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Radu Nicolae GOLOGAN

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Radu Nicolae GOLOGAN*)

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*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 77538 Bucharest, Romania

AN EXTENSION OF AKCOGLU'S ERGODIC THEOREM FOR A CLASS OF NONPOSITIVE L_2-CONTRACTIONS

by Radu-Nicolae Gologan

Abstract. Using a simple matrix representation we prove a pointwise ergodic theorem for some kind of order relatively bounded real contraction in L_2-spaces.

INTRODUCTION

In [1] Akcoglu succeded in proving the almost everywhere convergence of the ergodic ratios for positive L_p-contractions. $(p \geq 2)$

We know also from [2] that if the assumption of positivelly is dropped the ratios may diverge.

Using ([4]) our proof of Chacon's ergodic theorem for real contractions in L_1, we show that for some simple nonpositive contractions in L_2, Akcoglu's theorem works; for exemple for contractions of the form $f \mapsto gTf$ where g real, $|g| \leq 1$ and T is a positive L_2-contraction.

PRELIMINARIES

1. Let (X,μ) a σ-finite measure space.

Let T denote a real $L_2=L_2(X,\mu)$ contraction, that is f real implies Tf real, and $\int |Tf|^2 d\mu \leq \int |f|^2 d\mu$ for every $f \in L_2$.

Our assumption will be that for every $g \in L_2^+$

$$|T|g = \sup_{\substack{f \in L_2^{\mathbb{R}} \\ |f| \leq g}} Tf \text{ is } L_2 \text{ bounded such that:}$$

$$\int (|T| g)^2 d\mu \leq \int g^2 d\mu. \qquad (1)$$

It is then easy to check that $|T|$ extends to a L_2-contraction because of the Riesz-lattice structure of L_2 (see for exemple [3]). As usualy $|T|$ will be called the linear modulus of T.

Let is denote $A = \frac{T + |T|}{2}$, $B = \frac{|T| - T}{2}$. From the fact that $|Tf| \leq |T| \cdot f$ for $f \in L_2$, f real, we see that A and B are positive L_2-contractions.

The matrix

$$\tilde{T} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

defines a positive linear operator in $L_2(\tilde{X},\tilde{\mu})$ where $\tilde{X} = X \sqcup X$ and $\tilde{\mu}$ is the sum measure.

Let us check that under our assumption, $\tilde{T}$ defines a positive L_2-contraction.

It sufices to prove that:

$$\int \left| \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right|^2 d\tilde{\mu} \leq \int \begin{pmatrix} f \\ g \end{pmatrix}^2 d\tilde{\mu}$$

for f,g L_2^+, that is :

$$\int (|Af+Bg|^2 + |Bf + Ag|^2) d\mu \leq \int (|f|^2 + |g|^2) d\mu$$

Recalling the definitions of A and B we have :

$$|Af + Bg|^2 + |Bf + Ag|^2 =$$

$$= \left|T\left(\frac{f+g}{2}\right) + |T|\left(\frac{f-g}{2}\right)\right|^2 + \left|T\left(\frac{f+g}{2}\right) - |T|\left(\frac{f-g}{2}\right)\right|^2 =$$

$$= \frac{1}{2}\left|T\ (f+g)\right|^2 + \frac{1}{2}\left||T|\ (f-g)\right|^2$$

T and $|T|$ being L_2-contractions we have :

$$\int(|Af+Bg|^2+|Bf+Ag|^2)d\mu=$$
$$=\frac{1}{2}\int|T(f+g)|^2d\mu+\frac{1}{2}\int||T|(f-g)|^2d\mu\leq$$
$$\leq\frac{1}{2}\int|f+g|^2d\mu+\frac{1}{2}\int|f-g|^2d\mu=\int(f^2+g^2)d\mu$$

which proves our affirmation.

On the other hand we have T=A-B and and writing for $\tilde{T}^n$:

$$\tilde{T}^n=\begin{pmatrix}A_n & B_n\\ B_n & A_n\end{pmatrix}$$

we see that:

$$A_{n+1}=AA_n+BB_n$$
$$B_{n+1}=AB_n+BA_n$$

that is $T^n=A_n-B_n$.

Our result consists n simply applying Akcoglu's theorem for $\tilde{T}$.

2. Theorem. Let T be a real L_2-contraction satisfiing (1). Then for every $f\in L_2$

$$\lim_{n\to\infty}\frac{f+Tf+\ldots+T^{n-1}f}{n}$$

exists μ- a.e. in L_2.

Proof.

It sufices to prove the result for $f \in L_2^+$.

Let us apply Akoglu's theorem for $\tilde{f}=(f,0)$. We have that in $\tilde{L}_2$ the limit

$$\lim_{n\to\infty} \frac{\tilde{f} + \begin{pmatrix} A & B \\ B & A \end{pmatrix}\tilde{f} + \dots + \begin{pmatrix} A_n & B_n \\ B_n & A_n \end{pmatrix}\tilde{f}}{n+1}$$

exists $\tilde{\mu}$-a.e. Writing on components we have that boths:

$$\mu\text{-a.e.} \lim_{n\to\infty} \frac{f + Af + \dots + A_n f}{n+1}.$$

and

$$\mu\text{-a.e.} \lim_{n\to\infty} \frac{f + Bf + \dots + B_n f}{n+1}$$

are in L_2.

Taking the diference and recalling that $T^n = A_n - B_n$, the theorem is proved.

Remarks.

1. If $g \in L_2$ is real ang $|g| \leq 1$ and P is a positive L_2-contraction than $Tf = g \cdot Pf$ defines a real contraction satisfying 1. We see than that

$$\lim \frac{f + gPf + gP(gPf) + \dots + gP(g(\dots Pf)}{n}.$$

exists μ - a.e.

2. We do not know how restrictive is condition 1. Perhaps an alternative formulation will be useful.

3. It is easy to see that for other L_p spaces the proof dosen't work.

4. In [5] the same construction is used to extend in Riesz spaces for nonpositive linear operators the theorem of Ornstein and Brunel. The proof contains a regretable error but a reformulation of the initial assumptions makes it to work.

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