INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250-3638

A ROHLIN TYPE THEOREM FOR GROUPS ACTING ON VON NEUMANN ALGEBRAS

by

Adrian OCNEANU

PREPRINT SERIES IN MATHEMATICS
No.82/1979

BUCURESTI

A ROHLIN TYPE THEOREM FOR GROUPS ACTING ON VON NEUMANN ALGEBRAS

by

Adrian OCNEANU*)

December 1979

*) Department of Mathematics, National Institute for Scientific and Technical Creation, Bd.Pacii 220, 77538 Bucharest, Romania.

A ROHLIN TYPE THEOREM FOR GROUPS ACTING ON VON NEUMANN ALGEBRAS

by

Adrian Ocneanu

In classical ergodic theory one considers an ergodic automorphism of a measure space; a major problem is the classification of such structures. A first step towards such a result is the Rohlin tower theorem, according to which the space may be divided into any given number of measurable subsets, cyclically permuted by the automorphism modulo some small measure sets.

A first way of generalizing this result consists of the consideration of a loccaly compact group G acting freely by automorphisms of a measure space. The theorem was proved for $G=\mathbb{Z}^n$ by Katznelson and Weiss [3], for discrete abelian G by Conze [2], for $G=\mathbb{R}^n$ by Lind [4], for discrete solvable G by Ornstein and Weiss [5] and for solvable or almost connected amenable loccaly compact G by Series [6].

A new stage of generality appears in the work of A.Connes [1], where it is used for the classification of the automorphisms of a finite von Neumann algebra. The theorem is stated for an aperiodic automorphism of a von Neumann algebra which leaves fixed a faithful normal trace.

In the sequel we extend the result of Connes to several commuting automorphisms of a von Neumann algebra; in fact for finite extensions of $\mathbb{Z}^n$. From the quoted paper of Connes we use the theorem of characterisation of properly outer automorphisms, but for the rest our proof is different, even for one automorphism, of the proof given there, being partly inspired by the proofs in [4] , [5] for measure spaces.

Let M be a von Neumann algebra and Aut M its automorphisms. We recall ([1]) that for $g \in \mathrm{Aut}\, M$, there is a largest (central) projector $p(g)$, left fixed by g, on which g is inner; g is called properly outer if $p(g)=0$. A group G acting on a von Neumann algebra is said to act freely of $p(g)=0$ for any $g \neq 1$.

A nonvoid finite subset K of G is called a paving set of G if one can choose right translations of it to cover without overlappings G.

The main purposes of the paper are the following two theorems.

1. THEOREM. Let G be a group, finite extention of a finitely generated abelian group. Suppose M is a von Neumann algebra, τ is a normal trace on M, $\tau(1)=1$, and let G act freely on M preserving τ. Then for any paving set K of G and any $\delta>0$ there is a partition of unity $(f_k)_{\in k}$ in M such that

$$\|gf_k-f_{gk}\|_1 \leq \delta \text{ for all } k\in K,\ g\in G \text{ with } gk\in K$$

(where for $x\in M$, $\|x\|_1=\tau(|x|)$)

If G is a group and S is a subgroup of G, then G/S will denote the left quotient space of G modulo S.

2. THEOREM. Let G be a group, finite extension of a finitely generated abelian groups, and let S be a finite index subgroup of G. If M is a von Neumann algebra, τ a normal trace on M with $\tau(1)=1$ and if G acts freely on M preserving τ , then for any $\delta>0$ and any finite

subset G_1 of G there is a partition of unity $(f_i)_{i\in G/S}$ in M such that

$$\|gf_i - f_{gi}\|_1 \leq \delta \qquad \text{for all } g\in G_1,\ i\in G/S$$

3. COROLLARY. (A.Connes, [1]). Let M be a finite von Neumann algebra, τ a faithful normal trace on M, $\tau(1)=1$, and θ an aperiodic automorphism of M which preserves .

For any integer n and any $\delta>0$ there exists a partition of unity $(f_j)_{j\in 1,\dots,n}$ in M such that

$$\|\theta(f_1)-f_2\|_2\leq\delta\ ,\dots,\ \|\theta(f_j)-f_{j+1}\|_2\leq\delta\ ,\dots,\ \|\theta(f_n)-f_1\|_2\leq\delta$$

(where $\|x\|_2 = \tau(x^*x)^{1/2}$, $x\in M$).

Proof. We take $G=\mathbb{Z}$, $S=n\mathbb{Z}$, $K=\{1\}$ in Theorem 2 (where this time G is written additively) and we remark that
$\|x\|_2^2 = \|x^*x\|_1 \leq \|x\|\ \|x\|_1$ $\square$

In the applications of Rohlin type theorems it is required that the index set of the tower (K in theorem 1) can be chosen arbitrarily large and invariant. Lemma 6 shows that in our case such a choice is always possible.

We shall use the special form of the group in Theorem 1 only by means of one of its properties, given in the lemma 5 below. The result see to fail for general solvable groups. We recall the following.

4. DEFINITION. Let G be a group , K a finite subset of G and $\varepsilon>0$.

A finite subset G_1 of G will be called (ε, K) invariant if

$$\#(G_1 \cap \bigcap_{g\in K} g^{-1}G_1) \geq (1-\varepsilon)\#G_1$$

(where $\#$ denotes the cardinality).

5. LEMMA. Let G be a group, finite extension of a finitely generated abelian group. Then there exists $a_G>0$ such that G has arbitrarily large arbitrarily invariant subsets G_1 with

$$\#(G_1^{-1}G_1) \leq a_G \# G_1$$

Proof. For $m,n \in \mathbb{Z}$, $m\leq n$, we set $[m,n]=\{m,m+1,\dots,n\}$.

Any G as above is a finite extention of $\mathbb{Z}^N$, $N\in\mathbb{N}$. Indeed if $G'\in G$ is a finite extenstion, with finitely generated abelian G', and if $G'=T\oplus\mathbb{Z}^N$, where T is the torsion part of G', then $\mathbb{Z}^N$ is completely invariant in G'. So $\mathbb{Z}^N$ is a normal subgroup of G and $\mathbb{Z}^N \subset G$ is a finite extension.

If $N=0$ we can take $G_1=G$ for all K,ε and $a_G=1$. Suppose $N>0$ and let $K\subseteq G$ be the image of a section of the projection $G\to G/\mathbb{Z}^N$. For $m\in\mathbb{N}$ we let $C_m=[-m,m]^N \subset \mathbb{Z}^N$. Suppose we are given an arbitrary finite subset F of G. There exists $p\in\mathbb{N}$ such that

$$(1)\qquad F\cup KK\cup K^{-1}K\cup C_1K \subseteq KC_p$$

because $\bigcup_{p\geq 1} KC_p = K\mathbb{Z}^N = G$

We have inductively from (1)

$$C_nK \subseteq KC_{np}\quad, n\geq 1$$

and so

$$(KC_n)(KC_m) \subseteq KKC_{np}C_m \subseteq KC_pC_{np}C_m = KC_{m+(n+1)p}$$

Since $\#(KC_m) = (\#K)\ (\#C_m) = (\#K)(2m+1)^N$ we have

$$\lim_{m\to\infty}\left(\#(KC_{m+(n+1)p})/\#(KC_m)\right) = 1$$

So, for each n and any $\varepsilon > 0$, KC_m is (ε, KC_n)-invariant for large enough m; moreover, each finite subset of G is included in KC_n for some n. We also have

$$(KC_m)^{-1}KC_m = C_mK^{-1}KC_m \subseteq KC_{mp+p+1}$$

and we can take $a_G = (p+1)^N$, suitable for any $m \geq p+1$. For instance, if $G = \mathbb{Z}^2$ we can take $a_G = 4$. $\square$

The following result is true in fact for all solvable groups. Ornstein and Weiss have conjectured that it holds for any amenable group.

6. LEMMA. Let G be a group as in Theorem 1. Then there are arbitrary large arbitrary invariant paving sets K of G.

Proof. In the proof of lemma 5 remark that KC_m are paving sets, because

$$G = K\mathbb{Z}^N = \bigcup_h KC_mh$$

where h ranges in $((2m+1)\mathbb{Z})^N$ and the sets are disjoint. $\square$

In all the sequel, M will be a von Neumann algebra, $\mathcal{P}_M$ its lattice of projections and Aut M its group of automorphisms. We use the following fundamental result, due to A.Connes [1]:

7. THEOREM. Let M be countably decomposable and $g \in$ Aut M. Then g is properly outer if and only if for any non zero $e \in \mathcal{P}_M$ and any $\varepsilon > 0$, there is a non zero $f \in \mathcal{P}_M$, $f \leq e$ such that

$$\| f.gf \| \leq \varepsilon$$

having as consequence:

8. COROLLARY. Let G_1 be a finite set of properly outer automorphisms of M, $\varepsilon > 0$ and $0 \neq e \in \mathcal{P}_M$. Then there is $f \in \mathcal{P}_M$, $0 \neq f \leq e$ with $\| f.gf \| \leq \varepsilon$ for all $g \in G_1$.

From the same paper we use the following technical result.

9. LEMMA. If $\varepsilon > 0$, with $n!\varepsilon < 1$ and $(e_j)_{j \in [1,n]} \subset \mathcal{P}_M$ such that $\| e_j e_k \| \leq \varepsilon$ for all $j \neq k$, then there is a family $(f_j)_{j \in [1,n]} \subset \mathcal{P}_M$ such that (f_j) are mutually orthogonal, $f_j \sim e_j$, $\| e_j - f_j \| \leq n!\varepsilon$ for all $j \in [1,n]$ and $\bigvee_1^n e_j = \bigvee_1^n f_j$.

In what follows, we let τ denote a normal faithful trace on M, with $\tau(1) = 1$. If $e_1, e_2 \in \mathcal{P}_M$ then from the parallelogram law ([7], p.94) we easily infer

$$\tau(e_1 \vee e_2) = \tau(e_1) + \tau(e_2) - \tau(e_1 \wedge e_2).$$

In the conditions of Lemma 9

$$(2) \qquad \tau\left(\bigvee_1^n e_j\right) = \sum_1^n \tau(e_j)$$

10. DEFINITION. For finite $H \subseteq G$ and $\delta > 0$ we say that $f \in \mathcal{P}_M$, $f \neq 0$ is (δ, H)-invariant if

$$\tau(f \wedge \bigwedge_{g \in H} g^{-1} f) \geq (1-\delta)\tau(f)$$

11. DEFINITION. For finite $H \subseteq G$ and $\varepsilon > 0$ we say that $e \in \mathcal{P}_M$ is an (ε, H)-basis if $e \neq 0$ and

$$\| g_1 e \cdot g_2 e \| \leq \varepsilon \qquad \text{for } g_1, g_2 \in H,\ g_1 \neq g_2$$

In this case we call $(ge)_{g \in H}$ the H-tower with basis e.

The following Proposition shows, using Corollary 8, that under any sufficiently invariant projection f one can find an (ε, H)-basis e, such that the tower $(ge)_{g \in H}$ covers at least $(2a_G)^{-1}$ of f.

12. PROPOSITION. Let G be a group, finite extension of a finitely generated abelian group G, take a_G as in Lemma 5 and suppose that G acts freely on M. Then for any finite $K_o, K \subseteq G$ and $\delta > 0$ there is a finite $H \subseteq G$ satisfying

(3) H is (δ, K)-invariant and $K_o \subseteq H$

such that for any $(1/2, H^{-1}H)$ invariant $f \in \mathcal{P}_M$ and any $\varepsilon > 0$ there is an (ε, H) basis e such that

(4) $\bigvee_{g \in H} ge \leq f$

(5) $\tau(\bigvee_{g \in H} ge) \geq (2a_G)^{-1} \tau(f)$

It will be convenient to denote by $\Phi(K_o, K, \delta)$ the set of all H as above.

Proof. The idea of the proof is the following. We take H as in Lemma 5. Suppose first f is 1, and consider a maximal (ε, H)-basis e. If e′ was orthogonal to $e_1 = \bigvee_{g\in H^{-1}H} ge$, then $\bigvee_{g\in H} ge'$ would be orthogonal to $\bigvee_{g\in H} ge$, and from Corollary 8 we could find an (ε, H) basis $e'' \leq e'$. Then $e+e''$ would be an (ε, H)-basis, contradicting the maximality of e. So $e_1=1$ and from (2), e_1 is at most a_G times larger than $\bigvee_{g\in H} ge$. In the general case, if f is sufficiently invariant, the above reasoning can be done under f.

Let us go to the proper proof of the Proposition. We choose as in Lemma 5 a finite $H \subseteq G$ such that

(6) H is (δ, K)-invariant, $K_o \cup \{1\} \subseteq H$

(7) $(H^{-1}H) \leq a_G (\# H)$

Let $f \in \mathcal{P}_M$, $f \neq 0$ be a given projection such that

(8) f is $(1/2, H^{-1}H)$ invariant

Let $\varepsilon > 0$; we can suppose without loss of generality that

(9) $\varepsilon(\# H)! < 1$

Let $(e_i)_{i\in I}$ be a maximal family of nonzero projections such that

(10) e_i is an (ε, H)-basis, $i \in I$

(11) $\bigvee_{g\in H^{-1}H} ge_i \leq f$

(12) $\left(\bigvee_{g\in H^{-1}H} ge_i\right)_{i\in I}$ are mutually orthogonal

Under these circumstances, if $I\neq\emptyset$ then $e=\sum_{i\in I} e_i$ is an (ε,H)-basis and satisfies condition (4) in the Proposition. We now proceed to prove condition (5).

Let us take

$$f_1=\bigwedge_{g\in H^{-1}H} g^{-1}f$$

$$\bar{e}=\bigvee_{g\in H^{-1}H} ge$$

$$f_2=f_1\wedge(f-\bar{e})$$

We infer

(13) $\tau(f_2)=\tau(f_1)+\tau(f-e)-\tau(f_1\vee(f-e))\geqslant\tau(f_1)-\tau(\bar{e})$

As e is an (ε,H)-basis, from (9) and (2) we have

$$\tau\left(\bigvee_{g\in H} ge\right)=(\# H)\,\tau(e)$$

It results

$$\tau(\bar{e})\leqslant(\#(H^{-1}H))\,\tau(e)\leqslant a_G(\# H)\,\tau(e)=a_G\tau\left(\bigvee_{g\in H} ge\right)$$

and by means of (13) and (8)

$$\tau(f_2)\geqslant\tau(f_1)-\tau(\bar{e})\geqslant(1/2)\tau(f)-a_G\tau\left(\bigvee_{g\in H} ge\right)$$

If (5) was false, then $\tau(f_2)>0$ and f_2 would be nonzero.

According to Corollary 8 there would be an $e' \in \mathcal{P}_H$, $0 \neq e' \leq f_2$ such that $\|e'.ge'\| \leq \varepsilon$ for $g \in H^{-1}H \setminus \{1\}$. Then e' would be an (ε, H)-basis, e' would be orthogonal to $\bigvee_{g \in H^{-1}H} ge$ and so $\bigvee_{g \in H} ge'$ would be orthogonal to $\bigvee_{g \in H} ge$, thus contradicting the maximality of the family $(e_i)_{i \in I}$. Proposition 12 is proved.

<u>Proof of Theorem 1.</u> For convenience the proof will be divided into three parts. The idea of the proof is to apply succesively Proposition 12, taking into account the fact, proved in part (C), that the complement of a tower $(ge)_{g \in H}$ is arbitrarily invariant if H is sufficiently invariant and if the complement is not too small. The constants are choosen for at most n times of usage of the algorithm described in part (A), but we stop earlier if the tower arrives at the desired size in less than n steps. The towers obtained this way are then, after being orthogonalized by means of Lemma 9, put together and indexed by K; then, in part (B), the desired partition of unity is obtained.

We begin the proof making some choices

(14) $\beta = 1 - (2a_G)^{-1}(1 - \frac{\delta}{4})$, where we have supposed $\delta < 1$.

(15) $n \in \mathbb{N}$ with $\beta^n < \frac{\delta}{2}$

(16) $\delta_1 = \frac{\delta}{8}$

(17) $\gamma_1 = \frac{1}{2}$

and then for $k \in [2, n]$ we choose $\delta_k, \gamma_k > 0$ with

(18) $\delta_k \leq \frac{\delta}{8}$

(19) $\gamma_{k-1} - \gamma_k > (1 - \frac{\delta}{4})(1 - (1 - \gamma_{k-1})(1 - \delta_k))$

We put $H_0=L_0=1$ G, and take succesively according to Proposition 12 for $k\in[1,n]$

(20) $H_k\in\Phi(L_{k-1},L_{k-1}L_{k-1}K,\delta_k)$

(21) $L_k=L_k^{-1}=H_k^{-1}H_k$

We choose $\varepsilon>0$ such that

(22) $\varepsilon<\frac{\delta}{16}$

(23) $\varepsilon(\# H_k)<1$ for all $k\in[1,n]$

<u>Part (A)</u>. We use an algorithm, the step k of which is described below, for k=n,n-1,...,1 or until we stop in the meantime.

For k=n we take $F_k=1\in\mathcal{P}_M$. For general k we suppose inductively that we have a projection $F_k\in\mathcal{P}_M$ such that

(24) F_k is (γ_k, L_k) - invariant.

According to (20) and to the fact that $\gamma_k\leq\frac{1}{2}$ we can apply Proposition 12 in order to obtain an (ε, H_k) basis e_k with

(25) $\bigvee_{g\in H_k} ge_k\leq F_k$

(26) $(\# H_k)\tau(e_k)=\tau(\bigvee_{g\in H_k} ge_k)\geq(2a_G)^{-1}\tau(F_k)$

We define

(27) $G_k=\bigcap_{g\in L_{k-1}L_{k-1}} g^{-1}H_k\subseteq H_k$

From (20), as H_k is $(\delta_k, L_{k-1}L_{k-1}K)$-invariant, and making use of (23) and (2) we infer

$$(28)\quad (\# G_k)\,(1-\delta_k)\,(\# H_k)$$

$$\tau\Big(\bigvee_{g\in G_k} g e_k\Big) = (\# G_k)\,\tau(e_k)$$

If $k=1$ we stop. For $k>1$ there are two possibilites.

Case 1. If

$$(29)\quad (\# H_k)\,\tau(e_k) = \tau\Big(\bigvee_{g\in H_k} g e_k\Big) \geq \Big(1-\frac{\delta}{4}\Big)\,\tau(F_k)$$

then we stop. In this case, from (28) we have

$$(30)\quad \tau\Big(\bigvee_{g\in G_k} g e_k\Big) = (\# G_k)\,\tau(e_k) \geq (1-\delta_k)(\# H_k)\,\tau(e_k) \geq (1-\delta_k)\Big(1-\frac{\delta}{4}\Big)\tau(F_k) \geq \Big(1-\frac{\delta}{2}\Big)\tau(F_k)$$

Case 2. If (29) doesn't hold, that is if

$$(31)\quad (\# H_k)\,\tau(e_k) = \tau\Big(\bigvee_{g\in H_k} g e_k\Big) < \Big(1-\frac{\delta}{4}\Big)\,\tau(F_k)$$

we go to the step $k-1$ of the algorithm taking

$$(32)\quad F_{k-1} = F_k - \bigvee_{g\in G_k} g e_k$$

In part (C) we shall show that F_{k-1} is (γ_{k-1}, L_{k-1})-invariant and that

$$(33)\quad \tau(F_{k-1}) \leq \beta\,\tau(F_k).$$

Part (B). We denote by p the step at which we have stopped.

From part (A) we have obtained $e_p,\dots,e_n \in \mathcal{P}_M$ such that e_k is an (ε, G_k)-basis and the projectors $\bar{e}_k = \bigvee_{g\in G_k} ge_k$ are mutually orthogonal, for $k\in[p,n]$. We have

$$(34)\quad \tau\Big(\sum_{k=p}^{n}\bar{e}_k\Big)=\tau(1-F_{p-1})\geq 1-\frac{\delta}{2}$$

Indeed, if $p>1$, as a consequence of (31) we infer

$$\tau(F_{p-1})\leq\frac{\delta}{2}\tau(\bar{e}_p)\leq\frac{\delta}{2}$$

and if $p=1$, then (34) results from (33) for $k=n,n-1,\dots,1$ and (15).

We apply Lemma 9 under each e_k to obtain a family of mutually orthogonal projections $f_{k,g}$, $k\in[p,n]$, $g\in G_k$ with $\|f_{k,g}-ge_k\|\leq\varepsilon$ and $f_{k,g}\sim ge_k$ for k,g as above. They form, together with $F_{p-1}=1-\sum_{k=p}^{n}\bar{e}_k$ a partition of unity in M.

We also have

$$(35)\quad \|f_{k,g}-ge_k\|_1\leq\varepsilon\|f_{k,g}\vee ge_k\|_1\leq 2\varepsilon\tau(e_k)$$

K being a paving set of G we can choose a partition $G=\bigcup_{h\in H} Kh$; then $G=\bigcup_{l\in H} H$ is a partition too. For $k\in[p,n]$, $l\in K$ we let

$$G_{k,l}=G_k\cap lH$$

Then

$$gG_{k,l}\,\Delta\, G_{k,gl}\subseteq gG_k\,\Delta\, G_k$$

where Δ denotes the symmetric difference. As, from (20) and (27) G_k is (δ_k,K)-invariant, we infer

$$(36)\quad \#(gG_{k,l}\,\Delta\, G_{k,gl})\leq 2\delta_k(\#G_k)\leq\frac{\delta}{4}(\#G_k)$$

We take

$$f_l = \sum_{k=p}^{n} \sum_{g \in G_{k,l}} f_{k,g}$$

Then for any $l \in K$, $g \in G; gl \in K$ we obtain

(37) $$\|gf_l - f_{gl}\|_1 \leq \sum_{k=p}^{n} \Big\| \sum_{g_1 \in G_{k,l}} gf_{k,g_1} - \sum_{g_2 \in G_{k,l}} f_{k,g_2} \Big\|_1 \leq$$

$$\leq 4\varepsilon \sum_{k=p}^{n} (\# G_{k,l}) \tau(e_k) + \sum_{k=p}^{n} \# (gG_{k,l} \Delta G_{k,gl}) \tau(e_k)$$

where the first part of the inequality results remarking that for $g \in K$, $g_1 \in G_{k,i}$, $g_2 = gg_1$ G_{k,g_i} from (29) we have

$$\|gf_{k,g_1} - f_{k,g_2}\|_1 \leq \|gf_{k,g_1} - gg_1 e_k\|_1 + \|g_2 e_k - f_{k,g_2}\|_1 \leq 4\varepsilon\tau(e_k)$$

So, from (28), (37) and (36) we infer

$$\|gf_l - f_{gl}\|_1 \leq \sum_{k=p}^{n} \frac{\delta}{2} (\# G_k) \tau(e_k) = \frac{\delta}{2} \sum_{k=p}^{n} \tau(\bar{e}_k) \leq \frac{\delta}{2}$$

To make $(f_l)_{l \in K}$ a partition of unity, we just replace, for an arbitrary $l_o \in K$, f_{l_o} by $f_{l_o} + F_{p-1}$. As from (34) $\tau(F_{p-1}) \leq \frac{\delta}{2}$ the conclusion of Theorem 1 is satisfied.

Part (C). It remained to show that, in case 2 of part (A), F_{k-1} is (γ_{k-1}, L_{k-1})-invariant and satisfies (33). Take

(38) $$F'_k = \bigwedge_{g \in L_{k-1}} g^{-1} F_k \leq F_k$$

As, from (20), $L_{k-1} \subseteq L_k$, as a consequence of the induction hypothesis (24) we obtain

(39) $$\tau(F'_k) \geq (1 - \gamma_k) \tau(F_k)$$

From (27) $H_k \supseteq L_{k-1}L_{k-1}G_k$; so for $g_1, g_2 \in L_{k-1} = L_{k-1}^{-1}$, $g \in G_k$ we get succesively from (25) and the definition (32)

$$g_2^{-1} g e_k \leq g_1^{-1} F_k$$

Letting g, g_1, g_2 run we infer

$$\text{(40)} \quad \bigvee_{g \in L_{k-1}G_k} g e_k \leq \bigwedge_{g \in L_{k-1}} g^{-1} F_k = F_k'$$

From the definition of F_{k-1}, for $g_1 \in L_{k-1}$ we get

$$g_1^{-1} F_{k-1} = g_1^{-1} F_k - g_1^{-1}\Big(\bigvee_{g \in G_k} g e_k\Big) \geq F_k' - \bigvee_{g \in L_{k-1}G_k} g e_k$$

Letting g_1 run we obtain

$$\bigwedge_{g \in L_{k-1}} g^{-1} F_{k-1} \geq F_k' - \bigvee_{g \in L_{k-1}G_k} g e_k$$

and from (40) the right member is a projection. We have

$$\tau\Big(\bigwedge_{g \in L_{k-1}} g^{-1} F_{k-1}\Big) \geq \tau(F_k') - (\#(L_{k-1}G_k))\tau(e_k) \geq$$

$$\text{(41)} \qquad \geq (1-\gamma_k)\tau(F_k) - (\# H_k)\tau(e_k) \geq$$

$$\text{(42)} \qquad \geq (1-\gamma_{k-1})\tau(F_k) - (1-\gamma_{k-1})(1-\delta_k)(\# H_k)\tau(e_k) \geq$$

$$\text{(43)} \qquad \geq (1-\gamma_{k-1})\tau(F_{k-1})$$

where (41) results from (39), (42) from (19) and (31) and (43) from (32) and (28); hence F_{k-1} is (γ_{k-1}, L_{k-1})-invariant.

On the other hand

$$\tau(F_{k-1}) = \tau(F_k) - (\# G_k)\tau(e_k) \leq \tau(F_k) - (1-\delta_k)(\# H_k)\tau(e_k) \leq$$
$$\leq (1-(2a_G)^{-1}(1-\delta_k))\tau(F_k) \leq \beta\tau(F_k)$$

from (26), (28) and then (14). This way we have proved (33) and hence the proof of Theorem 1 is done. $\square$

Proof of Theorem 2. We may assume that $\delta < 1$. By Lemma 6 there is a $(\frac{\delta}{4}, G_1)$-invariant paving set K of G. By means of Theorem 1 we choose a partition of unity in M, denoted by $(e_k)_{k\in K}$, such that for all $g\in G, k\in K$ with $gk\in K$ we have

(44) $$\|ge_k - e_{gk}\|_1 \leq \frac{\delta}{4}(\# K)^{-1}$$

Fix $k\in K$. Adding for all $l\in K$ the inequality

$$\|e_k\|_1 = \|lk^{-1}e_k\|_1 \leq \|e_l\|_1 + \|lk^{-1}e_k - e_l\|_1 \leq \|e_l\|_1 + \frac{\delta}{4}(\# K)^{-1}$$

we get

(45) $$(\# K)\|e_k\|_1 \leq 1 + \frac{\delta}{4}$$

Let $p: G \to G/S$ be the natural projection and let for $i\in G/S$

$$A_i = \{k\in K \mid p(k) = i\} \quad \text{and } f_i = \sum_{k\in A_i} e_k$$

Then for any $g\in G_1$ we infer

$$\|gf_i - f_{gi}\|_1 \leq \sum_{k\in K_1} \|ge_k\|_1 + \sum_{k\in K_2} \|e_{gk}\|_1 + \sum_{k\in K_3} \|ge_k - e_{gk}\|_1$$

where $K_1 = \bigcup_{i\in G/S} (A_i \setminus g^{-1}A_{gi}) = K \setminus g^{-1}K$

$K_2 = \bigcup_{i\in G/S} (g^{-1}A_{gi} \setminus A_i) = g^{-1}K \setminus K$

$K_3 = K \cap g^{-1}K$

As K is $(\frac{\delta}{4}, G_1)$-invariant, $\#(K_1 \cup K_2) \leq \frac{\delta}{2}(\#K)$, so from (44) and (45) we obtain:

$$\|gf_i - f_{gi}\|_1 \leq \frac{\delta}{2}(1+\frac{\delta}{4}) + \frac{\delta}{4} < \delta$$

and the Theorem is proved. $\square$

AKNOWLEDGEMENTS. I would like to trank dr.Ş.Strătilă for pointing we out the subject, giving me valuable information concerning it and for carefully reading the manuscript.

B I B L I O G R A P H Y

[1] Connes A.: Outer conjugacy classes of automorphisms of factors. Ann.Sci.Ec.Norm.Sup., 4 eme serie, t.8, fasc.3(1975), 383-420.

[2] Conze, V.P.: Entropie d'un groupe abélien de transfromations. Z.Wahr.verw.Gebiete, 25(1972), 11-30.

[3] Katznelson Y. and Weiss B.: Commuting measure-preserving transformations. Israel J.Math.12(1972), 161-173.

[4] Lind D.A.: Loccally compact measure preserving flows. Adv. in Math.15(1975), 175-193.

[5] Ornstein D., Weiss P.: The Kakutani-Rohlin theorem for solvable groups, preprint.

[6] Series C.: The Rohlin tower theorem and hyperfiniteness for actions of continuous groups. Israel J.Math.30(1978), 99-122.

[7] Strătilă Ş., Zsidó L.: Lectures on von Neumann algebras. Turnbidge Wells, Abacus Press (1979).