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Weierstraß categories and the property of approximation

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0. Introduction

In his paper "On the solution of analytic equations" (cf. [2])
M. Artin proved the following famous approximation theorem:
Let k be a valued field of charactezstic 0 and
$f(X,Y) = (f_1(X,y), \ldots, f_m(X,Y))$ convergent power series in
$X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_N)$ with coefficients in k.
If the equation $f(X,Y) = 0$ has a formal solution
$\bar{y} = (\bar{y}_1, \ldots, \bar{y}_N)$, $\bar{y}_i$ $k[[X]]$ formal power serieses in X,
then there exists for any integer $c > 0$ a convergent solution
$y = (y_1, \ldots, y_N)$, $y_i \in k\{x\}$ convergent power serieses in X,
of the equation $f(X,Y) = 0$ such that $y_i \equiv \bar{y}_i \mod X^c$.
With similar methods M. Artin proved the analogous theorem for
algebraic power sereses (the case $n = 1$ was already considered
by M. J. Greenberg, cf. [7]) in [3].
The main idea of Artin's proofs is the application of the
Weierstraß preperation theorem and the implicit funktion theorem
(resp. Newton's lemma) in order to be able to apply induction
on n (the number of the indeterminates).
This gave us the idea to generaliz Artin's proof to classes of
rings with the preperation theorem and some other "good"
properties (cf. [12], [13]). The reason for such a generalization was to get a common proof for Artin's approximation

theorems in the algebraic and analytic case. Furthermore we wanted to prove the approximation theorem for convergent power serieses over a valued field of characteristic $p > 0$ (For the case of the field being completely valued such an approximation theorem was also proved by M. André [1], U. Jähner [8], M. van der Put [10] with different methods).

In [10] we developed the idea to consider classes of rings with the preperation theorem to the so-called Weierstraß categories and proved the approximation theorem for rings of these Weierstraß categories.

I. Denef and L. Lipshitz pointed out that some details of this proof were incomplete resp. incorrect and invented W-systems being similar to [13] but more general (i.e. families of regular local rings with the preparation theorem and some more good properties) and proved the approximation theorem for rings of a W-system (cf. [5]).[1]

The idea of this paper is slightly to generalize the notion of Weierstraß categories of [10] to get also a connection to [5] and to give a complete proof of the approximation theorem. Furthermore we want to prove Elkik's approximation theorem (cf. [6]) for Weierstraß categories.

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1) The authors would like to thank Mr. Denef and Mr. Lipshitz for their interest in our Weierstraß categories and for their hints with respect to some problems in our proof of the approximation theorem.

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In the first three chapters we will give a general definition of a Weierstraß category and its properties. Especially we will characterize smooth morphismus, prove the implicit function theorem and Newton's lemma.

In chapter 4 we will prove Elkik's approximation theorem for Weierstraß categories.

To get an idea of a Weierstraß category we will give the definition for the local case here.

Let R be a field or an henselian discrete valuation ring and by $\underline{\underline{C}}_R$ let us denote the category of all noetherian henselian local rings over R with the same residue field.

A full subcategory $\underline{\underline{H}}_R$ of $\underline{\underline{C}}_R$ is a Weierstraß category if it has the following properties:

1) Each morphism in $\underline{\underline{H}}_R$ is a Weierstraß-morphism, i. e. if for a morphism in $A \longrightarrow B$ in $\underline{\underline{H}}_R$ and a ~~closed~~ ideal $K \subseteq B$ ~~with respect to the m_A-adic topology~~ $A \longrightarrow B/K + m_A B$ is finite, then $A \longrightarrow B/K$ is A-finite.

2) $\underline{\underline{H}}_R$ contains free objects, i. e. if $A \in \underline{\underline{H}}_R$ and if $(T_1, \ldots, T_n)$ is a finite sequence of in determinates, there exists the free A-algebra $A \{\{ T_1, \ldots, T_n \}\}$ in $\underline{\underline{H}}_R$ (this means that for any A-algebra $B \in \underline{\underline{H}}_R$ and $t_1, \ldots, t_n \in m_B$ there exists exactly one A-morphism $A\{\{T_1, \ldots, T_n\}\} \longrightarrow B$ in $\underline{\underline{H}}_R$ mapping the T_i to the t_i).
Moreover, the kernels of the canonical morphisms $A \{\{ T_1, \ldots, T_n \}\} \longrightarrow A[T_1, \ldots, T_n]/(T_1, \ldots, T_n)$ are the ideals $(T_1, \ldots, T_n)$ $A \{\{ T_1, \ldots, T_n \}\}$.

3) Any $A \in \underline{\underline{H}}_R$ is a quotient of some $R \{\{T_1, \ldots, T_n\}\}$.

In [10] we called a Weierstraß category $\underline{\underline{H}}_R$ excellent iff for all $A \in \underline{\underline{H}}_R$ the morphism Spec $\hat{A} \longrightarrow$ Spec A is formally smooth.
In this paper we will consider a more general situation.
We call $\underline{\underline{H}}_R$ semi-excellent iff for all $A \in \underline{\underline{H}}_R$ and $T = (T_1, \ldots, T_n)$ the morphism Spec $\hat{A}[[T]] \longrightarrow$ Spec $A[T]$ is formally smooth in all $\mathfrak{p} \in$ Spec $A[T]$ being kernels of a suitable morphism $A[T] \longrightarrow \hat{B}$, $B \in \underline{\underline{H}}_R$.
The notation of a semi-excellent Weierstraß category is in principle a generalization of the excellent Weierstraß categories of [10] and the W-systems of Denef and Lipshitz [5].
We do not know if a semi-excellent Weierstraß category is already excellent. But it is useful to have this apparently more general notion because of the following example:

(1) Let k be a quasicomplete ~~ly~~ valued field (i.e. the completion $\bar{k}$ of k with respect to the valuation is a seperable extension of k) and $\underline{\underline{H}}_k$ the category of a analytic k-algebras, then $\underline{\underline{H}}_k$ is semi-excellent (cf. [17]). But we do not know if an analytic k-algebra is excellent in case of char $k = p > 0$. This only seems to be known up to now if k is already complete or in the case $[k : k^p] < \infty$.

Further examples of excellent Weierstraß categories are (cf. [10]):

(2) The category $\underline{\underline{H}}_R$ of henselian rings of finite typ over R, R a field or an excellent discrete valuation ring.

(3) The category $\underline{\underline{H}}_R$ of all noetherian henselian local R-algebras which are complete with respect to the m_R-adic topology such that, for all $A \in \underline{\underline{H}}_R$, $A/m_R^c A$ is henselian of finite typ over R/m_R^c for all $c \geq 1$, where R is a complete discrete valuation ring of charakteristic 0.

(4) Let $\{R_\alpha\}_{\alpha \in I}$ be a filtered system of fields or an complete discrete valuation ring such that the corresponding residue field extensions are separable and $R = \varinjlim R_\alpha$ be a field or an excellent discrete valuation ring,
$\underline{\underline{H}}_R$ category of all R-algebras $\varinjlim R_\alpha [[X_1, \ldots, X_n]]$ and their quotients.

The main result of this paper will be the following theorem (cf. Theorem 6.3.):
Semi-excellent Weierstraß categories have the property of approximation, i.e. Artin's approximation theorem holds:
Let (A, m) be a local ring from a semi-excellent Weierstraß category and $f = (f_1, \ldots, f_m)$ an arbitrary system of polynomials in some variables $Y = (Y_1, \ldots, Y_N)$ with coefficients in A (or more general $f_i \in A\{Y\}$). Then every solution $\bar{y}$ of f in $\hat{A}$ (the completion of A) can be well approximated in the m-adic topology by a solution of f in A (i.e. for every positive integer c there exists a solution y of f in A such that $y \equiv \bar{y} \bmod m^c A$).

We would like to express our gratitude to Miss Behrendt and Miss Subirge for typing the manuscript of this paper.

1. Basic definitions

We denote by $\underline{\underline{C}}$ the category of all Henselian ~~Henechim~~ pairs (A, I_A) such that

$$\bigcap_{\nu=0}^{\infty} I_A^{\nu} = 0$$

We call a morphisme $(A, I) \longrightarrow (B, J)$ in $\underline{\underline{C}}$ a <u>Weierstraß morphisme</u>, if it has the following property:

(W) For any closed ideal $K \subseteq B$ (with respect to the I-adic topology) such that the morphisme $A \longrightarrow B/K + IB$ is finite, the morphisme $A \longrightarrow B/K$ is A-finite.

<u>Remark</u> The property (W) implies in most cases the stronger property:

(W') For any separated B-module E (with respect to the I-adic topology) of finite type such that E/IE is of finite type over A, the module E is of finite type over A

<u>Proposition 1</u> Property (W) implies (W') in the following cases

(1) $\hat{A}$ and B are Noetherian rings

(2) A is complete with respect to the I-adic topology

More precisely, assume A is Noetherian, let $(A, I) \longrightarrow (B, J)$ be a Weierstraß morphisme in $\underline{\underline{C}}$ and let E be a B-module of finite type with annihilator ideal $N \subset B$, such that all ideals

$BI^{\nu} + N/N \subseteq B/N \subseteq \mathrm{End}_B\ (E)$ $(\nu = 1, 2, \ldots)$

are closed in the I-adic topology of $\mathrm{End}_B\ (E)$, the following properties are equivalent

(i) E is of finite type over A

(ii) E/IE is of finite type over A

Proof of (2): If $p_1, \ldots, p_N \in E$ and

$$E = Ap_1 + \ldots + Ap_N + IE$$

then $E = Ap_1 + \ldots + Ap_N$, since A is complete

For the last assertion we infere from (ii) the property

$$\hat{E} = \hat{A}p_1 + \ldots + \hat{A}p_N,$$

hence $\mathrm{End}_{\hat{A}}(\hat{E})$ is of finite type over $\hat{A}$ and by hypothesis $\hat{B}/N\hat{B} \subseteq \mathrm{End}_{\hat{B}}(\hat{E}) \subseteq \mathrm{End}_{\hat{A}}(\hat{E})$. There fore $\widehat{B/NB}$ is finite over $\hat{A}$ and $\hat{B}/N\hat{B} + I\hat{B} = B/N + IB$ is finite over A, hence B/N is finite over A by (W) and since E is of finite typ over B/N, it is also of finite type over A.

Statement (1) is a special case of the last assertion.

Now we consider a subcategory $\underline{\underline{H}}$ of $\underline{\underline{C}}$, and we define the notion of a free pair with respect to $\underline{\underline{H}}$ in the obvious way as follows:

If (A, I), (B, J) are pairs in $\underline{\underline{H}}$ and $T = (T_1, \ldots, T_n)$ a finite sequence of elements of J, we call (B, J) free over (A, I) with generators $T_1, \ldots, T_n$, if for any morphisme $(A, I) \to (C, K)$ in $\underline{\underline{H}}$ and any sequence $(t_1, \ldots, t_n)$, $t_i \in K$, there exists exactly one A-morphisme

$f: (B, J) \to (C, K)$ such that $f(T_i) = t_i$.

Obviously, (B, J) is uniquely determined by this property and we will denote it by $(A, I) \{\{ T_1, \ldots, T_n \}\} = (A \{\{T_1, \ldots, T_n\}\}, I \{\{ T_1, \ldots, T_n\}\})$ or simply by $A \{\{T_1, \ldots, T_n\}\}$ if there is no confusion obout the ideal J

Now the essential notion fo our paper is the following notion of a Weierstraß category

<u>Definition</u> A full subcategory $\underline{\underline{H}}$ of $\underline{\underline{C}}$ is called a Weierstraß category, W-category for short, if it satisfies the following axioms

(W0) For each morphisme $(A, I) \longrightarrow (B,J)$ in $\underline{\underline{H}}$ the rings $B/I^{\vee}B$ are Noetherian and $A/I \longrightarrow B/J$ is surjective

(W1) Each morphisme in $\underline{\underline{H}}$ is a Weierstraß morphisme

(W2) $\underline{\underline{H}}$ is closed with respect to finite morphismes in $\underline{\underline{C}}$, i. e. if $(A, I) \in \underline{\underline{H}}$ and if $(A, I) \longrightarrow (B, J)$ is a finite morphisme in $\underline{\underline{C}}$ and $A/I \longrightarrow B/J$ surjective then (B, J) belongs to $\underline{\underline{H}}$

(W3) $\underline{\underline{H}}$ contains free objects. If $(A, I) \in \underline{\underline{H}}$ and if $(T_1, \ldots, T_n)$ is a finite sequence of indeterminates, there exist the free pair

$(A, I)\{\{T_1, \ldots, T_n\}\}$ in $\underline{\underline{H}}$, which moreover satisfies the property: The kernels of the canonical morphismes

$$A\{\{T_1, \ldots, T_n\}\} \longrightarrow A[T_1, \ldots, T_n] / (T_1, \ldots, T_n)^{\vee}$$

(which containes $(T_1, \ldots, T_n)^{\vee}$) are the ideals

$(T_1, \ldots, T_n)^{\vee} A\{\{T_1, \ldots, T_n\}\}$.

Furthermore, we define an <u>excellent W-category</u> as a W-category, where all of its objects are local rings (A, I) such that the morphisme $\mathrm{Spec}\,(\hat{A}) \longrightarrow \mathrm{Spec}\,(A)$ is formally smooth.

The aim of this articel is to show, that excellent W-categories have the approximation property, if they contain a field or a discrete valuation ring.

2. Smooth morphismes in Weierstraß categories

In the following, $\underline{\underline{H}}$ denots a fixed W-category. We firstly determine the structure of formally smooth morphismes in $\underline{\underline{H}}$. Recall that a morphisme $(A, I) \longrightarrow (B, J)$ is called formally smooth, if for any local Artmién ring $(\bar{R}, \bar{m})$ and any (small) extension $(R, m) \longrightarrow (\bar{R}, \bar{m})$ of Artin-Rings there holds:

If

$$\begin{tikzcd} (A, I) \arrow[r] \arrow[d] & (R, m) \arrow[d] \\ (B, J) \arrow[r, "\bar{\mu}"'] \arrow[ur, dashed, "\mu"] & (\bar{R}, \bar{m}) \end{tikzcd}$$

is a commutative diagramm of morphismes (without the dotted arrow), the morphisme $\bar{\mu}$ can be lifted to an (A, I) - morphisme $\mu : (B, J) \longrightarrow (R, m)$.

We want to describe the structure of formally smooth morphismes in $\underline{\underline{H}}$. To do this, we firstly generalize the construction of free objects somewhat. Let (A, I) be a pair in $\underline{\underline{H}}$ and let E be a projective A-module of finite presentation

$$E = AT_1 + \dots + AT_n / \lambda_1(T) A + \dots + \lambda_r(T) A$$

$(\lambda_1, \dots, \lambda_r$ linear forms in $T)$

By $(A, I)\{\{E\}\} = (A\{\{E\}\}, I\{\{E\}\})$ we denote the following pair: $A\{\{E\}\} = A\{\{T\}\} / \lambda_1 A\{\{T\}\} + \dots + \lambda_r A\{\{T\}\}$

$$I\{\{E\}\} = I A\{\{E\}\} + T_1 A\{\{E\}\} + \dots + T_n A\{\{E\}\}$$

We want to show that $(A, I)\{\{E\}\} \in \underline{\underline{H}}$.

Clearly $I\{\{T\}\} = I A\{\{T\}\} + T_1 A\{\{T\}\} + \dots + T_n A\{\{T\}\}$ by aziom (W3) (observe $I \subseteq I\{\{T\}\}$, $T_\gamma \in I\{\{T\}\}$ and $A\{\{T\}\} / I\{\{T\}\} \longrightarrow A/I$ is surjective by axiome (W0)).

Furthermore $(A\{\{E\}\}, I\{\{E\}\})$ is Henselian and therefore it remains to show that
$B = A\{\{E\}\}$ is separated in the $J = I\{\{E\}\}$-adic topology. To prove this we embed $(A\{\{E\}\}, I\{\{E\}\})$ in $(A\{\{T\}\}, I\{\{T\}\})$. Because E is projective, there exists a projection operator

$$\pi : AT_1 + \dots + AT_n \longrightarrow AT_1 + \dots + AT_n$$

with kernel $A\lambda_1(T) + \dots + A\lambda_r(T)$.

The operator π can be lifted to a (A, I)-morphisme

$$\tilde{\pi} : A\{\{T\}\} \longrightarrow A\{\{T\}\}, \quad \tilde{\pi}(T_\nu) = \pi(T_\nu)$$

If $B' = \tilde{\pi}(A\{\{T\}\})$, $J' = \tilde{\pi}(I\{\{T\}\})$, the pair $(B', J') \subseteq (A\{\{T\}\}, I\{\{T\}\})$ is in $\underline{H}$ (by (W2))

If $U = (U_1, \dots, U_n)$ are indeterminates, we define a (B', J')-morphisme by

$p : B'\{\{U\}\} \longrightarrow A\{\{T\}\}$, $p(U_\nu) = T_\nu$.

If $f(T) \in A\{\{T\}\}$ and $\tilde{\pi}(f(T)) = 0$, the corresponding element $f(U) \in B'\{\{U\}\}$ is contained in the kernel of the morphisme $B'\{\{U\}\} \longrightarrow B'$, $U_\nu \longmapsto \pi(T_\nu)$, hence we can write (by (W3))

$$f(U) = \sum_{\nu=1}^{n} (U_\nu - \pi(T_\nu))\, \varepsilon'_\nu(U) \quad \cancel{(U)}$$

$(\varepsilon'_\nu(U) \in B'\{\{U\}\})$ and we obtain (applying the morphisme p)

$$f(T) = \sum_{\nu=1}^{n} (T_\nu - \pi(T_\nu))\, \varepsilon_\nu(T).$$

Therefore

$$f(T) = \sum_{\nu=1}^{n} (T_\nu - \pi(T_\nu))\, A\{\{T\}\} = \sum_{\rho=1}^{r} \lambda_\rho(T)\, A\{\{T\}\}$$

and $(B, J) \simeq (B', J') \in \underline{H}$.

Now we can describe the structure of formally smooth morphismes.

<u>Theorem 1</u> For any pair $(A, I) \in \underline{\underline{H}}$ there holds :

(1) If E is a projective A-module of finite typ, the morphisme $(A, I) \longrightarrow (A, I)\{\{E\}\}$ is formally smooth.

(2) If $(A, I) \longrightarrow (B, J)$ is a formally smooth morphisme in $\underline{\underline{H}}$ and if A is Henselian with respect to the ideal $I_1 = J \cap A$, the pair (A, I_1) is contained in $\underline{\underline{H}}$ and there exist a projective E-module of finite type such that

$$(B, J) \cong (A, I_1)\{\{E\}\}$$

To proof the first part of the theorem we can replace everything by the I-adic resp. $I\{\{E\}\}$-adic completion. But the algebra

$$\widehat{A\{\{E\}\}} = \hat{A}[[T]] / \lambda_1 \hat{A}[[T]] + \dots + \lambda_r \hat{A}[[T]]$$

is obviously formally smooth over $\hat{A}$.

Proof of assertion (2): We replace (A, I) by (A, I_1) (contained in $\underline{\underline{H}}$ by axiom (W2)), hence we can assume : $A/I \cong B/J$.

<u>Step I</u> <u>The module $\bar{E} = J/J^2 + IB$ is projective and of finite typ over $\bar{A} = A/I$.</u>

Obviously $\bar{E}$ is a $\bar{A}$-module of finite typ, since $\bar{B} = B/IB$ is Noetherian. We have to show that for any epimorphisme

$$M \longrightarrow N$$

Of $\bar{A}$-modules of finite typ any homomorphisme $\bar{p} : \bar{E} \longrightarrow N$ can be lifted to a homomorphisme $p : \bar{E} \longrightarrow M$.

Let m be any maximal ideal of $\bar{A}$ and assume $m^{\nu} M = m^{\nu} N = 0$ for suitable ν. Consider the epimorphisme of local Artinian algebras $R = \bar{A}/m \oplus M \longrightarrow \bar{R} = \bar{A}/m \oplus N$ $(M^2 = N^2 = 0)$ and the homomorphismes of rings

$s : \bar{B} \longrightarrow \bar{A}$, $s(b) = b \bmod JB$ (observe $\bar{B}/J\bar{B} \cong \bar{A}$)

and $\bar{t} : \bar{B} \longrightarrow \bar{R}$

$\bar{t}(b) = (s(b) \bmod m^{\gamma}, \ \bar{p}((b - s(b)) \bmod J^2\bar{B}))$

Because $\bar{B}$ is formally smooth over $\bar{A}$, we can lift $\bar{t}$ to an homomorphisme $t: \bar{B} \longrightarrow R$

and t induces a lifting $p: \bar{E} \longrightarrow M$ of the homomorphisme $\bar{p}$. This implies that for any maximal ideal m of $\bar{A}$ the m-adic completion of $\bar{E}$ is projective over the m-adic completion of $\bar{A}$, by faithfully flat decent we infere therefore that $\bar{E}$ is projective.

<u>Step II</u> <u>Construction of a surjection</u> $(A, I)\{\{E\}\} \longrightarrow (B, J)$

Since (A, I) is Henselian we can lift idempotent elements in any finite A-algebra $\mathcal{E}$ from $\mathcal{E}/I\mathcal{E}$ to $\mathcal{E}$.

Now $\bar{E}$ is a direct summand of a free $\bar{A}$ module $\bar{A}^N$ and we can lift the corresponding projection operator in $\mathrm{End}(A^N)/I\,\mathrm{End}(A^N)$ to $\mathrm{End}(A^N)$. Hence $\bar{E}$ can be lifted (uniquely up to an isomorphisme) to a projective A-module E of finite typ.

Since $\bar{E} = J/J^2 + IB$ we can lift the isomorphisme $E/IE \cong \bar{E}$ to an A-linear homomorphisme

$s: E \longrightarrow J$

and s induces a morphisme $(A, I)\{\{E\}\} \longrightarrow (B, J)$, also denoted by s, which is obviously surjective (by axiom (W1)).

<u>Step III</u> <u>The morphisme s is injective.</u>

It is sufficient to show: for any ideal Q in A such that A/Q is local Artinian, the induces homomorphisme

$\bar{s} : \ A\{\{E\}\} / QA\{\{E\}\} \to B/QB$

is injective (since the intersection of all this ideals Q is 0, by axiom (W0)).

Hence we can assume that A is local Artinian. Since B is formally smooth over A, we can construct step by step an homomorphisme $t : B \longrightarrow A\{\{E\}\}$ such that the diagramm

$$\begin{tikzcd} B \arrow[r, "t"] & A[[E]] \\ A\{\{E\}\} \arrow[u, "s"] \arrow[ur] & \end{tikzcd}$$

(with the canonical embedding $A\{\{E\}\} \subset A[[E]]$)

is commutative, therefore s is injective, and the proof of the theorem is finished.

We want to mention the following consequence.

<u>Corollary 1</u> If $(A, I) \longrightarrow (B, J)$ is a formally smooth morphisme in $\underline{\underline{H}}$ and $A/I \cong B/J$, the module $\mathrm{Hom}_A(E, I)$ acts transitively and free on the set of sections $\mathrm{Hom}_{(A,\ I)}((B, J), (A, I))$, and this set is not empty.

In the next section we derive the characterization of formal smoothness by the Jacobian criterian. For this reason we have to introduce the following facts

(1) For any morphisme $(A, I) \longrightarrow (B, J)$ in $\underline{\underline{H}}$ the pair (B, J) can be written up to isomorphy in the form

$$B = A\{\{T\}\}/K \quad , \quad J = IB + \sum_{\nu=1}^{n} T_\nu B$$

We have to choose representatives $t_1, \ldots, t_n$ of generators of the A-module $J/J^2 + IB$. Then we put $T = (T_1, \ldots, T_n)$ and define $A\{\{T\}\} \longrightarrow B$ by $T_\nu \longmapsto t_\nu$,

$K = \ker(A\{\{T\}\} \longrightarrow B)$.

By axiom (W1) we infere $B = A\{\{T\}\}/K$ and J is generated by the image of the ideal I and the elements t_ν.

(2) If E is any $A\{\{T\}\}$ - module, seperated with respect to the I-adic topology, the module of derivations is

$$\mathrm{Der}_A\,(A\{\{T\}\}, E) \simeq \mathrm{Hom}_{A\{\{T\}\}}\Big(\bigoplus_{\nu=1}^{n} A\{\{T\}\}\,dT_\nu, E\Big),$$

induced by a universal derivation

$$d : A\{\{T\}\} \longrightarrow \bigoplus_{\nu=1}^{n} A\{\{T\}\}\,dT_\nu$$

$$df = \sum_{\nu=1}^{n} \frac{\partial f}{\partial T_\nu}\,dT_\nu$$

Proof: The pair $B = A\{\{T\}\} + A\{\{T\}\}\varepsilon$, $J = I\{\{T\}\} + \varepsilon A\{\{T\}\}$ defined by $\varepsilon^2 = 0$ belongs to $\underline{\underline{K}}$ by axiom (W2). The $A\{\{T\}\}$-module

$$\Theta = \Big\{ \varphi \;\Big|\; \varphi \in \mathrm{Hom}_{(A,I)}((A\{\{T\}\}, I\{\{T\}\}), (B,J)),\ \varphi \bmod \varepsilon = id_{A\{\{T\}\}} \Big\}$$

is isomorphic to the modul $\mathrm{Der}_A\,(A\{\{T\}\}, A\{\{T\}\})$ by associating to a derivation θ the morphisme $\varphi : f \longmapsto f + \varepsilon\,\theta(f)$. Especially to any T_ν we can associate the morphisme $T_i \longmapsto T_i$ $(i \neq \nu)$, $T_\nu \longmapsto T_\nu + \varepsilon$, the corresponding derivation is denoted by $\frac{\partial}{\partial T_\nu}$.

If $D : A\{\{T\}\} \longrightarrow E$ is any derivation, we consider $D_o(f) = D(f) - \sum_{\nu=1}^{n} \frac{\partial f}{\partial T_\nu} D(T_\nu)$. From axiom (W3) we infer that $A\{\{T\}\}/(T)^\gamma A\{\{T\}\}$ is isomorphic to $A[T]/(T)^\gamma A[T]$, hence $D_o(f) \in (T)^\gamma E$ for any integer γ, and because E is separated, we get $D_o = 0$,

$$D = \sum_{\nu=1}^{n} D(T_\nu)\frac{\partial}{\partial T_\nu} \qquad \text{q.e.d.}$$

3. Quasiprojective schemes over Weierstaß categories

We want to study systems of equations of the typ $\{F(T, U) = 0\}$, where $T = (T_1, \dots, T_n)$; $U = (U_1, \dots, U_m)$ and $F(T,U) \in A\{\{T\}\}[U]^r$. In other words, in a slightly more general form(u)lation, we study schemes of the type

$X \longrightarrow \mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(A)$,

where $(A, I) \to (B, J)$ is a morphisme in $\underline{\underline{H}}$ and $\mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A)$ is induced by this morphisme, and where $X \longrightarrow \mathrm{Spec}\,(B)$ is a quasiprojective schema of finite presentation over B, i. e. X is a subschema of some projective scheme $P(E)$, where E is a B-module of finite presentation and where X is locally closed in $P(E)$ and locally defined in $P(E)$ over B by a finite number of equations.

If E is generated by say p elements, we can ~~fuid~~ find a closed embedding $P(E) \subseteq \mathbb{P}^p \times \mathrm{Spec}\,(B)$ of finite presentation, hence we can assume $X \subseteq \mathbb{P}^p \times \mathrm{Spec}\,(B)$.

More over, if $B = A\{\{T\}\}/K$, we can assume $X \subseteq \mathbb{P}^p \times \mathrm{Spec}\, A\{\{T\}\}$, locally closed and defined locally by K and finite many polynomial equations whith ceefficients in $A\{\{T\}\}$.

We prove the Jacobian criterium for this mixed situation, we then proof the existence of sections of X over A (theorem on implicite functions) and its generalization (the analoge of Newtons Lemma).

We denote by p the projection

$p : X \to \mathrm{Spec}\,(A)$.

If $x \in X$ is a point such that $p(x) \in V(I)$, we call p <u>formally smooth</u> at x (or X formally smooth over A in the point x), if the morphisme of pairs $(A, I) \to (\mathcal{O}_{X,x}, m_{X,x})$ (which is not in $\underline{\underline{H}}$) is formally smooth.

<u>Theorem 2</u> Assume $(A, I) \in \underline{\underline{H}}$, $T = (T_1, \ldots, T_n)$, and $X \subseteq P = \mathbb{A}^m \times \operatorname{Spec} A\{\{T\}\}$ is a locally closed subschema, $p : X \to \operatorname{Spec}(A)$ the corresponding morphisme. Let $U_1, \ldots, U_m$ be affine coordinates in $\mathbb{A}^m$. If $x \in X$, $p(x) \in V(I)$, the schema X is formally smooth over A in x, if and only if the following condition is satisfied:

(J) There exist functions $f_1, \ldots, f_k \in A\{\{T\}\}[U]$ which generate the kernel of the homomorphisme $\hat{\mathcal{O}}_{P,x} \to \hat{\mathcal{O}}_{X,x}$ and such that the Jacobian matrix $\left(\frac{\partial f_i}{\partial T_j}(x), \frac{\partial f_i}{\partial U_h}(x)\right)$ has the rank k.

(If $f \in \hat{\mathcal{O}}_{P,x}$ we denote by $f(x)$ the residual class of f in $\mathcal{O}_{P,x}/m_{P,x}$)

Proof: 1) Assume the condition (J) is satisfied and assume R, $\bar{R} = R/tR$ are local Artinian A-algebres such that $m_R t = 0$, and $\bar{u} : \mathcal{O}_{X,x} \to \bar{R}$ is a morphisme of local rings. We can lift the composition $\mathcal{O}_{P,x} \to \mathcal{O}_{X,x} \xrightarrow{\bar{u}} \bar{R}$ to a morphisme $v : \mathcal{O}_{P,x} \to R$ (by lifting the images of T_i, U_j). If the function f on P (is in the kernel of $\mathcal{O}_{P,x} \to \mathcal{O}_{X,x}$, the image $v(f)$ has the form $v(f) = t\,a(f)$, where $a(f) \in k = R/m$ (since t is annihilated by m_R).

For any choice of elements $a_1, \ldots, a_n, b_1, \ldots, b_m \in k$ the map

$$v'(f) = v(f) + t\sum_{j=1}^{n} a_j \frac{\partial f}{\partial T_j}(x) + t\sum_{h=1}^{m} b_h \frac{\partial f}{\partial U_h}(x)$$

is a morphisme of A-algebras $\mathcal{O}_{P,x} \to R$.

If condition (J) is satisfied we can choose $a_1, \dots, a_n$, $b_1, \dots, b_m$ in such a way that

$$v'(f_1) = \dots = v'(f_k) = 0$$

Since ker (v') contains some power of $m_{P,x}$ it contains therefore also the kernel of $\mathcal{O}_{P,x} \longrightarrow \mathcal{O}_{X,x}$ and induces a lifting $\mu : \mathcal{O}_{X,x} \longrightarrow R$ of $\bar{\mu}$.

2) Assume X is formally smooth over A in x. We will show that condition (J) is satisfied.

Define $C = \hat{\mathcal{O}}_{P,x}$, $B = \hat{\mathcal{O}}_{X,x}$, $K = \mathrm{Ker}\,(\mathcal{O}_{P,x} \to \mathcal{O}_{X,x})$ then $C/KC \simeq B$ and because of formal smoothness the canonical A-Morphisme

$$B \longrightarrow C/KC + m_C^2$$

can be lifted to a A-morphisme

$$\eta : B \longrightarrow C$$

If $\pi : C \longrightarrow B$ is the canonical morphisme, the composition $\pi \circ \eta : B \longrightarrow B$ conicides with id_B mod M_B^2, hense $\pi \circ \eta$ is an isomorphisme of B and $\varepsilon = \eta \circ (\pi \circ \eta)^{-1}$ is a section of π.

Any derivation $A\{\{T\}\}[U] \longrightarrow E$ over A with values in an I-adic separated C-modul E extends in a unique way to a derivation $C \longrightarrow E$, ~~thens~~ thus we have derivation

$$\frac{\partial}{\partial T_j} : C \longrightarrow C, \quad \frac{\partial}{\partial U_h} : C \longrightarrow C.$$

If $\vartheta : A\{\{T\}\}[U] \longrightarrow KC/K^2C =: \bar{K}$ is defined by $\vartheta(f) = f - \varepsilon \circ \pi(f)$ mod K^2C, it is a derivation, and we can extend it to C and have

$$\vartheta(f) = \sum_{j=1}^{n} \frac{\partial f}{\partial T_j}(T_j - t_j) + \sum_{h=1}^{m} \frac{\partial f}{\partial U_h}(U_h - u_h)$$

where $t_j = \varepsilon \circ \pi(T_j)$, $u_1 = \varepsilon \circ \pi(U_1)$. Then induces a B-linear map

$$v : \Omega =: \bigoplus_{j=1}^{n} B\,dT_j \oplus \bigoplus_{h=1}^{m} B\,dU_h \longrightarrow \bar{K}$$

$$v(dT_j) = (T_j - t_j) \bmod K^2C \qquad v(dU_h) = (U_h - u_h) \bmod K^2C$$

The derivation $A\{\{T\}\}[U] \longrightarrow \Omega$,

$$f \longmapsto \sum_{j=1}^{n} \pi\left(\frac{\partial f}{\partial T_j}\right) T_j + \sum_{h=1}^{m} \pi\left(\frac{\partial f}{\partial U_h}\right) dU_h$$

induces a B-linear map

$$w : \bar{K} \longrightarrow \Omega$$

(because of $\pi\left(\frac{\partial}{\partial T_j}(K^2)\right) = B$, $\pi\left(\frac{\partial}{\partial U_h}(K^2)\right) = 0$).

For $f \in KC$ there holds

$$v \circ w \;(f \bmod K^2C) = \left[\sum_{j=1}^{n} \pi\left(\frac{\partial f}{\partial T_j}\right)(T_j - t_j) + \sum_{h=1}^{m} \pi\left(\frac{\partial f}{\partial U_h}\right)(U_h - u_h)\right] \bmod K^2C$$

$$= \vartheta(f) = f - \varepsilon \circ \pi(f) \bmod K^2C$$

$$= f \bmod K^2C$$

Therefore $v \circ w = \mathrm{id}_{\bar{K}}$, $\bar{K}$ is a direct summand of Ω, hence a free B-modul. If the functions $f_1, \ldots, f_k \in A\{\{T\}\}[U]$ represent a free base of $\bar{K}$, the matrix corresponding to w is $\left(\frac{\partial f_i}{\partial T_j}, \frac{\partial f_i}{\partial U_h}\right)$ (evaluated on X), hence condition (J) is satisfied.

We consider the following question: Given a morphisme $(A, I) \longrightarrow (B, J)$ in $\underline{\underline{H}}$ and a quasiprojective B-schema $X \longrightarrow \mathrm{Spec}(B)$. Assume there is given a commutative diagramm of morphismes

$$(1) \qquad \begin{tikzcd} X \arrow[r] & \mathrm{Spec}\,(B) \arrow[d] \\ \mathrm{Spec}\,(A/I) \arrow[u, "\varepsilon_0"] \arrow[r, phantom, "\subset"] & \mathrm{Spec}\,(A) \end{tikzcd}$$

We want to extend ε_o to a section of X over Spec (A). To reduce this question to a slightly simpler situation we first prove the following:

<u>Lemma 1</u> Assume that in (1) set-theoretical there holds: $p_\emptyset \circ \varepsilon_o (V(I)) \subseteq V(J)$. If we denote the kernel of the homomorphisme $(p_\emptyset \circ \varepsilon_o)^* : B \longrightarrow A/I$ by J_1, the pair (B, J_1) belongs to $\underline{\underline{H}}$ (and $IB \subseteq J_1$).

Moreover there exists an embedding $X \subseteq \mathbb{P}^m \times \mathrm{Spec}\,(B)$ and a section $\eta : \mathrm{Spec}\,(B) \longrightarrow \mathbb{P}^m \times \mathrm{Spec}\,(B)$ which coincides on Spec (B/J_1) with the morphisme ε_o.

Proof: By assumption $V(J) \supseteq V(J_1) = p \circ \varepsilon_o (V(I)) \subseteq V(I)$ and by axiom (W0) the morphisme $(A, I) \longrightarrow (B, J)$ induces a closed embedding $V(J) \subseteq V(I)$.

Therefore $V(J_1) = V(J)$, hence (B, J_1) is Henselian and the inclusion $J_1^r \subseteq J \cap J_1$ holds for some $r \gg 0$, moreover, since by axiom (W0) the ring B/IB is Noetherian the ideals J, J_1 and $J \cap J_1$ are finitely generated modulo IB. If the elements $t_1, \ldots, t_r \in J \cap J_1$ represent a base of $J \cap J_1 / IB$ we can define an A-morphisme $A \{\{ T_1, \ldots, T_r \}\} \longrightarrow B$, $T_i \longmapsto t_i$. By axiom (W1) this morphisme is finite, hence we infere that (B, J_1) belongs to $\underline{\underline{H}}$ (by axiom (W2)). Now we construct the schema Y as follows:

The morphisme ε_o induces also a morphisme η_o : Spec $(B/J_1) \longrightarrow X$ and we can assume that $X \subseteq \mathbb{P}^m \times \mathrm{Spec}\,(B)$. The morphisme is given as a point of $\mathbb{P}^m \times \mathrm{Spec}\,(B)$ by an epimorphisme $(B/J_1)^{m+1} \longrightarrow \mathcal{L}$ onto an invertible (B/J_1)-module $\mathcal{L}$, and since (B, J_1) is Henselian we can lift $\bar{L}$ to an invertible B-module L and the epimorphisme to an epimorphisme of B-moduls, $B^{m+1} \longrightarrow L$. Therefore can be lifted to a B-morphisme

η : Spec (B) $\longrightarrow$ P^m x Spec (B) q.e.d.

<u>Lemma 2</u> Let S be an affin schema and $X \subseteq P^m \times S$ a projective S-schema defined by forms $F_\alpha(T_o, \dots, T_m) = 0$ with coefficients from $\Gamma(S)$. If η: S $\rightarrow$ X is a section, corresponding to an invertible sheaf L on S and m+1 sections $\eta_o, \dots, \eta_m$ generating L, let $U \subseteq S \times A^N$ $(N = (m+1)^2)$ be the schema defined by the equations

$$F_\alpha\Big(\eta_0 + \sum_{j=0}^{m} Y_{0j}\eta_j, \dots, \eta_m + \sum_{j=0}^{m} Y_{mj}\eta_j\Big) \otimes L^{-d_\alpha} = 0 \quad (d_\alpha = \deg F_\alpha)$$

and by the inequality $\det(\delta_{ij} + Y_{ij}) \neq 0$

Then $\xi(x) = (x, 0, \dots, 0)$ U defines a section of U over S and

$$\varepsilon(x, y_{ij}) \qquad \Big(x, \eta_0 + \sum_{j=0}^{m} y_{0j}\eta_j : \dots : \eta_0 + \sum_{j=0}^{m} y_{mj}\eta_j\Big)$$

defines an S-morphisme U $\longrightarrow$ X such that $\varepsilon \circ \xi = \eta$.

Moreover ε is a locally trivial fibration with fibre

$$Gl(m) \times A^m \times B_m$$

Proof: Let S_o by the open set where L is generated by η_o, by a lineare change of coordinates on $P^m \times S_o$ and $A^N \times S_o$ we can assume $(\eta_o, \dots, \eta_m) = (\eta_o, 0, \dots, 0)$ on S_o.

Then U is defined by

$$F_\alpha\big((1+Y_{oo})\eta_o, Y_{10}\eta_o, \dots, Y_{mo}\eta_o\big) = 0$$

and ε by

$$\varepsilon(x, y_{ij}) = (x, 1+y_{oo} : y_{10} : \dots : y_{mo}).$$

Hence ξ is a locally trivial fibration with fibre

$Gl(m) \times A^m \times G_m$ (stabilizer of a point of P^m under

the action of $Gl(m+1)$).

For later use we note the following

Lemma 3 If $\phi : E \longrightarrow A^n$ is a homomorphisme of A-moduls, where E is projective of rank k and if $x \in A$ such that

$$x \ \det(E^*) \subseteq \text{ image of } \wedge^k \phi^*$$

($\phi^* : A^n \longrightarrow E^*$ the dual map to ϕ)

there exist a homomorphisme $\gamma : A^n \longrightarrow E$

such that $\gamma \circ \phi = x \ id_E$

Note that the condition about x can also be written as

$x \in$ image of $(\det(E) \otimes \wedge^k A^{n*} \longrightarrow A)$ induced by ϕ^*.

In the case $k > n$ the element x must be 0, hence we can put $\gamma = 0$.

Assume $k \leq n$, the condition about x implies

$$x \ E^* \subseteq \text{ image of } \phi^*$$

If $p : A^m \longrightarrow E^*$ is an epimorphisme, we define a homomorphisme $\beta : A^m \longrightarrow (A^n)^*$ by $\beta(e_i) = v_i \quad (A^n)^*$ such that v_i are elements with the property $\phi^*(v_i) = x \ p(e_i)$

Since E^* is projective, p has a section $r : E^* \longrightarrow A^m$ and we define $\gamma^* = \beta \circ r$.

Then there holds $\phi^* \circ \gamma^* = x \ id_{E^*}$ and for the dual map

$$\gamma \circ \phi = x \ id_E$$

q.e.d.

<u>Theorem 2</u>- (theorem on implicite functions)

Assume that in the diagramm (1) there holds

(a) $p \circ \varepsilon_0 (V(I)) \subseteq V(J)$

(b) X is formally smooth over A in all points $\varepsilon_0(x)$

$(x \in \mathrm{Spec}\ (A(I))$

Then ε_0 can be extended to a section of $X \longrightarrow \mathrm{Spec}\ (A)$.

By lemma 1 we can replace (B,J) by (B, J_1), hence we can assume $p \circ \varepsilon_0 : \mathrm{Spec}\ (A/I) \longrightarrow \mathrm{Spec}\ (B/J)$.

Furthermore there exists an embedding $X \subseteq P^m \times \mathrm{Spec}\ (B)$, an invertible B-module L and m+1 sections $\eta_0, \dots, \eta_m$ of L generating L, such that the corresponding point

$\eta : \mathrm{Spec}\ (B) \longrightarrow P^m \times \mathrm{Spec}\ (B)$ conicides on $\mathrm{Spec}\ (B/J)$

with the given morphisme ε_0. Let the projective closure of X in $P^m \times \mathrm{Spec}\ (B)$ be defined by the family of forms

$$F_\alpha(U_0, \dots, U_m) \in B\,[U_0, \dots, U_m]$$

and consider indeterminates Y_{ij}, $i, j = 0, \dots, m$

and in the algebra $B\,\{\{Y_{00}, Y_{01}, \dots, Y_{mm}\}\} = B\,\{\{Y\}\}$

the I-adic closure K of the ideal generated by the functions corresponding to

$$F_\alpha(\eta_0 + \sum_{j=0}^{m} Y_{0j}\eta_j, \dots, \eta_m + \sum_{j=0}^{m} Y_{mj}\eta_j) \otimes L^{\otimes d_\alpha} \subseteq B\{\{Y\}\},\ d_\alpha = \deg F_\alpha$$

From lemma 2 we infere that the algebra $C = B\,\{\{Y\}\}/K$ is formally smooth over (A, I), hence by Corollary 1 the homomorphisme $C \longrightarrow A/I$ given by $(p \circ \varepsilon_0)^* : B \longrightarrow A/I$, and $Y_{ij} \longmapsto 0$ extends to a homomorphisme $C \longrightarrow A$, say by

$Y_{ij} \longmapsto y_{ij}$, $\varphi : B \longrightarrow A$.

Then by $(\varphi, \varphi^*\eta_0 + \sum_{j=0}^{m} y_{0j}\,\varphi^*\eta_j, \dots, \varphi^*\eta_m + \sum_{j=0}^{m} y_{mj}\,\varphi^*\eta_j)$

we get a section of X over Spec (A).

Our next aim is to formalate and to prove the so called Newton lemma.

If $\varphi = (\varphi_1, \dots, \varphi_k)$, $\varphi_i \in A\{\{T\}\}[U]$ and an A-morphisme $A\{\{T\}\}[U] \to A$, $T_i \mapsto t_i$, $i = 1, \dots, n$, $U_j \mapsto u_j$, $j = 1, \dots, m$ is given, in the case $k \leq n + m$, we define the following ideal in A :

$C(\varphi, t, u)$ = the ideal generated by the $(k \times k)$-minors of $(\partial\varphi_i/\partial T_j(t,u), \partial\varphi_i/\partial U_\nu(t,u))$ and by $\varphi_1(t,u), \dots, \varphi_k(t,u)$

If $Z \subseteq \operatorname{Spec}(A\{\{T\}\}[U])$ is the set of zeros of φ, the locus $V(C(\varphi, t, u))$ consist of all points of Spec (A), over which Z is not a smooth complete intersection of codimension k in $\operatorname{Spec}(A\{\{T\}\}[U])$.

Theorem 3 (Newtons Lemma, preliminary version)

Assume $\varphi = (\varphi_1, \dots, \varphi_k)$, $\varphi_i \in A\{\{T\}\}[U]$, $T = (T_1, \dots, T_n)$, $U = (U_1, \dots, U_m)$, and let H and I_1 be ideals in A such that $I_1 \subseteq I$ and A/I_1 is I-adic separated.

If the system of equations

$$\varphi(T, U) = 0$$

has a solution (t^o, u^o), $t_i^o \in I$, $u_j^o \in A$ modulo $H^2 I_1$ such that the ideal in A, generated by the $(k \times k)$-minors of $(\partial\varphi_i/\partial T_j(t^o, u^o), \partial\varphi_i/\partial U_\nu(t^o, u^o))$ contains the ideal H, then it has a solution (t, u) in A such that $t \equiv t^o \mod I_1 H$, $u \equiv u^o \mod I_1 H$

We can again consider a slightly more general situation: Given a morphisme $(A, I) \longrightarrow (B, J)$ in $\underline{\underline{H}}$, assume $p: X \longrightarrow \mathrm{Spec}\,(B)$ is a quasiprojective morphisme and X Spec (A) is formally smooth. Consider a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank and a homomorphisme $\varphi: \mathcal{E} \to \mathcal{O}_X$, then φ defines a closed subscheme Z of X by $\mathcal{O}_Z = \mathrm{coker}\,(\varphi)$. By $\Omega^1_{X/A}$ we denote the universal I-adic separated differential modul and by $d: \mathcal{O}_X \longrightarrow \Omega^1_{X/A}$ the universal derivation (for derivations over A with values in I-adic separated quasicoherent $\mathcal{O}_X$-moduls).

The universal I-adic separated derivation $d: \mathcal{O}_X \longrightarrow \Omega^1_{X/A}$ exists an can by described as follows:

If $X = \mathrm{Spec}\,(A \{\{T\}\} [U] / (F_1, \dots, F_m))$ then $\Omega^1_{X/A}$ corresponds to the module

$$\bigoplus_{j=1}^{n} \mathcal{O}_X\, dT_j \oplus \bigoplus_{h=1}^{m} \mathcal{O}_X dU_h \;/\; (dF_1, \dots, dF_m)$$

where $dF_i = \sum_{j=1}^{n} \frac{\partial F_i}{\partial T_j} dT_j + \sum_{h=1}^{m} \frac{\partial F_i}{\partial U_h} dU_h$, and

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial T_j} dT_j + \sum_{h=1}^{m} \frac{\partial f}{\partial U_h} dU_h .$$

The derivation d induces an $\mathcal{O}_Z$-linear map

$$\mathcal{E} \otimes \mathcal{O}_Z \longrightarrow \Omega^1_{X/A} \otimes \mathcal{O}_Z$$

(by $e \otimes 1 \mapsto d\,\varphi(e) \otimes 1$), and therefore an $\mathcal{O}_Z$-linear map

$$\det(\mathcal{E}) \otimes \mathcal{O}_Z \longrightarrow \Omega^{\mathrm{rk}(\mathcal{E})}_{X/A} \otimes \mathcal{O}_Z$$

(where $\Omega^r_{X/A}$ is defined as $\Lambda^r\, \Omega^1_{X/A}$).

If s_o : Spec (A) $\longrightarrow$ X is a section and if N is the ideal in A such that Spec (A/N) = s_o^{-1} (Z), we restrict this map to Spec(A/N). It's dual defines a homomorphisme

$$\mathrm{Hom}_A(s_o^{*} (S2^{rk(\mathcal{E})}_{X/A}) \otimes \mathcal{O}_Z), \; s_o^{*} (\det (\mathcal{E}) \otimes \mathcal{O}_Z)) \longrightarrow A/N.$$

Then we define the ideal C(φ , s_o) $\subseteq$ A by C(φ , s_o)/N = image of this homomorphisme.
In the special case X = Spec (A $\{\{T\}\}$ [U]), $\mathcal{E}$ = A $\{\{T\}\}$ $[U]^k$ and φ = (φ_1, ..., φ_k) it coincides with the ideal defined above.
We can define C(φ , s_o) in an alternative way as follows:
Consider the sheaf $\tilde{A}$ as $\mathcal{O}_X$-module via the section s_o and the map induced by φ

$$\mathrm{Der}_A(\mathcal{O}_X, \tilde{A}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \tilde{A} / N \tilde{A})$$

$$D \longmapsto D \circ \varphi \mod N \tilde{A}$$

If rk($\mathcal{E}$) = r , C(φ , s_o) is generated by N and the elements det(D_i(φ (e_j)), where (D_1, ..., D_r) runs through the set $\mathrm{Der}_A(\mathcal{O}_X, \tilde{A})^r$ and (e_1, ..., e_r) trough the (local) sections of $\mathcal{E}^r$,
i. e. C(φ, s_o)/N = image of

$$\Lambda^r \mathrm{Der}_A(\mathcal{O}_X, \tilde{A}) \otimes (\Lambda^r \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \tilde{A}/N\tilde{A}))^{-1} \longrightarrow A/N$$

<u>Theorem 3'</u> (Newtonsches Lemma)
Let (A, I) $\longrightarrow$ (B, J) be a morphisme in $\underline{\underline{H}}$, X $\xrightarrow{p}$ Spec(B) a quasiprojective schema over B and s_o: Spec(A) $\longrightarrow$ X a section. Assume X is formally smooth over A in the points of s_o(Spec(A))

and $p\,\zeta_0(V(I)) \subseteq V(I)$.

Let $\mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X$ be a morphisme of a vectorbundle of finite rank over X, $Z \subseteq X$ the schema of zeros of φ, and H and $I_1 \subseteq I$ ideals of A such that $(A/I_1, I/I_1) \in \underline{\underline{H}}$.

If (a) $\zeta_0(\mathrm{Spec}\ (A/H^2I_1)) \subseteq Z$

(b) $C(\varphi, \zeta_0) \supseteq H$

there exists a section ζ : Spec (A) $\longrightarrow$ Z, such that

$\zeta = \zeta_0$ on Spec (A/HI_1)

Proof: <u>Step I</u> Reduction to the affine case

By lemma 1 we can assume $A/I \cong B/J$.

If K denotes the kernel of $(p_\emptyset \circ \zeta_0)^* : B \longrightarrow A$, the pair (B, K) is Henselian (since $K \subseteq J$).

Hence we can assume that $X \subseteq P^m_B$ and there exists a section η : Spec (B) $\longrightarrow P^m_B$ given by $(L, \eta_0, \ldots, \eta_m)$ (L an invertible module over B generated by $\eta_0, \ldots, \eta_m$) which coincides on $(p_\emptyset \circ \zeta_0)$ (Spec (A)) with the given section ζ_0.

We consider again the equations $F_\alpha(U_0, \ldots, U_m) = 0$ defining the projective closure of X in $P^m \times$ Spec (B), the algebra $B\{\{Y\}\}$, $Y = (Y_{ij})$, ij = 0, ..., m, the I-adic closure K of the ideal generated by the functions corresponding to

$$F\left(\eta_0 + \sum_{j=0}^{m} Y_{0j}\eta_j, \ \ldots, \ \eta_m + \sum_{j=0}^{m} Y_{mj}\eta_j\right) \otimes L^{-d_\alpha} \subseteq B\{\{Y\}\}$$

($d_\alpha = \deg F_\alpha$) and the algebra $C = B\{\{Y\}\}/K$.

By $\left(\eta_0 + \sum_{j=1}^{m} Y_{0j}\eta_j : \ldots : \eta_m + \sum_{j=1}^{m} Y_{mj}\eta_j\right)$ there is defined a B-morphisme ξ : Spec (C) $\longrightarrow$ X, and by $Y_{ij} \longmapsto 0$ and $p_\emptyset \circ \zeta_0$

we get a section t_o of Spec (C) $\longrightarrow$ Spec (A).

By lemma 2 we see that (C, J $\{\{Y\}\}$ /K) is formally smooth over A.

We consider the homomorphisme of sheafs $\varepsilon^*\mathcal{E} \xrightarrow{\varepsilon^*(\varphi)} \mathcal{O}_{\mathrm{Spec}\,(C)}$ and we shall proof that $C(\varepsilon^*(\varphi), t_o) = C(\varphi, s_o)$. Since $\varepsilon^{-1}Z$ is the schema of zeros of $\varepsilon^*(\varphi)$, the assumptions (a), (b) are then satisfied for Spec (C), $\varepsilon^*(\varphi)$ and t_o.

If we prove the existence of a section

t: Spec(A) $\longrightarrow \varepsilon^{-1}(Z)$ satisfying $t = t_o$ on Spec(A/HI_1),

the section $s = \varepsilon \circ t$: Spec(A) $\longrightarrow$ Z has the required property.

Hence we have to prove $C(\varepsilon^*(\varphi), t_o) = C(\varphi, s_o)$ and then we have to prove the theorem in the special case where X = Spec(C).

Proof of $C(\varepsilon^*(\varphi), t_o) = C(\varphi, s_o)$:

Since $t_o^{-1}\,\varepsilon^{-1}\,Z = s_o^{-1}\,Z$, this subschemas of Spec(A) are defined by the same ideal N. Furthermore we have a canonical restriction map $\mathrm{Der}_A(C, A) \longrightarrow \mathrm{Der}_A(\mathcal{O}_X, \tilde{A})$.

By lemma 2 this map is surjective, therefore

$C(\varepsilon^*(\varphi), t_o) = C(\varphi, s_o)$.

Hence we have reduced the prove to the case

X = Spec (A $\{\{F\}\}$), F a projective A-module of finite typ and such that s_o corresponds to A $\{\{F\}\} \longrightarrow$ A, F $\longrightarrow$ 0 (by theorem 1 and corollary 1).

<u>Step II</u> Reduction of the case X = Spec (A $\{\{F\}\}$).

H^2I_1 + FA $\{\{F\}\}$ $\supseteq$ Image of (E $\xrightarrow{\psi}$ A $\{\{F\}\}$)

(where E $\longrightarrow$ A $\{\{F\}\}$ is the homomorphisme of A $\{\{F\}\}$ -modules corresponding to $\mathcal{E} \longrightarrow \mathcal{O}_X$) to A $\{\{F\}\}$ = A $\{\{T_1, \ldots, T_n\}\}$.

We can reduce the proof to the case, where F is free, i. e. $X = \operatorname{Spec} A\{\{T_1, \ldots, T_n\}\}$ as follows:
The module E can be written as $E = E_o \otimes_A A\{\{F\}\}$, where E_o is the projective A-module obtained from E by reduction mod $FA\{\{F\}\}$. An isomorphisme $E_o \otimes_A A\{\{F\}\} \to E$ is obtained by lifting the idendity of E_o to an A-linear map $j : E_o \longrightarrow E$ (observe that E_o is projective) and by $x \otimes f \longmapsto fj(x)$ $(x \in E_o,\ f \in A\{\{F\}\})$. This map reduces to the idendity mod $FA\{\{F\}\}$ and since $FA\{\{F\}\}$ is contained in the Jacobson radical of $A\{\{F\}\}$ and E is of finite type, it is surjective (by Nakayamas lemma) and therefore bijective (since E is projective).
Therefore $\varphi : E \to A\{\{F\}\}$ is determined by an A-linear map $\psi : E_o \longrightarrow H^2 I_1 + FA\{\{F\}\}$. There exist a projective A-modul F' such that $F \oplus F' = AT_1 \oplus \ldots \oplus AT_n$ is free.
We can lift ψ to an A-linear map $\psi' : E_o \longrightarrow H^2 I_1 + \sum_{v=1}^{n} T_v A\{\{T\}\}$ and extend it by the embedding $i : F' \longrightarrow A\{\{T\}\}$ to an A-linear map

$$\psi'' = \begin{pmatrix} \psi' & 0 \\ 0 & i \end{pmatrix} : E_o \oplus F' \longrightarrow H^2 I_1 + \sum_{v=1}^{n} T_v A\{\{T\}\}$$

Any A-derivation $D : A\{\{F\}\} \longrightarrow A$ can be extended to an A-derivation $\mathfrak{D} : A\{\{T\}\} \longrightarrow A$, hence $C(\varphi, 0) = C(\psi'', 0)$

<u>Step III</u> Proof in the case

$$X = \operatorname{Spec} A\{\{T_1, \ldots, T_n\}\}$$

$$\varphi : E_o \longrightarrow H^2 I_1 + \sum_{v=1}^{n} T_v\ A\{\{T\}\}$$

(E_o a projective A-module).

We have to determine elements $t_v \in HI_1 \subseteq I$ such that under the map $A\{\{T\}\} \to A$, $T_v \mapsto t_v$ the homomorphisme φ is mapped to 0. For any map $\varphi : E_o \to A\{\{T\}\}$ and $t_v \in I$ let us denote by $\varphi(t_1, \ldots, t_n) : E_o \to A$ the composition of φ with the A-homomorphisme $A\{\{T\}\} \to A$, $T_v \mapsto t_v$. We can write

$$(1) \quad \varphi = \sum_{i,j=1}^{r} X_i X_j \varphi_{ij} + \sum_{v=1}^{n} T_v \varphi_v + \sum_{v,\mu=1}^{n} T_v T_\mu \varphi_{v\mu}$$

$\varphi_{ij} : E_o \to I_1$, $\varphi_v : E_o \to A$

$\varphi_{v\mu} : E_o \to A\{\{T\}\}$, where $X_1, \ldots, X_r$ are suitable elements of H.

(This is possible since $\mathrm{Im}(\varphi) \subseteq H^2 I_1 + \sum_{v=1}^{n} T_v A\{\{T\}\}$ $= H^2 I_1 + \sum_{v=1}^{n} T_v A + \sum_{v,\mu=1}^{n} T_v T_\mu A\{\{T\}\}$).

Then we try to find elements t_v in the form

$$(2) \quad t_v = t_v(M) = \sum_{\rho=1}^{r} x_\rho u_{v\rho}$$

with elements $u_{v\rho} \in I_1 \subseteq I$.

The ideal $C(\varphi, 0)$ is the ideal generated by the image N of $\sum_{i,j=1}^{r} X_i X_j \varphi_{ij}$ and the element

$$\begin{vmatrix} \varphi_{v_1}(\omega_1) & \cdots & \varphi_{v_1}(\omega_k) \\ \cdots & & \\ \varphi_{v_k}(\omega_1) & \cdots & \varphi_{v_k}(\omega_k) \end{vmatrix} = \langle \varphi_{v_1} \wedge \cdots \wedge \varphi_{v_k}, \omega_1 \wedge \cdots \wedge \omega_k \rangle$$

if $k = \mathrm{rk}(E_o)$, where $(\omega_1, \ldots, \omega_k)$ runs tho through the set E_o^k.

Let Δ denote the ideal generated by the elements $\langle \varphi_1,\dots,\varphi_k, \omega_1,\dots,\omega_k\rangle$ we want to show $\Delta = C(\varphi, 0)$. We have by assumption (a) that $N \subseteq H^2 I_1$ and by (b) that $N + \Delta \supseteq H$, hence

$$N \subseteq (N + \Delta)\, HI_1 .$$

Therefore there holds the idendity $N + \Delta = NHI_1 + \Delta$. Since N is of finite type we infere from this idendity (by Nakayamas lemma) that $C(\varphi, 0) = \Delta + N = \Delta$.

Since the elements $x_j \in H \subseteq C(\ ,\ ) = \Delta$ are in Δ we can find for each index j an A-linear map

$$\gamma_j : AT_1 \oplus \dots \oplus AT_n \longrightarrow E_0$$

such that

$$(3) \quad \gamma_j \circ \Big(\sum_{v=1}^{n} T_v\, \varphi_v \Big) = x_j\, \mathrm{id}_{E_0}$$

(by lemma 3).

Substituting (2) into (1) we get

$$\varphi(t(u)) = \sum_{i,j=1}^{r} x_i x_j\, \psi_{ij} + \sum_{v=1}^{n} \sum_{j=1}^{r} x_j u_{vj}\, \varphi_v$$

$$+ \sum_{v,\mu=1}^{n} \sum_{i,j=1}^{r} x_i x_j u_{vi} u_{\mu j}\, \varphi_{v\mu}(t(u))$$

Using (3) we can write this as

$$\varphi(t(u)) = \sum_{i=1}^{r} x_i \sum_{\varkappa=1}^{n} \Big[\sum_j \psi_{ij}\, \gamma_j (T_\varkappa) + u_{\varkappa i} +$$

$$\sum_{v,\mu} \sum_j u_{vi} u_{\mu j}\, \varphi_{v\mu}(t(u))\, \gamma_j(T\ ) \Big] \varphi_\varkappa$$

Hence it is sufficient to determine the elements u_{vi} such that the terms in the square brackets vanish, i. e. we have to consider equations of the type

$$(4) \qquad a_{\varkappa i} + u_{\varkappa i} + \sum_{\nu, \mu, i, j} u_{\nu i} u_{\mu j} h_{\varkappa j \nu \mu}(U) = 0$$

$\varkappa = 1, \dots, n, \quad i = 1, \dots, r$

$a_{\varkappa i} \in I_1, \quad h_{\varkappa j \nu \mu}(U) \in A\{\{U\}\}.$

The Jacobian matrix of this system in $U = 0$ is the unit matrix, and for $U = 0$ the equations vanish modulo I_1, hence by theorem 2 they can be solved by elements $u_{\varkappa i} \in I$. Since $a_{\varkappa i} \in I_1$ we infere from (4) that $u_{\varkappa i} \in I_1 + (\sum_{\nu, j} Au_{\nu j})^2$ and since I_1 is I-adic closed, hence also closed with respect to the $(\sum_{\nu, j} Au_{\nu j})$-adic topology, this implies $u_{\varkappa i} \in I_1$ q.e.d.

4. Generalization of Elkiks theorem

In this section we consider a W-category where all pairs (A, I) are Noetherian.

We consider $A\{\{T\}\}$ $T = (T_1, \dots T_n)$ and a_n algebraic schema $P \longrightarrow \mathrm{Spec}A\{\{T\}\}$, which is smooth over $A\{\{T\}\}$.

For any closed subschema $X \subset P$, defined by a sheaf of ideals $K \subset \mathcal{O}_P$, we define the following sheaf of ideals $\mathcal{E}(X, P) = \mathcal{E}$:

For $U \subset P$ open and affine, $\mathcal{E}(U) = \sum_f [I(f) : K(U)] \Delta(f) + K(U)$ where f runs through the set of all tuple $(f_1, \dots, f_p) \in K(U)^p$, $p = 1, 2, \dots$, $I(f)$ demotes the ideal $\sum_{j=1}^{p} \mathcal{O}(U) f_j$ and $\Delta(f)$ the ideal in $\mathcal{O}_P(U)$, image of the map $\Lambda^p \mathrm{Der}_A(\mathcal{O}_P(U), \mathcal{O}_P(U)) \longrightarrow \mathcal{O}_P(U) = \Lambda^p \mathcal{O}_P(U)^p$ which is induced by the map

$$\mathrm{Der}_A(\mathcal{O}_P(U), \mathcal{O}_P(U) \to \mathcal{O}_P(U)^p$$
$$\theta \longmapsto (\theta(f_1), \dots, \theta(f_p))$$

Note the following properties of $\mathcal{E}$

(A) If $x \in X$, then $\mathcal{E}_x = \mathcal{O}_{P,x}$ if and only if $K_x = I(f)$ for a suitable p-tuple $f \in K_x^p$ satisfying $\Delta(f) = \mathcal{O}_{P,x}$

(B) If K_x is generated by a regulare sequence $f = (f_1, \dots, f_p)$ in $\mathcal{O}_{P,x}$, the ideals $\mathcal{E}_x$ and $\Delta(f) + K_x \subseteq \mathcal{E}_x$ have the same set of zeros in $\mathcal{O}_{P,x}$.

Now the formulation of Elkiks theorem is

Theorem 5. If $(A, I) \in \underline{\underline{H}}$, there exists a function $d : \mathbb{N}^3 \to \mathbb{N}$, $d(a, r, c) > \max(r, c)$, with the following property. Assume $P = \mathbb{P}^m \times \mathrm{Spec}\, A\{\{T\}\}$ and $X \subset P$ is a (quasi)projective subschema and $\mathcal{X} \subset \mathcal{O}_P$ a quasicoherent sheaf of ideals, such that for a suitable integer a there holds $\mathcal{X}^a \mathcal{O}_P \subseteq \mathcal{E}(X, P)$

Then for any A-morphisme

$S_0 : \mathrm{Spec}(A/I^d) \longrightarrow X$, $d = d(a, r, c)$

such that (if $V(\mathscr{X})$ denotes the closed subschema defined by the ideal $\mathscr{X}$)

s_0 (Spec $(A/I^r)) \supseteq V(\mathscr{X}) \cap X$

there exists a section s : Spec $(A) \longrightarrow X$ such that

$s = s_0$ on Spec (A/I^c)

The proof is easily reduced by lemma 1 and lemma 2 to the case $X \subset$ Spec $(A\{\{T\}\}) = P$ (the ideal H has to be replaced by its inverse image in lemma 2).

In this case, if $T = (T_1, \ldots, T_n)$, giving an A-morphisme s_0 : Spec $(A/I^d) \longrightarrow X$ is the same as to give a n-tuple $t^o = (t_1^o, \ldots, t_n^o) \in I^n$ such that $F(t^o) \equiv 0 \bmod I^d$, where $F = F(T)$ denotes a tuple of functions from $A\{\{T\}\}$, which generates the ideal of X. If $H \subset A\{\{T\}\}$ denotes the given ideal, the condition s_0 (Spec $(A/I^r)) \subseteq X \cap V(\mathscr{X})$ means in this case that

$I^r \subseteq H(t^o)$ (since $d > r$).

We consider in the following the pair (A, I), a q-tuple $F \in A\{\{T\}\}^q$, $T = (T_1, \ldots, T_n)$ and the ideal E corresponding to the embedding

$X = V(F) \subset P =$ Spec $A\{\{T\}\}$ as defined above.

<u>Lemma 4</u> Assume there are given $t \in I^{\oplus n}$, $x \in I$ and an ideal $N \subseteq A$ and the following conditions are satisfied

(i) $F(t) \equiv 0 \bmod x^s N$ for an integer $s > 0$

(ii) $x^r \in E(t)$ for an integer $r \geq 0$

(iii) $s > r$ and $0 : x^{s-r} = 0 : x^{s-r+1}$

In this case there exists an $y \in (x^{s-r} N)^{\oplus n}$ satisfying $F(t + y) \equiv 0 \bmod x^{2(s-r)} N$

Remarks: 1) The assertion is of course trivial if $2(s - r) \leq s$, i. e. if $s \leq 2r$, since we can take $y = 0$ in this case. However, if $s \geq 2r + 1$, the vector $t + y$ gives a better approximation of a solution of the equations $F = 0$ as the vector t.

2) If $x \in I$ is not nilpotent, we can always find an integer k such that $0 : x^k = 0 : x^{k+1} = \dots$.

Hence if in the lemma $s \geq \max(2r + 1, r + k)$, we can find a sequence.

$t = t^{(0)}, t^{(1)}, t^{(2)}, \dots, t^{(\nu)}, \dots$ of vectors satisfying

$$t^{(\nu+1)} \equiv t^{(\nu)} \mod x^{r+\nu+1} N$$

$$F(t^{(\nu)}) \equiv 0 \mod x^{s+\nu} N$$

(observe that the condition $x^r \in E(t^{(\nu)})$ and $t^{(\nu+1)} \equiv t^{(\nu)} \mod x^{r+\nu+1} N$, $F(t^{(\nu+1)}) \equiv 0 \mod x^{r+\nu+1} N$ implies $x^r \in E(t^{(\nu+1)})$)

Proof of the lemma: If $J(F, t)$ denotes the matrix with the rows $\frac{\partial F}{\partial T_i}(t)$, we can write for any $y = (y_1, \dots, y_n) \in I^{\oplus n}$

$$F(t + y) \equiv F(t) + y\, J(F, t) \mod \sum_{i,j} y_i y_j A$$

Hence we have to determine $y \in (x^{s-r} N)^{\oplus n}$ in such a way that

(1) $y\, J(F, t) \equiv -F(t) \mod x^{2(s-r)} N$

It is sufficient to solve the congruence

(2) $z\, J(F, t) \equiv -x^r F(t) \mod x^{2s} N$, $z \in (x^s N)^{\oplus n}$

If z is a solution of the congruence (2), we can write

$z = x^r y$, $y \in (x^{s-r} N)^{\oplus n}$, and $x^r [y\, J(F, t) + F(t)] = x^r (x^{2s-r} v)$ for a suitable vector $v \in N^{\oplus n}$, i. e.

$$y\, J(F, t) + F(t) - x^{2s-r} v \equiv 0 \mod x^{s-r} A \cap (0 : x^r)$$

But from condition (iii) we infere

$x^{s-r} A \cap 0 : x^r \subseteq x^{s-r} A \cap (0 : x^{s-r}) = 0,$

hence $y\, \mathcal{J}(F, t) \equiv - F(t) \mod x^{2s-r} N.$

We consider now the congruence (2). If $x^r = x_1 + x_2$ and if the vectors $z_i \in (x^s N)^{\oplus n}$ are solutions of the congruences

$z_i \quad \mathcal{J}(F, t) \equiv - x_i F(t) \mod x^{2s} N \qquad , i = 1, 2,$

the vector $z = z_1 + z_2$ is a solution from (2).

Now, since $x^r \in E(t)$, it is a finite summ of an element $x_1 \in I(F)(t) \subseteq x^r N$ and of elements $\delta(t)\, h(t)$, where δ is determined by a p-tuple $f \in I(F)^{\oplus p}$ as $\delta = \det\left(\frac{\partial f}{\partial T_{i_1}}, \ldots, \frac{\partial f}{\partial T_{i_p}}\right)$, and where h is an element of $I(f) : I(F)$

For x_1 we can take $z_1 = 0$ to solve the congruence

$z_1\ \mathcal{J}(F, t) \equiv - x_1 F(t) \mod x^{2s} N.$

Consider on the other hand elements $\delta(t)\, h(t)$, assume for example $\delta = \det\left(\frac{\partial f}{\partial T_1}, \ldots, \frac{\partial f}{\partial T_p}\right)$. There exists a $(p \times q)$ - matrix ψ over $A\langle\langle T\rangle\rangle$ such that

(3) $h F = f \psi$

and a $(p \times n)$ - matrix Γ over A such that

(4) $\Gamma\, \mathcal{J}(f, t) = \delta(t) I_p \qquad (I_p\ (p \times p)$-unite matrix)

(since $\delta(t)$ is a $(p \times p)$ - minor of $\mathcal{J}(f, t)$).

From (3) we infere (since $F(s) \equiv 0 \mod x^s N$)

$h(t)\ \mathcal{J}(F, t) \equiv \mathcal{J}(f, t)\ \psi(t) \mod x^s N,$

hence by (4)

$h(t)\, \Gamma\, \mathcal{J}(F, t) \equiv \delta(t)\ \psi(t) \mod x^s N$

and by (3)

$(h(t)\, f(t)\, \Gamma)\ \mathcal{J}(F, t) \equiv \delta(t)\, h(t)\, F(t) \mod x^{2s} N$

Therefore the vector $z = h(t)\, f(t)\, \Gamma \in (x^s N)^{\oplus n}$ solves the congruence

$$z\, J(F, t) \equiv \delta(t)\, h(t)\, F(t) \mod x^{2s} N, \text{ q.e.d.}$$

Proof of theoreme 5: We proceed by induction on $\dim(A)$.

If all elements of I are nilpotent, we can take the function $d(a, r, c) = \max(r, c, v) + 1$, if $I^v = 0$.

Assume $x \in I$ is not nilpotent, determine k such that $0 : x^k = 0 : x^{k+1} = \ldots$ and for each integer s an integer $c(s)$ such that $I^{c + c(s)} \cap x^s A \subseteq I^c x^s$ (lemma of Artin-Rees)

Define $s(a, r) = \max(2ar + 1, ar + 1)$ and $c(a, r) = \max\{c(s(a, r)), r + 1\}$.

Because of $\dim A / x^s A < \dim A$ we can assume that for each s there exist a function $N^3 \to N$ for $(A / x^s A, I/(x^s A))$, we shall demote it by $d(s, a, r, c)$.

Then we define

$d(a, r, c) =_{\text{def}} d(s(a, r), a, r, c + c(a, r))$ and we show that it satisfies the assertion of the theorem.

To do this, assume $F \in A\{\{T\}\}^n$ and $t^0 \in I^{\oplus n}$ are given, satisfying $F(t^0) \equiv 0 \mod I^d$ and $I^r \subseteq H(t^0)$, where $d = d(a, r, c)$.

By induction there exist a vector t such that

(1) $F(t) \equiv 0 \mod x^s A$, $s = s(a, r)$

(2) $t \equiv t^0 \mod I^{c + c(a, r)} + x^s A$

Changing t we can assume

(2') $t \equiv t^0 \mod I^{c + c(a, r)}$

From (1) and (2') we infere (since $F(t^0) \equiv 0 \bmod I^d$ and $c(a, r) \geqslant c(s(a, r))$

(3) $F(t) \equiv 0 \bmod x^s I^c$, $s = s(a, r)$ and

(4) $E^r \subseteq H(t)$, hence $I^{ar} \subseteq E(t)$

We can therefore apply lemma 4 (with $N = I^c$ and r replaced by $a r$) to determine a sequence

(5) $t, t^1, t^2, \dots, t^\nu, \dots,\quad F(t^\nu) \equiv 0 \bmod x^{s+\nu} I^c$

As in remark 2 (observe $s \geqslant \max(2ar + 1, ar + k)$)

We can write $x^r = h(t)$, $h \in H$, since $x^r \in I^r \subseteq H(t)$

<u>Case 1</u> Assume that the schema X of zeros of F on the open set $X_h = \{x \in X, h \neq 0 \text{ in } x\}$ is a complete intersection, i. e. defined by a regulare sequence $(f_1, \dots, f_p)$.

By the property (B) of the ideal E we infere

(5) $h^\mu \equiv 0 \bmod \Delta(f) + I(F)$ and $(h^\mu + g) I(F) \subseteq I(f)$

for $\mu \gg 0$ and for a suitable $g \in I(F)$.

Choose ν such that $s(a, r) + \nu = \max(2ar + 1, ar + k) + \nu \geqslant 2\mu r + 1$ and t^ν in (5), then

(6) $F(t^\nu) \equiv 0 \bmod x^{2\mu r + 1} I^c$

$t^\nu \equiv t \bmod x^{r+1} I^c$

These congruences imply

$$h(t^\nu) \equiv h(t) \bmod x^{r+1} I^c$$

hence $A\, h(t^\nu) = x^r A$

and $A\, h^\mu(t^\nu) = x^{r\mu} A$

Therefore the congruence (5), (6) yields

$$x^{r\mu} \in \Delta(f)(t^\nu)\,,\quad f(t^\nu) \equiv 0 \bmod x^{2r\mu + 1} I^c,$$

and we can apply theorem 3.

By this theorem there exists a solution t of the equations

$f(t) = 0$ such that

(7) $t \equiv t^{\nu} \bmod x^{r\mu+1} I^c$.

We claim that t is also a solution of $F(t) = 0$

By (6) and (7) we get $A(h^{\mu}(t) + g(t)) = x^{r\mu} A$,

and $f(t) \equiv 0 \bmod x^{r\mu+1} I^c$

By (5) it follows therefore

$$x^{\mu r} F(t) = 0$$

hence $F(t) \equiv 0 \bmod (0 : x^{\mu r}) \cap x^{r\mu+1} I^c$

Since we can choose μ arbitrary big, we can assume

$r\mu + 1 \geq k$, in this case there holds

$(0 : x^{\mu r}) \cap x^{r\mu+1} I^c \subseteq (0 : x^k) \cap x^k A = 0$,

hence $F(t) = 0$

<u>Case 2</u> The general case will be reduced to case 1.

Let $G_{\mu}(T, Z) \in A\{\{T\}\}Z_1 \oplus \ldots \oplus A\{\{T\}\}Z_q$ generate the module of relations of $F \bmod I(F)^2$.

Replace $A\{\{T\}\}$ by $A\{\{T, T', Z\}\}$

$T' = (T'_1, \ldots, T'_n)$, F by

$F' = (F, T_1', \ldots, T_n', G_1, \ldots, G_m)$

and H by $H' = H A\{\{T, T', Z\}\} + \sum_{\nu} T'_{\nu} A\{\{T, T', Z\}\} + \sum_{\mu} Z_{\mu} A\{\{T, T', Z\}\}$.

If $F(t^0) \equiv 0 \bmod I^d$, replace t^0 by

$t'^0 = (t^0, 0, F(t^0))$, then

$F'(t'^0) \equiv 0 \bmod I^d$

If furthermore $H(t^0) \supseteq I^r$ it follows $H'(t^0) \supseteq I^r$

(for $r \leq d$), and if $t' \equiv t'^0 \bmod I^c$ and $F'(t') = 0$

the first n components of t' satisfy $t \equiv t^0 \bmod I^c$ and

$F(t) = 0$.

If E' is the ideal corresponding to the embedding $X' = V(F') \subset P' = \operatorname{Spec} A\{\{T, T', Z\}\}$, then $E\, A\{\{T, T', Z\}\} \subseteq E'$.

Therefore we can replace (F) by F'.

We consider the schemas $S = \operatorname{Spec}(A)$ and

$X = \operatorname{Spec}(A\{\{T\}\}/I(F)) \subset Y = \operatorname{Spec}(A\{\{T\}\}) \subset Z = \operatorname{Spec}(A\{\{T, T'\}\})$

$\pi \uparrow$

$X' = \operatorname{Spec}(A\{\{T, Z\}\}/I(F, G)) \subset Y' = \operatorname{Spec}(A\{\{T, Z\}\}) \subset Z' = \operatorname{Spec}(A\{\{T, T', Z\}\})$ (where π demotes the projection)

We shall show that any affin open set $U' \subseteq X' - V(H')$ is a complete intersection of $n + q$ hypersurfaces in some open subschema of Z'.

The open set $X' - V(H')$ is mapped into $X - V(H)$ under the map π. Therefore, by property (A) of the ideals E resp. E', if we consider the universal separated differential modules (denoted by Ω^1) and the conormal sheafs (denoted by N), restricted on $X' - V(H')$ we have the following exact sequences of locally free sheafs

(1) $0 \longrightarrow N_{Y'|Z'} \otimes \mathcal{O}_{X'} \longrightarrow N_{X'|Z'} \longrightarrow N_{X'|Y'} \longrightarrow 0$

(2) $0 \longrightarrow \pi^* N_{X|Y} \longrightarrow \pi^* \Omega^1_{Y|S} \longrightarrow \pi^* \Omega^1_{X|S} \longrightarrow 0$

(3) $0 \longrightarrow \pi^* \Omega^1_{X|S} \longrightarrow \Omega^1_{X'|S} \longrightarrow \Omega^1_{X'|X} \longrightarrow 0$

(4) $0 \longrightarrow N_{X'|Y'} \longrightarrow \Omega^1_{Y'|S} \otimes \mathcal{O}_{X'} \longrightarrow \Omega_{X'|S} \longrightarrow 0$

and $N_{Y'|Z'} \otimes \mathcal{O}_{X'} \simeq \pi^* \Omega^1_{Y|S}$

$\pi^* N_{X,Y} \simeq \Omega^1_{X'|S}$.

5. The I-adic completion of W-categories

Let us start with a W-category $\underline{\underline{H}}$ and a pair $(A, I) \in \underline{\underline{H}}$.
We will construct a W-category $\underline{\underline{H}}_{\hat{A}}$ over the I-adic completion $\hat{A}$ of A which is in a certain sence minimal.
We will need this construction for the prove of the approximation theorem for W-categories. It is exactly at this stage where the theory of W-categories is still a little bit massy because we where not able to prove that this construction preserves the property of being Noetherian. That is why we had to develop the whole theory of W-categories also in the non-noetherian case.
For example, if we consider the construction in the category of Henselian algebras of finity type, the property of being Noetherian will be preserved - However, if we consider the category of analytic $\mathbb{C}$- algebras, and if $A \longrightarrow B$ is a morphisme of $\mathbb{C}$-algebras, we have to consider algebras of the typ

$$B_{\hat{A}} = \bigcup B\{u_1, \dots, u_m\} \subset \hat{B}$$

where $(u_1, \dots, u_m)$ runs through the set of all finite sequences in $m_{\hat{B}}$ and where $B\{u_1, \dots u_m\}$ is the image of the free algebra $B\{U_1, \dots, U_m\}$ in $\hat{B}$ under the B-morphisme defined by $U_i \longmapsto u_i$. We do not know, if $B_{\hat{A}}$ is Noetherian.
The general construction runs as follows:
Let (B, J) be a (A, I)- algebra of $\underline{\underline{H}}$ and $\mathcal{S}$ be the set of finite subsets of the image of $I\hat{A}$ in $J\hat{B}$ ($\hat{B}$ the I-adic completion of B).
For a $\mathfrak{s} = (s_1, \dots, s_N) \in \mathcal{S}$ we define $B_{\mathfrak{s}}$ to be the image of $B\{\{T_1, \dots, T_N\}\}$ in $\hat{B}$ via the B - homomorphism $T_i \longmapsto s_i$.

5.1. Definition: $B_{\hat{A}} = \bigcup_{s \in \mathcal{S}} B_s$

It is clear that $(B_{\hat{A}}, J\, B_{\hat{A}})$ is a Henselian $(\hat{A}, I\, \hat{A})$ - algebra and the functor of (A, I) - algebras $(B, J) \longmapsto (B_{\hat{A}}, J B_{\hat{A}})$ is a functor of the subcategory $\underline{\underline{H}}_A$ of all pairs (B, J) over (A, I) of $\underline{\underline{H}}$ into the category of Henselian pairs over $(\hat{A}, I\,\hat{A})$.

5.2. Lemma: The canonical morphism $B/\, I^q\, B \longrightarrow B_{\hat{A}}/\, I^q B_{\hat{A}}$, $q = 1, 2, \ldots$, is an isomorphism and $\hat{B}$ is the $J B_{\hat{A}}$-adic completion of $B_{\hat{A}}$.

Proof: We know that $I\, B_{\hat{A}} = \bigcup_{s \in \mathcal{S}} (I\, B_s + \sum_{s_\nu \in s} s_\nu\, B_s)$,

so the canonical morphism

$B/\, I\, B \longrightarrow B_{\hat{A}}/\, I\, B_{\hat{A}} = \varinjlim_{s} B_s/(I B_s + \sum_{s_\nu \in s} s_\nu B_s)$

is surjectiv, i. e. $B_{\hat{A}} = B + I\, B_{\hat{A}}$.

But this meens also $B_{\hat{A}} = B + I^q\, B_{\hat{A}}$, $q \geqslant 1$.

Since $I^q\, \hat{B} \cap B = I^q\, B$ the morphism

$B/\, I^q\, B \longrightarrow B_{\hat{A}}\, /\, I^q\, B_{\hat{A}}$ is also injective.

This gives us also the isomorphism $B/\, J^q \longrightarrow B_{\hat{A}}/\, J^q\, B_{\hat{A}}$

and the lemma is proved.

5.3. Definition/ Proposition: Let $\underline{\underline{H}}_{\hat{A}}$ be the category of all Henselian pairs $(\bar{B}, \bar{J})$ finite over a pair $(B_{\hat{A}}, J B_{\hat{A}})$ for an (A, I)- algebra $(B, J) \in \underline{\underline{H}}$ such that $\bigcap_{\nu=0}^{\infty} I^\nu\, \bar{B} = 0$ and $\hat{A} \longrightarrow \bar{B}/\, \bar{J}$ is surjective.

Then $\underline{\underline{H}}_{\hat{A}}$ is a W-category and $(B, J) \longmapsto (B_{\hat{A}}, J B_{\hat{A}})$ is a functor $\underline{\underline{H}}_A \longrightarrow \underline{\underline{H}}_{\hat{A}}$.

<u>Proof:</u> (W 0) and (W 2) are fullfilled by definition.
To prove (W 3) we choose a finite sequence $\overline{T} = (T_1, \dots, T_N)$ of indeterminates and $(\overline{B}, \overline{J}) \in \underline{\underline{H}}_{\hat{A}}$.
We may suppose that $(\overline{B}, \overline{J})$ is the quotient of a pair $(B_{\hat{A}}, JB_{\hat{A}})$ of $\underline{\underline{H}}_{\hat{A}}$ with cernel N (if (B, J) is finite over $(B_{\hat{A}}, JB_{\hat{A}})$, then for some $(X_1, \dots, X_\ell) = X$ there is a surjective map $B\langle\langle X\rangle\rangle_{\hat{A}} \to \overline{B}$).
We will show that $B\langle\langle T\rangle\rangle_{\hat{A}} / \overline{N}$ with
$\overline{N} = \bigcap_{\nu=1}^{\infty} (N\, B\langle\langle T\rangle\rangle_{\hat{A}} + I^{\nu}\, B\langle\langle T\rangle\rangle_{\hat{A}})$ is the free pair $\overline{B}\langle\langle T\rangle\rangle$.
Let $(\overline{C}, \overline{K})$ be a $(\overline{B}, \overline{J})$- algebra of $\underline{\underline{H}}_{\hat{A}}$ and $\overline{t}_1, \dots, \overline{t}_N \in \overline{K}$.
We have to show that there exists exactly one $(\overline{B}, \overline{J})$- morphism
$f : B\langle\langle T\rangle\rangle_{\hat{A}} / \overline{N} \longrightarrow \overline{C}$ with $f(T_i) = \overline{t}_i$.
We choose a (A, I)- algebra $(C, K) \in \underline{\underline{H}}$ such that $(\overline{C}, \overline{K})$ is the quotient of $(C_{\hat{A}}, K\, C_{\hat{A}})$ with cernel N'.
Now $\overline{C} = \bigcup_{\varsigma \in \mathcal{S}} C_\varsigma / N' \cap C_\varsigma$ and so the $\overline{t}_1, \dots, \overline{t}_N$ are already in some $C_\varsigma / N' \cap C_\varsigma$.
By construction there holds $C_\varsigma / N' \cap C_\varsigma \in \underline{\underline{H}}$ and this is a (B, J)- algebra. We obtain a unique (B, J)- morphism
$f_0 : B\langle\langle T\rangle\rangle \longrightarrow C_\varsigma / N' \cap C_\varsigma$ with $T_\nu \longmapsto t_\nu$ and the following commutative diagramm.

$$\begin{array}{ccc} B\langle\langle T\rangle\rangle & \longrightarrow & \overline{C} \\ \downarrow & & \downarrow \\ \hat{B}[[T]] & \longrightarrow & \hat{\overline{C}}. \end{array}$$

For this reason we can lift f_0 to a $(\overline{B}, \overline{J})$- morphism
$B\langle\langle T\rangle\rangle_{\hat{A}} \longrightarrow \overline{C}$ annullating $\overline{N}$, i. e. $B\langle\langle T\rangle\rangle_{\hat{A}} / \overline{N} = \overline{B}\langle\langle T\rangle\rangle$ is free in $\underline{\underline{H}}_{\hat{A}}$.

Now we have to prove that the canonical morphisms

$$\overline{B}\{\{T\}\} / (T_1, \ldots, T_N)^\vee \longrightarrow \overline{B}[T] / (T_1, \ldots, T_N)^\vee$$

are injective, or the canonical morphisms

$$\phi_\vee : \overline{B}[T] \longrightarrow \overline{B}\{\{T\}\} / (T_1, \ldots, T_N)^\vee$$

are surjective.

However the canonical morphism $\varinjlim_{\gamma} B_\gamma \{\{T\}\} \longrightarrow \overline{B}\{\{T\}\}$

being surjectiv, we obtain a commutativ diagram

$$\begin{tikzcd}
\overline{B}[T] \arrow[r, "\phi_\vee"] & \overline{B}\{\{T\}\} / (T_1, \ldots, T_N)^\vee \\
\varinjlim_{\gamma} B_\gamma [T] \arrow[u] \arrow[r] & \varinjlim_{\gamma} B_\gamma \{\{T\}\} / (T_1, \ldots, T_N)^\vee \arrow[u]
\end{tikzcd}$$

Then $\phi_\vee$ is also surjectiv.

To prove (W1) let $(\overline{B}, \overline{J}) \longrightarrow (\overline{B}', \overline{J}')$ be a morphism in $\underline{\underline{H}}_{\hat{A}}$ and $\overline{K} \subseteq \overline{B}'$ a closed ideal (with respect to the $\overline{J}$-adic topology) such that $\overline{B}' / \overline{K} + J \overline{B}'$ is B-finite. We will prove that $\overline{B}' / \overline{K}$ is $\underline{B}$-finite. Now $(B, \overline{J})$ is finite over some $(B_{\hat{A}}, J B_{\hat{A}})$ for a $(B, J) \in \underline{\underline{H}}$ and (B, J) is the quotient of some $A\{\{T\}\}$ in $\underline{\underline{H}}$. So we may suppose that $(\overline{B}, \overline{J})$ is the free object $\hat{A}\{\{T\}\}$ in $\underline{\underline{H}}_{\hat{A}}$. On the other hand $(\overline{B}', \overline{J}')$ is the quotient of some pair $(B'_{\hat{A}}, J' B'_{\hat{A}})$ for a suitable pair $(B', J') \in \underline{\underline{H}}$.

Since $\overline{B}$ is free we may assume that $(\overline{B}', \overline{J}') = (B'_{\hat{A}}, J' B'_{\hat{A}})$.

Let $\overline{K} + I B'_{\hat{A}} = f_1 B'_{\hat{A}} + \ldots + f_r B'_{\hat{A}} + I B'_{\hat{A}}$, $f_1, \ldots, f_r \in \overline{K}$

$(B'_{\hat{A}} / I B'_{\hat{A}} = B' / I B'$ is noetherian).

We consider the free algebra $\hat{A}\{\{T, T'\}\}$, $T' = (T_1', \ldots, T')$, and choose an $\hat{A}\{\{T\}\}$-morphism $\hat{A}\{\{T, T'\}\} \longrightarrow B'_{\hat{A}}$, $T_i \longmapsto f_i$. Now the algebra $B'_{\hat{A}} / (I\hat{A})\{\{T, T'\}\} B'_{\hat{A}} = B'_{\hat{A}} / \overline{K} + (I\hat{A})\{\{T\}\}$ is $\hat{A}\{\{T, T'\}\}$-finite and it is sufficiant to show that the algebra $B'_{\hat{A}}$ is $\hat{A}\{\{T, T'\}\}$-finite.

We are therefore in the following situation:

Let $(B'_{\hat{A}}, J' B'_{\hat{A}})$ be an $\hat{A}\{\{T\}\}$ algebra such that $B'_{\hat{A}} / (I \hat{A})\{\{T\}\} B'_{\hat{A}}$ is $\hat{A}\{\{T\}\}$ finite. We have to show that $B'_{\hat{A}}$ is $\hat{A}\{\{T\}\}$-finite.

Since $B'_{\hat{A}} / I B'_{\hat{A}} = B' / I B' \in \underline{\underline{H}}$ we infere that $B'_{\hat{A}} / I B'_{\hat{A}}$ is $\hat{A}\{\{T\}\}$-finite too. We choose a set of generators modulo $I B'_{\hat{A}}$ $W_1, \ldots, W_q \in B'_{\hat{A}}$ and consider for suitable $s = (s_1, \ldots, s_r)$ (such that $W_1, \ldots, W_q$ and the images of $T_1, \ldots, T_N$ in $B'_{\hat{A}}$ are in B'_s) the algebra-morphism.

$A\{\{T\}\}_s \longrightarrow B'_s$. It is clear that $B'_s / I B'_s + \sum_{s_i \in s} s_i B'_s$ is $A\{\{T\}\}_s$-finite generated by $W_1, \ldots, W_q$. But the algebras B'_s, $A\{\{T\}\}_s$ are contained in $\underline{\underline{H}}$ and consequently $B'_s = \sum_i A\{\{T\}\}_s W_i$. Thos implies $B'_{\hat{A}} = \sum_i \hat{A}\{\{T\}\} W_i$.

Proposition 5.3. is proved.

6. Generalization of Artin's theorem

In this chapter we will prove for excellent Weierstraß-categories over a field or an excellent discrete valuation ring the famos approximation theorem of M. Artin (cf. [2], [3]).

In this way we give a common proof for the analytiv and the algebraic case (cf. examples of excellent Weierstraßcategories in chapter 1).

One of the basic tools to prove the approximation theorem is the following lemma:

6.1. Approximation principle:

Let $\underline{\underline{H}}$ be a W-category, $(A, I) \rightarrow (B, J) \rightarrow (C, K)$ morphisms in $\underline{\underline{H}}$ and let be A a noetherian and I-adic complete ring.

Let $X \xrightarrow{p}$ Spec C be quasiprojectiv B-scheme and s_0: Spec $\hat{B} \rightarrow X$ a formal section of the I-adic completion $\hat{B}$ of B such that X is formally smooth in s_0 (Spec $\hat{B}$) and $p \circ s_0$ (V (J)).

Let $\mathcal{E} \xrightarrow{\varphi} \mathcal{O}_X$ be a morphisme of a vector bundle of finite rank over X. $Z \subseteq X$ the scheme of zeros of φ and C (φ, s_0) be defined as in theorem 3.

If there holds: (a) s_0 (Spec $\hat{B}$) $\subseteq Z$

(b) $\hat{B}$ / C (φ, s_0) is finite over A

then there exist a (B, J)-algebra (D, L) $\in \underline{\underline{H}}$ a quasiprojective D-scheme Y smooth over B a B-morphism s: $Y \rightarrow Z$ and a formal section $\bar{t}$: Spec $\hat{B} \rightarrow Y$ such that $s \circ \bar{t} = s_0$.

As in the proof of theorem 3 resp. Elkik's theorem we can reduce the proof to the following lemma:

6.2. Lemma:

With the same $\underline{\underline{H}}$, (A, I), (B, J) as in 6.1. Let $F = (F_1, \ldots, F_m) = 0$ be a system of equations, $F_i \in B\langle\langle T\rangle\rangle[T']$, $T = (T_1, \ldots, T_N)$, $T' = (T_1', \ldots, T'_{N'})$, $N + N' \geq m$ and $(\bar{t}, \bar{t}')$ a formal solution of $F = 0$, $\bar{t} \in J\hat{B}^N$, $\bar{t}' \in \hat{B}^{N'}$ such that for the ideal $\Delta_m (F, \bar{t}, \bar{t}')$ generated by the (mxm)-minors of the matrix $\partial(F_1, \ldots, F_m) / \partial(T, T')(\bar{t}, \bar{t}')$ in $\hat{B}$ the A-algebra $\hat{B} / \Delta_m (F, \bar{t}, \bar{t}')$ is finite.

Then there exist a free B-algebra $B\{\{Z\}\}$ in $\underline{H}$,

$Z = (Z_1, \ldots, Z_q)$, $t(Z) \in J\{\{Z\}\}B\{\{Z\}\}^N$, $t'(Z) \in B\{\{Z\}\}^{N'}$

and $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_q) \in J\hat{B}^q$ such that $F(t(z), t'(z)) = 0$

and $t(\bar{z}) = \bar{t}$, $t'(\bar{z}) = \bar{t}'$.

Moreover if $K \subseteq J$ is a finitely generated ideal in B such that B/K is noetherian and $(\bar{t}, \bar{t}' \bmod K\hat{B}) \in B/K$ then one can choose $\bar{z}$ to be from $K\hat{B}^q$.

Proof: Let $K \subseteq J$ be an ideal such that B/K is noetherian and $(\bar{t}, \bar{t}' \bmod K\hat{B}) \in B/K$. We choose $h_1, \ldots, h_a \in K$ such that $(\bar{t}, \bar{t}') = (t^o, t'^o) + \bar{M}h$, $h = \begin{pmatrix} h_1 \\ \vdots \\ h_a \end{pmatrix}$, $(t^o, t'^o) \in B^{N+N'}$ and $\bar{M}$ a $N + N' \times a$-matrix over $\hat{B}$. Now $\hat{B} = B + \Delta_o^2\hat{B}$,

$\Delta_o = \Delta_m(F, \bar{t}, \bar{t}') \cap B$: $\hat{B}/\Delta_m(F, \bar{t}, \bar{t}')$ is a finite A-modul and also a finite B-modul. $B/J \longrightarrow B/J \otimes_B \hat{B}/\Delta_m(F, \bar{t}, \bar{t}')$ is surjectiv and by the lemma of Nakayama also $B \rightarrow \hat{B}/\Delta_m(F, \bar{t}, \bar{t}')$ is surjectiv, i. e. $B/\Delta_o \xrightarrow{\sim} \hat{B}/\Delta_m(F, \bar{t}, \bar{t}')$. Especially B/Δ_o is noetherian and complet with respect to the I-adic topology, i. e. $B/\Delta_o \simeq \hat{B}/\Delta_o\hat{B}$. This means $\Delta_o\hat{B} = \Delta_m(F, \bar{t}, \bar{t}')$ and $\hat{B} = B + \Delta_o\hat{B}$. Especially we get $\hat{B} = B + \Delta_o^2\hat{B}$.

Using this fact we can write $\bar{M} = M + \sum d_i \bar{M}_i$, $d_i \in \Delta_o^2$, M a $(N + N') \times a$-matrix over B, $\bar{M}_i$ $(N + N') \times a$-matrixes over $\hat{B}$.

So we get $(\bar{t}, \bar{t}') = (t^o, t'^o) + Mh + \sum d_i (\bar{M}_i h)$.

Now let $B\{\{Z\}\} \in \underline{H}$, $Z = (Z_i)_i$, $Z_j = (Z_{j1}, \ldots, Z_{j\,N+N'})$, be the free B-algebra.

We will consider the system of equations

$G(T, T') = F((T, T') + \sum d_i Z_i) = 0$. The ~~ideal~~ is to apply Newtons lemma to this system and $(\tilde{t}, \tilde{t}') = (t^0, t'^0) + M h$.

First we will show that $\Delta_m(F, \tilde{t}, \tilde{t}') = \Delta_0$ and $\Delta_m(G, \tilde{t}, \tilde{t}') = \Delta_0 B\{\{Z\}\}$.

Obviously $(\tilde{t}, \tilde{t}') \equiv (t, t') \bmod \Delta_0^2 K B$ implies

$\Delta_m(F, \tilde{t}, \tilde{t}')\hat{B} \subseteq \Delta_m(F, \bar{t}, \bar{t}') = \Delta_0 \hat{B} \subseteq \Delta_m(F, \tilde{t}, \tilde{t}')\hat{B} + \Delta_0^2 K \hat{B}$,

i. e. $\Delta_m(F, \tilde{t}, \tilde{t}') \subseteq \Delta_0 \subseteq \Delta_m(F, \tilde{t}, \tilde{t}') + \Delta_0^2 K$

$B / \Delta_0^2 K$ is A finite we have (because $H \hat{B} \cap B = H$ for all ideals $H \supseteq \Delta_0^2 K$), but this meens $\Delta_m(F, \tilde{t}, \tilde{t}') = \Delta_0$.

Now we know $(\tilde{t}, \tilde{t}') \equiv (\tilde{t}, \tilde{t}') + \sum d_i Z_i$ mod $\Delta_0^2 \sum_{i,v} B\{\{Z\}\} Z_{iv}$ and recerve similarly $\Delta_m(G, \tilde{t}, \tilde{t}') = \Delta_0 B\{\{Z\}\}$ (note that $\partial G / \partial (T, T')(\tilde{t}, \tilde{t}') = \partial F / \partial (T, T')((\tilde{t}, \tilde{t}') + \sum d_i Z_i)$).

Now $G(\tilde{t}, \tilde{t}') = F((\tilde{t}, \tilde{t}') + \sum d_i Z_i) \equiv F(\tilde{t}, \tilde{t}') \bmod \Delta_0^2 \sum_{i,v} Z_{iv} B\{\{Z\}\}$

and $0 = F(\bar{t}, \bar{t}') \equiv F(\tilde{t}, \tilde{t}') \bmod \Delta_0^2 K \hat{B}$, i. e.

$G(\tilde{t}, \tilde{t}') \equiv 0 \bmod \Delta_0^2 (K B\{\{Z\}\} + \sum_{i,v} Z_{iv} B\{\{Z\}\})$.

We can apply Newton's lemma and get a solution $(\tilde{\tilde{t}}, \tilde{\tilde{t}}') \in B\{\{Z\}\}^{N+N'}$ of the system of equations $G(T, T') = 0$. Then $(t(Z), t'(Z)) = (\tilde{\tilde{t}}, \tilde{\tilde{t}}') + \sum d_i Z_i$ is a solution for $F(T, T') = 0$.

6.3. Artin's theorem: Let $\underline{\underline{H}}$ be an excellent Weierstraß category over an excellent discrete valuation ring R, $(A, I) \longrightarrow (B, J)$ a morphism in $\underline{\underline{H}}$ and let $X \xrightarrow{P}$ Spec B be a quasiprojectiv B-scheme. If $\bar{\bar{s}}$: Spec $\hat{A} \longrightarrow X$ is a formal section of the I-adic completion $\hat{A}$ of A then there exist a free (A, I)-algebra $(C, K) \in \underline{\underline{H}}$, an A-algebra-morphism s: Spec $C \longrightarrow X$ and a formal section $\bar{\bar{s}}_{\bar{s}}$: Spec $\hat{A} \longrightarrow$ Spec C such that $s \circ \bar{\bar{s}}_{\bar{s}} = \bar{\bar{s}}$.

The proof of this theorem breaks into several steps:

I. step: Reduction to the case that A is regular

6.3.' Theorem: Let $\underline{\underline{H}}$ be an excellent Weierstraß category over an excellent discrete valuation ring R with prim element π

Let $F = (F_1, \ldots, F_m) = 0$ be a system of equations,

$F_i \in R\{\{X, Y\}\}[T]$, $X = (X_1, \ldots, X\ )$, $Y = (Y_1, \ldots, Y_N)$,

$T = (T_1, \ldots, T_{N'})$ and $(\bar{y}, \bar{t})$ a formal solution of $F = 0$,

$\bar{y} \in (\pi, X)\, \hat{R}[[X]]^N$, $\bar{t} \in \hat{R}[[X]]^{N'}$.

Then there exist $y(Z) \in (\pi, X, Z)\, R\{\{X, Z\}\}^N$, $t(Z) \in R\{\{X, Z\}\}^{N'}$ for a suitable $Z = (Z_1, \ldots, Z_\ell)$ and $\bar{z} \in (\pi, X)\, \hat{R}[[X]]^\ell$ such that $F(y(Z), t(Z)) = 0$ and $y(\bar{z}) = \bar{y}$, $t(\bar{z}) = \bar{t}$.

In this step we will show that theorem 6.3.' implies theorem 6.3.

We may start with a system $(F_1^o, \ldots, F^o) = F^o = 0$ of equations, $F_i^o \in A\{\{Y\}\}[T]$, $A \in \underline{\underline{H}}$ and a formal solution $(\bar{y}^o, \bar{t}^o) \in \hat{A}^{N+N'}$ of $F^o = 0$.

Let $A = R\{\{X\}\}/K$ and let K be generated by $b_1, \ldots, b_h$.

We choose $F_i \in R\{\{X, Y\}\}[T]$, $(\bar{y}, \bar{t}) \in \hat{R}[[X]]^{N+N'}$ su such that $F_i \bmod K\, R\{\{X, Y\}\}[T] = F_i^o$ and $(\bar{y}, \bar{t}) \bmod K\, \hat{R}[[X]]^{N+N'} = (\bar{y}^o, \bar{t}^o)$.

Then $F_i(\bar{y}, \bar{t}) = \sum_{v=1}^{h} b_v \bar{W}_{vi}$ for suitable $\bar{W}_{vi} \in \hat{R}[[X]]$.

We apply now theorem 6.3.' to the system $F_i - \sum_{v=1}^{h} b_v W_{vi} \in R\{\{X, Y\}\}[T, W]$ and receive $W_{vi}(Z)$, $y(Z)$, $t(Z)$ from $R\{\{X, Z\}\}$ for a suitable $Z = (Z_1, \ldots, Z_\ell)$ and a $\bar{z} \in \hat{R}[[X]]^\ell$ such that $y(\bar{z}) = \bar{y}$, $t(\bar{z}) = \bar{t}$ and $W_{vi}(\bar{z}) = \bar{W}_{vi}$ and $F_i(y(Z), t(Z)) = \sum_{v=1}^{h} b_v W_{vi}(Z)$.

Now $(y^o(Z), t^o(Z)) = (y(Z), t(Z)) \bmod K\, R\{\{X, Z\}\}\, A\{\{Z\}\}$ is the required solution of $F^o = 0$.

II. step: Reduction to the case that the $F_1, \ldots, F_m$ of 6.3.' generate the kernel I of

$$R\{\{X, Y\}\}[T] \longrightarrow \hat{R}[[X]] \text{ and } I \cap R\{\{X, Y\}\} = (0).$$

To prove theorem 6.3.' it is clear that we may suppose that $F_1, \ldots, F_m$ generate the kernel I of the morphism

$$\delta : R\{\{X, Y\}\}[T] \longrightarrow \hat{R}[[X]] \text{ defined by } Y \longmapsto \bar{y},\ T \longmapsto \bar{t}$$

(otherwise we may add some equations). If $I \cap R\{\{X, Y\}\} = (0)$, we are ready. In the other case we choose an automorphism

$$\varphi : R\{\{X, Y\}\}[T] \longrightarrow R\{\{X, Y\}\}[T] \text{ with the following}$$

properties:

(i) $\varphi / R\{\{X, Y\}\} \equiv \mathrm{id}_{R\{\{X, Y\}\}} \bmod (X, Y)^2\, R\{\{X, Y\}\}$

(ii) $\varphi(R\{\{X, Y\}\}) \subseteq R\{\{X, Y\}\}$

(iii) $\varphi(I)$ is generated by $G_1, \ldots, G_m \in$
$R\{\{X, Y_1, \ldots, Y_t\}\}[Y_{t+1}, \ldots, Y_N, T]$ and
$(G_1, \ldots, G_m)\, R\{\{X, Y_1, \ldots, Y_t\}\}[Y_{t+1}, \ldots, Y_N, T] \cap R\{\{X, Y_1, \ldots, Y_t\}\} = 0$

(We may choose φ to be the composition of automorphisms of the type $X_i \longmapsto X_i + Y_k^{n_{ki}}$, $Y_i \longmapsto Y_i + Y_k^{m_{ki}}$ and apply the preparation theorem to get the $G_1, \ldots, G_m$).

Now $\delta \circ \varphi^{-1} : R\{\{X, Y\}\}[T] \longrightarrow \hat{R}[[X]]$ is a morphism mapping $\varphi(I)$ to zero, but in general $\delta \circ \varphi^{-1}$ is not a $R\{\{X\}\}$-morphism. Because of (i) and (ii) $\delta \circ \varphi^{-1} / R\{\{X\}\}$ induces a R-automorphism of $\hat{R}[[X]]$. Let ψ be the inverse of this automorphism, then $\psi \circ \delta \circ \varphi^{-1} : R\{\{X, Y\}\}[T] \longrightarrow \hat{R}[[X]]$ is a $R\{\{X\}\}$-morphism mapping $\varphi(I)$ to zero, i.e. if we denote by $\tilde{y} = \psi \circ \delta \circ \varphi^{-1}(Y)$, $\tilde{t} = \psi \circ \delta \circ \varphi^{-1}(T)$, then $G_i(\tilde{y}, \tilde{t}) = 0$, $i = 1, \dots, m$.

Now let us suppose theorem 6.3.' holds for the G_i.

Then for a suitable $Z = (Z_1, \dots, Z_s)$ there exist $\tilde{y}(Z)$, $\tilde{t}(Z)$ $R\{\{X, Z\}\}^{N+N'}$ and $\tilde{z} \in \hat{R}[[X]]^s$ such that $G_i(\tilde{y}(Z), \tilde{t}(Z)) = 0$ and $\tilde{y}(\tilde{z}) = \tilde{y}$, $\tilde{t}(\tilde{z}) = \tilde{t}$. Let us denote by λ the $R\{\{X, Z\}\}$-morphism $R\{\{X, Z, Y\}\}[T] \rightarrow R\{\{X, Z\}\}$ defined by $\tilde{y}(Z)$, $\tilde{t}(Z)$ and $\mu : R\{\{X, Z\}\} \rightarrow \hat{R}[[X]]$ the $R\{\{X\}\}$-morphism defined by $\tilde{z}$, then $\mu \circ \lambda / R\{\{X, Y\}\}[T] = \psi \circ \delta \circ \varphi^{-1}$.

Let us denote the canonical prolongation of φ to $R\{\{X, Z, Y\}\}[T]$ also by φ, then $\lambda \circ \varphi$ (especially $\mu \circ \lambda \circ \varphi$) maps I to zero, but in general $\lambda \circ \varphi / R\{\{X, Z\}\}$ is not the identity (note $\lambda \circ \varphi(Z) = Z$), especially $\mu \circ \lambda \circ \varphi / R\{\{X\}\}$ may not be the canonical injection. Because of (i) and (ii) $\lambda \circ \varphi / R\{\{X, Z\}\}$ is a R-automorphism (and $\mu \circ \lambda \circ \varphi / R\{\{X\}\}$ induces an R-automorphism of $\hat{R}[[X]]$).

Let σ be the inverse of $\lambda \circ \varphi / R\{\{X, Z\}\}$, then $\sigma \circ \lambda \circ \varphi$ is a $R\{\{X, Z\}\}$-morphism of $R\{\{X, Z, Y\}\}[T]$ into $R\{\{X, Z\}\}$ mapping I to zero, i.e. if $y(Z) = \sigma \circ \lambda \circ \varphi(Y)$ and $t(Z) = \sigma \circ \lambda \circ \varphi(T)$ then $F_i(y(Z), t(Z)) = 0$, $i = 1, \dots, m$.

If we put $\overline{z} = \psi^{-1}(\hat{z})$ it is not difficult to see that $y(\overline{z}) = \overline{y}$ and $t(\overline{z}) = \overline{t}$.

III. step: Reduction to the finiteness condition of the approximation principle (Neron's blowing up)

We may start with the following situation (with the notations of 6.3.'):

$I = (F_1, \ldots, F_e)$ is the kernel of the $R\{\{X\}\}$-morphism $R\{\{X, Y\}\}[T] \longrightarrow \hat{R}[[X]]$ defined by $\overline{y}$ and $\overline{t}$ and $I \cap R\{\{X, Y\}\} = 0$.

6.4. Lemma: Let $F_1, \ldots, F_m \in I$ be a minimal set of generators of $I\, R\{\{X, Y\}\}[T]_I$ and $\Delta_m(F_1, \ldots, F_m, \overline{y}, \overline{t})$ the ideal generated by the m-minors of the Jacobian matrix $\partial(F_1, \ldots, F_m) / \partial(Y, T)(\overline{y}, \overline{t})$, then $\Delta_m(F_1, \ldots, F_m, \overline{y}, \overline{t}) \neq 0$.

6.4.' Remark: Lemma 6.4. holds also if we replace in the assumtions of 6.4. the valuation ring R by a field.

Proof: Let $B = : R\{\{X\}\}$ and without restriction of generality we may suppose that the $\overline{t}_i$ are not units, then we can lift the morphism $B\{\{Y\}\}[T] \longrightarrow \hat{B}$ to a morphism $\hat{B}[[Y, T]] \longrightarrow \hat{B}$ with kernel H generated by the $Y_i - \overline{y}_i$, $T_j - \overline{t}_j$. Let us denote $Q = \hat{B}[[Y, T]]_H$, $\underline{m}_Q$ the maximal ideal of Q and $\overline{K}$ the residue field, then we have $\underline{m}_Q \cap P = \underline{m}_P$ the maximal ideal of $P = \{\{X, Y\}\}[T]_I$. Q is a formally smooth local P-algebra ($\hat{B}[[Y,T]] \mid B\{\{Y\}\}[T]$ is formally smooth)

Now let us consider the canonical morphism

$\varphi : \mathrm{Hom}_Q(\underline{m}_Q, \overline{K}) \longrightarrow \mathrm{Hom}_P(\underline{m}_P, \overline{K})$ induced by the inclusion $\underline{m}_P \longrightarrow \underline{m}_Q$. With respect to the base $U_1, \ldots, U_{N+N'}$ of $\mathrm{Hom}_Q(\underline{m}_Q, \overline{K})$, $U_i(Y_j - \overline{y}_j) = \delta_{ij}$, $U_i(T_j - \overline{t}_j) = 0$ if $i \leq N$

and $U_i(Y_j - \bar{y}_j) = 0$, $U_i(T_j - \bar{t}_j) = \delta_{ij}$ if $i > N$, and the base $v_1, \ldots, v_m$ of $\mathrm{Hom}_P(\underline{m}_P, \bar{K})$, $v_i(F_j) = \delta_{ij}$ the matrix associated to φ is just the Jacobian matrix $\partial(F_1, \ldots, F_m) / \partial(Y, T)$ $(\bar{y}, \bar{t})$. To prove the lemma we have to show that φ is surjective.

Let us consider the following commutative diagramm:

$$\begin{array}{ccc} \mathrm{Der}_K(Q, \bar{K}) & \longrightarrow & \mathrm{Hom}_Q(\underline{m}_Q, \bar{K}) \\ \rho \downarrow & & \downarrow \varphi \\ \mathrm{Der}_K(P, \bar{K}) & \xrightarrow[\sigma]{} & \mathrm{Hom}_P(\underline{m}_P, \bar{K}) \end{array}$$

given by the restriction maps ($\bar{K} = Q(\bar{B})$ the fraction field of $\bar{B}$).

We will prove that ρ and σ are surjective.

Let $v \in \mathrm{Hom}_P(\underline{m}_P, \bar{K}) = \mathrm{Hom}_P(\underline{m}_P / \underline{m}_P^2, \bar{K})$ and $P/\underline{m}_P =: L \hookrightarrow P/\underline{m}_P^2 =: \bar{P}$ an embedding of the coefficient field of $\bar{P}$ which extends the canonical embedding $K \hookrightarrow \bar{P}$.

Such an embedding $L \hookrightarrow \bar{P}$ exists because L/K is a separable extension ($L \subseteq Q(\hat{B})$ and $Q(\hat{B}) / Q(B)$ is separable and $\bar{P}$ is a complete local ring. (Theorem of Cohen, cf. [6A]).

Now $\bar{P} \cong L \oplus \underline{m}_P / \underline{m}_P^2$ and clearly the morphism v is the restriction of the derivation θ_v, $\theta_v(x + y) := v(y)$ $(x \in L, y \in \underline{m}_P/\underline{m}_P^2)$, i.e. σ is surjective.

The morphism ρ is surjective because of the formal smothness of the P-algebra Q (cf. [6A]):

Let $\theta : P \longrightarrow \bar{K}$ be a K-derivation. In order to get a lifting $\tilde{\theta} : Q \longrightarrow \bar{K}$ we consider the ring $E = Q[Y]/(Y^2, Y\underline{m}_Q) = Q \oplus y\bar{K}$ together with the P-algebra structure $p \cdot (q + ay) = pq + (\bar{p}a + \theta(p)\bar{q})y$, $p \in P$, $q \in Q$, $a \in \bar{K}$, $\bar{p}, \bar{q}$ the residue classes of p resp. q in $\bar{K}$. Since the canonical morphism $E \longrightarrow E/yE = Q$ is a

P-algebra morphism and because of the formal smoothness of Q we can lift the identity in $\mathrm{Hom}_P(Q, Q)$ to a morphism

$\hat{\psi} : Q \longrightarrow \hat{E} = \hat{Q} \oplus y\bar{K}$:

$$\begin{array}{ccc} P & \longrightarrow & \hat{Q} \\ \downarrow & & \| \\ \hat{E} & \longrightarrow & \hat{E}/y\hat{E} = \hat{Q}. \end{array}$$

Now it is clear that $\hat{\psi}(Q) \subseteq E$ and $\hat{\psi}(q) = q + y\tilde{\theta}(q)$.

It is not difficult to verity that $\tilde{\theta} : Q \longrightarrow \bar{K}$ is a derivation and $\tilde{\theta}/P = \theta$. The lemma is proved.

To apply the approximation principle we are intrested to have $\hat{R}[[X]]/\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ to be a finite $\hat{R}[[X_1, \ldots, X_{n-1}]]$-modul. The next lemma will arrange this situation.

6.5. Lemma: We use the same notations and assumptions as in lemma 6.4. The B-morphism $B\{\{Y\}\}[T] \longrightarrow \hat{B}$ given by $Y_i \longmapsto \bar{y}_i$, $T_i \longmapsto \bar{t}_i$ extends to a B-morphism $B\{\{Y\}\}[T, Z] \longrightarrow \hat{B}$, $Z = (Z_1, \ldots, Z_s)$, $Z_i \longmapsto \bar{z}_i$, with kernel K and a minimal set of generators $G_1, \ldots, G_t$ of $B\{\{Y\}\}[T, Z]_K$ such that the ideal

$\Delta_t(G_1, \ldots, G_t, \bar{y}, \bar{t}, \bar{z}) \subseteq / \pi B$.

Proof: We know that $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \neq 0$ (lemma 6.4.)

If $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \subseteq / \pi B$ we are ready.

Otherwhise let $\Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t}) \subseteq / \pi^k B$ $k > 1$ and we will drop this k step by step. Let $\ell(F_1, \ldots, F_m, \bar{y}, \bar{t}) = \mathrm{ord}_\pi \Delta_m(F_1, \ldots, F_m, \bar{y}, \bar{t})$ and $\ell(I, \bar{y}, \bar{t}) = \min\{\ell(F_1, \ldots, F_m, \bar{y}, \bar{t}), F_1, \ldots, F_m \in I$ generating $I\, B\{\{Y\}\}[T]_I\}$.

To prove the lemma it is sufficient to prove the following lemma:

Neron's π-desingularization: With the ~~same~~ notations and assumptions of lemma 6.4. the B-morphism $B\{\{Y\}\}[T] \longrightarrow \hat{B}$ extends to a B-morphism $B\{\{Y\}\}[T, Z] \longrightarrow \hat{B}$, $Z = (Z_1, \ldots, Z_e)$, $Z_i \longmapsto \bar{z}_i$, with kernel K such that $\ell(K, \bar{y}, \bar{t}, \bar{z}) < \ell(I, \bar{y}, \bar{t})$.

To prove this lemma, we may suppose that $\ell(I, \bar{y}, \bar{t}) = k > 0$. Let $F_1, \ldots, F_m \in I$ such that $\mathrm{ord}_\pi \Delta_m (F_1, \ldots, F_m, \bar{y}, \bar{t}) = k$, i.e. there exists a m-minor of the Jacobian matrix $\partial(F_1, \ldots, F_m) / \partial (Y, T)(\bar{y}, \bar{t})$ which is exactly divisible by π^k.

Let $K = \{ f \in B\{\{Y\}\}[T], \pi / f(\bar{y}, \bar{t})\}$ and $G_1, \ldots, G_e \in B\{\{Y\}\}[T]$ such that $\pi, G_1, \ldots, G_e$ is a minimal set of generators of $B\{\{Y\}\}[T]_K$. The residue classes of the G_i mod π generate $\bar{K}(B\{\{Y\}\}[T]/\pi B\{\{Y\}\}[T])_{\bar{K}} =: m$, $\bar{K}$ is the kernel of the B/π B-morphism $B/\pi B\{\{Y\}\}[T] \longrightarrow \hat{B}/\pi\hat{B}$ induced by the residue classes of the $\bar{y}_j$ and $\bar{t}_j$ mod π. Using lemma 6.4.' we receive that the ideal generated by the e-minars of the Jacobian matrix of the G_i mod π is not in m, i.e.

(i) $\Delta_e (G_1, \ldots, G_e, \bar{y}, \bar{t}) \not\subseteq \pi\hat{B}$.

Furthermore the residue classes of $F_1, \ldots, F_m$ mod π are linearly dependent in m/m^2, i.e. especially

(ii) $I \cap K^2 \not\subseteq K \cdot I$.

If the $F_1, \ldots, F_m$ would be linearly independent we could find $F_{m+1}, \ldots, F_e \in K$ such that $F_1, \ldots, F_e$ mod π would be a base of m / m^2. By the above lemma the ideal generated by the e-minars of the Jacobian of $F_1, \ldots, F_e$ mod π is not in m. But this is not possible because the ideal generated by the m-minors of the Jacobian of the $F_1, \dots, F_m$ mod π is already in m.

Now let us consider the B-morphism $B\{\{Y\}\}[T, Z] \longrightarrow \hat{B}_1$

$Z = (Z_1, \dots, Z_e)$ and $Y_i \longmapsto \bar{y}_i$, $T_i \longmapsto \bar{t}_i$, $Z_i \longmapsto \frac{G_i(\bar{y}, \bar{t})}{\pi}$.

Let J be the kernel of this morphism, then $\pi Z_i - G_i$ and I are in J and because of the assumptions at the beginning of (III) we have $J \cap B\{\{Y\}\} = 0$ and $ht\, J = m+e$.

We will show that $\ell(J, \bar{y}, \bar{t}, \bar{z}) < k$. We have to choose a "good" minimal set of generators $H_1, \dots, H_{m+e}$ of $B\{\{Y\}\}[T, Z]_J$ such that $\mathrm{ord}_\pi \Delta_{m+e}(H_1, \dots, H_{m+e}, \bar{y}, \bar{t}, \bar{z}) < k$.

By definition of the G_i we can choose a $H \in B\{\{Y\}\}[T]$ $H \notin K$ and $D_i, K_{ij} \in B\{\{Y\}\}[T]$ such that

$$H F_i = \pi D_i + \sum_j K_{ij} G_j, \text{ i.e.}$$

(1) $H F_i = \pi \tilde{H}_i + \sum_j K_{ij} (G_j - \pi Z_j)$, $\tilde{H}_i \in B\{\{Y\}\}[T, Z]$.

Using (ii) we can find $L_1, \dots, L_m \in B\{\{Y\}\}[T]$, $L_k \notin K$ for some $k \in \{1, \dots, m\}$ such that $\sum_{i=1}^{m} L_i F_i \in K^2$, i.e.

(2) $\sum_{i=1}^{m} L_i F_i = \sum_{j=1}^{e} M_j (G_j - \pi Z_j) + \pi^2 \tilde{H}$ for suitables $\tilde{H}, M_j \in B\{\{Y\}\}[T, Z]$.

Now $H_1 := \tilde{H}_1, \dots, H_{k-1} := \tilde{H}_{k-1}, H_k := \tilde{H}, H_{k+1} := \tilde{H}_{k+1}, \dots, H_m := \tilde{H}_m$,

$H_{m+1} := \pi Z_1 - G_1, \dots, H_{m+e} := \pi Z_e - G_e$ generate

$J B\{\{Y\}\}[T, Z]_J$.

Using (1) and (2) we will show that.

(3) $\mathrm{ord}_\pi \Delta_{m+e}(H_1, \dots, H_{m+e}, \bar{y}, \bar{t}, \bar{z}) < k$.

Diffrentiating (1) and (2) we receive obtainc

(4) $H\, \partial F_i / \partial(Y, T) \equiv \pi\, \partial H_i / \partial(Y, T) + \sum_{j=1}^{e} (K_{ij}) \cdot \partial G_j / \partial(Y, T) \mod J$

$\frac{\partial H_i}{\partial Z_t} = K_{it} \mod J$

$i = 1, \dots, k-1, k+1, \dots, m$

(5) $\sum_i L_i\, \partial F_i/\partial(Y, T) \equiv \sum_{j=1}^{e} M_j\, \partial G_j/\partial(Y, T) +$
$\pi^2 \partial H_k/\partial(Y, T) \bmod J$

$\pi \dfrac{\partial H_k}{\partial Z_t} \equiv M_t \mod J$

(6) $\partial H_{m+i}/\partial(Y, T) = -\partial G_i/\partial(Y, T)$

$\partial H_{m+i}/\partial Z_j = \pi \cdot \delta_{ij}$

$i = 1, \ldots, e$

Using (4), (5) and (6) we can see that the Jacobian matrix.

(7) $\partial(\pi H_1, \ldots, \pi H_{k-1}, \pi^2 H_k, \pi H_{k+1}, \ldots, \pi H_m, H_{m+1}, \ldots, H_{m+e})/$
$\partial(Y, T, Z)(\bar{y}, \bar{t}, \bar{z})$ is equivalent to the matrix

(8) $\begin{pmatrix} \partial(F)/\partial(Y, T)(\bar{y}, \bar{t}) & 0 \\ -\partial(G)/\partial(Y, T)(\bar{y}, \bar{t}) & \pi I_e \end{pmatrix}$.

I_e the e x e ~~identity~~ unite matrix.

By definition of $\ell(I, \bar{y}, \bar{t}) = k$ the matrix $\partial(F)/\partial(Y, T)(\bar{y}, \bar{t})$ has a m-minor not divisible by π^{k+1}. Then it is clear that the matrix (8) has a $m+e$ -minor not divisible by $\pi^{k+1+d-(d-m)} = \pi^{k+m+1}$, because $\Delta_e(G, \bar{y}, \bar{t}) \not\subseteq \pi B$ (cf. (i)).

Finally we have a $m+e$ -minor of (7) not divisible by π^{k+m+1}, i.e. there is a $m+e$ -minor of $\partial(H_1, \ldots, H_{m+e})/\partial(Y, T, Z)(\bar{y}, \bar{t}, \bar{z})$ not divisible by π^k, i.e. (3) holds and consequently $l(K, \bar{y}, \bar{t}, \bar{z}) < k$.

Neron's desingularization step is finished.

IV. step: Induction on n using the approximation principle.

We may start with the following situation (with the notations of 6.3.'):

$I = (F_1, \ldots, F_e)$ is the kernel of the $R\{X\}$-morphism $R\{X, Y\}[T] \longrightarrow \hat{R}[[X]]$ defined by $\bar{y}$ and $\bar{t}$, and

$\Delta_m (F_1, \dots, F_m, \overline{y}, \overline{t}) \subseteq\!\!/\ \pi \hat{R} [[X]]$.

Clearly it is enough to look for a solution of $F_1 = \dots = F_m = 0$ ($F_1, \dots, F_m$ generate $I R\{\{X, Y\}\} [T]_I$):

Let us suppose for a moment that for suitable $Z = (Z_1, \dots, Z_t)$, $y(Z), t(Z) \in R\{\{X, Z\}\}$ and $\overline{z} \in \hat{R} [[X]]^t$

$F_i (y(Z), t(Z)) = 0 \quad i = 1, \dots, m$ and $y(\overline{z}) = \overline{y}$, $t(\overline{z}) = \overline{t}$.

If $r > m$, then $G \cdot F_r = \sum_{j=1}^{m} H_{rj} F_j$ for a suitable $G \in\!\!/\ I$, i.e.

$G(y(Z), t(Z)) \cdot F_r (y(Z), t(Z)) = 0$.

Now it is clear, that $G \in\!\!/\ I$ implies $G(\overline{y}, \overline{t}) \neq 0$ and especially $G(y(Z), t(Z)) \neq 0$, i.e. $F_r (y(Z), t(Z)) = 0$.

1. Case $n = 0$

In this case $\Delta_m (F_1, \dots, F_m, \overline{y}, \overline{t})$ is a unit (for simplificity we may suppose the $\overline{t}_i$ to be not units). Let $\det \partial (F_1, \dots, F_m) / \partial (Y_{r1}, \dots, Y_{rs}, T_{j\,s+1}, \dots, T_{jm}) (\overline{y}, \overline{t})$ be a unit. Then we consider the system

$F_1 = 0, \dots, F_m = 0, F_{m+1} =: Y_{r\,s+1} - Z_1 = 0, \dots, F_{N+N'} = T_{js} - Z_{N+N'-m} = 0$

defined over $R\{\{Z_1, \dots, Z_{N+N'-m}, Y\}\} [T]$.

We have $F_i (0, 0) \equiv 0 \mod (\pi, Z_1, \dots, Z_{N+N'-m})$

and $\det (\partial (F_1, \dots, F_{N+N'}) / \partial (Y, T) (0,0))$ = unit in $R\{\{Z_1, \dots, Z_{N+N'-m}\}\}$.

Using the implicit function theorem we get $y(Z), t(Z) \in R\{\{Z_1, \dots, Z_{N+N'-m}\}\}$ such that $F_i (y(Z), t(Z)) = 0 \quad i = 1, \dots, m$

$$y_{r,\,s+j} (Z) = Z_j \qquad j = 1, \dots, N' - s$$

$$t_{jk} (Z) = Z_{N'-s+k} \qquad k = 1, \dots, N+s-m$$

Using again the implicit function theorem for the system

$$F_i\ (y\ (Z),\ t\ (Z))\ /\ \begin{array}{ll} Z_j = \bar{y}_{rs+j} & j = 1, \ldots, N' - s \\ Z_{N'-s+k} = \bar{t}_{jk} & \end{array}$$

defined over $\hat{R}[[Z]]$ we get the required $\bar{z} \in \hat{R}^{N+N'-m}$ such that $y\ (\bar{z}) = \bar{y}$ and $t\ (\bar{z}) = \bar{t}$.

2\. Case: $n > 0$.

We suppose, the theorem 6.3.' is true for $R\langle\langle X_1, \ldots, X_{n-1}\rangle\rangle$. Further, we may suppose (after applying a suitable $\hat{R}$-automorphism of $\hat{R}[[X]]$ and using the preparation theorem) that $\hat{R}[[X]]\ /\ \Delta_m\ (F_1, \ldots, F_m, \bar{y}, \bar{t})$ is a finite $\hat{R}[[X_1, \ldots, X_{n-1}]]$ - modul. Now we may apply the approximation principle 6.3. to the pairs $(\hat{R}[[X_1, \ldots, X_n]],\ (\pi, X_1, \ldots, X_n)\ )$ and $R\langle\langle X\rangle\rangle_{\hat{R}}[[X_1, \ldots, X_n]]$.

There exists a free algebra $R\langle\langle X\rangle\rangle_{\hat{R}}[[X_1, \ldots, X_{n-1}]]\langle\langle Z\rangle\rangle$, $Z = (Z_1, \ldots, Z_s)$ and $\tilde{y}\ (Z),\ \tilde{t}\ (Z) \in R\langle\langle X\rangle\rangle_{\hat{R}}[[X_1, \ldots, X_{n-1}]]$
$\langle\langle Z\rangle\rangle^{N+N'}$ and $\bar{z} \in \hat{R}[[X]]^s$ such that

$F_i\ (\tilde{y}\ (Z),\ \tilde{t}\ (Z)) = 0 \quad i = 1, \ldots, m$ and $\tilde{y}\ (\bar{z}) = \bar{y}$, $\tilde{t}\ (\bar{z}) = \bar{t}$.

Now $R\langle\langle X\rangle\rangle_R[[X_1, \ldots, X_{n-1}]]\langle\langle Z\rangle\rangle = R\langle\langle X, Z\rangle\rangle_R[[X_1, \ldots, X_{n-1}]]$ and looking carefully to the construction of $R\langle\langle X, Z\rangle\rangle_{\hat{R}}[[X_1, \ldots, X_{n-1}]]$ we can choose a $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_k)$, $\bar{u}_i \in (\pi\ , X_1, \ldots, X_{n-1})\ \hat{R}[[X_1, \ldots, X_{n-1}]]$, such that $(\tilde{y}\ (Z),\ \tilde{t}\ (Z)) \in R\langle\langle X, Z\rangle\rangle_{\bar{u}}^{N+N'}$.

Let K to be the kernel of the $R\langle\langle X_1, \ldots, X_{n-1}\rangle\rangle$-morphism $R\langle\langle X_1, \ldots, X_{n-1}, U\rangle\rangle \longrightarrow R\langle\langle X_1, \ldots, X_{n-1}\rangle\rangle_{\bar{u}}$ defined by $U \longmapsto \bar{u}$, $u\ (Z') \in R\langle\langle X_1, \ldots, X_{n-1}, Z'\rangle\rangle$ be a zero of K

$Z' = (Z_1', \ldots, Z_b')$ and $\overline{z}' \in R [[X_1, \ldots, X_{n-1}]]^b$

such that $u(\overline{z}') = \overline{u}$.

We consider the $R\{\{X_1, \ldots, X_{n-1}\}\}$-morphism

$$R\{\{X_1, \ldots, X_{n-1}\}\}_{\overline{u}} \{\{X, Z\}\} \longrightarrow R\{\{X, Z, Z'\}\}$$

defined by $\overline{u} \longmapsto u(Z')$, $X \longmapsto X$, $Z \longmapsto Z$ and

denote by $y(Z, Z')$ resp. $t(Z, Z')$ the image of $\widetilde{y}$ resp. $\widetilde{t}$

this morphism.

Then $F_i(y(Z, Z'), t(Z, Z')) = 0$ and $y(\overline{z}, \overline{z}') = \overline{y}$, $t(\overline{z}, \overline{z}') = \overline{t}$.

The theorem is proved.

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