INSTITUTUL
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MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250-3638

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PREPRINT SERIES IN MATHEMATICS
No.11/1980

BUCURESTI

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March 1980

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DAN VUZA

In our paper [4] we studied a class of ordered rings and gave a theorem of Hahn-Banach type for modules over Such rings. We intend now to complete the formulation by discussing the case of maps invariant with respect to the action of a semigroup.

If G is a lattice - ordered group we shall use the standard notations

$$G^+ = \{g \in G \mid g \geq 0\}$$
$$g^+ = \sup (g,0)$$
$$g^- = \sup (-g,0)$$
$$|g| = \sup (g, -g).$$

Definition 1 ([4]). A G-ring is a ring A with an unit e and an ordering $\leq$ satisfying the axioms:

i) A is a boundedly complete lattice-ordered group.

ii) for every $a \in A^+$ and $b \in A$ we have

$(ab)^+ = ab^+$,

$(ba)^+ = b^+a$

iii) 2e is invertible.

Using a theorem of Birkhoff and Pearce one can show that a G-ring is associative and commutative.

The class of G-rings contains rings such as $L^\infty([0,1])$ or the space of equivalence classes of measurable functions on $[0,1]$.

If we require that every non-divisor of 0 is invertible we find the class of F-rings which was studied by A.Ghika ([1]).

From [4] we quote the following results on G-rings:

On every G-ring A can be given a unique structure of vector lattice over $\mathbb{R}$.

If $a,b \in A$, $0 \leq a \leq b$ and a is invertible then b is invertible and $0 \leq b^{-1}$.

For every $a \in A$ there is a $v \in A^+$ such that

$$v = v^2$$
$$a^+ = av$$
$$a^- = a(v-e).$$

<u>Definition 2.</u> Let A be a G-ring. An ordered A-module is an A-module H such that H is an ordered group and for every $a \in A^+$ and $h \in H^+$ we have $ah \in H^+$.

Every ordered A-module H can be turned into an ordered vector space putting

$$\alpha h = (\alpha e)h, \quad \alpha \in \mathbb{R}, \ h \in H.$$

<u>Definition 3.</u> Let A be a G-ring, E an A-module, H an ordered A-module. A map $p: E \longrightarrow H$ is A-sublinear iff

i) $p(x+y) \leq p(x)+p(y)$, $x,y \in E$

ii) $p(ax) = ap(x)$, $a \in A_+$, $x \in E$.

For reader's convenience we reproduce from [4] the following results with proofs:

Proposition 1. Let A be a G-ring, E an A-module, H an ordered A-module, $p:E\longrightarrow H$ a map such that

$$p(x+y)\leq p(x)+p(y), \quad x,y\in E$$
$$p(ax)\leq ap(x) \quad , a\in A^{+}, x\in E.$$

Then p is A-sublinear.

Proof. If $a\in A^{+}$ is invertible then

$$p(ax)\leq ap(x)=ap(a^{-1}(ax))\leq aa^{-1}p(ax)=p(ax).$$

If $a\in A^{+}$ then a+e is invertible. We have

$$ap(x)\geq p(ax)=p((a+e)x-x)\geq p((a+e)x)-p(x)=$$
$$=(a+e)p(x)-p(x)=ap(x).$$

Theorem 1. Let A be a G-ring, E an A-module, H an boundedly complete lattice - ordered A-module, F a submodule of E, $f:F\longrightarrow H$ an A-linear map, $p:E\longrightarrow H$ an A-linear map, $p:E\longrightarrow H$ an A-sublinear map such that

$$f(x)\leq p(x), \quad x\in F.$$

Then there exists an A-linear map $g:E\longrightarrow H$ such that $g|F=f$ and

$$g(x)\leq p(x), \quad x\in E.$$

Proof: By Zorn's lemma it is sufficient to consider the case $E = F + Ay$ with $y \notin F$. For $x_1, x_2 \in F$ we have

$$f(x_1-x_2) \leq p(x_1-x_2) \leq p(x_1+y)+p(-y-x_2)$$

so there is a $k \in H$ such that

$$f(x)+k \leq p(y+x),\ x \in F \tag{1}$$

$$f(-x)-k \leq p(-y-x),\ x \in F \tag{2}$$

Let $a \in A$. There is a $v \in A^+$ such that $a^+=av$, $a=a(v-e)$. ne is invertible, so $a^+ + (ne)^{-1}$ and $a^- + (ne)^{-1}$ are invertible. Let

$$b_n=(a^+ + (ne)^{-1})v,$$
$$c_n=(a^- + (ne)^{-1})(e-v),$$
$$a_n=b_n-c_n$$

We have

$$a_n[(a^+ + (ne)^{-1})^{-1}v-(a^- + (ne)^{-1})^{-1}(e-v)]= e,$$
$$b_n=a_n v,$$
$$c_n=a_n(v-e).$$

In particular, a_n is invertible.

For $z \in E$ we have

$$vp(a_n z)=p(a_n vz)=p(b_n z)=b_n p(z),$$
$$(e-v)p(a_n z)=p(a_n(e-v)z)=p(-c_n z)=c_n p(-z)$$

so

$$p(a_n z)=b_n p(z)+c_n p(-z) \tag{3}$$

From (1) and (2)

$$f(b_n x)+b_n k\leq b_n p(y+x),$$

$$f(-c_n x)-c_n k\leq c_n p(-(y+x)).$$

Using (3) it results

$$f(a_n x)+a_n k\leq p(a_n y+a_n x).$$

Replacing x by $a_n^{-1}x$ we get

$$f(x)+a_n k\leq p(a_n y+x), \qquad x\in F. \tag{4}$$

We have

$$p(a_n y+x)=p(ay+x+(ne)^{-1}(2v-e)y)\leq p(ay+x)+(ne)^{-1}(2v-e)p(y).$$

According to (4) we get

$$f(x)+ak\leq p(ay+x)+(ne)^{-1}(2v-e)\ (p(y)-k)^{+}.$$

As H is Archimedian it results

$$f(x)+ak\leq p(ay+x),\ a\in A,\ x\in F \tag{5}.$$

From (5) we have that the map $g:E\longrightarrow H$ given by

$$g(x+ay)=f(x)+ak$$

is well-defined and satisfyies the requirements of the theorem.

<u>Definition 4</u>. Let E be an A-module and K a semigroup An A-linear action of K on E is a representation of K into the semi-group of the A-linear endomorphisms of E.

<u>Definition 5.</u> Let E,F be A-modules and K a semi-group acting A-linearly on E. A map $f:E \longrightarrow F$ is K invariant iff

$$f(kx)=f(x), \quad k\in K, \ x\in E.$$

<u>Definition 6.</u> Let E be an A-module, H an ordered A-module and K a semi-group acting A-linearly on E. A map $p:E \longrightarrow H$ is A-decreasing iff

$$p(kx)\leq p(x), \quad k\in K, \ x\in E.$$

For a semi-group K and an ordered A-module H, let $\mathcal{B}_A(K,H)$ be the set of maps $t:K\longrightarrow H$ such that $t(K)$ is order-bounded. $\mathcal{B}_A(K,H)$ is an ordered A-module by defining

$$(t_1+t_2)(k) = t_1(k)+t_2(k),$$
$$(at_1)(k) = at_1(k),$$
$$t\geq 0 \Longleftrightarrow t(k)\geq 0 \ \forall \ k\in K.$$

If $h\in H$ we denote by t_h the map given by

$$t_h(k)=h.$$

Let $\widetilde{K}$ be the semigroup given by

$$\tilde{K} = K \times K,$$

$$(k_1,l_1)(k_2,l_2)=(k_1k_2,l_2l_1)$$

We have a linear action of $\tilde{K}$ on $\mathcal{B}_A(K,H)$ given by

$$(ktl)(m) = t(lmk).$$

Definition 7. An invariant mean on the semi-group K is a positive linear K-invariant map $\lambda : \mathcal{B}_{\mathbb{R}}(K,\mathbb{R}) \longrightarrow \mathbb{R}$ such that

$$\lambda(t_1)=1.$$

Theorem 2. Let A be a G-ring, H boundedly complete lattice-ordered A-module and let K be a semigroup which admits an invariant mean. Then there is a $\tilde{K}$-invariant A-linear map $\mu : \mathcal{B}_A(K,H) \to H$ such that

$$\mu(t_h) = h, \qquad h \in H,$$
$$\mu(t) \geq 0 \quad \text{if } t \geq 0.$$

Proof. Let

$$E = \mathcal{B}_A(K,H)$$
$$V = \left\{ \sum_{i=1}^{n}(k_i t_i l_i - t_i) \,\middle|\, n \in \mathbb{N}, t_i \in E, k_i, l_i \in K, 1 \leq i \leq n \right\}$$
$$F = \left\{ t_h + v \,\middle|\, h \in H, v \in V \right\}.$$

Define $p : E \longrightarrow H$ by

$$p(t) = \sup_{k \in K} t(k).$$

By proposition 1, p is an A-sublinear map. We want to prove that

$$h \leq p(t_h+v), \ h \in H, \ v \in V \qquad (6)$$

Suppose that

$$v = \sum_{i=1}^{n} (k_i t_i l_i - t_i).$$

Put

$$z_i = \sup_{k \in K} |t_i(k)|.$$

Let $z \in H$ be such that

$$h + v(k) \leq z, \quad k \in K.$$

Consider the order ideal I of H spanned by h,z and z_i, $1 \leq i \leq n$. Let λ be an invariant mean on K and let $\varphi : I \longrightarrow \mathbb{R}$ a positive linear map . If $u:K \longrightarrow \mathbb{R}$ is given by $u(k)=\varphi(v(k))$ then $u \in \mathcal{B}_{\mathbb{R}}(K,\mathbb{R})$ and $\lambda(u)=0$. It follows that $\varphi(k) \leq \varphi(z)$. By Kakutani's representation theorem we have that $h \leq z$.

(6) shows that the map $f:F \longrightarrow H$ given by

$$f(t_h+v)=h, \ h \in H, \ v \in V$$

is well-defined and satisfyies

$$f(x) \leq p(x), \ x \in F.$$

Applying theorem 1 we can extend f to μ: $E \longrightarrow H$ such that

$$\mu(t) \leq p(t), \quad t \in E.$$

It follows that $t \geq 0$ implies $\mu(t) \geq 0$. As $\mu(v)=0$ for $v \in V$, μ is $\tilde{K}$-invariant.

Theorem 3. Let A be a G-ring, E an A-module, H a boundedly complete lattice - ordered A-module, K a semi-group acting A-linearly on E and admitting an invariant mean, F a submodule of E such that $kx \in F$ if $k \in K$ and $x \in F$, $p: E \longrightarrow H$ a K-decreasing A-sublinear map, $f: F \longrightarrow H$ a K-invariant A-linear map such that

$$f(x) \leq p(x), \quad x \in F.$$

Then there exists a K-invariant A - linear map $g: E \rightarrow H$ such that $g|F = f$ and

$$g \leq (x) \leq p(x), \quad x \in E.$$

Proof. By theorem 1 there exists an A-linear map $g_1: E \longrightarrow H$ such that $g_1|F = f$ and $g_1(x) \leq p(x)$, $x \in F$. Define $T: E \rightarrow \mathcal{B}_A(K, H)$ by

$$T(x)(k) = g_1(kx), \quad x \in E, \ k \in K.$$

Let $\mu: \mathcal{B}_A(K,H) \longrightarrow H$ be the map given by theorem 2. The map $g: F \longrightarrow H$ given by

$$g(x) = \mu(T(x))$$

satisfyies the requirements of the theorem.

If we put $A = \mathbb{R}$ in theorem 3 we obtain a result of Silverman ([2],[3]).

As an application, let A be the G-ring of classes of measurable real-valued functions on $[0,1]$. A sequence $(f_n)_{n\in\mathbb{N}}$ of elements of A is said to be convergent to f A iff for a choice of representants $\varphi_n \in f_n, \varphi\in f$ we have $\lim_{n\to\infty}\varphi_n(t)=\varphi(t)$ for almost every $t\in[0,1]$. Let E be the set of order-bounded sequences of elements of A. E is an A-module by defining

$$(f_n)_{n\in\mathbb{N}}+(g_n)_{n\in\mathbb{N}} = (f_n+g_n)_{n\in\mathbb{N}},$$
$$f(f_n)_{n\in\mathbb{N}}= (ff_n)_{n\in\mathbb{N}}.$$

The semigroup $\mathbb{N}$ acts on E by

$$k(f_n)_{n\in\mathbb{N}}= (f_{n+k})_{n\in\mathbb{N}}.$$

Let F be the submodule of order-bounded convergent sequences of elements of A and $l:F\longrightarrow A$ the map which associates to every convergent sequence its limit. Define $p:E\longrightarrow A$ by

$$p((f_n)_{n\in\mathbb{N}}) = \inf_{n\in\mathbb{N}} \sup_{m\geq n} f_m$$

p is A-sublinear and $\mathbb{N}$ - decreasing, l is $\mathbb{N}$- invariant and we have $l(x)\leq p(x)$, $x\in F$.

By theorem 3 we obtain an invariant positive A-linear map $L:E\longrightarrow A$ such that $L|F = l$, that is an invariant A-valued limit of Banach type. This limit cannot be obtained " pointwise ", i.e. by applying a limit of Banach type for scalar sequences to the sequence $\varphi_n(t)$ $(\varphi_n\in f_n)$ because we do not know if the limit function is measurable.

R e f e r e n c e s

1 A.GHIKA " Prelungirea funcţionalelor generale liniare în module seminormate ".
St.cercet.matem., 1950, 1,2,251-269.

2 R.J.SILVERMAN " Means on semi-groupsand the Hahn-Banach extension property ".
Trans.Amer.Math.Soc., 83, 222-237 (1956).

3 - " Invariant means and cones with vector interiors ".
Trans.Amer.Math.Soc., 88, 75-79 and 327-330 (1958).

4 D.VUZA "O generalizare a inelelor Ghika ".
St.cercet.matem., 1979, 31, 2, 251-274.