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THE ADDITION OF LOCAL OPERATORS ON PRODUCT SPACES

by

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The Addition of Local Operators on Product Spaces

by

L.Stoica

The study of a product space is a classical theme in potential theory. While the early papers [4], [8] study functions on the product space which are related to the structures of the terms of the product, the present paper following the idea of the probabilistic work [3] constructs a structure on the product space and studies this structure. Namely we construct local operators which fulfil the requirements from [13] on a product space. This subject is a particular aspect from the recent program of N.Boboc which asks for the construction of the notion of product in potential theory.

In section 2 we consider two local operators L^1, L^2 on locally compact spaces X_1, X_2 which possess bases of regular sets. Then we construct the sum L^1+L^2 on $X_1 \times X_2$ and prove that the product of two regular sets is regular for L^1+L^2.

In section 3 we prove a similar result for the sum of a series of local operators on the product of a sequence of compact spaces.

Section 4 considers a local operator L and constructs the operator L-d/dt. A similar construction within a different frame was made in [11].

In section 5 we are interested in those local operators which yield Bauer spaces and in properties which imply that the sum of two such local operators also/yields a Bauer space.

Section 6 shows that the sum of a series of local operators preserves this properties under suitable conditions. This result extends to compact spaces the result of C.Berg, although more precise, which constructs a Brelot space on the infinite dimensional torus [1]. It should be also noted that harmonic spaces in the sense of Constantinescu and Cornea are constructed on product spaces by E.Popa and V.Schirmeier.

All terminology and notation which is not specifically explained here will be that of [13].

1. A simple lemma

Let X be a locally compact space with a countable base and L a local operator on X. Suppose that L is locally dissipative and locally closed.

1.1. Lemma

Let U be a relatively compact open set such that $\partial U \neq \emptyset$ and

1^{o} for any $x \in \partial U$ there exists a finite family $\{\varphi_1, \ldots \varphi_k\} \subset D(U,L)$ such that $\varphi_i > 0$, $L\varphi_i < -1$, $i=1,\ldots k$ and

$$\lim_{\substack{y \to x \\ y \in U}} (\inf_{i \leq k} \varphi_i(y)) = 0,$$

2^{o} there exists $\varphi \in D(U,L)$ such that $L\varphi < -1$, $\varphi \geq 0$ and $\|\varphi\| < \infty$,

3^{o} the spaces $D_o(U)$ and $LD_o(U)$ are dense in $\mathcal{C}_o(U)$, where

$$(1.1) \qquad D_o(U) = \{f \in \mathcal{C}_o(U) \cap D(U,L) / Lf \in \mathcal{C}_o(U)\}.$$

Then U is P-regular.

Proof

First we are going to prove the following property:

(1.2) if $\{f_n / n \in N\}$ is a sequence in $D_o(U)$ such that $Lf_n \longrightarrow 0$ uniformly on each compact subset of U and $\sup_n \|Lf_n\| < \infty$, then $f_n \longrightarrow 0$ uniformly.

Let $\varepsilon > 0$. We choose a finite family $\{\varphi_1, \ldots, \varphi_k\} \subset D(U,L)$ such that $\varphi_i > 0$, $L\varphi_i \leqslant -1$, $i=1,\dots,k$ and the set $K = \bigcap_{i \leqslant k} \{\varphi_i \geqslant \varepsilon\}$ is compact.

If $f \in D_o(U)$, $|Lf| \leqslant \varepsilon$ on K and $\|Lf\| \leqslant 1$, then from [13] 1.4 we deduce

$$\sup_{\mathring{K}} (f-\varepsilon\varphi) \leqslant \max (0, \sup_{\partial K} (f-\varepsilon\varphi)),$$

because $Lf-\varepsilon L\varphi \geqslant 0$ on K. Hence

$$\sup_{\mathring{K}} f \leqslant \varepsilon\|\varphi\| + \max (0, \sup_{\partial K} f).$$

On the other hand $L(f-\varphi_i) \geqslant 0$ implies $f \leqslant \varphi_i$, $i=1,\dots,k$, and hence $\sup_{U \setminus K} f \leqslant \varepsilon$, which leads to

$$\|f\| \leqslant \varepsilon (\|\varphi\| + 1).$$

(1.2) results from this inequality.

Let now $f \in \mathscr{C}_b(U)$. Condition 3° allows us to choose a sequence $\{\varphi_n / n \in N\} \subset D_o(U)$ such that $\sup_n \|L\varphi_n\| < \infty$ and $L\varphi_n \longrightarrow f$ uniformly on the compact subsets of U. Next we assert that $\{\varphi_n\}$ is a Cauchy sequence in $\mathscr{C}_o(U)$. If it is not, we have a $\delta > 0$ and a subsequence $\{\varphi_{n_k} / k \in N\}$ such that $\|\varphi_{n_k} - \varphi_{n_{k+1}}\| \geqslant \delta$, $k \in N$. On the other hand $L\varphi_{n_k} - L\varphi_{n_{k+1}} \longrightarrow 0$ uniformly on the compact subsets of U and this contradicts (1.2). We conclude that $\varphi_n \longrightarrow u$ uniformly, $u \in \mathscr{C}_o(U) \cap D(U,L)$, and $Lu=f$. This proves the lemma.

1.2. Corollary

Let us suppose that the family of all P- and D-regular sets forms a base of X. If U,V are two P-regular sets then $U \cap V$ is P-regular too.

Proof

Conditions 1° and 2° from the lemma are obviously fulfilled. In order to check condition 3° one uses [13] 3.4.

2. The sum of two local operators

Let X_1, X_2 be locally compact spaces with countable bases and let L_i be (a) locally dissipative local operator on X_i such that the family of P-and D-regular sets is a base of X_i, $i=1,2$. We denote by $X=X_1 \times X_2$; if $U \subset X$ and $x \in X_1$, $y \in X_2$ then we put $U_x=\{z \in X_2 / (x,z) \in U\}$, $U_y=\{z \in X_1 / (z,y) \in U\}$, if f is a function on U, then $f_x=f(x,.)$ is defined on U_x and similarly $f_y=f(.,y)$ is defined on U_y. We define a local operator on X, $L=L(L_1,L_2)$, as follows: if U is an open set in X, then $D(U,L)$ is the family of all functions $f \in \mathscr{C}(U)$ such that:

1° for any $x \in X_1$, $f_x \in D(U_x, L_2)$

2° for any $y \in X_2$, $f_y \in D(U_y, L_1)$

3° $Lf \in \mathscr{C}(U)$ where $Lf(x,y)=L_1 f_y(x)+L_2 f_x(y)$, $(x,y) \in U$.

L is obviously locally dissipative. If U_i is an open set in X_i and $f_i \in D(U_i, L_i)$, $i=1,2$, then $f_1 \otimes f_2 \in D(U_1 \times U_2, L)$ and $L(f_1 \otimes f_2)=L_1 f_1 \otimes f_2 + f_1 \otimes L_2 f_2$. Thus one can prove the property from [13] 1.6:

(2.1) $(\forall)$ $x \in X$, $(\forall)$ V neighbourhood of x, $(\exists)$ U open, $x \in U$, $\overline{U} \subset V$, $(\exists)$ $g \in \mathscr{C}_0(U) \cap D(U,L)$ such that $g(x) > 0$ and $\|Lg\| < \infty$.

Then $\tilde{L}$ the local closure of L exists, is locally dissipative and locally closed. We denote by $L_1+L_2=\tilde{L}$. Next we are going to prove that the family of P-and D-regular sets in a base of X.

2.1. Proposition

Let U_i be a P-regular set in X_i such that there exists $\varphi_i\in D(U_i,L_i)$, $\varphi_i\geq 1$, $L_i\varphi_i\leq 0$, i=1,2. Then $U=U_1\times U_2$ is P-regular (with respect to L_1+L_2).

Proof

We are going to apply 1.1. Thus we remark that $\varphi_1 \otimes G^{U_2}1$ fulfils the requirements of 1.1, 2^o. Condition 1.1, 1^o may be checked using the functions $\varphi_1 \times G^{U_2}1$ and $G^{U_1}1 \times \varphi_2$. In the reminder proof we are going to check 1.1, 3^o. First we introduce some notations We denote by $G^i_\lambda=G^{U_i}_\lambda$. The Hille-Yosida theorem applied on the space $\mathscr{C}_o(U_i)$ gives us a C_o-class semigroup $\{P^i_t/t>0\}$ such that $G^i_\lambda=\int_0^\infty \exp(-\lambda t)P^i_t dt$, $\lambda\geq 0$. $\{P^i_t/t>0\}$ extends also as sub-Markov semigroup of kernels on U_i. The product semigroup $P_t=P^1_t \otimes P^2_t$ is the natural tensor product of kernels, i.e. if $(x,y)\in U$, then $P^{(x,y)}_t=P^{1x}_t \otimes P^{2y}_t$ is a product measure on U. $\{P_t/t>0\}$ is also a C_o-class semigroup on the space $\mathscr{C}_o(U)=\mathscr{C}_o(U_1)\otimes\mathscr{C}_o(U_2)$. Now we remark that $G_\lambda=\int_0^\infty \exp(-\lambda t)P_t dt, \lambda\geq 0$ define a family of kernels on U and $G_o 1\leq G^1_o 1 \otimes 1$. If $f_i\in\mathscr{C}_o(U_i)$, i=1,2, then

$$\left|\int_s^\infty P^1_t f_1 \times P^2_t f_2 dt\right| \leq \int_s^\infty P^1_t 1 dt \, \|f_1\|\cdot\|f_2\|$$

But $\int_s^\infty P_t^1 1dt \longrightarrow 0$ $(s\to\infty)$ uniformly because $\int_0^\infty P_t^1 1dt = G_o^1 1 \in \mathcal{C}_o(U_1)$.

Since $\int_0^s P_t^1 f_1 \otimes P_t^2 f_2 dt \in \mathcal{C}_o(U)$ we deduce $G_o(f_1 \otimes f_2) \in \mathcal{C}_o(U)$. Further the relation $\|G_o\| \leq \|G_o^i 1\|$ shows that G_o is a linear operator on $\mathcal{C}_o(U)$.

Now we remark that $G_o^1(\mathcal{C}_o(U_1)) \otimes G_o^2(\mathcal{C}_o(U_2)) \subset D_o(U)$. Namely if $f_i \in \mathcal{C}_o(U_i)$, $i=1,2$, then $G_o^1 f_1 \otimes G_o^2 f_2 \in D(U,L)$ and $L(G_o^1 f_1 \otimes G_o^2 f_2) = -G_o^1 f_1 \otimes f_2 + f_1 \otimes G_o^2 f_2$. This shows that $\overline{D_o(U)} = \mathcal{C}_o(U)$. Further we need the following equalities:

(2.2) $$G_o(G_o^1 f_1 \otimes G_o^2 f_2)(x_1,x_2) = G_o^1(G_o(f_1 \otimes G_o^2 f_2)(\cdot,x_2))(x_1) =$$
$$= G_o^2(G_o(G_o^1 f_1 \otimes f_2)(x_1,\cdot))(x_2),\quad f_i \in \mathcal{C}_o(U_i),\ x_i \in U_i,\ i=1,2,$$

(2.3) $$G_o^1 f_1 \otimes G_o^2 f_2 = G_o(G_o^1 f_1 \otimes f_2 + f_1 \otimes G_o^2 f_2),\quad f_i \in \mathcal{C}_o(U_i),\ i=1,2.$$

The first results by strightforward computations. The second equality results from

$$\int_0^\infty\int_0^\infty (P_s^1 f_1)(P_t^2 f_2)dsdt = \int_0^\infty \left(\int_0^\infty (P_u^1 f_1)(P_t^2 f_2)du\right)dt +$$
$$+ \int_0^\infty \left(\int_s^\infty (P_s^1 f_1)(P_u^2 f_2)du\right)ds.$$

From (2.2) and (2.3) we deduce $L(G_o(G_o^1 f_1 \otimes G_o^2 f_2)) = -G_o^1 f_1 \otimes G_o^2 f_2$, and hence $G_o^1(\mathcal{C}_o(U_1)) \otimes G_c^2(\mathcal{C}_o(U_2)) \subset LD_o(U)$, $\overline{LD_o(U)} = \mathcal{C}_o(U)$. The operator L_1+L_2 being an extention of L we conclude that 1.1, 3^o is fulfilled and U is P-regular.

If the set U_i satisfies $\overline{U}_i \subset V_i$ for some P-regular set, V_i, then $\varphi_i = \alpha G^{V_i} 1$ fulfils $\varphi_i \geq 1$ on U_i for suitable α. Then we deduce that the sets $U = U_1 \times U_2$ which fulfil the requirements from 1.2 form a base of X. Hence $L_1 + L_2$ has a base of P-regular sets or (equivalently on account of 1.8, 4^o [13]) it has a base of P-and D-regular sets.

Now let X_1, X_2, X_3 be three locally compact spaces with countable bases. Let L_i be a locally dissipative local operator on X_i such that the family of P-and D-regular sets is a base of X_i, $i=1,2,3$. On $X_1 \times X_2 \times X_3$ we define a local operator, L^o, as follows: if U is an open set in $X_1 \times X_2 \times X_3$ then $D(U, L^o)$ is the family of all functions $f \in \mathcal{C}(U)$ such that for each $x = (x_1, x_2, x_3) \in U$,

a) $f_{(x_i, x_j)} \in D(U_{(x_i, x_j)}, L_k)$ for $i \neq j \neq k \neq i$, $i, j, k \in \{1, 2, 3\}$,

b) $L^o f \in \mathcal{C}(U)$, where $L^o f(x_1, x_2, x_3) = L_1 f_{(x_2, x_3)}(x_1) + L_2 f_{(x_1, x_3)}(x_2$

$$+ L_3 f_{(x_1, x_2)}(x_3).$$

L^o is locally dissipative and fulfils (2.1). Hence its local closure $\tilde{L}^o$ exists. We also note that the proof of 2.1 may be repeated here word by word, and hence we deduce that $\tilde{L}^o$ has a base of P- and D-regular sets. On the other hand from $L_1 + L_2$ and L_3 we get another local operator $L^{oo} = L(L_1 + L_2, L_3)$ on $(X_1 \times X_2) \times X_3$ defined by 1^o, 2^o, 3^o, such that its local closure, $\tilde{L}^{oo}$, is $(L_1 + L_2) + L_3$. It is easy to see that $D(U, L^o)$ $D(U, L^{oo})$ for each open set U and L^{oo} extends L^o hence $\tilde{L}^{oo}$ extends $\tilde{L}^o$. If U is P- regular or D-regular with respect to $\tilde{L}^o$ then it is alive with respect to $\tilde{L}^{oo}$ and the kernels H^U, G^U associated to $\tilde{L}^o$ coincide with those associated to $\tilde{L}^{oo}$. If U is P- and D-regular and $\varphi\in D(V,\tilde{L}^{oo})$, $\overline{U}\subset V$, $\|\tilde{L}^{oo}\varphi\|<\infty$, then $G^U(-\tilde{L}_{oo}\varphi)+H^U\varphi=\varphi$. Hence $\varphi\in D(U,\tilde{L}_o)$ and $\tilde{L}^o\varphi=\tilde{L}^{oo}\varphi$. Further one deduces $\tilde{L}^o=\tilde{L}^{oo}$. Thus we may put the notation $L_1+L_2+L_3=(L_1+L_2)+L_3=L_1+(L_2+L_3)$ and conclude that the addition of local operators is well defined for each finite family.

3. The sum of a series of local operators

Let $\{X_i / i \in N\}$ be a sequence of compact spaces with countable bases and for each $i \in N$ let L^i be a locally dissipative local operator on X_i such that $1 \in D(X_i, L^i)$, $L^i 1 = 0$ and the family of P- and D-regular sets on X_i is a base. We know from [13] Corollary 3.3 that there exists a Markov semigroup of kernels $\{P^i_t / t > 0\}$ on X_i which is also a C_o-class semigroup of contractions on the Banach space $\mathscr{C}(X_i)$ whose infinitesimal generator has $D(X_i, L^i)$ as domain and coincides with L^i as linear operator on this last space. We denote by $X = \prod_{i \in N} X_i$.
For $J \subset N$ we put $X(J) = \prod_{i \in J} X_i$. If $A \subset X$, $f : A \longrightarrow R$ and $x \in X(J)$ we put $A_x = \{ y \in X(N \setminus J) / (x,y) \in A \}$ and $f_x : A_x \longrightarrow R$, $f_x(.) = f(x,.)$. If J is finite we put $L(J) = \sum_{i \in J} L^i$.

We define a local operator L on X as follows: if U is an open set $U \subset X$, then $D(U, L)$ is the family of all functions f which fulfil the following property:

1^o there exists a finite set $J = J(f) \subset N$ such that for each pair $(x,y) \in U$, $x \in X(J)$, $y \in X(N \setminus J)$, the function f_x is constant on U_x and $f_y \in D(U_y, L(J))$.

For a function f and (x,y) as above we define $Lf(x,y) = L(J) f_y(x)$. L is locally dissipative and fulfils (2.1). Its local closure $\tilde{L}$ is locally dissipative and locally closed. We denote by $\sum_{i \in N} L^i = \tilde{L}$. The next proposition implies that $\sum_{i \in N} L^i$ has a base of P- and D-regular sets.

3.1. Proposition

Let $\mathcal{J}$ be a finite subset of N and U a P-regular set in $X(\mathcal{J})$ (with respect to $L(\mathcal{J})$). Then $U_o = U \times X(N \setminus \mathcal{J})$ is P-regular (with respect to $\sum_{i \in N} L^i$).

Proof

The proof is similar to that of 2.1, therefore we only sketch it. Let $\{P_t^* / t > 0\}$ be the semigroup of kernels on U such that $G_\lambda^U = \int_0^\infty \exp(-\lambda t) P_t^* dt$, $\lambda \geq 0$. For a fixed finite set $K \subset N \setminus \mathcal{J}$ we define $P_t^K = P_t^* \otimes (\bigotimes_{i \in K} P_t^i)$, $G_\lambda^K = \int_0^\infty \exp(-\lambda t) P_t^K dt$, $\lambda \geq 0$ and for $i \in K$ put $G_\lambda^i = \int_0^\infty \exp(-\lambda t) P_t^i dt$, $\lambda > 0$. First we are going to prove that $U_K = U \times X(K)$ is P-regular with respect to $L(\mathcal{J} \cup K)$. Since $G_o^i = \sup_{\lambda > 0} G_\lambda^i$ is not finite for $i \in K$ we have no equalities analogous to (2.2) and (2.3). Therefore we consider $\lambda > 0$ and put $\alpha = \lambda / n$ where n is the number of elements from K. Then for $f^* \in \mathcal{C}_o(U)$ and $f_i \in \mathcal{C}(X_i)$, $i \in K$ we have

$$g = G_o^U f^* \otimes (\bigotimes_{i \in K} G_\alpha^i f_i) \in D(U_K, L(\mathcal{J} \cup K)) \text{ and } L(\mathcal{J} \cup K)_\lambda g = (\sum_{i \in \mathcal{J}} L^i + \sum_{i \in K} L_\alpha^i) g \in \mathcal{C}_o(U_K).$$

For $L(\mathcal{J} \cup K)_\lambda$ we have some equalities similar to (2.2) and (2.3), namely

$$(3.1) \qquad G_\lambda^K (G_o^U \otimes (\bigotimes_{i \in K} G_\alpha^i)) = (G_o^U \otimes I_{\mathcal{C}(X(K))}) G_\lambda^K (I_{\mathcal{C}_o(U)} \otimes (\bigotimes_{i \in K} G_\alpha^i)) =$$

$$= (I_{\mathcal{C}_o(U)} \otimes G_\alpha^j \otimes I_{\mathcal{C}(X(K \setminus \{j\}))}) G_\lambda^K (G_o^U \otimes I_{\mathcal{C}(X_j)} \otimes (\bigotimes_{\substack{i \in K \\ i \neq j}} G_\alpha^i)),$$

(3.2) $$G_o^U \otimes (\bigotimes_{i\in K} G_\alpha^i) = G_\lambda^K (I_{\mathcal{C}_o(U)} \otimes (\bigotimes_{i\in K} G_\alpha^i) + \sum_{j\in K} G_o^U \otimes I_{\mathcal{C}(X_j)} \otimes$$

$$\otimes (\bigotimes_{\substack{i\in K\\ i\neq j}} G_\alpha^i)).$$

One deduces $L(J\cup K)_\lambda G_\lambda^K g=-g$ and from 1.1 it results that U_K is P-regular with respect to $L(J\cup K)_\lambda$, just like in the proof of 2.1. Further one deduces that U_K is P-regular with respect to $L(J\cup K)$ by using the result from [13] 1.8, 2^o.

Further one deduces that U_o is P-regular relative to $\sum_{i\in N} L^i$ by applying 1.1 again.

4. The operator L-d/dt

Let X be a locally compact space with a countable base and L a locally dissipative, local operator such that the family of P- and D-regular sets with respect to L is a base. In this section we shall construct another local operator, L-d/dt, on the space X x R, which fulfils the same properties. It should be noted that in [11] ch.IV a similar operator was constructed within a different framework.

We denote by T=(0,1] the torus and consider it as a differentiable manifold. On the space T x R we consider L^o, the local closure of $\partial^2/\partial x^2$-d/dt. L^o is locally dissipative has a base of P- and D-regular sets (see [13] section 6) and is translation invariant. The sum $L+L^o$ is also locally dissipative and has a base of P- and D-regular sets. We shall use also the following property: if U is an open set in X x T x R then

$$(4.1)\qquad f\in D(\tau_x U,\ L+L^o) \text{ iff } f\circ\tau_{-x}\in D(U,L+L^o) \text{ and}$$

$$L+L^of=(L+L^o(f\circ\tau_x))\circ\tau_{-x},\ \text{for each } x\in T,$$

where τ_x:X x T x R $\to$ X x T x R is defined by $\tau_x(z,y,s)=(z,x+y,s)$. We denote θ:X x T x R $\to$ X x R, the map defined by $\theta(z,x,s)=(z,s)$. Then a local operator denoted by $\check{L}$ is defined on X x R: if U⊂X x R is an open set, a function f belongs to $D(U,\check{L})$ if and only if $f\circ\theta\in D(\theta^{-1}(U)$, $L+L^o)$ and $\check{L}f=g$, where g is the unique function in $\mathcal{C}(U)$ such that $L+L^o(f\circ\theta)=g\circ\theta$ (the existence of g is a consequence of (4.1)).

4.1. Proposition

Let $W \subset X$ be a P-regular set (with respect to L) and $h \in D(W,L)$ such that $Lh=0$, $h \geq 1$ on W and $\|h\| < \infty$. For $t_o \in R$ we define $p: W \times R \longrightarrow R$, by $p(x,t)=(t-t_o)G^W 1(z)-(t-t_o)^2 h(z)$, and put $U=\{(x,t)\ W \times R / t > t_o,\ p(x,t) > 0\}$. Then U is P-regular with respect to $\check{L}$.

Proof

We have $p \circ \theta \in D(\theta^{-1}(U), L+L^o) \cap \mathscr{C}_o(\theta^{-1}(U))$ and $L+L^o(p \circ \theta)(z,x,t) = -G^W 1(z)-(t-t_o)(1-2h(z)) < 0$. Moreover for each neighborhood A of the compact set $\partial W \times T \times \{t_o\}$ there exists $a \in R$, $a > 0$ such that $L+L^o(p \circ \theta) \leq -a$ on $\theta^{-1}(U) \setminus A$. Now we may use 3.4, 2° [13] and get a kernel V on $\theta^{-1}(U)$ such that

$Vf \in \mathscr{C}_o(\theta^{-1}(U)) \cap D(\theta^{-1}(U), L+L^o)$; $L+L_o Vf=-f$, for each $f \in \mathscr{C}_b(\theta^{-1}(U))$ provided $\operatorname{supp} f \cap (\partial W \times T \times \{t_o\}) = \emptyset$.

If $f \in \mathscr{C}_b(\theta^{-1}(U))$, $0 \leq f \leq 1$, and $\operatorname{supp} f \cap (\partial W \times T \times \{t_o\}) = \emptyset$, then $L+L_o G^W 1 = -1 \leq -f = L+L_o Vf$ and $Vf=0$ on $\partial \theta^{-1}(U)$. It follows $Vf(z,x,s) \leq G1(z)$ for each $(z,x,s) \in \theta^{-1}(U)$.

Further let $\{\varphi_n / n \in N\} \subset \mathscr{C}_c(W)$ be a sequence such that $0 \leq \varphi_n \leq \varphi_{n+1} \leq 1$ and for any compact set $K \subset W$ there exists $n \in N$ such that $\varphi_n = 1$ on K. If $f \in \mathscr{C}(\theta^{-1}(U))$, $0 \leq f \leq 1$, and $m \geq n$, then

$$0 \leq V((\varphi_m - \varphi_n)f) \leq \sup\{ V((\varphi_m-\varphi_n)f)(z,x,s)/(z,x,s) \in \theta^{-1}(U),\ \varphi_n(z) < 1\}$$

$$\leq \sup\{G^W 1(z)/\varphi_n(z) < 1\} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

We deduce that $V(\varphi_n f) \longrightarrow Vf$ uniformly and $Vf \in \mathscr{C}_o(\theta^{-1}(U))$. Hence $V_o f \in D(\theta^{-1}(U), L+L^o)$ and $L+L^o Vf=-f$. Thus $\theta^{-1}(U)$ is P-regular with respect to $L+L^o$ and $V=G^{\theta-1(U)}$. Further on account of (4.1), straightforward computations show that U is P-regular with respect to $\check{L}$ and G^U satisfies

(4.2) $(G^U f)\circ\theta = V(f\circ\theta)$ for each $f \in \mathscr{C}_b(U)$.

4.2. Corollary

$\check{L}$ has a base of P- and D-regular sets.

Proof

Let $x \in X$ and $\{V_n / n \in N\}$ a sequence of D-regular neighbourhoods which tends to $\{x\}$. The proof of 2.5 [13] shows that $H^{V_n} 1(x) \longrightarrow 1$. Thus for large n, $H^{V_n} 1=h$ satisfies $h>0$ on a neighbourhood of x. Then by 4.1 we can construct a base of sets similar to U.

Next we define $\hat{L}$, another local operator on X x R: if U is an open set in X x R, then D(U,L) is the family of all functions $f \in \mathscr{C}(U)$ which fulfil

1^o for each $x \in X$, $f_x \in \mathscr{C}^1(U_x)$,

2^o for each $s \in R$, $f_s \in D(U_s, L)$,

3^o $\hat{L}f \in \mathscr{C}(U)$, where $\hat{L}f(x,s) = Lf_s(x) - d/dt f_x(s)$, $(x,s) \in X \times R$.

Once again condition (2.1) is easy to check. We denote by L-d/dt the local closure of $\hat{L}$. Obviously $\check{L}$ extends $\hat{L}$. Since $\check{L}$ is locally closed there results that it extends L-d/dt.

4.3. Proposition

$\check{L}=L-d/dt$

Proof

Let $U\subset X$ be a P-regular set with respect to L and let $\{P_t/t>0\}$ the semigroup of kernels on U such that $G_\lambda^U=\int_0^\infty \exp(-\lambda t)P_t dt$, $\lambda\geqslant 0$. We define on U x R a semigroup $\{Q_t/t>0\}$ by putting $Q_t f(x,s)=(P_t(f(.,s-t))(x)$, for each $f\in\mathcal{C}_b(U \times R)$, i.e. $Q_t=P_t\otimes P'_t$, where $\{P'_t/t>0\}$ is the left-translation semigroup. Further $G=\int_0^\infty Q_t dt$ is a kernel on U x R and $G1(x,s)=G^U 1(x)$.

Let $f\in\mathcal{C}_o(U)$ and $\varphi\in\mathcal{C}^1(R)\cap\mathcal{C}_c(R)$. Then we have $G(G^U f\otimes\varphi)(x,s)=G^U(G(f\otimes\varphi)(.,s))(x)$ for $(x,s)\in U \times R$ and $G(G^U f\otimes\varphi)(x,.)\in\mathcal{C}^1(R)$, $d/dt\, G(G^U f\otimes\varphi)(x,.)=G(G^U f\otimes\varphi')(x,.)$. An equality analogous to (2.3) holds and it allows to deduce that, $G(G^U f\otimes\varphi)\in D(U \times R,\hat{L})$ and $\hat{L}G(G^U f\otimes\varphi)=-G^U f\otimes\varphi$. On the other hand since $P_t G^U 1 \longrightarrow 0$ uniformly when $t\to\infty$, one deduces that $G(G^U f\otimes\varphi)\in\mathcal{C}_o(U \times R)$. Thus $Gf\in D(U \times R, L-d/dt)\cap\mathcal{C}_o(U \times R)$ and $(L-d/dt)Gf=-f$ for each $f\in\mathcal{C}_o(U \times R)$.

Let $\{\varphi_n\}$ be a sequence in $\mathcal{C}_o(U \times R)$ such that $0\leqslant\varphi_n\leqslant\varphi_{n+1}\leqslant 1$ and $\lim_{n\to\infty}\varphi_n=1$ on U x R. The relation $G1=\lim_{n\to\infty}G\varphi_n$ implies that $G(1-\varphi_n)\longrightarrow 0$ uniformly on each compact subset of U x R. If

$f \in \mathscr{C}_b(U \times R)$ then $G(f\varphi_n) \longrightarrow Gf$ uniformly on each compact subset of $U \times R$, which shows that $Gf \in D(U \times R, L-d/dt)$ and $(L-d/dt)Gf=-f$.

Next, for each open set $V \subset X \times R$ we denote by $\mathcal{H}(V) = \{f \in D(V, \check{L}) / \check{L}f=0\}$. Then Corollary 4.2 and [13] 2.6 show that $\mathcal{H} = \{\mathcal{H}(V)/V$ open set$\}$ defines a quasiharmonic space on $X \times R$. From [12] 1.4 we deduce that Gf is a potential for each $f \in \mathscr{C}_{o+}(U \times R)$. From [12] 1.2, 2° it follows that $G1 = \sum_{n \in N} G(\varphi_{n+1} - \varphi_n)$ is also a potential. Then [12] 2.8 shows that $Gf = f \cdot G1$ for each $f \in \mathscr{C}_b(U \times R)$. Finally we deduce that $\check{L} = L-d/dt$ by using [12] 6.5, b).

5. Bauer Spaces and Strong Feller Semigroups

Let X be a locally compact space with a countable base and L a local operator which is locally dissipative and suppose that the family of all P- and D-regular sets is a base of X. We denote by

$$\mathcal{H} = \{\mathcal{H}(U) = \{f \in D(U,L) / L f = 0\} / U \text{ open set}\}.$$

It is known [13] 2.6 that $(X,\mathcal{H})$ is a quasiharmonic space.

Now we recall that a kernel R on a locally compact space T is said to be strong Feller if $Rf \in \mathcal{C}(T)$ for each $f \in \mathcal{B}_b(T)$.

We say that L satisfies the property of "strong Feller resolvents" if: (SFR) There exists a covering $\mathcal{U}$ of P-regular sets such that G^U is strong Feller for each $U \in \mathcal{U}$.

We say that L satisfies the property of "strong Feller semigroups" if:

(SFS) There exists a covering $\mathcal{U}$ of P-regular sets such that if $U \in \mathcal{U}$ and $\{P_t^U / t>0\}$ is a semigroup of kernels on U which fulfils $G_\lambda^U = \int_0^\infty \exp(-\lambda t) P_t^U dt$, for each $\lambda \geq 0$, then each kernel P_t^U, $t>0$ is strong Feller.

The following proposition is essentially known in a stronger variant [6].

5.1. Proposition

The following properties are equivalent:

1^o L satisfies (SFR),

2^o $(X,\mathcal{H})$ is a Bauer space in the meaning of [5].

3^o For each P-regular set, U, the kernels $G_\lambda^U, \lambda \geq 0$ are strong Feller.

4^o The global resolvent $\{G_\lambda / \lambda > 0\}$ constructed in [13] 2.3 is strong Feller.

Proof

We prove only "$1^o \Longrightarrow 2^o$" because the reminder proof is obvious. The existence of a strong base of regular sets results from 1.2. For each point $x \in X$ we have $H^U 1(x) \longrightarrow 1$ when $U \searrow \{x\}$, and U is taken to be a D-regular neighbourhood of x. This implies that $\mathcal{H}$ is non-degenrate at x. Let now V be a P- and D-regular set such that $\overline{V} \subset U$ for some $U \in \mathcal{U}$. Since $G^V = G^U - H^V G^U$ one deduces that G^V is strong Feller. Therefore each excessive function is lower semi-continuous. One deduces the Bauer convergence property.

Next we are going to prove the main result of this section:

5.2. Theorem

The following properties are equivalent:

1^o L satisfies (SFS).

2^o L-d/dt satisfies (SFR).

3^o If X^o is a locally compact space with a countable base and L^o is a locally dissipative local operator such that the family of all P- and D-regular sets is a base of X^o and L^o satisfies (SFR), then $L+L^o$ satisfies (SFR) too.

4^o If $X^o = T \times R$ and L^o is the local closure of $\partial^2/\partial x^2 - \partial/\partial t$ (as in section 4), then $L+L^o$ satisfies (SFR).

5^o If U is an open set, $U \subset X$, and $\{P_t / t > 0\}$ is a sub-Markov semigroup of kernels which is also a (C_o)-class semigroup of operators on a Banach space $F\subset\mathcal{C}_b(U)$ whose infinitesimal generator, Δ, has a domain, $D(\Delta)$, such that $D(\Delta)\subset D(U,L)$, $\Delta = L$ as linear operators on $D(\Delta)$, and $\mathcal{C}_o(U)\subset\overline{D(\Delta)}=F$, then P_t is strong Feller for each $t>0$.

In order to prove this theorem we need the following lemmas.

5.3. Lemma

Let X be a locally compact space with a countable base and V_1, V_2 two sub-Markov kernels on X which are strong Feller. Then V_1V_2 is a compact operator from $\mathcal{C}_b(X)$ into $\mathcal{C}(X)$, i.e. the family $\{V_1V_2f/f\in B(X), |f|\leqslant 1\}$ is equicontinuous.

Proof

As in [10] Lemma 9 we deduce that for each sequence $\{f_n/n\in N\}\subset\mathcal{C}_b(X)$ with $|f_n|\leqslant 1$, $n\in N$, there exists a subsequence $\{f_{n_k}/k\in N\}$ and $f\in\mathcal{B}_b(X)$ such that $V_2f_{n_k}(x)\longrightarrow V_2f(x)$ for each $x\in X$.

Let now $g_k=V_2(f_{n_k}-f)$. Since $\{\sup_{k\geqslant\rho} g_k/\rho\in N\}$ is decreasing to zero we have

$$V_1(\sup_{k\geqslant\rho} g_k)\searrow 0(\rho\to\infty) \text{ uniformly on each compact set.}$$

Analogous $V_1(\inf_{k\geqslant\rho} g_k)\searrow 0(\rho\to\infty)$, and hence $V_1V_2f_n\longrightarrow V_1V_2f$ uniformly on each compact set.

5.4. Lemma

Let X_1, X_2 be two locally compact spaces with countable bases and $\{P^i_t / t > 0\}$ a sub-Markov semigroup of kernels on X_i such that $t \longrightarrow P^i_t f(x)$ is right continuous for each $x \in X_i$, $f \in \mathcal{C}_b(X_i)$. Suppose that $G^i_o = \int_0^\infty P^i_t dt$, i=1,2 are (finite) kernels and $\{P^1_t / t > 0\}$, G^2_o are strong Feller.

Then $G = \int_0^\infty P^1_t \otimes P^2_t dt$ is a strong Feller kernel on $X_1 \times X_2$.

Proof

From 5.3 there results that $\{P^1_t f / f \in B(X_1), |f| \leq 1\}$ is an equicontinuous family for each $t > 0$.

First we are going to prove that for a given $f \in \mathcal{B}(X_1 \times X_2)$ such that $|f| \leq 1$ and a given compact set $K \subset X_2$ the family

$$\{Gf(.,y_2) / y_2 \in K\} \quad \text{is equicontinuous.}$$

If $y_2 \in X_2$, then μ^{y_2} will denote the measure on $X_2 \times R_+$ which fulfils

$$\int g(x_2,t) d\mu^{y_2}(x_2,t) = \int_0^\infty P^2_t(g(.,t))(y_2)dt, \quad g \in \mathcal{B}_b(X_2 \times R_+).$$

Since $\mu^{y_2}(1) = G^2_o 1(y_2)$ we deduce that there exists a real number $c > 0$ such that $\mu^{y_2}(1) \leq c$ for each $y_2 \in K$. Let $y_2 \in X_1$ and $\varepsilon > 0$. The family $\{ P^1_t f_{x_2} / x_2 \in X_2, t \geq \varepsilon/4$ being equicontinuous we can choose a neighbourhood W of y_1 such that

$$|P^1_t f_{x_2}(y_1) - P^1_t f_{x_2}(y_1')| < \varepsilon/2c$$

for each $y_1' \in W$, $x_2 \in X_2$ and $t \geqslant \varepsilon/4$. This implies

$$|G_o f(y_1, y_2) - G_o f(y_1', y_2)| \leqslant \varepsilon/4 + \varepsilon/4 +$$

$$+ \int_{\{t \geqslant \varepsilon/4\}} |P^1_t f_{x_2}(y_1) - P^1_t f_{x_2}(y_2')| \, d\mu^{y_2}(x_2, t) \leqslant \varepsilon$$

for each $y_1' \in W$ and $y_2 \in K$, which proves our assertion.

Next we are going to prove that for $f \in \mathcal{B}(X_1 \times X_2)$, $|f| \leqslant 1$, and $y_1 \in X_1$ the function $G_o f(y_1, .)$ is continuous on X_2. The properties of the semigroup $\{P^1_t / t > 0\}$ and the result 10.VIII from [9] allow us to construct a finite measure, μ, on X_1 and a function $g \in \mathcal{B}((0,\infty) \times X_1)$ such that

$$P^{1y_1}_t = g(t, .) \cdot \mu \quad \text{for each} \quad t > 0.$$

If we put $h(x_1, y_2) = \int_0^\infty g(t, x_1) P^2_t f_{x_1}(y_2) dt$, $x_1 \in X_1$, $y_2 \in X_2$, then

(5.1) $\qquad Gf(y_1, y_2) = \int h(x_1, y_2) d\mu(x_1).$

Now we assert that $h(x_1, .)$ is continuous provided $\int_0^\infty g(t, x_1) dt < \infty$. Let $x_1 \in X_1$ and $\varepsilon > 0$. We choose $\varphi \in \mathcal{C}_o((0,\infty))$ such that

$$\int_0^\infty |g(t, x_1) - \varphi(t)| \, dt < \infty$$

and a finite number of constants $\alpha_1, \alpha_2, \ldots, \alpha_n \geqslant 0$, and $c_1, c_2, \ldots c_n$ such that

$$\left|\varphi(t)-\sum_{i=1}^{n} c_i e^{-\alpha_i t}\right|<\varepsilon \text{ on } R_+$$

Then

$$\left|h(x_1,y_2)-\sum_{i=1}^{n} c_i G^2_{\alpha_i} f_{x_1}(y_2)\right|\leq \varepsilon+\varepsilon\int_0^\infty P^2_t 1(y_2)=\varepsilon+\varepsilon G^2 1(y_2),$$

where $G^2_\alpha=\int_0^\infty \exp(-\alpha t)P^2_t dt, \alpha\geq 0$ are the kernels of the resolvent associated to $\{P^2_t/t>0\}$. Since $G^2_o=G^2$ is strong Feller one deduces that for each $\alpha\geq 0$ the kernel G^2_α is strong Feller. For a compact set $K\subset X_2$, there exists a constant c such that $G^2 1(y_2)\leq c$ ($y_2\in K$). We deduce that $h(x_1,.)$ can be uniformly approximated on K by continuous functions of the form $\sum_{i=1}^{n} c_i G^2_{\alpha_i} f_{x_1}$. Hence $h(x_1,.)$ is continuous on X_2.

Since $\int_0^\infty P_t^{1y_1} 1 dt$ is a finite measure on X_1 we deduce $\int_0^\infty g(t,.)dt\in\mathcal{L}^1(\mu)$, and hence $\int_0^\infty g(t,.)<\infty$, μ-a.e. Since $|h(.,y_2)|\leq\int_0^\infty g(t,.)dt$ for each $y_2\in X$ one deduces from (5.1) that $G_o f(y_1,y_2^n)\longrightarrow G_o f(y_1,y_2^o)$, provided $y_2^n\longrightarrow y_2^o$, i.e. $G_o f(y_1,.)$ is continuous on X_2. Further it is easy to deduce that $G_o f$ is continuous.

<u>Proof of Theorem 5.2</u>

$1^o\Longrightarrow 3^o$ The proof of 2.1 and Lemma 5.4 show that the kernel G^U is strong Feller if $U=U_1\times U_2$, $U_1\subset X$ is P-regular and the kernels of its associated semigroup are strong Feller, and $U_2\subset X^o$ is P-regular and G^{U_1} is strong Feller.

$3^o\Longrightarrow 4^o$ It is obvious.

$4^o\Longrightarrow 2^o$ The kernel G^U associated to the open set U from

Proposition 4.1 is strong Feller. This results from relation (4.2).

$2^o \Longrightarrow 5^o$ Let $f \in D(\Delta)$ and define $\varphi : U \times R_+ \longrightarrow R$ by

(5.2) $\qquad \varphi(x,t) = P_t f(x)$.

Then

(5.3) $\qquad \varphi \in D(U\times(0,\infty), L-d/dt)$ and $(L-d/dt)\varphi = 0$.

Since $\mathcal{C}_o(U) \subset D(\Delta)$ and $L-d/dt$ is locally closed one deduces that (5.3) is still valid when φ is defined by (5.2) with $f \in \mathcal{C}_o(U)$. Further a monotone class theorem together with Proposition 5.1, 2^o shows that (5.3) is still valid when φ is defined by (5.2) with $f \in \mathcal{B}_b(U)$. Particularly $P_t f \in \mathcal{C}(U)$ for each $f \in \mathcal{B}_b(U)$ and $t > 0$.

$5^o \Longrightarrow 1^o$ It is obvious.

Example

Let $X = (0,\infty)\times(-\infty,\infty)$ and let L^1 be the local operator associated to $(\partial/\partial x_1)^2 + x_1 \partial/\partial x_2$. Then L^1-d/dt satisfies (SFR) on account of Proposition 5.1 and of Corollary from p.101 [2], and hence L^1 satisfies (SFS). However ker L is not a Brelot space. If we denote by L^2 the local operator associated to $(1/x_1)(\partial/\partial x_1)^2 + \partial/\partial x_2$ we see that L^2 does not satisfies (SFS) while ker L^1=ker L^2. We do not know whether for a local operator L the property (SFS) follows from the assumption that ker L is a Brelot space.

6. (SFS) for the sum of a series of local operators

Let $\{X_i / i \in N\}$ be a sequence of compact spaces with countable bases. Suppose that for each $i \in N$, L^i is a locally dissipative local operator on X_i such that $1 \in D(X_i, L^i)$, $L^i 1 = 0$, there exists a base of P- and D-regular sets, and L^i satisfies (SFS). In this section we are going to prove that $\sum_{i \in N} L_i$ satisfies (SFS) under suitable circumstances. We are going to use the notations from section 3. For a kernel V defined on a compact space, T, we put

$$(6.1) \qquad M(V) = \sup\{ |Vf(x) - Vf(y)| / x, g \in T,\ f \in \mathcal{B}_b(T),\ |f| \leqslant 1\}$$

Obviously we have $M(V) \leqslant 2$.

6.1. Lemma

Suppose that P_n is a sub-Markov kernel on X_n such that $P_n 1 > 0$ and the family $\{P_n f / f \in \mathcal{B}(X_n),\ |f| \leqslant 1\}$ is equicontinuous, for each $n \in N$. If $\sum_{n \in N} M(P_n) < \infty$, then $P = \bigotimes_{n \in N} P_n$ is a kernel on X such that $\{Pf / f \in \mathcal{B}(X_n),\ |f| \leqslant 1\}$ is equicontinuous.

Proof

Let $x = (x_o, x_1, \ldots) \in X$ and $\varepsilon > 0$. We choose $n_o \in N$ such that $\sum_{n \geqslant n_o} M(P_n) < \varepsilon/2$ and for each $k < n_o$ we choose V_k a neighbourhood of x_k such that

$$|P_k f(x_k) - P_k f(y_k)| < \varepsilon/2n_o, \quad \text{if } y_k \in V_k,\ f \in \mathcal{B}(X_k),\ |f| \leqslant 1.$$

We are going to prove that, for $y\in \prod_{k=0}^{n_o-1} V_k \times \prod_{i=n_o}^{\infty} X_i$, $f\in \mathcal{B}(X)$, $|f|\leq 1$,

we have

(6.2) $$|Pf(x)-Pf(y)|<\varepsilon .$$

Let $y=(y_o,y_1,\ldots)\in X$, with $y_k\in V_k$ for $k<n_o$. We put

$z_k=(x_o,x_1,\ldots,x_k, y_{k+1},y_{k+2},\ldots)$, for each $k\in N$. If $f\in\mathcal{C}(\prod_{i=0}^{p} X_i)$, $p\in N$ is regarded as a function $f\in\mathcal{C}(X)$, then $Pf=(\bigotimes_{i=1}^{p} P_i)f \otimes (\bigotimes_{i=p+1}^{\infty} P_i 1)$.

Then for $k>p$ we have

$$Pf(z_k)=(\bigotimes_{i=1}^{p} P_i)f(x_1\ldots x_p).\prod_{i=p+1}^{k} P_i 1(x_i)\prod_{i=k+1}^{\infty} P_i 1(y_i).$$

On account of Lemma 6.2 (stated below) we deduce

$$\lim_{k\to\infty} Pf(z_k)=Pf(x).$$

On the other hand, if $|f|\leq 1$, one deduces

$$|Pf(z_k)-Pf(z_{k+1})|<\varepsilon/2n_o, \text{ for } k<n_o-1,$$

$$|Pf(z_k)-Pf(z_{k+1})|<M_k, \text{ for } k\geq n_o-1.$$

Thus we get $|Pf(y)-Pf(x)|<\varepsilon$. Further it is easy to deduce relation (6.2) for each $f\in\mathcal{C}(X)$ such that $|f|<1$ and using a monotone class theorem to deduce the relation for $f\in\mathcal{B}(X)$ such that $|f|\leq 1$.

6.2. Lemma

Let $0<a_n$, $b_n\leq 1$, $n\in N$, be such that $\sum_{n\in N}|a_n-b_n|<\infty$.

Then $\lim_{k\to\infty} \prod_{n=k}^{\infty} a_n = \lim_{k\to\infty} \prod_{n=k} b_n$.

6.3. Theorem

If the semigroup $\{P_t^i/t>0\}$, $i\in N$ satisfy the condition $\sum_{i\in N} M(P_t^i) < \infty$ for each $t>0$, then $\sum_{i\in I} L^i$ satisfies (SFS).

Proof

Since the infinitesimal generator of $\{P_t^i/t>0\}$ is L^i considered as a linear operator on $D(X_i, L^i)$ by Theorem 5.2, 5^o one deduces that P_t^i is strong Feller.

Further Lemma 5.3 and Lemma 6.1 allow us to deduce that the kernels $P_t^{U_o}$, $t>0$ defined by $P_t^{U_o} = P_t^* \otimes (\bigotimes_{i\in N\setminus J} P_t^i)$, where U_o and P_t^* have the meaning from 3.1, are strong Feller. This implies (SFS) for $\sum_{i\in N} L_i$.

Next we are going to show that Theorem 6.3 applies for a large class of examples.

Lemma 6.4

Let T be a compact space with a countable base and let $\{P_t/t>0\}$ be a Markov semigroup of kernels.

1^o For any $s,t>0$,

$$M(P_{s+t})\ (1/2)M(P_s)M(P_t) \leq M(P_s).$$

2^o If the kernels P_t, $t>0$ are strong Feller and there exists a measure μ on T such that $P_t^* \to \mu$ $(t\to\infty)$ for each $x\in T$, then $\lim_{t\to\infty} M(P_t) = 0$.

Proof

1° For $f\in\mathcal{B}_b(T)$, $|f|\leqslant 1$ we put $\alpha=(1/2)(\sup P_s f+\inf P_s f)$ and $g=(2/M(P_s))(P_s f-\alpha)$. Since $P_t\gamma=\gamma$ we deduce

$$\sup P_{s+t}f-\inf P_{s+t}f\leqslant (1/2)M(P_s)(\sup P_t g-\inf P_t g)\leqslant$$

$$\leqslant (1/2)M(P_s)M(P_t).$$

2° From 5.3 we know that P_t is a compact operator on $\mathcal{C}(T)$. If $f\in\mathcal{B}_b(T)$ and $s>0$ then $\{P_t f/t\geqslant s\}$ is equicontinuous, and hence $P_t f \longrightarrow (f)$ uniformly. On the other hand $K=\{P_s f/f\in\mathcal{B}(T),\ |f|\leqslant 1\}$ is a compact subset of $\mathcal{C}(T)$ and the family of operators $\{P_t/t\geqslant s\}$ is equicontinuous. Then $P_t \longrightarrow \mu\otimes 1$ uniformly on K, where the operator $\mu\otimes 1$ associates to each function $f\in\mathcal{C}(T)$ the constant function $\mu(f)$. This implies the assertion.

Let T be a compact space with a countable base, L a local operator on T, and $\{P_t/t>0\}$ a semigroup on $\mathcal{C}(T)$ whose infinitesimal generator has D(T,L) as domain and coincides with L as linear operator on this domain.

We suppose that $X_i=T$ and $L^i=a_iL$, $a_i\in R_+$, for each $i\in N$. Then $P^i_t=P_{a_it}$ for any $t>0$, $i\in N$. We also suppose that there exists a measure μ on T such that $P^x_t \longrightarrow \mu\ (t\to\infty)$ for each $x\in T$.

6.5. Proposition

If the sequence $\{a_i/i\in N\}$ satisfies

$$\sum_{i\in N} e^{-\theta a_i}<\infty \text{ for each } \theta>0, \tag{6.3}$$

then $\sum_{i\in N} L_i$ satisfies (SFS).

Proof

From Lemma 6.4, 1^o we deduce

$$M(P_t) \leqslant 2(M(Pr)/2)^{[t/r]}$$

for each $t,r>0$, where $[t/r]\in N$ and satisfies $0\leqslant t-[t/r]r<r$. Using 6.4, 2^o we choose r such that $M(Pr)<2$ and denote by $\theta=-\ln(M(Pr)/2)$. Then we deduce

$$M(P_{a_i t}) \leqslant 2e^{\theta}e^{-(\theta t/r)a_i}$$

Hence $\sum_{i\in N} M(P_t^i)<\infty$ and the proposition results from Theorem 6.3.

7. The sum of two local operators associated to harmonic spaces

While the author was prepearing for publication the previous sections of the present paper E.Popa and U.Schirmeier have independently communicated to him a result very similar to Theorem 5.2. Namely they considered harmonic spaces in the sense of Constantinescu and Cornea and used the Markov process associated to a strict potential instead of the local operator used in Theorem 5.2. Next we present the result for harmonic spaces in the sense of Constantinescu and Cornea by using local operators associated to them in the sense of [13] section 4.

Let $(X,\mathcal{U})$ be a harmonic space such that $1\in\mathcal{U}(X)$ and X has a countable base and let L be a local operator associated to $\mathcal{U}$. The property (SFR) is obviously satisfied by L. The property (SFS) for L is defined as in section 5. The sum of two local operators associated to harmonic spaces is defined as in section 2 and the operator L-d/dt is defined as in section 4. Then we have a result parallel to Theorem 5.2.

7.1. Theorem

The following properties are equivalent:

1^{o} L satisfies (SFS).

2^{o} L-d/dt is associated to a harmonic space.

3^{o} If $(X^{o},\mathcal{U}^{o})$ is a harmonic space such that $1\in\mathcal{U}^{o}(X^{o})$, X^{o} has a countable base, and L^{o} is a local operator associated to $\mathcal{U}^{o}$, then $L+L^{o}$ is associated to a harmonic space on $X \times X^{o}$.

4^{o} If X^{o}=TxR, L^{o} is the local closure of $\partial^2/\partial x^2 - \partial/\partial t$, and $\mathcal{U}^{o}$ is the hyperharmonic sheaf associated to L^{o}, then $L+L^{o}$ is associated to a hrmonic space.

5^{o} If U is an open set, $U \subset X$, and $\{P_t / t>0\}$ is a sub-Markov semigroup of kernels which is also a(C_o)-class semigroup of operators on a Banach space $F \in \mathcal{C}_b(U)$ whose infinitesimal generator, Δ , has a domain, $D(\Delta)$, such that $D(\Delta) \subset D(U,L)$, $\Delta = L$ as linear operators on $D(\Delta)$, and $\mathcal{C}_o(U) \subset \overline{D(\Delta)} = F$, then P_t is strong Feller for each $t>0$.

The proof of this theorem is quite similar to the proof of Theorem 5.2 on account of [13] section 4 and especially Lemma 4.1.

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