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DUALIZING DIVISORS OF TWO-DIMENSIONAL SINGULARITIES

by

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Introduction

This paper is a continuation of our previous paper [3], from which we shall borrow in general the terminology. We shall fix a normal non-regular two-dimensional local ring R essentially of finite type over an algebraically closed field k of arbitrary characteristic throughout. Let $f: X \longrightarrow Y = \mathrm{Spec}(Y)$ be the minimal desingularization of R, E the reduced exceptional fibre of f (i.e. the fibre of f over the closed point of Y), $U = X - E$ and $\omega_X = \Omega^2_{X/k}$. Our aim is to study the m-dualizing divisor of R, where $m \geqslant 1$ is a fixed integer. This divisor, denoted by D_m, is by definition the smallest effective divisor Δ on X with support in E such that for every $\varphi \in \Gamma(U, \omega_X^m)$ one has $(\varphi) + \Delta \geqslant o$, where (φ) stands for the divisor on X associated to φ. Of particular interest is the divisor $D = D_1$, called simply the dualizing divisor of R, which coincides with the Gorenstein divisor of R if R is Gorenstein (see [3]), and with the minimally elliptic cycle of E if the geometric genus of R is one (see [8] for the definition of the minimally elliptic cycle of R).

The paper has three sections. The first one contains a list of the known definitions and results which will be used later. Section 2 deals with some general properties of the m-dualizing divisors of R, the key result being theorem (2.7). In the last section we apply this theory to elliptic singularities.

§1. Preliminaries

(1.1) Let $f:X \longrightarrow Y = \mathrm{Spec}(R)$ be an arbitrary desingularization of R, where R is as in the introduction. Denote by E the reduced fibre of f over the closed point y of Y, of irreducible components $E_1,\dots,E_n$, and by $U = X-E \cong Y - y$. By the theorem of connectedness of Zariski (see EGA III (4.3.1)) E is a connected curve, and by a result proved in [10] the intersection matrix $\|(E_i.E_j)\|$ is negative definite.

(1.2) The fundamental cycle of E (or of f) is the smallest divisor $Z > o$ with support in E such that $(Z.E_i) \leqslant o \ \ \forall i = 1,\dots,n$ (see [2]). Z can be calculated by the help of a computation sequence (see [7]), which is a sequence

(1.2.1) $\quad Z_o = o, \quad Z_1 = E_{i_1}, \quad Z_{j+1} = Z_j + E_{i_{j+1}}$,

where E_{i_1} is arbitrary and $(Z_j.E_{i_{j+1}}) > o$ if $j \geqslant 1$. Because the matrix $\|(E_i.E_j)\|$ is negative definite this sequence must terminate, say at Z_s, and then $Z = Z_s$ is just the fundamental cycle of E (see [7]).

(1.3) A desingularization $f:X \longrightarrow Y = \mathrm{Spec}(R)$ is said to be minimal if E does not contain any exceptional curve of the first kind as component. Such a desingularization exists and is unique up to an isomorphism.

(1.4) The arithmetic genus $p_a(R)$ of R is by definition (see [11])

$$p_a(R) = \sup\left\{p_a(\Delta) \ / \ \Delta > o \ \text{ and } \ \mathrm{Supp}(\Delta) \subseteq E\right\},$$

where $p_a(\Delta) = 1/2.(\Delta + K.\Delta) + 1 = 1 - \chi(O_\Delta)$ is the arithmetic genus of Δ (K being a canonical divisor of X). One knows that $p_a(R)$ is independent of the desingularization (see [11]). The geometric genus $p_g(R)$ of R is by definition $\dim H^1(O_X)$ and it is also independent of the desingularization. In general we have the inequality $p_a(R) \leqslant p_g(R)$. We say with Artin [2] that R has a rational singularity if $p_g(R) = o$. The condition " $p_a(R) = o$ " also characterizes rational singularities (see [2]). We say with Wagreich [11] that R has an elliptic singularity if $p_a(R) = 1$.

(1.5) Theorem. ([3]) Let $f: X \longrightarrow Y$ be the minimal desingularization of R. Then R is Gorenstein if and only if one of the following conditions holds:

i) R has a rational double singularity.

ii) There exists a divisor $D > o$ with $Supp(D) = E$ whose dualizing sheaf $\omega_D = (\omega_X \otimes O_X(D)) \otimes O_D$ is isomorphic to O_D.*)

Moreover, if ii) holds, the divisor D (refered in the sequel as the Gorenstein divisor of R) with the above property is unique, $D \geqslant Z$ (Z being the fundamental cycle of f), $p_g(R) = \dim H^o(O_D)$ and $\omega_X \cong O_X(-D)$.

If R has a rational double singularity then the Gorenstein divisor of R is zero by convention.

(1.6) Theorem. ([3]) If $f: X \longrightarrow Y$ is an arbitrary desingularization of R, then for every invertible O_X-module M there is a canonical exact sequence

(1.6.1) $$o \longrightarrow \Gamma(X,M) \xrightarrow{res} \Gamma(U,M) \longrightarrow H^1(M^{-1} \otimes \omega_X)' \longrightarrow H^1(M),$$

where $H^1(.)'$ denotes the dual of the k-vector space $H^1(.)$.

(1.7) Vanishing theorem of Laufer-Ramanujam. (see [3]) If $f: X \longrightarrow Y$ is desingularization of R and L an invertible O_X-module such that $(L.E_i) \geqslant (\omega_X.E_i)$ $\forall i = 1,\dots,n$, then $H^1(L) = o$.

(1.8) Consider the numerical invariants of R (introduced in the complex-analytic case by Knöller in [6]):

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*) In [3] the condition ii) in theorem (4.2) is incorrectly stated because the condition " Supp(D) = E " is omited. The proof of the corrected version of theorem (4.2) is in fact exactly the proof given in [3]. All the other results from [3] which are corollaries of theorem (4.2) remain unaffected.

$$r_m(R) = \dim \Gamma(U, \omega_X^m)/\Gamma(X, \omega_X^m), \quad \forall m \geq 1.$$

It turns out that $r_m(R)$ is independent of the desingularization for every $m \geq 1$ (see [3]), and by (1.6) $r_1(R)$ coincides with the geometric genus of R. These numerical invariants can be computed via:

(1.9) Proposition. ([3]) Let $f: X \longrightarrow Y$ be the minimal desingularization of R. Then $(\omega_X . E_i) \geq 0$ $\forall i = 1, \ldots, n$, $H^1(\omega_X^m) = 0$ and $r_m(R) = \dim H^1(\omega_X^{1-m})$ for every $m \geq 1$.

(1.10) Definition. (Laufer [6]) A divisor $E' > 0$ with support in E is minimally elliptic if $\chi(O_{E'}) = 0$ and $\chi(O_\Delta) > 0$ for every Δ such that $0 < \Delta < E'$. (In other words, $p_a(E') = 0$ and $p_a(\Delta) \leq 0$ for every Δ such that $0 < \Delta < E'$.)

(1.11) Theorem-definition. (Laufer [6]) R has a minimally elliptic singularity if the minimal desingularization $f: X \longrightarrow Y$ of R satisfies one of the following equivalent conditions:

a) The fundamental cycle Z of f is a minimally elliptic divisor.

b) $(Z.E_i) = -(\omega_X . E_i)$ $\forall i = 1, \ldots, n$.

c) $\chi(O_Z) = 0$ and any proper subvariety of Z is the exceptional set for a rational singularity.

(1.12) Proposition. (Laufer [8]) Assume that R has an elliptic singularity and let $f: X \longrightarrow Y$ be the minimal desingularization of R. Then there exists a unique minimally elliptic divisor E' on X. E' is the smallest divisor $F > 0$ with support in E such that $\chi(O_F) = 0$. There exists a computation sequence (1.2.1) for the fundamental cycle Z of E such that $E' = Z_t$ for a suitable $1 \leq t \leq s$ (and in particular, $E' \leq Z$). Finally, $(E'.E_i) = -(\omega_X . E_i)$ for every i such that $E_i \subseteq \mathrm{Supp}(E')$.

Although Laufer works in [8] over the complex field $\mathbb{C}$, his proofs of (1.11) and (1.12) remain valid in arbitrary characteristic.

§2. Dualizing divisors

(2.1) In the situation of (1.1) (with $f:X \longrightarrow Y$ an arbitrary desingularization of R), let L be an arbitrary O_X-module and Δ an effective divisor on X with support in E. Then the map $s \rightsquigarrow s/u$ defines an isomorphism between $L\otimes O_X(\Delta)/U$ and L/U, where $s\in\Gamma(V,L\otimes O_X(\Delta))$ $(V\subseteq U)$ and $u\in\Gamma(X,O_X(\Delta))$ is a section of $O_X(\Delta)$ such that the divisor of u, denoted by (u), coincides with Δ. Applying (1.6) to $M = L\otimes O_X(\Delta)$ and taking into account of the above identification we get the exact sequence:

$$(2.1.1)\qquad o \longrightarrow \Gamma(X,L\otimes O_X(\Delta)) \xrightarrow{\alpha_{L,\Delta}} \Gamma(U,L) \longrightarrow$$
$$\longrightarrow H^1(X,O_X(-\Delta)\otimes L^{-1}\otimes \omega_X)' \longrightarrow H^1(X,O_X(\Delta)\otimes L).$$

Note that

$$(2.1.2)\qquad \mathrm{Im}(\alpha_{L,\Delta}) = \{s\in\Gamma(U,L)-\{o\} \ / \ (s)+\Delta \geqslant o\} \cup \{o\},$$

where as above (s) stands for the divisor of s over X. We are interested in studying the divisors $\Delta \geqslant o$ with support in E such that the (injective) map $\alpha_{L,\Delta}$ be an isomorphism. Denote by F_L the set of all such divisors.

From the exact sequence (2.1.1) we see that $\Delta\in F_L$ if $H^1(O_X(-\Delta)\otimes L^{-1}\otimes\omega_X) = o$. Theorem (1.7) says that one has this vanishing if

$$(2.1.3)\qquad -(\Delta.E_i) \geqslant (L.E_i) \quad \forall i = 1,\dots,n.$$

Divisors Δ satisfying (2.1.3) do exist. In fact, denote by $d = \det\|(E_i.E_j)\|$ and choose n positive integers $d_1,\dots,d_n$ such that $d_i \geqslant (L.E_i)$ and d divides d_i $\forall i = 1,\dots,n$. Then by Cramer's rule there is a unique Δ with $\mathrm{Supp}(\Delta)\subseteq E$ such that $-(\Delta.E_i) = d_i \geqslant (L.E_i)$ $\forall i = 1,\dots,n$. Since $d_i > o$ $\forall i = 1,\dots,n$, an easy argument of [2] shows that $\Delta > o$ (and in fact $\mathrm{Supp}(\Delta) = E$). In other words we have shown that F_L is a non-void set for every desingularization f and for every invertible O_X-module L.

By (2.1.2) the surjectivity of the map $\alpha_{L,\Delta}$ is equivalent with saying that every $s \in \Gamma(U,L)-\{o\}$ has a pole on E_i of order at most r_i, if Δ has the form $\Delta = \sum_{i=1}^{n} r_i E_i$.

(2.2) <u>Lemma.</u> <u>There exists a divisor D_L such that for every other divisor $\Delta \in F_L$ we have $D_L \leq \Delta$.</u>

<u>Proof.</u> It is sufficient to show that if $\Delta_i = \sum_{j=1}^{n} r_{ij} E_j$, $i = 1,2$, are two divisors in F_L then, denoting by $\Delta = \sum_{j=1}^{n} \min(r_{1j}, r_{2j}) \cdot E_j$, for every non-zero section $s \in \Gamma(U,L)$ one has $(s) + \Delta \geqslant o$. But this inequality is an obvious consequence of the following ones: $(s) + \Delta_1 \geqslant o$ and $(s) + \Delta \geqslant o$. Q.E.D.

(2.3) <u>Definition.</u> Let $f: X \longrightarrow Y$ be the minimal desingularization of R. For every $m \geqslant 1$ the divisor $D_{\omega_X^m}$, denoted simply D_m, is called the <u>m-dualizing divisor</u> of R. If $m = 1$ we shall write D instead of D_1, and the divisor D will be refered as the <u>dualizing divisor</u> of R (instead of the 1-dualizing divisor of R).

(2.4) <u>Proposition.</u> <u>Assume that there exists a nowhere vanishing section $s \in \Gamma(U,L)$. Then</u>

$$D_L = -\sum_{i=1}^{n} \min(\mathrm{order}_{E_i}(s), o) \cdot E_i .$$

<u>In particular, if R is Gorenstein and if f is minimal then D coincides with the Gorenstein divisor of R (see (1.5)).</u>

<u>Proof.</u> The hypothesis implies that $L/U \cong O_X/U$. Therefore $\Gamma(U,L) \cong \Gamma(U,O_X) \cong R$ (R is normal) and s is a basis of $\Gamma(U,L)$ as R-module. Hence for every $s' \in \Gamma(U,L)-\{o\}$ there is a function $\alpha \in R-\{o\}$ such that $s' = \alpha \cdot s$. Whence

$$(s') = (\alpha) + (s) \geqslant (\alpha) + \sum_{i=1}^{n} \min(\mathrm{order}_{E_i}(s), o) \cdot E_i ,$$

or else, $D_L \leqslant -\sum_{i=1}^{n} \min(\mathrm{ord}_{E_i}(s), o) \cdot E_i$. The opposite inequality is obvious because $\mathrm{Supp}((s)) \subseteq E$.

If R is Gorenstein and f is minimal there is a nowhere vanishing 2-form $\omega \in \Gamma(U, \omega_X)$, and by (1.9) we have $((\omega).E_i) = (\omega_X.E_i) \geqslant o \ \forall i = 1,\dots,n$. If R has not a rational singularity $-(\omega) \geqslant Z$, where Z is the fundamental cycle of E, and in particular, $\mathrm{ord}_{E_i}(\omega) < o$ for every $i = 1,\dots,n$. By the first part of the proposition $D = -(\omega)$, and thus D is the Gorenstein divisor of R by (1.5).

Q.E.D.

(2.5) Proposition. Assume that $(L.E_i) \geqslant (\omega_X.E_i)$ for every $i = 1,\dots,n$. Then $D_L = o$ if and only if $H^1(L^{-1} \otimes \omega_X) = o$. In particular, $D = o$ if and only if R has a rational singularity.

Proof. The hypothesis and (1.7) imply that $H^1(L) = o$, and therefore the exact sequence (2.1.1) (with $\Delta = o$) becomes:

$$o \longrightarrow \Gamma(X,L) \xrightarrow{\text{res}} \Gamma(U,L) \longrightarrow H^1(X, L^{-1} \otimes \omega_X)' \longrightarrow o.$$

Therefore $D_L = o$ is equivalent to $H^1(L^{-1} \otimes \omega_X) = o$. The last part of the proposition follows taking $L = \omega_X$. Q.E.D.

(2.6) Proposition. Assume that R has not a rational singularity. Then

$$p_g(R) = \dim H^1(O_D).$$

Proof. The exact sequence

$$o \longrightarrow \omega_X \longrightarrow \omega_X \otimes O_X(D) \longrightarrow \omega_D \longrightarrow o$$

yields the exact sequence of cohomology

$$o \longrightarrow \Gamma(X, \omega_X) \longrightarrow \Gamma(X, \omega_X \otimes O_X(D)) \longrightarrow \Gamma(D, \omega_D) \longrightarrow H^1(\omega_X) = o \ \text{(by (1.7))}.$$

Hence applying duality on D we get:

$$\dim H^1(O_D) = \dim \Gamma(D, \omega_D) = \dim \Gamma(X, \omega_X \otimes O_X(D))/\Gamma(X, \omega_X),$$

and recalling that D is the dualizing divisor, $\Gamma(X, \omega_X \otimes O_X(D)) = \Gamma(U, \omega_X)$ (via the map $\alpha_{\omega_X, D}$), and therefore we get:

$$\dim H^1(O_D) = \dim \Gamma(U, \omega_X)/\Gamma(X, \omega_X).$$

The last dimension is precisely $p_g(R)$ by (1.8). Q.E.D.

(2.7) Theorem. Let $m \geqslant 1$ be a positive integer. If $f: X \longrightarrow Y$ is the minimal desingularization of R and D_m is the m-dualizing divisor of R, then

$$(2.7.1) \qquad (D_m . E_i) + m(\omega_X . E_i) \geqslant o \quad \forall i = 1, \dots, n.$$

Moreover the inequalities (2.7.1) become all equalities if and only if $O_X(-D_m) \simeq \omega_X^m$, and in this case we have either $D_m = o$ and R has a rational double singularity, or else $D_m \geqslant Z$, with Z the fundamental cycle of f. In particular

$$(2.7.2) \qquad (D.E_i) + (\omega_X . E_i) \geqslant o \quad \forall i = 1, \dots, n,$$

with equalities everywhere if and only if R is Gorenstein. If R has not a rational singularity then $p_a(D) \geqslant 1$, and $p_a(D) = 1$ and $\mathrm{Supp}(D) = E$ if and only if R is Gorenstein.

Proof. Write $D_m = \sum_{j=1}^{n} r_j E_j$. In order to prove (2.7.1) one distinduishes two cases:

a) $r_i = o$. Then $(D_m . E_i) \geqslant o$, and by (1.9) $(\omega_X . E_i) \geqslant o$ as well. Hence $(D_m . E_i) + m(\omega_X . E_i) \geqslant o$ because $m \geqslant 1$.

b) $r_i > o$. By the definition of D_m there is a section $s_i \in \Gamma(U, \omega_X^m)$ such that

$$(2.7.3) \begin{cases} (s_i) = \sum_{j=1}^{n} r_{ij} E_j + \Delta_i, \\ r_{ij} \leqslant r_j \quad \forall j = 1, \dots, n \quad \text{and} \quad r_{ii} = r_i \\ \Delta_i \text{ effective divisor not containing any } E_j \text{ as component.} \end{cases}$$

Then we have

$$(D_m . E_i) = r_i (E_i^2) + \sum_{j \neq i} r_j (E_i . E_j),$$

$$m(\omega_X . E_i) = ((s_i) . E_i) = -r_{ii}(E_i^2) - \sum_{j \neq i} r_{ij}(E_i . E_j) + (\Delta_i . E_i),$$

and using relations (2.7.3) one gets:

$$(2.7.4) \qquad (D_m . E_i) + m(\omega_X . E_i) = \sum_{j \neq i} (r_j - r_{ij}) + (\Delta_i . E_i) \geqslant o.$$

It remains to see what happens if all (2.7.1) become equalities. We distinguish also two cases:

a') <u>There is an index i such that</u> $r_i = o$. Ordering the components of E convenably, we may assume that $r_i = o$ if and only if $i \leq t$, where t is such that $1 \leq t \leq n$. Then $D_m = \sum_{j=t+1}^{n} r_j E_j$ with $r_j > o$ for $j \geq t+1$. For $i \leq t$ we have seen from case a) that $(D_m . E_i) = (\omega_X . E_i) = o$. We claim that $t = n$, i.e. $D_m = o$ in this case. In fact if $t < n$ then one component of $E_1, \dots, E_t$, say E_i, must intersect $Supp(D_m) = E_{t+1} \cup \dots \cup E_n$ (otherwise E should be not connected), and then $(D_m . E_i) > o$, which is absurd. Hence $t = n$ and thus $(\omega_X . E_i) = o$ (i.e. $p_a(E_i) = o$ and $(E_i^2) = -2$) for every $i = 1, \dots, n$. In other words, if there is an index i such that $r_i = o$ and one has equalities in all (2.7.1) then $D_m = o$ and R has a rational double singularity (and hence R is Gorenstein).

b') $r_i > o$ <u>for every</u> $i = 1, \dots, n$. Then for every i we can choose a section $s_i \in \Gamma(U, \omega_X^m)$ satisfying (2.7.3). From (2.7.4) we see that one has only equalities in (2.7.1) if and only if $r_j = r_{ij}$ for every $j \neq i$ such that $E_i \cap E_j \neq \phi$ and $(\Delta_i . E_i) = o$. In particular we can take $s_j = s_i$ for every $j \neq i$ such that $E_i \cap E_j \neq \phi$. Since E is connected it follows that $s_i = s_j$ for every i and j. Hence there is a section $s \in \Gamma(U, \omega_X^m)$ such that

$$(s) = -D_m + \Delta,$$

where $E \cap Supp(\Delta) = \phi$. Since f is a proper morphism and R is a local ring we infer that $\Delta = o$. Thus we get $\omega_X^m \cong O_X(-D_m)$, and since $(D_m . E_i) = -m(\omega_X . E_i) \leq o$ for every i (by (1.9)), we have also $D_m \geq Z$.

If $m = 1$ and R has not a rational singularity (i.e. $D > o$ by (2.5)) we have:

$$p_a(D) = 1/2.(D^2) + 1/2.(\omega_X . D) + 1 = 1/2. \sum_{i=1}^{n} r_i \left[(D.E_i) + (\omega_X . E_i) \right] + 1 \geq 1.$$

If moreover $Supp(D) = E$ (i.e. $r_i > o$ for every $i = 1, \dots, n$) then $p_a(D) = 1$ if and only if one has <u>only</u> equalities in (2.7.1), that is if and only if R is Gorenstein by theorem (1.5) and the first part of this theorem. Q.E.D.

(2.8) Corollary. Let $f:X \longrightarrow Y$ be the minimal desingularization of R and D the dualizing divisor of R. Assume that $D>o$ (i.e. R has not a rational singularity). Then $p_a(D)=1$ if and only if $\omega_D \cong O_D$, and the number of connected components of D does not exceed the geometric genus of R if $p_a(D)=1$.

Proof. By the definition of ω_D it is clear that if $\omega_D \cong O_D$ then $p_a(D) = 1$ (because $\chi(O_D) = \dim H^o(O_D) - \dim H^1(O_D) = \dim H^o(O_D) - \dim H^o(\omega_D) = o$).

According to the proof of theorem (2.7), the condition " $p_a(D)=1$ " means that

$$(D.E_i) + (\omega_X.E_i) = o \quad \text{for every } i \text{ such that } E_i \subseteq \mathrm{Supp}(D).$$

Then from (2.7.3) and (2.7.4) we infer that $r_j = r_{ij}$ for every $j \neq i$ such that $E_i, E_j \subseteq \mathrm{Supp}(D)$ and $E_i \cap E_j \neq \phi$, and that $(\Delta_i.E_i) = o$. In particular, for every connected component D' of D (such that $\mathrm{ord}_{E_i}(D') = \mathrm{ord}_{E_i}(D)$ if $E_i \subseteq \mathrm{Supp}(D')$) we can take $s_i = s_j = s'$ for every i and j such that E_i and E_j are contained in the support of D'. Thus we get

$$(s') = -D' + \Delta', \text{ with } \mathrm{Supp}(\Delta') \cap \mathrm{Supp}(D') = \phi .$$

Therefore $\omega_{D'} = (\omega_X \otimes O_X(D')) \otimes O_{D'} \cong O_X(\Delta') \otimes O_{D'} \cong O_{D'}$ since $\mathrm{Supp}(\Delta')$ and $\mathrm{Supp}(D')$ have no common points. Since D' was an arbitrary connected component of D then we get $\omega_D \cong O_D$.

Assume now that $p_a(D)=1$; we have seen that this means that $\omega_D \cong O_D$. By proposition (2.6) and duality on D we have $p_g(R) = \dim H^o(\omega_D) = \dim H^o(O_D)$. If t is the number of connected components of D we have obviously $t \leq \dim H^o(O_D)$.

Q.E.D.

(2.9) By a characterization of the rational double singularities due to Knöller [6] if $k = \mathbb{C}$ (and also [3] if k is arbitrary), R has a rational double singularity if and only if $r_m(R) = o$ for every $m \geq 1$. This last condition may be obviously expressed by saying that $D_m = o$ for every $m \geq 1$. Therefore if R has not a rational double singularity then $D_m > o$ for some $m \geqslant 2$.

(2.1o) Proposition. Assume that on the minimal desingularization $f: X \longrightarrow Y$ of R we have $\omega_X^m = O_X(-D_m)$ and $D_m > o$ for some $m \geqslant 1$. Then

$$r_m(R) = p_g(R) - (D_m^2)\cdot(m-1)/2m.$$

Proposition (2.1o) is an extension of corollary (4.7) of [3], where one assumes moreover that R is Gorenstein and has not a rational singularity. Because the proof of (2.1o) is an easy extension of the proof of this corollary, we shall not give it. If $k = \mathbb{C}$ this formula results also from [5].

(2.11) Remarks. 1) Besides the situation where R is Gorenstein and has not a rational singularity, the hypotheses of proposition (2.1o) are also fulfilled in the following two important cases:

1_a) R has a rational singularity of multiplicity > 2. Indeed, if $\hat{U}$ is the punctured spectrum of the completion of R with respect to its maximal ideal, then by [9] $Pic(\hat{U})$ is a finite group and the canonical homomorphism $Pic(U) \longrightarrow Pic(\hat{U})$ is injective. Thus $Pic(U)$ is also finite and therefore ω_X/U has a finite order m (necessarily > 1 because R is not Gorenstein by (1.5)). Hence there is a nowhere vanishing section $s \in \Gamma(U, \omega_X^m)$. By proposition (2.4) $D_m = -(s)$ (since f is minimal, $((s).E_i) = m(\omega_X.E_i) \geqslant o$ $\forall i = 1,\dots,n$ and by [2] we can deduce that $-(s) \geqslant Z$, with Z the fundamental cycle of E), and hence $O_X(-D_m) \cong \omega_X^m$.

1_b) R does not have a rational double singularity but k is the algebraic closure of a finite field. Indeed, by [1] $Pic(U)$ is then a torsion group, and hence ω_X/U has again a finite order in $Pic(U)$.

2) Fix m and t two positive integers and write $D_m = \sum_{i=1}^{n} r_i E_i$ and $D_{mt} = \sum_{i=1}^{n} r_i' E_i$. If $r_i > o$ there is a section $s \in \Gamma(U, \omega_X^m)$ such that

$$(s) = r_i E_i - \sum_{j \neq i} r_j'' E_j + \Delta ,$$

where $r_j'' \leqslant r_j$ for $j \neq i$ and $\Delta \geqslant o$ does not contain any component of E. Then we get

$$(s^t) = -tr_i E_i - \sum_{j \neq i} tr''_j E_j + t\Delta ,$$

and therefore (recalling the definition of D_{mt}) $r'_i \geq tr_i$. In particular,

$D_{mt} \geq tD_m$.

(2.12) Proposition. <u>Assume that R is Gorenstein and has not a rational singularity. Then for every $m \geq 1$ the homomorphism $Pic(D_{m+1}) \longrightarrow Pic(D_m)$ induced by the inclusion $D_m \subset D_{m+1}$ of subschemes of X, is an isomorphism, and the map $Pic(X) \longrightarrow Pic(D)$ induced by the inclusion $D \subset X$ is injective.</u>

Proof. The hypothesis implies that $D > o$, $Supp(D) = E$ and $D_m = mD$ (by (2.5) and (2.4)). The standard exact sequence

$$o \longrightarrow O_X(-mD) \otimes O_D \xrightarrow{\ u\ } O^*_{D_{m+1}} \longrightarrow O^*_{D_m} \longrightarrow 1$$

(in which u is the map $a \rightsquigarrow 1+a$) yields the exact sequence of cohomology:

$$H^1(D, O_X(-mD) \otimes O_D) \longrightarrow Pic(D_{m+1}) \longrightarrow Pic(D_m) \longrightarrow H^2(D, O_X(-mD) \otimes O_D) = o.$$

Now by (1.7) $H^1(O_X(-mD)) = o$ because $(-mD.E_i) = m(\omega_X.E_i) \geq (\omega_X.E_i)\ \forall i$ (the first equality comes from theorem (2.7) and the inequality from the fact that f is minimal). From this we infer that $H^1(D, O_X(-mD) \otimes O_D) = o$ because the natural homomorphism $H^1(O_X(-mD)) \longrightarrow H^1(D, O_X(-mD) \otimes O_D)$ is surjective. Thus we have proved the first part of the proposition.

Let L be an invertible O_X-module such that $L_D = L \otimes O_D$ is isomorphic to O_D. Then $(L.E_i) = deg(L_D/E_i) = deg(O_{E_i}) = o$ for every $i = 1,\dots,n$. The exact sequence

$$o \longrightarrow L \otimes O_X(-D) \longrightarrow L \longrightarrow L_D \cong O_D \longrightarrow o$$

yields the exact sequence of cohomology

$$\Gamma(X,L) \longrightarrow \Gamma(D, O_D) \longrightarrow H^1(L \otimes O_X(-D)).$$

But (always via (1.7)) the last group is zero because $(L \otimes O_X(-D).E_i) = (L.E_i) - (D.E_i) = -(D.E_i) = (\omega_X.E_i)$ for every $i = 1,\dots,n$. We get that the map $\Gamma(X,L) \longrightarrow \Gamma(D,O_D)$ is surjective, and therefore there is a section $s \in \Gamma(X,L)$ whose restriction to D is 1. Consequently $s(x) \neq o$ for every $x \in D$, and since Supp(D) = E, $s(x) \neq o$ for every $x \in X$ (f is a proper morphism and R a local ring), i.e. $L \cong O_X$. Therefore the map $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(D)$ is injective.

Q.E.D.

(2.13) We now define a sequence $\{p_m(R)\}_{m \geq 1}$ of numerical invariants of R by

$$p_m(R) = \dim \Gamma(U,\omega_X^m)/\Gamma(X,\omega_X^m \otimes O_X(D_{m-1})), \quad \forall m \geq 1,$$

where we put $D_o = o$, and where $f: X \longrightarrow Y$ is the minimal desingularization of R. For every $m \geq 1$ we have the inclusions

$$\Gamma(X,\omega_X^m) \subseteq \Gamma(X,\omega_X^m \otimes O_X(D_{m-1})) \subseteq \Gamma(U,\omega_X^m),$$

and thus

(2.13.1) $\quad p_m(R) \leq r_m(R)$ for every $m \geq 1$.

Moreover $p_1(R) = r_1(R) = p_g(R)$.

(2.14) Proposition. If $f: X \longrightarrow Y$ is the minimal desingularization of R then

a) $\quad p_m(R) = \dim H^1(X,\omega_X^{1-m} \otimes O_X(-D_{m-1}))$ $\forall m \geq 1$.

b) If R is Gorenstein then for every $m \geq 1$ $p_m(R) = p_g(R)$.

c) R has a rational double singularity if and only if $p_m(R) = o$ for every $m \geq 1$.

Proof. a) If in (2.1.1) we take $L = \omega_X^m$ and $\Delta = D_{m-1}$ we get the exact sequence

$$o \longrightarrow \Gamma(X,\omega_X^m \otimes O_X(D_{m-1})) \longrightarrow \Gamma(U,\omega_X^m) \longrightarrow H^1(X,\omega_X^{1-m} \otimes O_X(-D_{m-1}))' \longrightarrow$$

$$\longrightarrow H^1(X,\omega_X^m \otimes O_X(D_{m-1})).$$

We claim that $H^1(X,\omega_X^m \otimes O_X(D_{m-1})) = o$. To prove this it will be sufficient by (1.7) to see that

$$(\omega_X^m \otimes O_X(D_{m-1}).E_i) \geqslant (\omega_X.E_i) \quad \forall i = 1,\ldots,n.$$

But this follows from theorem (2.7), and thus the above exact sequence proves the assertion a).

b) If R is Gorentein, by (1.5) we have two possibilities:

- R has a rational double singularity. Then $D_m = o$ for every $m \geqslant 1$, and hence $p_m(R) = r_m(R) = o$ for every $m \geqslant 1$; on the other hand $p_g(R)$ is also zero.

- R has not a rational singularity. Then by (2.4) and (2.5) $D > o$ and $D_m = mD$. Therefore for every $m \geqslant 1$ we have $\omega_X^m \cong O_X(-mD)$. In this case b) follows by applying the formula of a).

c) We have already seen that $p_m(R) = o$ for every $m \geqslant 1$ if R has a rational double singularity. Conversely, assume that $p_m(R) = o$ for every $m \geqslant 1$. Since $p_1(R) = p_g(R) = o$, R has a rational singularity. By remark (2.11) 1_a) Pic(U) is then finite and hence there is a positive integer $s \geqslant 1$ such that $\omega_X^s/U \cong O_U$, or else, $\omega_X^s \cong O_X(-D_s)$. By proposition (2.4) we get that $D_{ts} = tD_s$ for every $t \geqslant 1$.

Now, $p_m(R) = o$ for every $m \geqslant 1$ means that the injective map

$$\Gamma(X,\omega_X^m \otimes O_X(D_{m-1})) \longrightarrow \Gamma(U,\omega_X^m)$$

is an isomorphism, and recalling the definition of D_m, we get that $D_{m-1} \geqslant D_m$ for every $m \geqslant 1$. Therefore $D_p \geqslant D_m$ for every p and m such that $1 \leqslant p \leqslant m$. In particular $D_s \geqslant D_{2s} = 2D_s$, and since $D_s \geqslant o$, we get $D_s = o$. This means that $\omega_X^s \cong O_X$, which implies that $(\omega_X.E_i) = o$ for every $i = 1,\ldots,n$, or else, $p_a(E_i) = o$ and $(E_i^2) = -2$ for every $i = 1,\ldots,n$. In other words, R has a rational double singularity. Q.E.D.

§3. Applications to elliptic and minimally elliptic singularities

(3.1) Theorem. Assume that R has an elliptic singularity, and let $f:X \longrightarrow Y$ be the minimal desingularization of R and D the dualizing divisor of R. Then $\omega_D \cong O_D$. Moreover, R is Gorenstein if and only if Supp(D) = E.

Proof. Since R is elliptic (and hence not rational) $D > o$ (by (2.5)), and thus $p_a(D) \leqslant 1$. By theorem (2.7) we have also $p_a(D) \geqslant 1$. The first part of the theorem follows from corollary (2.8), and the second from the last part of theorem (2.7). Q.E.D.

(3.2) Proposition. In the notations of (1.1) let $Z_o, Z_1, \ldots, Z_s = Z$ be a computation sequence for the fundamental cycle Z of the desingularization $f:X \longrightarrow Y$ of R (where R is as in the introduction). For a fixed index t such that $1 \leqslant t \leqslant s$ denote by $Z' = Z_t$, and let L be an invertible $O_{Z'}$-module such that $\Gamma(L) \neq o$ and $\deg_{E_i}(L_{E_i}) = o$ for every i such that $E_i \subseteq \mathrm{Supp}(Z')$, where we set $L_\Delta = L \otimes O_\Delta$ for every Δ such that $o < \Delta \leqslant Z'$. Then $L \cong O_{Z'}$.

Proof. The exact sequence $(1 \leqslant j < t)$

$$o \longrightarrow L \otimes [O_X(-Z_j)/O_X(-Z_{j+1})] \longrightarrow L_{Z_{j+1}} \longrightarrow L_{Z_j} \longrightarrow o$$

yields the exact sequence

$$(3.2.1) \qquad o \longrightarrow \Gamma(L \otimes [O_X(-Z_j)/O_X(-Z_{j+1})]) \longrightarrow \Gamma(L_{Z_{j+1}}) \longrightarrow \Gamma(L_{Z_j}).$$

On the other hand, $O_X(-Z_j)/O_X(-Z_{j+1})$ is an invertible $O_{E_{i_{j+1}}}$-module of degree $-(E_{i_{j+1}}.Z_j)$, and since $\deg(L_{E_{i_{j+1}}}) = o$ we get

$$\deg(L \otimes [O_X(-Z_j)/O_X(-Z_{j+1})]) = -(E_{i_{j+1}}.Z_j) < o.$$

Hence $\Gamma(L \otimes [O_X(-Z_j)/O_X(-Z_{j+1})]) = o$, and the exact sequence (3.2.1) becomes:

$$(3.2.2) \qquad o \longrightarrow \Gamma(L_{Z_{j+1}}) \longrightarrow \Gamma(L_{Z_j}), \quad 1 \leqslant j < t.$$

If $a \in \Gamma(L)$ is a non-zero global section of L one deduces that the restriction $a_j = a/Z_j$ is again non-zero for every $1 \leq j \leq t$. In particular, $a_1 \neq o$ and since $Z_1 = E_{i_1}$ is an integral curve and $\deg(L_{Z_1}) = o$, $a(x) = a_1(x) \neq o$ for every $x \in Z_1 = E_{i_1}$.

Since $Z_2 = E_{i_1} + E_{i_2}$ and $(E_{i_1}.E_{i_2}) > o$ we have $E_{i_1} \neq E_{i_2}$ and $E_{i_1} \cap E_{i_2} \neq \emptyset$. Taking into account that $a(x) = a_1(x) \neq o$ for every $x \in E_i$ and that the set $\{x \in Z_2 \,/\, a_2(x) \neq o\}$ is open in Z_2, it follows that $a(x) = a_2(x) \neq o$ for every $x \in Z_2$. Repeating this procedure we see by induction that $a(x) \neq o$ for every $x \in Z'$, i.e. $L \cong O_{Z'}$. Q.E.D.

(3.3) Remark. If in the exact sequence (3.2.2) we take $L = O_{Z'}$ one sees by induction that $H^o(O_{Z'}) = k$, and in particular, $H^o(O_Z) = k$. If moreover R has an elliptic singularity and E' is the minimally elliptic divisor on X (with $f:X \longrightarrow Y$ the minimal desingularization of R), then by (1.12) there is a computation sequence (1.2.1) such that $E' = Z_t$ for a suitable $1 \leq t \leq s$. We deduce that $H^o(O_{E'}) = k$, and since $\chi(O_{E'}) = o$, we have also $\dim H^1(O_{E'}) = 1$.

(3.4) Corollary. Assume that R has an elliptic singularity and let E' be the minimally elliptic divisor on the minimal desingularization X of R. Then $\omega_{E'} \cong O_{E'}$.

Proof. By (1.12) $(E'.E_i) + (\omega_X.E_i) = o$ for every i such that $E_i \subseteq \mathrm{Supp}(E')$ and $E' = Z_t$ for a suitable computation sequence (1.2.1). Since $\omega_{E'} = \omega_X \otimes O_X(E') \otimes O_{E'}$ we get

$$\deg_{E_i}(\omega_{E'} \otimes O_{E_i}) = o \quad \text{for every } i \text{ such that } E_i \subseteq \mathrm{Supp}(E').$$

By duality on E' and the above remark we have $\dim \Gamma(\omega_{E'}) = \dim H^1(O_{E'}) = 1$. Now the conclusion follows applying proposition (3.2). Q.E.D.

(3.5) Proposition. Assume that R has an elliptic singularity and let D and E' be the dualizing and minimally elliptic divisor of R respectively (on the minimal desingularization X of R). Then $D \geq E'$ and D has a connected support. Moreover the following conditions are equivalent:

a) $D = E'$.

b) $Supp(D) = Supp(E')$.

c) $p_g(R) = 1$.

Proof. By theorem (3.1) $p_a(D) = 1$, or equivalently $\chi(O_D) = o$. Therefore applying (1.12) we get $D \geq E'$. Let now $D^{(1)}, \dots, D^{(t)}$ be the connected components of D (such that for every $i = 1, \dots, t$ and $E_j \subseteq Supp(D^{(i)})$, $ord_{E_j}(D^{(i)}) = ord_{E_j}(D)$). Then $\omega_{D^{(i)}} \cong O_{D^{(i)}}$ and hence $\chi(O_{D^{(i)}}) = o$. Again by (1.12) we get that for every i, $D^{(i)} \geq E'$, and hence necessarily $t = 1$.

By theorem (2.7), theorem (3.1) and proposition (1.12) we have:

$$(D.E_i) = (E'.E_i) = -(\omega_X.E_i) \quad \text{for every i such that } E_i \subseteq Supp(E').$$

Taking into account that $\|(E_i.E_j)\|$ is negative definite we deduce that a) $\Longleftrightarrow$ b).

a) $\Longrightarrow$ c). By remark (3.3) $\dim H^1(O_{E'}) = 1$. Since $D = E'$ this implication follows from proposition (2.6).

c) $\Longrightarrow$ a). The exact sequence

$$o \longrightarrow O_X(-E') \longrightarrow O_X \longrightarrow O_{E'} \longrightarrow o$$

yields the exact sequence of cohomology

$$H^o(O_X) \xrightarrow{u} H^o(O_{E'}) \longrightarrow H^1(O_X(-E')) \longrightarrow H^1(O_X) \longrightarrow H^1(O_{E'}) \longrightarrow o.$$

Since $H^o(O_{E'}) = k$ (cf. the proof of (3.4) and remark (3.3)) the map u is surjective, and since we have also $\dim H^1(O_X) = \dim H^1(O_{E'}) = 1$, we get $H^1(O_X(-E')) = o$. Then the exact sequence (2.1.1) (with $L = \omega_X$ and $\Delta = E'$) shows that the natural map $\Gamma(X, \omega_X \otimes O_X(E')) \longrightarrow \Gamma(U, \omega_X)$ is an isomorphism. In other words $E' \in F_{\omega_X}$, and recalling the definition of D we get that $E' \geq D$, and finally $D = E'$ by the first part of the proposition. Q.E.D.

(3.6) Corollary. Let $f: X \longrightarrow Y$ be the minimal desingularization of R, and Z the fundamental cycle of R. The following conditions are equivalent:

a) R has a minimally elliptic singularity.

b) R is Gorenstein and its Gorenstein divisor coincides to Z.

c) R is Gorenstein and $p_g(R) = 1$.

Proof. a) $\Longrightarrow$ b). This implication follows from (3.4), the fact that $E' = Z$ and theorem (1.5).

b) $\Longrightarrow$ c). By theorem (1.5) and remark (3.3) $p_g(R) = \dim H^o(O_Z) = 1$.

c) $\Longrightarrow$ a). Let D be the Gorenstein divisor of R. By proposition (2.4) D coincides to the dualizing divisor of R, and $D \geqslant Z$. Since $p_g(R) = 1$, proposition (3.5) shows that $D = E'$. But $E' \leqslant Z$ by proposition (1.12). Thus $E' = Z$, that is R has a minimally elliptic singularity (see (1.11)). Q.E.D.

(3.7) Note. The equivalence between conditions a) and c) of corollary (3.6) is a result due to Laufer if $k = \mathbb{C}$ (see [8]). Laufer's proof makes use of some analytic arguments at some points.

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