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by

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§ 0. Preliminaries

In this paper all rings are supposed to be commutative with 1-element and containing the field $\mathbb{Q}$ of the rational numbers. A differential ring will mean a ring A together with m derivations $D_1,..,D_m \in \mathrm{Der}(A,A)$. If $a = (a_1,..,a_m) \in \mathbb{N}^m$ and $x \in A$ we shall write $x^{(a)}$ instead of $D_1^{a_1}..D_m^{a_m}x$. If $m = 1$ A will be called an ordinary differential ring. An ideal $L \subseteq A$ is said to be a differential ideal iff $D_i L \subseteq L$ for all $i=1,..,m$. If L is an ideal in A we denote by $[L]$ the smallest differential ideal which contains L and we put $\{L\} = \sqrt{[L]}$ which is also differential. If L_1, L_2 are ideals in A then $\{L_1L_2\} = \{L_1\} \cap \{L_2\}$. Let us consider

$$\mathrm{Sp}\,A = \{P \in \mathrm{Spec}\,A \mid P \text{ is differential}\}$$

together with the topology induced by Spec A. One knows that :

(0.1) Sp A is a dense subspace in Spec A

We say that A is differentially noetherian iff Sp A is a noetherian topological space. If A is a differential ring we may consider the ring of differential polynomials

$$A\{Y\} = A[Y_a, a \in \mathbb{N}^m]$$

with the derivation rule $Y_a^{(b)} = Y_{a+b}$ for all $b \in \mathbb{N}^m$.

We define $A\{Y_1,..,Y_n\} = A\{Y_1,..,Y_{n-1}\}\{Y_n\}$. Then :

(0.2) Theorem ([7]) If A is differentially noetherian then $A\{Y_1,..,Y_n\}$ is also differentially noetherian.

A morphism of differential rings will mean a morphism of rings $u: A \longrightarrow B$ which commute with each of the m derivations. We say that u is differentially of finite type iff u is of the form $u: A \xrightarrow{v} A\{Y_1,..,Y_n\} \xrightarrow{w} B$ with w surjective. We write then $B = A\{y_1,..,y_n\}$, $y_i = w(Y_i)$.

An extension of differential fields $K \subset \Omega$ is said to be differentially of finite type iff there exist $y_1,..,y_n \in \Omega$ such that Ω is the smallest subfield of Ω which contains K and $y_1,..,y_n$. The extension $K \subset \Omega$ is called universal iff for all fields K_1, K_2

$$\begin{array}{ccccc} K & \xrightarrow{u} & K_1 & \xrightarrow{v} & \Omega \\ & & {\scriptstyle w}\ \cap & \nearrow {\scriptstyle f} & \\ & & K_2 & & \end{array}$$

such that u and w are differentially of finite type extensions of fields, there exists a morphism $f : K_2 \to \Omega$ such that $fw = v$.

(0.3) Theorem ([7]) Every differential field has a universal extension.

A field K is called universal if it is a universal extension of $\mathbb{Q}$. Let K be a universal field. For all ideal $\mathbb{L}$ in $K\{Y_1,..,Y_n\}$ put $V(L) = \{\eta \in K^n \mid F(\eta)=0 \text{ for all } F \in L\}$ and for all $X \subseteq K^n$ put $I(X) = \{F \in K\{Y_1,..,Y_n\} \mid F(X)=0\}$ The sets of the form $V(I)$ will be considered closed in K^n and so we have a topology on K^n.

(0.4) Theorem.([7]) $I(V(L)) = \{L\}$ for all ideal L in $K\{Y_1,..,Y_n\}$ and $V(I(X)) = \overline{X}$ for all subset X of K^n.

Let A be a differential ring and $B = A\{Y_1,..,Y_n\}$. We put $Y_i^{(a)} \leq Y_j^{(b)}$ iff $(i,|a|,a) \leq (j,|b|,b)$ in the sense of the lexicographic order on $\mathbb{N}^{m+2}$ (here $|a| = a_1+..+a_m$ if $a = (a_1,..,a_m)$). If $F \in B$ let u_F be the largest $Y_i^{(a)}$ which occurs in F and put $S_F = \frac{\partial F}{\partial u_F}$ (called separant of F).

(0.5) Theorem ([7]) If K is a differential field and $F, G \in K\{Y_1, \ldots, Y_n\}$, G irreducible, such that $F \in \{G\} : S_G$ and F is free of every proper derivative of u_G then G divides F.

It is important to consider the closed subsets of K^n as "geometric objects" i.e. to consider morphisms between them in order to decide when two systhems of differential polynomials have "isomorphic" sets of solutions. This point of vue leads to our definition of a Ritt scheme (see § 1) The main problem which we solve in § 1 is : when are two affine Ritt schemes isomorphic?. In § 2 we discuss a cohomological property of the affine space $\mathbb{A}^n$. In § 3 we discuss morphisms differentially of finite type between Ritt schemes and we prove Chevalley's constructibility theorem for such morphisms (in the case of a single derivation).

§ 1 Ritt schemes. Classification of affine Ritt schemes

(1.0) Definitions. Let X be a topological space. A sheaf of differential rings on X will mean a sheaf of rings $\mathcal{O}$ such that for any open set $U \subseteq X$ the ring $\mathcal{O}(U)$ is a differential ring and for any open sets $U \subseteq V \subseteq X$ the restriction maps $\mathcal{O}(V) \longrightarrow \mathcal{O}(U)$ is a morphism of differential rings.

A locally ringed space $(X, \mathcal{O})$ will be called a differential locally ringed space iff $\mathcal{O}$ is a sheaf of differential rings. A morphism of differential locally ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ will mean a morphism of locally ringed spaces such that for all open set $U \subseteq Y$,

$\Gamma(U,\mathcal{O}_Y) \longrightarrow \Gamma(f^{-1}(U),\mathcal{O}_X)$ is a morphism of differential rings.

Now let A be a differential ring, $X = \mathrm{Sp}\,A$, $Y = \mathrm{Spec}\,A$, $j : X \hookrightarrow Y$ the inclusion map and M an A-module. Then we may consider on Y the sheafs $\tilde{A}$ and $\tilde{M}$ in the sense of [5]. Let us define

$$\mathcal{O}_X = \hat{A} = j^{-1}(\tilde{A}) = \text{structure sheaf on } \mathrm{Sp}\,A$$

and also put $\hat{M} = j^{-1}(\tilde{M})$. Obviously $\hat{A}$ is a sheaf of differential rings and $\hat{M}$ is a sheaf of $\hat{A}$-modules.

A differential locally ringed space $(X,\mathcal{O})$ will be called a <u>Ritt scheme</u> iff for any $x \in X$ there exists an open set $U \ni x$ such that $(U,\mathcal{O}|_U)$ is isomorphic to $(\mathrm{Sp}\,A, \hat{A})$ for some differential ring A. A Ritt scheme of the form $(\mathrm{Sp}\,A, \hat{A})$ will be called affine and will be denoted simply by $\mathrm{Sp}\,A$. The scheme $\mathrm{Sp}(A\{Y_1,..,Y_n\})$ will be called the n-affine space over A and will be denoted by $\mathbb{A}^n_A$.

(1.1) <u>Connection with K^n</u> If K is a universal field and X is a closed subset in K^n, a function $f : X \longrightarrow K$ will be called regular at $P \in X$ if there exists an open neighbourhood U of P in X and there exist $F,G \in K\{Y_1,..,Y_n\}$ such that $G(Q) \neq 0$ for all $Q \in U$ and $f(Q) = F(Q)/G(Q)$ for all $Q \in U$. The function f will be called regular on an open set $U \subseteq X$ if f is regular at any point of U. We get a "sheaf of regular functions" which we denote by $\mathcal{O}_1$. On the other hand, let A be the ring $K\{Y_1,...,Y_n\}/I(X)$. Then the sheaf $\hat{A}$ defined on $\mathrm{Sp}\,A$ naturally induces a sheaf $\mathcal{O}_2$ on X. It is apparent that $\mathcal{O}_1$ and $\mathcal{O}_2$ are naturally isomorphic.

(1.2) We may describe $\hat{M}$ as follows : For all $f \in A$ consider the mutiplicative systhem $|f| = \{s \in A \mid f \in \{s\}\}$

and then M is the sheaf associated to the presheaf

$$D(f) \longmapsto M_{|f|} \quad , \quad D(f) = \{P \in \mathrm{Sp}\, A \mid P \not\ni f\}$$

Obviously for all $P \in \mathrm{Sp}\, A$ the stalk of $\hat{M}$ at P is M_P. The presheaf $D(f) \longmapsto M_{|f|}$ is not generally a sheaf as one may deduce from §2. However the following is true:

(1.3) <u>Proposition</u>. Suppose A is a differential ring which is factorial and L is a differential radical ideal in A. Then the presheaf $D(f) \longmapsto L_{|f|}$ is a sheaf.

<u>Proof</u>. It is sufficient to show that for any $f, f_1, .., f_n \in A$ such that $\{f\} = \{f_1, .., f_n\}$ we have $L_{|f|} = \bigcap_{i=1}^{n} L_{|f_i|}$ the intersection being taken in K= field of quotiens of A. "$\subseteq$" is obvious. To prove "$\supseteq$" take $x \in \bigcap_{i=1}^{n} L_{|f_i|}$ $x = g/h$ such that g and h have no common prime factors in A. On the other hand $x = g_i/t_i$ with $g_i \in L$ and $t_i \in |f_i|$. It follows that $t_i g \in hL$. From $t_i g \in hA$ we get $t_i \in hA : gA = hA$ and so $\{f\} = \{f_1, .., f_n\} \subseteq \{t_1, .., t_n\} \subseteq \{h\}$ hence $h \in |f|$. From $t_i g \in L$ we get $t_i \in L : gA$ which is also a differential radical ideal (see [7]) and so $\{f\} \subseteq \{t_1, .., t_n\} \subseteq L : gA$. We get $gf \in L$ and writing $x = gf/hf$, $gf \in L$, $hf \in |f|$ it follows that $x \in L_{|f|}$.

(1.4) <u>Definition</u> A differential ring A such that the canonical morphism $A \longrightarrow A_1 := \Gamma(\mathrm{Sp}\, A, \hat{A})$ is an isomorphism will be called an <u>irredundant</u> ring.

Irredundant rings cannot be found in general among rings of the classical algebraic geometry. Indeed we have :

(1.5) <u>Proposition</u>. Let A be an irredundant domain which is not a field. Then A cannot be an algebra of finite type over a differential non-constant field.

<u>Proof</u>. Suppose A is an algebra of fihite type over K = a nonconstant differential field. Then A is integral

over a polynimial ring $K[t_1,..,t_s]$ and all we have to show is that $s=0$. Suppose $s \geqslant 1$ and put $t=t_1$. Since K is nonconstant, there exist $x \in K$ and i such that $D_i x \neq 0$. Since $\mathrm{degtr}_K L < \infty$ (L=field of quotiens of A) it follows that there exists a polynomial $F \in K[Y, D_iY, D_i^2Y, ..]$ such that $F(t)=0$. By [7], Ch II, §6 there exists $c \in K$ such that $F(c) \neq 0$. Then the equality $F(t-c+c)=0$ shows that $\{t-c\}=A$. Since A is irredundant, we have $A = A_{|1|} = A_1$ and since $t-c \in |1|$ it follows that $t-c$ must be invertible in A. But $t-c$ is a prime element in $K[t_1,..,t_s]$ and by lying over theorem it must lie in a prime ideal of A : contradiction. So $s=0$ and the proposition is proved.

In order to give an important example of an irredundant ring, we first prove the following:

(1.6) Lemma. Let A be a differential domain, $B=A\{Y_1,..,Y_n\}$, $F \in B$ and $D(F)=\{P \in \mathrm{Sp}\,A \mid P \not\ni F\}$. Then every element $x \in \bigcap_{P \in D(F)} B_P$ may be written as $x=\frac{H}{G}$, $H,G \in B$ with

$$u_G \leq u_F$$

Proof. Let K be the field of quotiens of A. There exist $F_i, G_i \in B$, $i=1,..,r$, such that $\{G_1,..,G_r\} \ni F$ and $x = F_i/G_i$ for all i. Let $H,G \in B$ such that $x=H/G$ and H,G have no common prime divisors in the ring $K\{Y_1,..,Y_n\}$, which is factorial. Since $HG_i = F_iG$ it follows that G divides G_i in $K\{Y_1,..,Y_n\}$ so there exist $a_1,..,a_r \in A$ such that $a_iG_i \in (G)B$. Putting $a=a_1..a_r$ it follows that $\{G\} \supseteq \{aG_1,..,aG_r\} = \{a\} \cap \{G_1,..,G_r\} \ni aF$. Now let $G = E_1^{i_1} \ldots E_k^{i_k}$, where E_i are prime elements in the ring $K\{Y_1,..,Y_n\}$. Since $aF \in \{E_i\}$ we get by (0.5) that $u_F = u_{aF} \geqslant u_{E_i}$. Since $u_G = \max_{i=1,..,r} u_{E_i}$ we get that $u_F \geqslant u_G$.

(1.7) Proposition. Let A be a differential domain. Then there is an isomorphism

$$(A\{Y_1,..,Y_n\})_1 \simeq A_1\{Y_1,..,Y_n\}$$

Consequently, if A is irredundant, so is $A\{Y_1,..,Y_n\}$.

Proof. Both rings are subrings in the field of quotiens of $B = A\{Y_1,...,Y_n\}$. Obviously, $A_1\{Y_1,..,Y_n\} \subseteq B_1$. Conversely, if $x \in B_1 = \bigcap_{P \in Sp\, B} B_P$ then by (1.6) $x = F/g$ with $F \in B$ and $g \in A$. Let us prove that $F/g \in A_1\{Y_1,..,Y_n\}$ by induction on the number of monomials in F. Let $p \in Sp\, A$ and $P = pB + [Y_1,..,Y_n] \in Sp\, B$. Since $F/g \in B_P$ we get that $FW = gH$ with $W,H \in B$ and $W(0,..,0) = w \notin p$. Let $fM = f\prod_{i,a}(Y_i^{(a)})^{k_{ia}}$, $f \in A$, be a monomial of minimum degree in F. Identifying the coefficients of M we get $fw = gh$ with $h \in A$ and so $f/g \in A_p$. Since p runs through $Sp\, A$ it follows that $f/g \in A_1$ and so $fM/g \in A_1\{Y_1,..,Y_n\}$. Applying the induction hypothesis to $F - fM$ we get that $(F - fM)/g \in A_1\{Y_1,..,Y_n\}$ and so $F/g \in A_1\{Y_1,..,Y_n\}$.

In order to prove our main result (1.10) we have to prove a technical lemma (1.9) Let us resume some facts about modules of quotiens . Let A be a ring, $\mathcal{T}$ a Serre class of A-modules closed under direct infinite sums and put $F = F_{\mathcal{T}} = \{I \text{ ideal in } A \mid A/I \in \mathcal{T}\}$. (F is called "the additive topology associated to $\mathcal{T}$"). Consider $M \in Mod(A)$. One says that $x \in M$ is F-torsioned iff $Ann(x) \in F$. Then one associates to $\mathcal{T}$ a "radical" defined $t(M) = t_F(M) = \{x \in M \mid x \text{ is F-torsioned}\}$. One defines then $M_F = \varinjlim_{I \in F} Hom(I, M/t(M))$. The functor $M \longmapsto M_F$ is left exact. One has $\ker(M \longrightarrow M_F) = t_F(M)$ and $\operatorname{coker}(M \longrightarrow M_F) \in \mathcal{T}$

For any ideal I in A, one defines $Sat(I) = \{x \in A \mid I:x \in F\}$ and put $C_F(A) = \{I \text{ ideal in } A \mid I = Sat(I)\}$. Now the set $F^e = \{J \text{ ideal in } A_F \mid J \cap A \in F\}$ is an additive topology and there is a one to one correspondence between $C_F(A)$ and $C_{F^e}(A_F)$, given by $I \longmapsto I_F$ and $J \longmapsto J \cap A$. This correspondence induces a one to one correspondence between $\operatorname{Spec} A \cap C_F(A)$ and $\operatorname{Spec} A_F \cap C_{F^e}(A_F)$. Note that $\operatorname{Spec} A \cap C_F(A)$ and $\operatorname{Spec} A \cap F$ form a partition of Spec A. (see [2], [4], [9])

To formulate our lemma, let us consider a ring A and X a subset of $Y = \operatorname{Spec} A$. Let $j : X \hookrightarrow Y$ denote the inclusion map. For all $P \in X$ consider $\mathcal{T}_P = \{M \in Mod(A) \mid M_P = 0\}$ and for all $f \in A$ put $D(f) = \{P \in X \mid f \notin P\}$ and $\mathcal{T}_f = \bigcap_{P \in D(f)} \mathcal{T}_P$. Let $F(f)$ and t_f be the additive topology and the radical associated to $\mathcal{T}_f$. For $D(f) \subseteq D(g)$ we have $F(g) \subseteq F(f)$ and so we get a presheaf on X defined by:

$$D(f) \longmapsto M_{F(f)}$$

which we denote by $\check{M}$. On the other hand we may consider the sheaf $\hat{M} = j^{-1}(\tilde{M})$. Then

(1.9) <u>Lemma</u>. Suppose that every $D(f)$ is quasi-compact and that $Ass(M) \subseteq X$. Then there exists a natural isomorphism $\hat{M} \longrightarrow \check{M}$ of presheafs. Consequently, $\check{M}$ is a sheaf.

<u>Proof</u>. For any subset $S \subseteq A$ put $\{S\} = \bigcap_{P \in X,\, P \supseteq S} P$.

Obviously, $x \in N$ is $F(f)$-torsioned iff there exists a finite set $S \subseteq Ann(x)$ such that $\{f\} = \{S\}$ (N being any A-module). Now $x \in M$ is $F(f)$-torsioned iff there exists n such that $f^n x = 0$. Indeed, since $Ass(M) \subseteq X$, it follows that $\sqrt{Ann(x)}$ is an intersection of prime ideals belonging to X, hence $\sqrt{Ann(x)} = \{Ann(x)\}$ and so $f^n x = 0$ for some n.

Let us indicate two natural morphisms of A-modules u, v,

$\Gamma(D(f),\hat{M}) \underset{v}{\overset{u}{\rightleftarrows}} M_{F(f)}$ which evidently satisfy $uv=id$ and $vu=id$. Let us define u. Take $s \in \Gamma(D(f),\hat{M})$; by quasi-compacity of $D(f)$, there exist $f_1,..,f_k \in A$ and $x_1,..,x_k \in M$ such that $\{f_1,...,f_k\}=\{f\}$ and $s|_{D(f_i)}$ is given by $x_i/f_i \in M/f_i$ for all $i=1,..,k$. Since $x_i/f_i = x_j/f_j$ in any M_P with $P \in D(f_i f_j)$ we get that $f_i x_j - f_j x_i \in \bigcap_{P \in D(f_i f_j)} \mathcal{T}_P = \mathcal{T}_{f_i f_j}$ and so there exist N such that

$$(f_i f_j)^N (f_i x_j - f_j x_i) = 0 \quad \text{for all } i \text{ and } j.$$

Replacing x_i/f_i by $x_i f_i^N/f_i^{N+1}$ we may suppose that $N=0$. Consider the morphisms of A-modules $e : A^k \longrightarrow I=f_1A+\cdots+f_kA$ and $r : A^k \longrightarrow M$ sending the elements of a basis of A^k into f_i and x_i respectively. Notice that $r(\ker(e)) \subseteq t_f(M)$. Indeed if $a_1,...,a_k \in A$ such that $\sum_i a_i f_i = 0$ we get that $f_j(\sum_i a_i x_i) = \sum_i a_i f_i x_j = 0$ for all j and so $\sum_i a_i x_i \in t_f(M)$. So r induces a morphism $\tilde{r} \in Hom(I, M/t_f(M))$. Now we define $u(s)$ to be the image of $\tilde{r}$ in $M_{F(f)}$. Let us define v. Take $x \in M_{F(f)}$; since its image in $\mathrm{coker}(M \xrightarrow{e_f} M_{F(f)})$ is $F(f)$--torsioned, there exist $\{f_1,...,f_k\}=\{f\}$ and $x_1,...,x_k \in M$ such that $e_f(x_i)=f_i x$ for all i. It follows that $e_f(x_i f_j - x_j f_i)=0$ and so x_i/f_i "stick" together and give a section $s \in \Gamma(D(f),\hat{M})$.

(1.10) <u>Theorem</u>. Let A be a differential ring, differentially noetherian and without embedded primes. Then :

1) A_1 is differentially noetherian and without embedded primes.

2) The canonical morphism $A \xrightarrow{e} A_1$ is injective.

3) The canonical morphism $Sp\, A_1 \xrightarrow{r} Sp\, A$ is an isomorphism of Ritt schemes ; consequently, A_1 is irredundant.

<u>Proof</u>. <u>Step</u> 1. $Ass(A) \subseteq Sp\, A$ and every $D(f) \subseteq Sp\, A$ is quasi-compact. Consequently, $A_1 = A_{F(1)}$ and $e : A \to A_1$ is injective.

The quasi-compacity follows immediately from the equality $\{L\} = \sqrt{[L]}$. Now if $P \in Ass(A)$, by our hypothesis P is minimal. Since every radical differential ideal in a differentially noetherian ring is a finite intersection of differential prime ideals, we have $P \supseteq nil(A)=P_1 \cap \cdots \cap P_k$, $P_i \in Sp\,A$. and so $P = P_i$ for some i.

Step 2. The morphism of topological spaces $r : Sp\,A_1 \longrightarrow Sp\,A$ is a homeomorphism.

By lemma (1.9) and Step 1. we get that $A_1 = A_{F(1)}$ .Put $F=F(1)$. Note that $Sp\,A \subseteq C_F(A)$ because if we found an $x \in Sat(P) \setminus P$, $P \in Sp\,A$, we should get $P:x=I \in F$ and so $Ix \subseteq P$ hence $I \subseteq P$ hence $A=\{I\} \subseteq P$, contradiction. On the other hand we have $F^e=F(e(1))$. Indeed if $J \in F^e$ we get $J \cap A=I \in F(1)$ and so there exist $f_1,..,f_k \in I$ and $g_{ia} \in A$ such that $\sum_{i,a} g_{ia}f_i^{(a)}=1$ This equality holds also in A_1 and so $J \in F(e(1))$. Conversely, if $J \in F(e(1))$ there exist $y_1,\dots,y_q \in J$ and $z_{ia} \in A_1$ such that $1 = \sum_{i,a} z_{ia}y_i^{(a)}$. There exist $\{f_1,\dots,f_p\} = A$ and $z_{iak}, y_{ik} \in A$ such that $f_k z_{ia}=z_{iak}$ and $f_k y_i=y_{ik} \in J \cap A \subseteq$ $\subseteq \{I\}$, hence $y_i \in \{I\} : f_k$ wich is a differential ideal and so $f_k y_i^{(a)}=y_{iak} \in \{I\}$, for all i,a,k. We get then $f_k^2 = \sum_{i,a} z_{iak}y_{iak} \in$ $\in \{I\}$. Consequently, $A=\{f_1^2,..,f_p^2\} \subseteq \{I\}$, hence we get that $I \in F$ and so $J \in F^e$. So we deduce that $Sp\,A_F \subseteq C_{F^e}(A_F)$. Obviously, if $Q \in Sp\,A_F$ then $Q \cap A \in Sp\,A$. Now if $P \in Sp\,A$ then $P_F \in Sp\,A_F$ (because since $Ass(P) \subseteq Ass(A) \subseteq Sp\,A$, applying lemma (1.9) to P we get $P_F= \Gamma(Sp\,A,\hat{P})$ which has a natural structure of an $\Gamma(Sp\,A,\hat{A})=A_F$-differential module, so P_F is a differential ideal in A_F). From all the above considerations we deduce that the one-to-one correspondence between $Spec A \cap C_F(A)$ and $Spec\,A_F \cap C_{F^e}(A_F)$ gives us a one to one correspondence between $Sp\,A$ and $Sp\,A_F$. Obviously $Q \longmapsto Q \cap A$ is continous. To prove that $P \xrightarrow{H} P_F$ is continous, take $y \in A_F$;there exist $\{f_1,\dots,f_k\} = A$ and $y_i \in A$ such that $f_i y=y_i$. Then $H^{-1}(D(y)) = \bigcup_i H^{-1}(D(f_i y)) = \bigcup_i D(y_i)$.

Step 3. The morphism of sheafs $r^{\#} : \hat{A} \longrightarrow r_*(\hat{A}_1)$ is an isomorphism.

We only have to show that $r^{\#}$ is an isomorphism on the stalks. Since r is a homeomorphism, the stalk of $r_*(\hat{A}_1)$ at $P \in \mathrm{Sp}\,A$ is equal to the stalk of $\hat{A}_1$ at $r^{-1}(P)$ i.e. it is equal to $(A_F)_{P_F}$. Consider the canonical morphisms $A \xrightarrow{\alpha} A_F \xrightarrow{\beta} A_P$. The ideal PA_P, being differential has the property that $Q = PA_P \cap A_F$ is also differential and since $Q \cap A = P$ we must have $Q = P_F$, by Step 2. We have the following diagram :

$$\begin{array}{ccccc} A & \xrightarrow{\alpha} & A_F & \xrightarrow{\beta} & A_P \\ \downarrow & & \downarrow & & \parallel \\ A_P & \xrightarrow{\alpha'} & (A_F)_{P_F} & \xrightarrow{\beta'} & (A_P)_{PA_P} \end{array}$$

We get $\beta'\alpha' = ID$, since $\beta'\alpha'$ is a morphism of A-algebras. It is sufficient to prove that β' is injective. But if $x \in A_F$ such that $\beta(x)=0$, then x must be annihilated by an element of $A \setminus P \subseteq A_F \setminus P_F$.

Step 4. A_1 has no embedded primes.

Indeed, if $Q \in \mathrm{Ass}(A_1)$, then Q consists only of zero-divisors We claim that the same is true for $Q \cap A$. (If $x \in Q \cap A$ then $xy = 0$ for some $0 \neq y \in A_1$. There exist $\{f_1, \dots, f_k\} = A$ such that $f_i y = y_i \in A$ and we get $xy_i = 0$ for all i. But there exists at least one i such that $y_i \neq 0$). We deduce that $Q \cap A \subseteq \bigcup_{i=1}^{k} P_i$, P_i being differential ideals which are minimal. So we get $Q \cap A = P_i$ for some i. Consequently $Q \notin F^e$ and so $Q \in C_{F^e}(A_F)$ hence $Q = (Q \cap A)_F = (P_i)_F$. This shows that we cannot have $Q_1, Q_2 \in \mathrm{Ass}(A_1)$ with $Q_1 \subsetneqq Q_2$. The theorem is proved.

We shall say that a Ritt scheme is noetherian iff its topological space is noetherian. We shall say that an affine Ritt scheme has no embedded components iff it is of the form Sp A where A is a differential ring without embedded primes.

(1.11) <u>Corollary</u>. (Classification) There is an equivalence between the following categories :

$$\left\{\begin{array}{l}\text{Differentially noetherian}\\ \text{affine Ritt schemes without}\\ \text{embedded components}\\ \qquad\qquad + \\ \text{morphisms of Ritt schemes}\end{array}\right\} \begin{array}{c}\mapsto\\ \leftarrow\!\dashv\end{array} \left\{\begin{array}{l}\text{Diff. noetherian}\\ \text{irredundant rings}\\ \text{without embedded primes}\\ \qquad\qquad + \\ \text{morphisms of diff. rings}\end{array}\right\}$$

Let K be a universal field and $X \subseteq K^n$ a closed subset. Let $A=K\{Y_1,\ldots,Y_n\}/I(X)$ its "coordinate ring". Then the Ritt scheme $Sp\,A$ will be denoted also by X.

(1.12) <u>Corollary</u>. (Classification in K^N) Let $X \subseteq K^n$ and $Y \subseteq K^m$ closed subsets in affine spaces. The Ritt schemes X and Y are isomorphic iff the differential rings $\Gamma(X,\mathcal{O}_X)$ and $\Gamma(Y,\mathcal{O}_Y)$ are isomorphic. *This comes from the following:*

(1.13) <u>Remark</u>. Both (1.10) and (1.11) hold if we replace "differentially noetherian without embedded components" by "reduced", (we say that an affine Ritt scheme is reduced iff it is of the form $Sp\,A$ with A reduced). This is true because in a reduced differential ring any annihilator ideal is a differential ideal and in fact this is a property *which is* sufficient in all the proofs we gave.

(1.14) <u>Corollary</u>. Let A be a reduced differential ring and $u : A \longrightarrow B$ a morphism of differential rings, B being reduced and irredundant. Then A_1 is irredundant and there exists a unique morphism making commutative the diagram :

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ & \searrow & \swarrow \\ & B & \end{array}$$

(The proof is standard after using the fact that $A_1=A_{F(1_A)}$ and $B_1 = B_{F(1_B)}$).

As a consequence of (1.13) and (1.14) one may prove the existence of the product of any two objects in certain subcategories of the category of Ritt schemes. An application will be also given in §3. Let us make also the following :

(1.15) Remark. For any differential ring A the morphism $A \longrightarrow A/t_1(A)$ induces an isomorphism of Ritt schemes $Sp\ A/t_1(A) \longrightarrow Sp\ A$.

(The proof is standard, using only the definitions).

A Ritt scheme X will be called reduced (or integral) iff for any open set $U \subseteq X$ the ring $\Gamma(U, \mathcal{O}_X)$ is reduced (or integral). By (1.15) it follows that an affine Ritt scheme is reduced (or integral) iff it is of the form $Sp\ A$, A being a reduced (or integral) differential ring.

2. Non-vanishing cohomology of $\mathbb{A}^n$

For any topological space X and for all open subset $U \subseteq X$ $H^i(U, \ )$ will denote the derived functors of $\Gamma(U, \ ) : Ab(X) \to Ab$. The following result shows that there is a great difference, from the cohomological point of vue, between schemes and Ritt schemes.

(2.1) Theorem. Let A be a differential domain, $n \geq 1$ and $\mathbb{A}^n_A$ the n-affine space over A, $\mathcal{O}$ being the structure sheaf of $\mathbb{A}^n_A$. Then we have :

$$H^1(U, \mathcal{O}) \neq 0$$

for all nonempty open subsets U of $\mathbb{A}^n_A$.

Proof. Suppose $H^1(U, \mathcal{O}) = 0$ where $U = D(I)$, I being a nonzero ideal in $B = A\{Y_1, .., Y_n\}$. Replacing $A\{Y_1, .., Y_{n-1}\}$ by A we may suppose that $n=1$ and put $Y=Y_1$. Consider $F \in I$ $F \notin A$, $u_F = Y^{(b)}$, $b \in \mathbb{N}^m$. Take $a \in \mathbb{N}^m$, $a > b$ in the lexicographic order and take $c \in \mathbb{N}^m$, $c \neq (0,..,0)$. Put $y = Y^{(a)} - Y^{(a+c)}$

and consider the exact sequence

$$0 \longrightarrow B \xrightarrow{w} B \longrightarrow B/yB=M \longrightarrow 0$$

where w is the mutiplication by $y \in B$. This is a sequence of B-modules (but not of differential B-moduled) and induces an exact sequence of $\hat{B} = \mathcal{O}$-modules on $\mathbb{A}^n_A$: $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \hat{M} \longrightarrow 0$. We get an exact sequence

$$\Gamma(U,\mathcal{O}) \xrightarrow{p} \Gamma(U,\hat{M}) \longrightarrow H^1(U,\mathcal{O})=0$$

and we only have to prove that p is not surjective. Put $F_1 = Y^{(a)}-1$ and $F_2 = Y^{(a+c)}-1$, $F_1, F_2 \in B$. We have $D(F_1) \cup D(F_2) = \mathrm{Sp}\, B$ because $F_1^{(c)} - F_2 = 1$. Put $s_1 = \bar{1}/F_1 \in M_{|F_1|}$ and $s_2 = \bar{1}/F_2 \in M_{F_2}$ (for all $x \in B$ we write $\bar{x} = x \bmod yB$). Since $\bar{1}/F_1 = \bar{1}/F_2$ in $M_{|F_1F_2|}$, s_1 and s_2 "stick together" and give a section $s \in \Gamma(U,\hat{M})$. If p is surjective, there exists $t \in \Gamma(U,\mathcal{O})$ such that $p(t)=s$. But $\Gamma(U,\mathcal{O}) = \bigcap_{P \in U} B_P \subseteq \bigcap_{P \in D(F)} B_P$ and so, by lemma (1.6) t may be written $t=W/H$ $W,H \in B$ and $u_H \leq u_F$, i.e. $H \in A\left[Y^{(e)} \mid e \leq b \text{ in the lexicographic order}\right]$. Since we have $(\overline{W/H})/1 = \bar{1}/F_1$ in any M_P, $P \in D(F_1) \cap U = D(F_1 I)$, we get that for any such P there exists $T_P \in B \setminus P$ and $G_P \in B$ such that

$$T_P((Y^{(a)}-1)W - H) = G_P(Y^{(a)} - Y^{(a+c)})$$

But $Y^{(a)} - Y^{(a+c)}$ cannot divide the polynomial $(Y^{(a)}-1)W-H = E$ because if it did, making in E the substitution $Y^{(a)} = Y^{(a+c)} = 1$ we would get $H = 0$ (since H does not change under this substitution). Consequently, $Y^{(a)} - Y^{(a+c)}$ divides T_P, and so $T_P \in [Y^{(a)}]$ for all $P \in D(F_1 I)$. Let J be the ideal $\{(T_P)_{P \in D(F_1 I)}\}$; obviously we have $F_1 I \subseteq J$. Hence, $F_1 F \in [Y^{(a)}]$ which is a prime ideal. On the other hand $F_1 = Y^{(a)} - 1 \notin [Y^{(a)}]$ and $F \notin [Y^{(a)}]$ because $u_F = Y^{(b)}$, $b < a$, contradiction. Our theorem is proved.

3. The constructibility theorem

(3.1) Definition. A morphism of Ritt schemes $X \xrightarrow{f} Y$ will be called differentially of finite type iff it is quasi-compact and locally, on both X and Y it is of the form $\mathrm{Sp}\, B \longrightarrow \mathrm{Sp}\, A$, where $A \longrightarrow B$ is a morphism differentially of finite type.

(More precisely, iff f is quasi-compact and for every $x \in X$ and $y \in Y$ with $f(x)=y$, there exist open neighbourhoods V and U of x and y respectively, $f(V) \subseteq U$ and there exists a morphism $A \longrightarrow B$ differentially of finite type such that we have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \wr\!\wr & & \wr\!\wr \\ \mathrm{Sp}\, B & \longrightarrow & \mathrm{Sp}\, A \end{array}$$

)

(3.2) Proposition. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms differentially of finite type between integral Ritt schemes, then gf is also differentially of finite type.

Proof. We prove the propositiom in several steps. To make our formulations shorter let us give the following definition : a morphism $A \longrightarrow B$ between two irredundant domains will be called an $\mathcal{F}$-morphism iff it is of the form $A \xrightarrow{u} C \longrightarrow C_1 = B$ where u is differentially of finite type and C is a domain.

Step 1. If $A \xrightarrow{f} B$ is a morphism differentially of finite type between domains, then $A_1 \xrightarrow{f_1} B_1$ is an $\mathcal{F}$-morphism.

Indeed we have the following diagram

$$\begin{array}{ccccc} A & \longrightarrow & A\{Y_1,..,Y_n\} & \longrightarrow & A\{Y_1,..,Y_n\}/P = B \\ \downarrow & & \downarrow & & \downarrow \\ A_1 & \longrightarrow & (A\{Y_1,..,Y_n\})_1 & \longrightarrow & (A\{Y_1,..,Y_n\}/P)_1 = B_1 \end{array}$$

Using (1.14) one may check that $(A\{Y_1,..,Y_n\}/P)_1 = (C/P_1)_1$, where $P_1 = P_{F(1)}$ and $C = (A\{Y_1,..,Y_n\})_1 = A_1\{Y_1,..,Y_n\}$, by (1.7).

<u>Step</u> 2. If $A \xrightarrow{u} B \xrightarrow{v} C$ are $\mathcal{F}$-morphisms then vu is also an $\mathcal{F}$-morphism.

Using again (1.7) we get the following diagram :

$$vu : A \to A\{Y_1,..,Y_n\} \twoheadrightarrow E \to E_1 = B \to E_1\{Z_1,..,Z_m\} \twoheadrightarrow E_1\{Z_1,..,Z_m\}/Q = F \to F_1 = C$$

$$A \searrow A\{Y_1,..,Y_n,Z_1,..,Z_m\} \overset{p}{\twoheadrightarrow} E\{Z_1,..,Z_m\} \xrightarrow{w} (E\{Z_1,..,Z_m\})_1$$

Now using (1.14) one may check that $C \cong (A\{Y_1,..,Y_n,Z_1,..,Z_m\}/J)_1$ where $J = p^{-1}(w^{-1}(Q))$.

<u>Step</u> 3. Let $A \to B$ be an $\mathcal{F}$-morphism and $h \in A$. Then $(A_h)_1 \longrightarrow (B_h)_1$ is also an $\mathcal{F}$-morphism.

Indeed there exists a domain C and a factorization $A \longrightarrow$ $\longrightarrow C \longrightarrow C_1 = B$, the first morphism being differentially of finite type. It follows that $A_h \longrightarrow C_h$ is differentially of finite type and by Step 1 $(A_h)_1 \longrightarrow (C_h)_1$ is an $\mathcal{F}$-morphism. On the other hand one may easily check that $(C_h)_1 \longrightarrow (B_h)_1$ is an isomorphism.

<u>Step</u> 4. A quasi-compact morphism $X \xrightarrow{f} Y$ of integral Ritt schemes is differentially of finite type iff for any affine subset $U \subseteq Y$ we have $f^{-1}(U) = \bigcup_i V_i$, V_i being open affine subsets in X such that $\Gamma(U, \mathcal{O}_Y) \longrightarrow \Gamma(V_i, \mathcal{O}_X)$ are $\mathcal{F}$-morphisms.

Indeed, "if" is obvious via (1.15) and (1.10). To prove "only if" remark that we may suppose A and B from the definition (3.1) to be domains (replacing them by $A/t_1(A)$ and $B/t_1(B)$ which are domains). Now let $U = Sp\ E$, $E = E_1$ be an open affine subset in Y. By Step 1., for all $y \in U$ there exists $W = Sp\ F \ni y$, $F = F_1$ such that $f^{-1}(W) = \bigcup W_i$, $W_i = Sp\ G_i$, $G_i = (G_i)_1$ and $F \longrightarrow G_i$ are $\mathcal{F}$-morphisms. We may suppose $W \subseteq U$ (because otherwise there exists $s \in F$ such that $y \in Sp(F_s) \subseteq Sp\ E$ and by Step 3, we get that $(F_s)_1 \longrightarrow ((G_i)_s)_1$ are $\mathcal{F}$-morphisms). Now there exists $t \in E$ such that $y \in Sp(E_t) \subseteq Sp\ F$. We get the diagram:

$$\begin{array}{ccccc} E & \longrightarrow & F & \xrightarrow{\beta} & G \\ {\scriptstyle\delta}\downarrow & & & & \downarrow \\ (E_t)_1 & \xrightarrow{\alpha} & (F_t)_1 & \xrightarrow{\gamma} & (G_t)_1 \end{array}$$

G being any G_i. Since β is an $\mathcal{F}$-morphism, by Step 3 we get that γ is an $\mathcal{F}$-morphism. Obviously α is an isomorphism and since δ is an $\mathcal{F}$-morphism, it follows from Step 2 that $E \longrightarrow (G_t)_1$ is an $\mathcal{F}$-morphism. Since the family $Sp(G_t)$ cover $f^{-1}(U)$, our statement follows.

(3.3) <u>Theorem</u>. Let $X \xrightarrow{f} Y$ a morphism of ordinary Ritt schemes, differentially of finite type. Suppose Y is differentially noetherian. Then f is constructible.

<u>Proof</u>. We may obviously reduce ourselves to the case of a morphism of the form $Sp\ B \xrightarrow{f} Sp\ A$, where $A \xrightarrow{u} B$ is differentially of finite type.

<u>Step</u> 1. The case when u is of finite type (in the usual sense).

It is sufficient to prove that $f(Sp\ B)$ is constructible. Since $Sp\ A$ is a noetherian space, it is sufficient by a classical criterion ([8], 6.C) to prove that whenever a morphism $Sp(B/PB) \xrightarrow{g} Sp(A/P)$ is dominant for some $P \in Sp\ A$, it follows that the image of g contains a nonempty open subset. But if g is dominant one may check that $A/P \longrightarrow B/PB$ is injective, as in the non-differential case. So we may suppose that A is a domain and $A \subseteq B$ and we have to show that $f(Sp\ B)$ contains a nonempty open subset in $Sp\ A$. But $f(Spec\ B)$ contains a nonempty open set $U \subseteq Spec\ A$ (see [8], proof of 6.E which holds without noetherian hypothesis) and our statement follows from (0.1) and from the general formula $f(Sp\ B) = f(Spec\ B) \cap Sp\ A$.

<u>Step</u> 2. General case.

It is sufficient to prove that $f(\mathrm{Sp}\, B)$ is constructible and applying [8], 6.C again, we reduce ourselves to the following problem : if $u : A \longrightarrow B$ is injective and differentially of finite type, A being a differentially noetherian domain, then the image of $f:\mathrm{Sp}\, B \longrightarrow \mathrm{Sp}\, A$ contains a nonempty open set. Suppose $B = A\{y_1,\dots,y_n\}$. Let $y_1,\dots,y_N$ be a maximal family of differentially algebraic independent elements over A. Put $C = A\{y_1,\dots,y_N\}$. Since B is differentially noetherian by (0.2), we may write $\mathrm{nil}(B) = P_1 \cap \dots \cap P_r$, $P_i \in \mathrm{Sp}\, B$ and so we get $(0)=\mathrm{nil}(B) \cap C = (P_1 \cap C) \cap \dots \cap (P_r \cap C)$ hence there exists i such that the morphism $C \longrightarrow B/P_i = E$ is injective. Put $z_j = y_j \bmod P_i$ for all $j \geqslant N+1$. For any such j take $F_j \in C\{Y\}$, $F_j \neq 0$, $F_j(z_j)=0$. Suppose that we have chosen F_j of minimum order n_j and of minimum degree among those of order n_j. Consider $S_j = \partial F_j / \partial Y^{(n_j)}$ the separant of F_j. We have $S_j \neq 0$ (because of the characteristic) and $S_j(z_j) \neq 0$ by the minimality of F_j. Put $S = \prod_{j=N+1}^{n} S_j(z_j)$ which is a nonzero element in E. We claim that E_S is an C-algebra of finite type. Indeed for each j we have

$$F_j = \sum_k G_{kj}(Y^{(n_j)})^k \quad , \quad G_{kj} \in C\left[Y, Y', \dots, Y^{(n_j - 1)}\right]$$

We get

$$0 = (F_j(z_j))' = \sum_k (G_{kj}(z_j))'(z_j^{(n_j)})^k + S_j(z_j) z_j^{(n_j+1)}$$

We get then by induction that for any $q \geqslant 0$

$$z_j^{(q)} \in C\left[z_{N+1}, \dots, z_{N+1}^{(M)}, \dots, z_n, \dots, z_n^{(M)}, 1/S\right] \quad , M = \max_j n_j$$

By Step 1 the morphism $\mathrm{Sp}(E_S) \longrightarrow \mathrm{Sp}\, C$ is constructible and since it is dominant we get that its image contains a principal open set $D(H) \subseteq \mathrm{Sp}\, C$, $H \neq 0$. Now if h is a nonzero coefficient of H it follows that $D(h) \subseteq \mathrm{Sp}\, A$ is contained in $f(\mathrm{Sp}((B/P_i)_S)) \subseteq f(\mathrm{Sp}\, B)$. The theorem is proved.

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