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by

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POTENTIAL IN STANDARD H-CONES

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The aim of this paper is to extend, in the framework of standard H-cones, the notions of "potential" and "harmonic". More precisely in any standard H-cone $\mathcal{S}$ we distinguish the elements h, which are specifically dominated by an element $s \in \mathcal{S}$ if it is naturally dominated by s, called substractibles and the elements p which have no specific substractible nonzero minorant, called pure potentials. We caracterise, in terms of balayages, the substratibles and pure potential elements. Some remarkable results are obtained in the case when any universally continuous element of $\mathcal{S}$ is a pure potential. In this situation, if the dual H-cone of $\mathcal{S}$ satisfies the axiom of polarity, then the pure potentials in $\mathcal{S}$ are exactly the elements which can be writen as a sum of a sequence of universally continuous elements.

0. Preliminaries and notations. Throughout this paper $\mathcal{S}$ will be a standard H-cone (see [1]).

We denote by $\leq$ (resp. $\preceq$) the natural (resp. specific) order relation in $\mathcal{S}$ and by $\mathcal{S}_0$ the convex cone of all universally continuous elements of $\mathcal{S}$. Whenever s is a strictly positive element of $\mathcal{S}$ we right $s>0$. We denote by $\mathcal{S}^*$ the dual H-cone of $\mathcal{S}$. It is known that $\mathcal{S}^*$ is also a standard H-cone and we denote by $\mathcal{S}^*_0$ convex cone of all universally continuous elements of $\mathcal{S}^*$. We denote also by $\leq$ (resp. $\preceq$) the natural (resp. specific) order relation in $\mathcal{S}^*$. We remember that $\mathcal{S}$ is embeded canonically in $\mathcal{S}^{**}$ as a convex cone which is solid and dense in order from below in $\mathcal{S}^{**}$ with respect to the natural order.

The coarsest topology on $\mathcal{S}$ for which the functions on $\mathcal{S}$ defined by

$$s \longrightarrow \mu(s) \quad , \mu \in \mathcal{S}_o^*$$

are continuous is called natural topology on $\mathcal{S}$. A similar topology may be defined on $\mathcal{S}^*$ (see also [2]). The fine topology on $\mathcal{S}$ will be the coarsest topology on $\mathcal{S}$ for which the functions on $\mathcal{S}$ defined by

$$s \longrightarrow \mu(s) \quad , \mu \in \mathcal{S}^*$$

are continuous. Analogously we define the fine topology on $\mathcal{S}^*$.

For any $u \in \mathcal{S}$, $u > 0$ we denote

$$K_u = \{\mu \in \mathcal{S}^* \mid \mu(u) \leq 1\}$$

The set K_u is convex and compact in the natural topology on $\mathcal{S}^*$. We denote by X_u the set of all non-zero extrem points of K_u. Any $s \in \mathcal{S}$ may be identified with the function on X_u defined by

$$s(x) =: x(s),$$

and in this way $\mathcal{S}$ becomes a standard H-cone of functions on the set X (see [1]). The natural (resp. fine) topology on X_u is that induced by the natural (resp. fine) topology on $\mathcal{S}^*$.

In general we say that $\mathcal{S}$ is represented as an H-cone of functions on a set X (see [1]) if there exists $u \in \mathcal{S}$, $u>0$ such that $X \subset X_u$ and for any $s, t \in \mathcal{S}$ we have $s \leq t$ whenever $s(x) \leq t(x)$ for any $x \in X$. Obviously in this case we have $u = 1$ on X.

We remember that a balayage on $\mathcal{S}$ is a map $B: \mathcal{S} \to \mathcal{S}$, which is additive, increasing and continuous in order from below with respect to the natural order, contractive (i.e. $Bs \leq s, (\forall) s \in \mathcal{S}$) and idempotent (i.e. $B(Bs)=Bs\ (\forall)\ s \in \mathcal{S}$).

We denote by $[\mathcal{S}]$ the ordered real vector space generated by $\mathcal{S}$ and by the order relation given by the convex cone of all elements $s-t$ where $s,t \in \mathcal{S}$, $s \geq t$. The ordered vector space $[\mathcal{S}]$ is a vector latice and for any $f, g \in [\mathcal{S}]$ we denote their infimum by $f \wedge g$. For any $f \in [\mathcal{S}]$, we put

$$Rf =: \bigwedge \{ s \in \mathcal{S} \mid s \geq f \}$$

and for any $f \in [\mathcal{S}]$, $f \geq 0$ we denote by B_f the balayage defined by

$$B_f s = \bigvee_n R(s \wedge nf).$$

For any subset A of X_u we denote by B_s^A the function

$$B_s^A = \bigwedge \{ s' \in \mathcal{S} \mid s' \geq s \text{ on } A \}.$$

Generally the map

$$s \longrightarrow B_s^A$$

is additive, increasing and continuous in order from below, contractive but it is not idempotent. If it is idempotent then this map is a balayage called the balayage on A. This is the case when A is fine open or more general when A is a subbasic subset of X_u (see[3]).

If $u \in \mathcal{S}, u > 0$ then for any $s \in \mathcal{S}$ we denote (see[3]) Carr s the subset of the natural closer $\overline{X_u}$ of X in K_u defined by

$$x \in \text{Carr } s \overset{def}{\Longleftrightarrow} B^{X_u \setminus V} s \neq s \quad \text{for any natural closed neighbourhood } V \text{ of } x.$$

The set Carr s is called the carrier of s and it is proved that for any $s \in \mathcal{S}_o$ we have Carr $s \neq \emptyset$ and $s' \geq s$ whenever this inequality hols only on the set $X_u \cap$ Carr s.

We remember that $\mathcal{S}$ satisfies the axiom of polarity (see[1]) if for any $s \in \mathcal{S}_o$ and any decreasing sequence of balayages $(B_n)_n$ on $\mathcal{S}$ we have

$$B_k(\bigwedge_n B_n s) = \bigwedge_n B_n s.$$

One can prove (see[3]) that $\mathcal{S}^*$ satisfies the axiom of polarity iff any element $s \in \mathcal{S}$ which is dominated by an element $s_o \in \mathcal{S}_o$ is of the form

$$s = \sum_n s_n \quad \text{where } s_n \in \mathcal{S}_o.$$

1. Substractible elements in a standard H-cone

An element $h \in \mathcal{S}$ is called substractible if for any $s \in \mathcal{S}$ such that $h \leq s$ we have

$$h \lesssim s.$$

We remark that the set of all substractible elements is a convex subcone of $\mathcal{S}$ which is solid in $\mathcal{S}$ with respect to the specific order. Moreover any element $s \in \mathcal{S}$ of the from

$$s = \sum_n h_n, \quad h_n \text{ substractible}$$

is substractible.

Theorem 1. Let $(B_n)_n$ be a decreasing sequence of balayages on $\mathcal{S}$ such that there exists $u \in \mathcal{S}$, $u > 0$ for which

$$\bigwedge_n B_n u = 0$$

Then any element $h \in \mathcal{S}$ such that

$$n \in N \Longrightarrow B_n h = h$$

is substractible and $u \curlywedge h = 0$.

Let $h \in \mathcal{S}$ such that

$$n \in N \Longrightarrow B_n h = h$$

and let $s \in \mathcal{S}$ be such that

$$h \leqslant s.$$

Since $B_n h = h$ it follows from [3], that the element

$$f_n =: (s-h) \wedge (u - B_n u)$$

is of the form t-Bt where

$$t = (s+B_n u)\wedge(u+h)$$

Let now $\psi_1, \psi_2 \in \mathcal{S}_o^*$ such that $\psi_1 \leq \psi_2$. Since

$$t-B_n t \leq u-B_n u$$

it follows that

$$t-Bt+B_n u \in \mathcal{S}$$

i.e

$$f_n + B_n u \in \mathcal{S}.$$

We have

$$\psi_1((s-h)\wedge u) \leq \psi_1(f_n+B_n u) \leq \psi_2(f_n+B_n u) \leq$$
$$\leq \psi_2(f_n)+\psi_2(B_n u) \leq \psi_2((s-h)\wedge u)+\psi_2(B_n u)$$

and therefore

$$\psi_1((s-h)\wedge u) \leq \psi_2((s-h)\wedge u).$$

Hence

$$(s-h)\wedge u \in \mathcal{S}$$

Since we can change u by any element nu where $n \in N$ and since $u > 0$ we deduce that

$$s - h \in \mathcal{S}.$$

If we denote

$$p = u \wedge h$$

we have

$$B_n p = p \qquad (\text{since } p \gtrsim h)$$

$$B_n p \leq \bigwedge_n B_n u = 0, \ p = 0.$$

<u>Theorem 2. Let</u> h <u>be a substractible element of</u> $\mathcal{S}$ <u>and</u> p <u>such that</u> $p \wedge h = 0$. <u>Then the balayage</u> $\mathcal{B}_f$ <u>where</u> $f = (h-p)^+$ <u>satisfies the properties</u>

$$B_f h = h, \qquad B_f p \leq h.$$

Let $s \in \mathcal{S}$ be such that

$$s \geq (nf) \wedge h \quad (\forall)\ n \in \mathbb{N}$$

We have

$$h - s \leq p, \ p \geq R(h-s).$$

Since

$$R(h-s)\lesseqgtr h$$

it follows that R(h-s) is substractible and therefore

$$R(h-s)\gtreqless p.$$

Using the hypothesis we get

$$R(h-s) = 0, \quad h\leq s.$$

Obviously we have

$$h\geq B_f h = \bigvee_n R(h\wedge nf)\geq h, \quad B_f h=h$$

and

$$B_f p = \bigvee_n R(p\wedge nf)\leq h \ .$$

<u>Corollary. Let</u> h <u>be a substractible element in</u> $\mathcal{S}$ <u>and let</u> p <u>be a strictly positive element of</u> $\mathcal{S}$ <u>such that</u> $p\wedge h = 0$. <u>Then the decreasing sequence of balayages</u> $(B_{f_n})_n$ <u>where</u>

$$f_n = (h-np)^+$$

<u>satisfies the following properties</u>

1) $\bigwedge_n B_{f_m} p = 0$

2) $B_{f_n} h = h \quad (\forall)\ n\in \mathbb{N}$

Let u be a strictly positive element of $\mathcal{S}^*$ and let Y_{u^*} be the set of all non-zero extrem points of the compact convex set

$$K_{u^*} = \{\theta \in \mathcal{S}^{**} \mid \theta(u^*) \leq 1\}$$

We say that an element $s \in \mathcal{S}$ is <u>u - representable</u> if there exists a Borel measure μ_s on Y_{u^*} such that for any $\psi \in \mathcal{S}_o^*$ we have

$$\psi(s) = \int \psi(\theta) d\mu_s(\theta)$$

The measure μ_s is uniquely determined by the preceding equality. We remark that any element $s \in \mathcal{S}$ such that $u^*(s) < \infty$ is u^*- representable and the associated measure is finite. Generally since $\psi(s) < \infty$ we deduce that the associated measure μ is σ- finite and therefore an element $s \in \mathcal{S}$ is u - represen-table iff it is of the from

$$s = \sum_{n \in N} s_n$$

where $u^*(s_n) < \infty$.

<u>Theorem 3. Let</u> $u^* \in \mathcal{S}^*$, $u^* > 0$ <u>and</u> $h \in \mathcal{S}$ <u>an</u> u^* - <u>representable element such that the associated measure is carred by a polar subset</u> A <u>of</u> Y_{u^*}. <u>Then</u> h <u>is substractible.</u>

Let $s^* \in \mathcal{S}^*$, $s^* > 0$ such that $s^* = +\infty$ on A. Let $(B_n)_{n \in N}$ be the sequence of balayages on $\mathcal{S}$ such that

$$n \in N \Rightarrow B_n^* = B^{G_n} \quad , \; G_n = \{ y \in Y_{u^*} \mid s^*(y) > n \}$$

Since for any $\psi \in \mathcal{S}_o^*$ we have

$$B_n^* \psi = \psi \text{ on } G_n$$

it follows that

$$\psi(B_n h) = (B_n^* \psi)(h) = \int B_n^* \psi \, d\mu_h = \int \psi \, d\mu_h = \psi(h)$$

and therefore $B_n h = h$ for any $n \in N$.

We remark also that for any $p \in \mathcal{S}_o$ and any $\psi \in \mathcal{S}_o^*$ we have $\psi \leq \alpha u^*$ for a suitable $\alpha > 0$ and

$$\psi(\bigwedge_n B_n p) = \inf_n \psi(B_n p) = \inf_n (B_n^* \psi)(p) \leq \inf_n \frac{\alpha}{n} u^*(p) = 0$$

Hence

$$\bigwedge B_n p = 0$$

and therefore

$$\bigwedge_n B_n q = 0$$

for any $q \in \mathcal{S}$ of the forme

$$q = \sum_{n \in N} p_n \; , \quad p_n \in \mathcal{S}_o$$

The assertion follows now from theorem 1.

Proposition 4. Let $u^* \in \mathcal{S}^*$, $u^* > 0$ and let $h \in \mathcal{S}$ be such that any specific minorant of h which is u^*-representable is equal to zero. Then h is substractible.

Let $v^* \in \mathcal{S}^*$ be such that $v^* > 0$, $v^* \leq u^*$ and $v^*(h) < 1$. We denote by μ_h the representing measure on Y_{v^*} associated to h. We shall show that μ_h is carred by the set

$$\left\{ y \in Y_{v^*} \,\middle|\, u^*(y) = +\infty \right\}$$

We denote by μ_n the restriction of μ_h to the Borel set

$$\left\{ y \in Y_{v^*} \,\middle|\, u^*(y) \leq n \right\},$$

and let h_n be the element of $\mathcal{S}$ defined by

$$\psi(h_n) = \int \psi \, d\mu_n \qquad , \ (\forall) \ \psi \in \mathcal{S}_0^*$$

Obviously $u^*(h_n) \leq n \mu_h(v^*) < n$

and therefore h_n is u^*-representable and $h_n \prec h$.

Using the hypotheses we deduce $h_n = 0$ for any n and

$$\mu_h\left(\left\{ y \in Y_{v^*} \,\middle|\, u^*(y) < \infty \right\}\right) = 0.$$

2. Potentials in standard H-cones

Suppose that $\mathcal{S}$ is represented as an H-cone of functions on the set X. A point $x \in X$ is called absorbent with respect to $\mathcal{S}$ if there exists $s \in \mathcal{S}$ such that $s(y)>0$ for any $y \in X$, $y \neq x$ and $s(x)=0$.

<u>Proposition 5. For any $x_o \in X$ the following assertions are equivalent:</u>

1) x_o <u>is absorbent with respect to $\mathcal{S}$</u>

2) $B_1^{\complement\{x_o\}}(x_o) = 0$

3) <u>there exists a substractible element</u> $p \in \mathcal{S}_o$, $p \neq 0$ <u>such that</u> $B_p^{\{x_o\}} = p$.

1) $\Leftrightarrow$ 2) is immediate.

2) $\Rightarrow$ 3) Since $\{x_o\}$ is fine open it follows that $B_1^{\{x_o\}}(x_o)=1$. It is easy to see that the element $p=:B_1^{\{x_o\}}$ belongs to $\mathcal{S}_o$ and

$$B_p^{\{x_o\}} = p.$$

We shall show that p is substractible. Let $s \in \mathcal{S}$ be such that

$$p \leq s.$$

We have

$$B_s^{\complement\{x_o\}} = \begin{cases} s(x) & \text{for any } x \neq x_o \\ 0 & \text{for } \quad x = x_o. \end{cases}$$

Since $\{x_o\}$ is fine open we have $s(x_o)<\infty$,

$$R(s-B_s^{\complement\{x_o\}}) = s(x_o).p,$$

and therefore

$$s(x_o).p \precsim s,$$

$$p \precsim \frac{s}{s(x_o)} \precsim s.$$

3) $\Rightarrow$ 2). Let p be a substractible element of $\mathcal{S}_o$, $p\neq 0$ such that

$$B_p^{\{x_o\}} = p.$$

Obviously $p(x_o) \neq 0$ and we may suppose that $p(x_o)=1$. If $B_1^{C\{x_o\}}(x_o)>0$ then there exists a natural closed neighbourhood V of x_o such that

$$B_1^{X\setminus V}(x_o) > 0.$$

Since p is substractible it follows that

$$p \precsim \alpha B_1^{X\setminus V}$$

for a suitable $\alpha > 0$ and therefore

$$R_p^{X\setminus V} = p.$$

We may embeded X in a natural way in the set X_1 of all extrem points of the convex compact set

$$K_1 = \{\mu \in \mathcal{S}^* \mid \mu(1) \leq 1\}$$

Let U be a natural open neighbourhood of x_o in $\overline{X}$ such that

$$U \cap X \subset V$$

We have

$$B_p^{X_1 \setminus U} = B_p^{X \setminus U} = p.$$

From this equality it follows (see [3]) that

$$\mathrm{Carr}\ p \subset \overline{X} \setminus U$$

which contradicts (see [3]) the relation

$$B_p^{\{x_0\}} = p.$$

<u>Proposition 6.</u> <u>Suppose</u> $\mathcal{S}$ <u>represented as an H-cone of functions on the set X.</u> <u>Then the set of all absorbent points from</u> X <u>with respect to</u> $\mathcal{S}$ <u>is discrete in the natural topology.</u>

If x is an absorbent point of X with respect to $\mathcal{S}$ we denote by p_x the substractible element of $\mathcal{S}_o$ such that

$$B_{p_x}^{\{x\}} = p_x, \quad p_x(x) = 1.$$

Any point $y \in X$, $p_x(y) > 0$ is not absorbent because in the contrary caz we have

$$p_y \leq \alpha p_x$$

for a suitable $\alpha > 0$ and therefore

$$p_y \lneq \alpha p_x$$

which contradicts the fact that

$$\text{carr } p_y = \{y\}, \text{ carr } p_x = \{x\}.$$

An element $p \in \mathcal{S}$ is called pure potential if for any substractible element $h \in \mathcal{S}$ we have

$$p \wedge h = 0.$$

Proposition 7. Any pure potential $p \in \mathcal{S}$ is u^*-representable for any $u^* \in \mathcal{S}^*$, $u^* > 0$.

Let $p \in \mathcal{S}$ be a pure potential and let $u^* \in \mathcal{S}^*$ such that $u^* > 0$. Since the set of all specific minorauts of p which are u^*-representable containes its least upper bound it is sufficient to suppose that any specific minorant of p which is u^*-representable is equal to zero. In this case we want to prove that $p=0$. Indeed this last assertion is a direct consequence of proposition 4 and of the definition of a pure potential.

In the sequel we denote by $\mathcal{S}_h$ the set of all substractible elements of $\mathcal{S}$.

Theorem 8. Suppose that $\mathcal{S}$ is represented as an H-cone of functions on X such that any element of $\mathcal{S}_o^*$ is represented by a borel measure on X. Then any element of $\mathcal{S}_o \cap \mathcal{S}_h$ is of the form

$$\sum_x \alpha_x p_x$$

where x runs the set of all absorbent points of X with respect to $\mathcal{S}$ and for any such point x

$$p_x = B_1^{\{x\}}$$

Let $p \in \mathcal{S}_0 \cap \mathcal{S}_h$ and suppose that

$$p \curlywedge p_x = 0$$

for any absorbent point x of X with respect to $\mathcal{S}$. For the proof of the theorem it is sufficient to show that any such element p is equal to zero. Suppose that $p \neq 0$.

We denote by X_1 the set of all non zero extrem points of the compact convex set

$$K_1 = \{\mu \in \mathcal{S}^* \mid \mu(1) \leq 1\}$$

Since any universally continuous element of $\mathcal{S}^*$ is represented by a Borel measure on X it follows that any compact part of $X_1 \setminus X$ is semipolar. We want to prove that there exists a point

$$x_0 \in \operatorname{carr} p \cap X$$

which is not absorbent with respect to $\mathcal{S}$.

In the contrary case if we denote by A the set of all absorbent points of X with respect to $\mathcal{S}$, then A is countable and therefore carr $p \cap A$ is a G_δ - set contained in $\overline{X_1} \setminus X$. Since

$$x \in A \Rightarrow p \curlywedge p_x = 0$$

it follows that [see 3]

$$p = \chi_{(\operatorname{carr} p \setminus A) \cap X_1} p = \bigvee_{\substack{K \subset (\operatorname{carr} p \setminus A) \cap X_1 \\ k \text{ compact}}} \chi_K \cdot p$$

Any compact part k of $X_1 \setminus X$ being semipolar we deduce p=0.

From the fact that x_o is not an absorbent point we deduce that there exists a naturally closed neighbourhood V of x_o in K_1 such that

$$B_p^{X\setminus V}(x_o) > 0$$

We choose now W a naturally closed neighbourhood of x_o in K_1 such that

$$W \subset \mathring{V}$$

and such that

$$p \leq \theta B_1^{X\setminus V} \quad \text{on } W$$

for a suitable θ. If follows that

$$q = :R(p - B_p^{X\setminus \bar{W}}) \leq \theta B_1^{X\setminus V}$$

Since x_o Carr q we deduce that p B_p and q 0. Obviously q p. Using the fact that p is substractible we have

$$q \precsim \theta B_1^{X\setminus V}$$

which contradicts the fact

$$q \neq 0, \text{ carr } q \subset W.$$

From the preceding theorem it follows that the following assertions are equivalent:

1) any element of $\mathcal{S}_o$ is a pure potential

2) any extrem element of $\mathcal{S}_o$ is a pure potential

3) there exists a set X such that $\mathcal{S}$ is represented as an H-cone of functions on X such that any $\mu \in \mathcal{S}_o^*$ is representable as a Borel measure on X and such that X doesn't possess absorbent points.

4) if $\mathcal{S}$ is represented as an H-cone of functions on X such that any element of $\mathcal{S}_o^*$ is representable as a Borel measure on X then X doesn't possess absorbent points.

5) There exists a pure potential p, p>0.

We say that the standard H-cone $\mathcal{S}$ satisfies the axiom A if one of the preceding assertions 1-5 holds for $\mathcal{S}$.

Theorem 9. Assume that $\mathcal{S}$ satisfies the axiom A and let h be an element of $\mathcal{S}$. Then the following assertions are equivalent:

1) h is substractible

2) There exists $u \in \mathcal{S}$, $u > 0$ and a decreasing sequence of balayages $(B_n)_n$ on $\mathcal{S}$ such that

$$\bigwedge_n B_n u = 0, \quad B_n h = h$$

for any $n \in N$.

3) There exists $u \in \mathcal{S}$, $u>0$, u pure potential and a decreasing sequence of balayages $(B_n)_m$ on $\mathcal{S}$ such that

$$\bigwedge_n B_n u = 0, \quad B_n h = h \quad \text{for any } n \in N.$$

4) For any $u \in \mathcal{S}$, $u>0$, u pure potential there exists a decreasing sequence of balayages $(B_n)_n$ on $\mathcal{S}$ such that

$$\bigwedge_{n\in N} B_n u = 0, \quad B_n h = h \quad \text{for any } n\in N.$$

1) $\Rightarrow$ 4) follows from the corollary of theorem 2

4) $\Rightarrow$ 3) follows using the axiom A

3) $\Rightarrow$ 2) is immediately

2) $\Rightarrow$ 1) follows from theorem 1.

Theorem 10. Assume that $\mathcal{S}$ satisfies the axiom A. Then the following assertions are equivalent:

1) The standard H-cone $\mathcal{S}^*$ satisfies the axiom of polarity.

2) Any pure potential p is of the form

$$p = \sum_{n\in N} p_n, \quad p_n \in \mathcal{S}_o .$$

3) For any $u^* \in \mathcal{S}^*$, $u^*>0$ and any pure potential $p \in \mathcal{S}$ the corresponding representation measure μ_p on Y_{u^*} doesn't charge any semipolar subset of Y_{u^*}.

4) For any $u^* \in \mathcal{S}^*$, $u^*>0$ an u^*-representable element $h \in \mathcal{S}$ is substractible whenever the corresponding representation measure μ_h is carred by a semipolar set.

1) $\Rightarrow$ 3) Let $p \in \mathcal{S}$ be a pure potential and $u^* \in \mathcal{S}^*$, $u^*>0$. From proposition 7 there exists a measure μ_p on Y_{u^*} such that

$$\psi(p) = \int \psi \, d\mu_p$$

for any $\psi \in \mathcal{S}_o^*$. From theorem 3 it follows that the measure μ_p

doesn't charge any polar subset of Y_{u^*} and so, using 1), the measure μ_p doesn't charge any semipolar subset of Y_{u^*}.

3) $\Rightarrow$ 2) follows from [3], theorem 3.7.

2) $\Rightarrow$ 1) follows from [3], theorem 3.8.

1) $\Rightarrow$ 4) Let $u^* \in \mathcal{S}^*$, $u^* > 0$ and let h be a substractible element of $\mathcal{S}$ which is u^*-representable. The corresponding representation measure μ_p may be decomposed in the form

$$\mu_p = \mu' + \mu''$$

where μ' is carred by a polar subset of Y_{u^*} and μ'' doesn't charge any polar set. The element of $\mathcal{S}$ associated with μ'' is a pure potential (see 1) $\Leftrightarrow$ 3)) and being a specific minorant of h is equal to zero. Hence μ_p is carred by a polar subset of Y_{u^*}.

3) $\Rightarrow$ 4) Indeed any specific minorant p of h is represented on Y_{u^*} by a measure carred by a semipolar set and therefeore p can't be a nonzero pure potential.

4) $\Rightarrow$ 3) If p is a pure potential and μ_p is its corresponding representation measure then μ_p doesn't charge any semipolar set.

Corollary. Suppose that $\mathcal{S}$ satisfies axiom A and that its dual $\mathcal{S}^*$ satisfies the axiom of polarity. Then we have:

a) for any $u^* \in \mathcal{S}^*$, $u^* > 0$ and any substractible element $h \in \mathcal{S}$ which is u - reprezentable the corresponding reprezentation measure μ_h is carred by a polar subset al Y_{u^*}.

b) an element $p \in \mathcal{S}$ is a pure potential iff it is u^* - representable for any $u^* \in \mathcal{S}^*$, $u^* > 0$.

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