq 2$. Consequently, if $h=(h_1,\ldots,h_s)$ is a system of generators for $\underline{a}$, then the system

$$(*)\quad h_i(\bar{u})=Z.\,Y_i\ ,\quad i=1,\ldots,r$$

has no solutions in $\hat{A}$; Z,Y_i being variables. Using theorem 2.5 [10] (or theorem 1.4 chapter [9]), there exists a natural number $c\in\mathbb{N}$ such that the system $(\bar{*})$ has also no solutions in $\hat{A}/\underline{m}^c\hat{A}$, where $\underline{m}$ is the maximal ideal of A. Choose $u\in\underline{m}.A^n$ such that $u\equiv\bar{u}$ mod $\underline{m}^cA$. Then the system

$$(*)\quad h_i(u)=Z.\,Y_i,\quad i=1,\ldots,r$$

has still no solutions in $\hat{A}$ (Otherwise $(*)$ has solutions in $\hat{A}/\underline{m}^c\hat{A}$ and so $(\bar{*})$ has solutions in $\hat{A}/\underline{m}^c\hat{A}$. Contradiction!).
Thus the map $\tau_u:\hat{A}\langle U\rangle\longrightarrow\hat{A}$ given by $S\rightsquigarrow S(u)$ yields $ht(\tau_u(\underline{a}))\geq 2$.

Step 3. <u>There exists an ideal $\underline{a} \subset A$ such that $ht(\underline{a})=2$ and $\Delta(g,\bar{y}) \supset \underline{a}\,\hat{A}$.</u>

Clearly, it is enough to show that for every prime ideal $\underline{b} \subset A$ of height two there exists an ideal $\underline{a} \subset A$ such that ht $\underline{a}=2$ and $\underline{b} \supset \underline{a}\,\hat{A}$.

Now, let $\underline{b} \subset \hat{A}$ be a prime ideal of height two. If $\underline{b} \subset (p,X).\hat{A}$, then $\underline{b} = (p,X)\hat{A}$ and we take $\underline{a}=(p,X)$, p being a local parameter of R. If $\underline{b} \not\subset (p,X)\hat{A}$, then $\underline{a}$ contains a T-regular power series h. Thus the canonical map $R[[X]] \to \hat{A}/_{(h)}$ is finite by Weierstrass Preparation Theorem and $\underline{a}$ must have a nonzero intersection with $R[[X]]$ because of $ht(\underline{b}/_{(h)})=1$. Let $\tilde{h}$ be the monic polynomial from $R[[X]][T]$ which is a multiple of h. We may take for $\underline{a}$ the ideal generated in A by $\tilde{h}$ and $\underline{b}\, R[[X]]$.

Step 4. <u>Preparation for Newton lemma</u>

Let $c \in \mathbb{N}$. In this step we shall prove the existence of some $y \in A^N$ such that $g(y) \in (p,X)^c \Delta^2(g,y)$ and $y \equiv \bar{y} \bmod (p,X,T)^c.\hat{A}$. Let $\underline{a} \subset A$ be an ideal of height two such that $\Delta(g,\bar{y}) \supset \underline{a}\hat{A}$. Let $v=(v_1,\ldots,v_t)$ be a system of generators of $\underline{a}$, let us consider $v_i = \sum_{j=1}^{m} M_j(\bar{y}).\bar{u}_{ij}$, $i=1,\ldots,t$ where $\bar{u}_{ij} \in \hat{A}$ and $M_1,\ldots,M_s$ are the r x r-minors of $(\frac{\partial g}{\partial y})$. We remark that $A/_{X^c.\underline{a}^2}$ is an AE-ring (cf. (1.1)).

Now, consider the following system of equations over A:

$$(+)\quad g = 0$$
$$\sum_{j=1}^{s} M_j(Y).U_{ij} = v_i \quad , i=1,\ldots,t$$

where $U=(U_{ij})$ are some variables. The pair $(\bar{y},\bar{u})$ induces a solution of (+) in $\hat{A}/X^c\underline{a}^2\hat{A}$, which can be approximated modulo $(p,X,T)^c$ by a solution of (+) from $A/X^c\underline{a}^2$, i.e. there exists (y,u) in A such that $g(y)\equiv 0$, $\sum_{j=1}^{s} M_j(y)\, u_{ij}\equiv v_i \mod X^c\underline{a}^2$ and $y\equiv\bar{y}$, $u\equiv\bar{u} \mod (X,Z,T)^c\hat{A}$. Consequently, there exist $d_{ik}\in X^c\underline{a}$, $i,k=1,\dots,t$ such that $\sum_{j=1}^{s} M_j(y)\, u_{ij}=v_i+\sum d_{ik}\, v_k$ and so $v_i\in\Delta(g,y)$ for all $i=1,\dots,t$. Thus $g(y)\in X^c\Delta^2(g,y)$.

Step 5. Newton lemma

Applying the Newton lemma we get a solution $\tilde{y}\in A^N$ for g such that $\tilde{y}\equiv y \mod X^c\Delta(g,y)$.

Step 6. The solutions of g obtained for c sufficiently large are solutions for the whole q

Let q_i $i=1,\dots,t$ be the minimal prime ideals of (g). Clearly q is one of these, let us say $q=q_1$. Let $\tilde{\tau}: A[Y]\longrightarrow A$ be the map given by $P\rightsquigarrow P(\tilde{y})$. We have $\operatorname{Ker}\tilde{\tau}\supset\sqrt{(g)}$ and thus we are ready if $t=1$. If $t>1$, then consider a polynom $S\in\bigcap_{i=2}^{t} q_i$, which is not in q. Thus $S(\bar{y})\neq 0$ and there exists a natural number c' such that $S(\bar{y})\not\equiv 0 \mod (p,X\,T)^{c'}A$.

If $c>c'$, then we get $S(\tilde{y})\not\equiv 0 \mod (p,X\,T)^{c'}$ by Taylor's formula and so $S\notin\operatorname{Ker}\tilde{\tau}$. It results that $q\subset\operatorname{Ker}\tilde{\tau}$, the second ideal being prime.

(3.3) Remark. One can prove theorem 1.8 with the same methods used in the proof of 1.3. Questions could only arise in Step 3, but also here one can use the same idea because

$K\{T\}[[X,Z]]\supset K\langle T\rangle[[X,Z]]\supset K[[X,Z]]\langle T\rangle$.

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