INSTITUTUL DE MATEMATICA

INSTITUTUL NATIONAL PENTRU CREATIE STIINTIFICA SI TEHNICA

ISSN 0250-3638

ALGEBRAIC MAPPING GERMS AND GENERIC PROJECTIONS

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PREPRINT SERIES IN MATHEMATICS

No.13/1980

BUCURESTI

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March 1980

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Algebraic mapping germs and generic projections

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Introduction and formulation of the main results

It is well-know that generic projections of smooth algebraic curves into the projective plane can have only nodes as singularities and that generic projections of smooth algebraic surfaces into the 3-dimensional space can, up to analytic equivalence, have only 3 types of singularities, namely (1) 2 smooth branches crossing normally, (2) 3 smooth branches crossing normally or (3) a pinch point (or Whitney umbrella) defined up to equivalence either by the equation $xy^2 - z^2 = 0$ or by the parameterization $x = u^2$, $y = v$, $z = uv$.

J. Mather, using the concept of stability of $\mathcal{C}^\infty$- mapping germs, was able to determine all singularities which can appear as singularities under generic projections $X^n \subset \mathbb{P}^N \longrightarrow \mathbb{P}^p$ for suitable dimensions p and n (cf. [15]), working over the field $\mathbb{C}$ of complex numbers.

Here we want to present a purely algebraic approach to these results which include varieties over arbitrary algebraically closed ground fields. The basic idea is to use localizations in the sense of the etal topology of schemes, which leads to the notion of algebraic equivalence of schemes or morphisms of schemes in a point, instead of analytic equivalence. Thus the notion of algebraic power series will play an important role in our paper; algebraic power series over a field K are those formal power series $f(T_1, \ldots, T_n)$ which are algebraic over the field $K(T_1, \ldots, T_n)$.

If the field K is, for example, the field $\mathbb{C}$ of complex numbers, these are exactly those power series which represent branches of algebraic functions of $\mathbb{C}^n$ at the origin. The set of all algebraic power series $f(T_1, \ldots, T_n)$ forms a local ring which we will denote by $K\langle T_1, \ldots, T_n\rangle$ or by $\mathcal{O}_n$ and which is the Henselian closure of the local ring $\mathcal{O}_{\mathbb{A}^n,0}$ or, what is the same, the limit of all rings $\mathcal{O}(U)$, where U runs through the family of etal neighbourhoods of 0 in $\mathbb{A}^n$.

The basic material about Henselian rings and algebraic power series can be found, for example, in [11] or [12].

To formulate the main results of our paper we have to introduce some definitions.

We consider smooth algebraic varieties X, Y over the ground field K and a finite morphism $\varphi : X \longrightarrow Y$ and by X' we denote the image $\varphi(X) \subset Y$ of φ and suppose that φ induces a birational morphism $X \longrightarrow X'$ (in other words, X is the normalization of the variety $X' \subset Y$).

Consider a point $y \in X'$ and $\varphi^{-1}(y) = \{x_1, \ldots, x_r\}$ as a set.

It is well-known that the points $x_1, \ldots, x_r$ are in 1,1-correspondence with the branches of X' at y ; more precisely, by $x_j \longmapsto \mathrm{Ker}(\widehat{\mathcal{O}}_{X',y} \longrightarrow \widehat{\mathcal{O}}_{X,x_j}) = \bar{P}_j$ and $\bar{P}_j \longmapsto V(\bar{P}_j) \subseteq \mathrm{Spec}(\widehat{\mathcal{O}}_{X',y})$ we get a 1,1-correspondence of the set $\varphi^{-1}(y)$ with the set of irreducible components of $\mathrm{Spec}(\widehat{\mathcal{O}}_{X',y})$ ($\widehat{}$ denotes the completion of local rings).

It is also known that there exists an etal neighbourhood $U \longrightarrow X'$ of y, which exactly splits into r irreducible components U_j through the point $\tilde{y} \in U$ corresponding to $y \in X'$ such that $\hat{\mathcal{O}}_{U_j,\tilde{y}} \simeq \hat{\mathcal{O}}_{X',y} / P_j$, i.e. we can already find the analytic branches of X' at y in an etal neighbourhood of y (cf. [11]); hence in the Henselian closure

$$(X',y) = \mathrm{Spec}(\mathcal{O}^h_{X',y}) = \varprojlim_U U$$

(U etal neighbourhood of y in X').

<u>Definition 1</u> The branches of X' (or of φ) at $y \in X'$ are <u>crossing normally</u>, if we can find a decomposition of germs $(Y,y) = (\mathbb{A}^{n_1} \times \ldots \times \mathbb{A}^{n_r} \times \mathbb{A}^{n_{r+1}}, 0)$ and closed subgerms $(Z_i, 0) \subset (\mathbb{A}^{n_i}, 0)$ such that the germ (X',y) is the union (in (Y,y)) of the "branches"

$$(X'_i,y) =_{\text{def.}} (\mathbb{A}^{n_1} \times \ldots \times \mathbb{A}^{n_{i-1}} \times Z_i \times \mathbb{A}^{n_{i+1}} \times \ldots \times \mathbb{A}^{n_{r+1}}, 0)$$

(for $i = 1, \ldots, r$) and such that φ induces finite morphisms

$$\varphi_{x_i} : (X,x_i) \longrightarrow (X'_i,y) .$$

Remark: If all branches (X',y) are smooth this is nothing but the "normal crossing" in the usual sense.

<u>Definition 2</u> A mapping germ $\varphi : (X,x) \longrightarrow (Y,y)$ ((X,x) and (Y,y) smooth germs of varieties) is called <u>stable</u> if, for any germ $(S,0)$ of an algebraic scheme, any prolongation ϕ of φ to a germ of a $(S,0)$-morphism

$$\begin{array}{ccc} (X \times S, x \times 0) & \xrightarrow{\phi} & (Y \times S, y \times 0) \\ \cup & & \cup \\ (X, x) & \xrightarrow{\varphi} & (Y, y) \end{array}$$

is equivalent to the trivial prolongation ($\varphi \times id_S$),

i.e. there exist prolongations

$$\alpha : (X \times S, x \times 0) \simeq (X \times S, x \times 0) \quad \text{of} \quad id_X$$

and

$$\beta : (Y \times S, y \times 0) \simeq (Y \times S, y \times 0) \quad \text{of} \quad id_Y$$

such that for $p \in X$, $s \in S$:

$$\phi(\alpha(p,s), s) = \beta(\varphi(p), s) .$$

These two definitions are the basic concepts of our paper.

Let us call a morphism $\varphi : X \longrightarrow Y$ of smooth varieties X, Y a <u>general morphism</u> if it satisfies the following three points:

(1) The morphism is finite and $\varphi : X \longrightarrow X' = \varphi(X) \subset Y$ is birational onto X'.

(2) The branches of φ are crossing normally everywhere.

(3) The germs of $\varphi : (X,x) \longrightarrow (Y, \varphi(x))$ are stable for all $x \in X$.

Then we can state the <u>main results</u> as follows

(I) If $\varphi : X \longrightarrow Y$ is a general morphism, it holds that

(i) The germ (X',y) and the multi-germ

$$\varphi_y : (X, x_1, \ldots, x_r) \longrightarrow (Y,y)$$

(where $\varphi^{-1}(y) = \{x_1, \ldots, x_r\}$)

are uniquely determined up to equivalence by the Artinian K-algebra

$$Q_y(\varphi) =_{def.} (\varphi_* \mathcal{O}_X)_y \;/\; m_{Y,y}(\varphi_* \mathcal{O}_X)_y$$

$$= \prod_{i=1}^{r} Q_{x_i}(\varphi)$$

where

$$Q_x(\varphi) =_{def.} \mathcal{O}_{X,x} \;/\; m_{Y,\varphi(X)} \mathcal{O}_{X,x}$$

(ii) $\dim Q_y(\varphi) \leq \frac{\dim Y - g(Q_y(\varphi))}{p - n}$

(if $p > n$)

where $g(Q) = \sum_{i=1}^{r} g(Q_i)$ is a certain invariant

of Artininan K-algebras ($g(Q_i) \geq$ embedding dimension of Q_i in the local case) (cf. §9)

(iii) $\dim (X',y)^{sing} = \dim Y - \dim (Q)\ \mathrm{codim}(X') - g(Q)$

(II) If $\phi : X \times S \longrightarrow Y$ is a morphism, where X, Y are smooth varieties and S is any algebraic scheme, the set $S' \subset S$ of all s, such that $\phi_s : X \times \{s\} \longrightarrow Y$ is general, contains a Zariski open subset, if it is not empty.

(III) If $\dim(X) \leq 6\ (p - \dim(X)) + 8$ or if $p - \dim(X) > 3$ and $\dim(X) \leq 6\ (p - \dim(X)) + 7$, and if $d \geq p + 1$ (where p and d are integers) and if X is a projective variety with a very ample sheaf $\mathcal{L}$, then there exists a non-empty Zariski-open subset $U \subseteq \mathrm{Grass}(p, |\mathcal{L}^{\otimes d}|)$ of p-dimensional linear systems in $|\mathcal{L}^{\otimes d}|$ such that for $\Lambda \in U$ the corresponding projection

$$\varphi_\Lambda : X \longrightarrow \mathbb{P}^p$$

is general.

(IV) Under the same restrictions on dim(X) and $p = \dim(Y)$ as in (III) for any general morphism $\varphi : X \longrightarrow Y$ there are exactly 54 local Artinian algebras which can appear as the algebras $Q_x(\varphi)$.
The table of this algebras and their relevant invariants is given in §10. Hence, by (I), for a given ample (p, n) there are only finite many types of singularities in $\varphi(X)$.

Part I of the paper is devoted to local considerations. The main results are a characterization of stable mapping germs (proposition 13 and 14, §7) and of the equivalence of stable mapping germs (proposition 17 and its corollary, §8) and the construction of "normal forms" of stable mapping germs φ, starting with the local Artinian algebra Q, such that $Q(\varphi) \cong Q$ (§9 theorem). Furthermore the relationship between unfoldings and deformations is clarified.

Part II is devoted to the determination of the simple contact classes (represented by their local Artinian algebra) and of their stable representations as well as to the global application of the local results, based on an algebraic construction of the jet bundle.

A preliminary version of this paper appeared in the preprint series of the "Forschungsinstitute für Mathematik der ETH Zürich". The first author wants to thank the ETH Zürich for the exellent working conditions during the fall term 1977, where parts of the results were obtained.

§1. Some preliminaries

A morphism of local rings $A \to B$ with the same residual class field K will be called a Weierstrass morphism if it has the following property:

If M is any finite B-module such that $M/m_A M$ is of finite length as A-module, then M is also a finite A-module.

We will frequently use the following result about Weierstrass morphisms

Proposition 1: If $M \supset M'$ are finite B-modules and there exists an integer h such that $m^h M \subseteq M'$, then for any A-submodule $P \subset M$ there holds:

If $M' \subseteq P + m_B^{c+1} M'$ for $c \geqslant e_A(P)(h+e_A(m_A))$ then

$M' \subset P \qquad (e_A(P) \underset{\text{def.}}{=} \dim_K(P \otimes_A K))$

Proof: If $P' = P \cap M'$, then $M' = P' + m_B^{c+1} M'$ and we have to proof, that the residual morphism $P' \to \bar{M}' = M'/m_A M'$ is surjective, then the result will follow by the property of Weierstrass-morphisms and Nakayama's Lemma. Now $P' \to \bar{M}'/m_B^{c+1}\bar{M}'$ is surjective by hypothesis, hence $l(\bar{M}'/m_B^{c+1}\bar{M}') = \sum_{j=0}^{c} l(m_B^j \bar{M}'/m_B^{j+1}\bar{M}') \leqslant l(P'/m_A P')$ and therefore the result follows if we prove $l(P'/m_A P') \leqslant c$, because then $m_B^c \bar{M}' = m^{c+1}\bar{M}'$ and from Nakayama's lemma again we infer $m_B^c \bar{M}' = 0$.

From $m_B^h M \subseteq M'$ follows $m_A P \subseteq P \cap M' = P'$, hence

$l(P'/m_A P') \leqslant l(P/m_A^{h+1} P) \leqslant l(P/m_A P)\, l(A/m_A^{h+1}) \leqslant e_A(P) \binom{h+e_A(m_A)}{h} = c$

q.e.d.

§2. Equivalence of mapping germs

A mapping germ $\varphi : (\mathbb{A}^n, 0) \to (\mathbb{A}^p, 0)$ is equivalent to a p-tupel of algebraic power series without constant term. By $J(n,p)$ we will denote the set of all such p-tupels and by $J_c(n,p)$ the affin space of all p-tupels of trunced power series without constant term, of degree c and by

$$j_c : J_{c'}(n,p) \to J_c(n,p), \ (\infty \geq c' > c, \ J_\infty = J)$$

the corresponding truncations (it corresponds to mapping germs restricted to the c^{th} infinitesimal neighbourhood of 0 in $\mathbb{A}^n$). We will also consider the set $\hat{J}(n,p) = \varprojlim_c J_c(n,p)$ (formal power series). By $\mathfrak{E}(n,p)$ we denote the group of equivalences of $J(n,p)$, i.e. the group of automorphisms $\mathrm{Aut}(\mathbb{A}^n,0) \times \mathrm{Aut}(\mathbb{A}^p,0) \subset J(n,n) \times J(p,p)$, acting on $J(n,m)$ by

$$(g,r)(\varphi) = g \circ \varphi \circ r^{-1}$$

The linear algebraic groups $\mathfrak{E}_c(n,p)$ = image of $\mathfrak{E}(n,p)$ in $J_c(n,n) \times J_c(p,p)$ acts algebraically on $J_c(n,p)$, and we have again canonical truncation maps, which are surjective

$$j_c : \mathfrak{E}_{c'}(n,p) \to \mathfrak{E}_c(n,p) \qquad (\infty \geq c' > c)$$

We define $\hat{\mathfrak{E}}(n,p) = \varprojlim_c \mathfrak{E}_c(n,p)$, this group is contained in $\hat{J}(n,n) \times \hat{J}(p,p)$ and acts on $\hat{J}(n,p)$, compatible with the action of $\mathfrak{E}(n,p) \subset \hat{\mathfrak{E}}(n,p)$ onto $J(n,p) \subset \hat{J}(n,p)$.

The groups $\mathfrak{E}_c(n,p)$ are linear algebraic groups acting algebraically on the affin spaces $J_c(n,p)$.

If $\varphi \in J(n,p)$ (resp. $\hat{J}(n,p)$) we denote by $I(\varphi)$ the ideal of

$\mathcal{O}_n = \mathcal{O}^h_{\mathbb{A}^n,0}$ generated by the components of φ (resp. by $\hat{I}(\varphi)$ the ideal of $\hat{\mathcal{O}}_n = \hat{\mathcal{O}}_{\mathbb{A}^n,0}$ generated by the components of φ), by $Q(\varphi)$ (resp. $\hat{Q}(\varphi)$) the K-algebra $\mathcal{O}_n/I(\varphi)$ (resp. $\hat{\mathcal{O}}_n/\hat{I}(\varphi)$) and by $X(\varphi)$ (resp. $\hat{X}(\varphi)$) the scheme $\mathrm{Spec}(Q(\varphi))$ resp. $\mathrm{Spec}(\hat{Q}(\varphi))$, which is the fibre of the mapping germ φ.

It is useful to consider the somewhat larger group $C(n,p)$ of contact equivalences. $C(n,p)$ consists of all automorphisms γ of $(\mathbb{A}^n \times \mathbb{A}^p, 0)$ such that the composition $p_1 \circ \gamma : (\mathbb{A}^n \times \mathbb{A}^p, 0) \to (\mathbb{A}^n, 0)$ depends only on the first component x of a point $(x,y) \in \mathbb{A}^n \times \mathbb{A}^p$ and such that γ induces an automorphism $(\mathbb{A}^n \times 0, 0) \to (\mathbb{A}^n \times 0, 0)$. It acts on $J(n,p)$ such that the graph of $\gamma\varphi$ is the transform of the graph of φ under γ, i.e. if we write $\gamma(x,y) = (\alpha(x), \beta(x,y))$ (the first component of $(\mathbb{A}^n,0)$ the second one of $(\mathbb{A}^p,0)$), then

$$(\gamma\varphi)(x) = \beta(\alpha^{-1}(x), \varphi(\alpha^{-1}(x)))$$

Considering truncated power series we get a projective system of linear algebraic groups

$$j_c : C_{c'}(n,p) \to C_c(n,p) \qquad (c' > c)$$

acting algebraically on the projective system of affin spaces $J_c(n,p)$. Therefore the projective limit $\hat{C}(n,p)$ acts on $\hat{J}(n,p)$. Mapping germs in the same $\hat{C}(n,p)$- orbit are called formally contact equivalent. The following proposition is obvious (cf. appendix).

<u>Proposition 2</u> Mapping germs $\varphi, \varphi' \in J(n,p)$ (resp. $\hat{J}(n,p)$) are contact equivalent (resp. formally contact equivalent) if and only if $X(\varphi) \simeq X(\varphi')$ (resp. $\hat{X}(\varphi) \simeq \hat{X}(\varphi')$), where $X(\varphi)$ denotes the fibre $\varphi^{-1}(0)$, $\hat{X}(\varphi)$ the formal fibre.

If we consider 1-parameter families $\gamma_t \in \mathfrak{C}(n,p)$ resp. $C(n,p)$

such that $\gamma_0 = \mathrm{id}$, and if we calculate $\gamma_t(\varphi) - \varphi \bmod (t^2)$ we get what we call the tangent space to the orbit at φ:

$$T(\varphi,\mathfrak{E}) = m_n\Delta(\varphi) + \varphi^*(m_p)^{\oplus p} \subseteq m_n^{\oplus p} = J(n,p)$$

$$T(\varphi,C) = m_n\Delta(\varphi) + I(\varphi)^{\oplus p} \subseteq m_n^{\oplus p} = J(n,p)$$

where $\Delta(\varphi)$ is the $\mathcal{O}^h_{\mathbb{A}^n,0}$-module generated by $\frac{\partial\varphi}{\partial x_1},\ldots,\frac{\partial\varphi}{\partial x_n}$ and, $\varphi^*(m_p)$ is the maximal ideal of $K\langle\varphi_1,\ldots,\varphi_p\rangle \subset K\langle x_1,\ldots,x_n\rangle = \mathcal{O}^h_{\mathbb{A}^n,0}$.

They are mapped onto the tangent spaces

$$T(j_c\varphi,\mathfrak{E}_c) \text{ resp. } T(j_c\varphi,C_c)$$

of the orbits of $\mathfrak{E}_c$ resp. C_c at $j_c\varphi$ ($\mathfrak{E}_c$ and C_c are algebraic groups acting algebraically on the affin space $J_c(n,p)$). We call φ $\mathfrak{E}$-finite resp. C-finite if $T(\varphi,\mathfrak{E})$ resp. $T(\varphi,C)$ are open in the Krull topology of $J(n,p)$, more precisely:

The morphism $\varphi \in J(n,p)$ is called <u>k-$\mathfrak{E}$-finite</u> resp. <u>k-C-finite</u> if $T(\varphi,\mathfrak{E})$ resp. $T(\varphi,C)$ contains $m_n^k J(n,p)$.

We will also consider the subgroups $\mathfrak{E}^\nu = \mathrm{Ker}(\mathfrak{E}\to\mathfrak{E}_\nu)$ of $\mathfrak{E}$, the tangent spaces to the $\mathfrak{E}^\nu$-orbits are

$$T(\varphi,\mathfrak{E}^\nu) = m_n^{\nu+1}\Delta(\varphi) + m_p^\nu(\varphi^* m_p^{\oplus p})$$

and the mapping germ φ is called k-$\mathfrak{E}^\nu$-finite if $m_n^k J \subset T(\varphi,\mathfrak{E}^\nu)$.

We will show that $\mathfrak{E}$-finiteness implies $\mathfrak{E}^\nu$-finiteness and that it depends only on a sufficient high jet of φ.

<u>Proposition 3</u>: (1) For any integer k there holds:

If a mapping germ $\varphi \in J(n,p)$ is k-C-finite and $\varphi' \in J(n,p)$ is a mapping germ such that $j_{k+1}(\varphi') = j_{k+1}(\varphi)$, then φ' is k-C-finite.

(2) There exist functions $a(k,\nu)$ with the following property:

If a mapping germ $\varphi \in J(n,p)$ is k-$\mathfrak{E}$-finite, then φ is

$a(k,\nu)$ - $\mathfrak{E}^{\nu}$-finite

(3) If $c(k) = k+1 + p^2 \binom{k+p}{p}$ then there holds -.

If a mapping germ φ is k-$\mathfrak{E}$-finite and if φ' is a mapping germ ~~given~~ such that $j_c(\varphi') = j_c(\varphi)$ and $c \geq c(k)$ then φ' is k-$\mathfrak{E}$-finite.

The proof of (1) is a straight forward application of Nakayama's lemma.

To prove (2), we define $a(k,\nu)$ by induction on ν. Put $a(k,0) = k$ and assume $a(k.\mu)$ has been defined for $\mu < \nu$ such that φ is $a(k,\mu)$ - $\mathfrak{E}^{\mu}$-finite.

From $T(\varphi,\mathfrak{E}^{\nu}) = m_n^{\nu+1}\Delta(\varphi) + m_p^{\nu}(\varphi^* m_p^{\oplus p})$ we infer

$$\ell(T(\varphi,\mathfrak{E}^{\nu-1}) / (T(\varphi,\mathfrak{E}^{\nu})) \leq n\binom{n+\nu-1}{n-1} + p\binom{p+\nu-1}{p-1},$$

hence

$$\ell(J/T(\varphi,\mathfrak{E}^{\nu})) \leq \ell(J/T(\varphi,\mathfrak{E}^{\nu-1})) + n\binom{n+\nu-1}{n-1} + p\binom{p+\nu-1}{p-1}$$

$$\ell(J/T(\varphi,\mathfrak{E}^{\nu-1})) \leq \ell(J/m_n^{a(k,\nu-1)}J) \leq p[\binom{n+a(k,\nu-1)}{n}-1]$$

therefore, if k, n, p are fixed

$$\ell(J/T(\varphi,\mathfrak{E}^{\nu})) \leq \ell(\nu) \qquad \text{(depending only on } \nu)$$

Now we define integers $q_j(\nu)$ by

$$q_o(\nu) = 0, \quad q_{j+1}(\nu) = q_j(\nu) + \binom{q_j(\nu)+p}{p} p\binom{p+\nu}{p-1} + 1$$

The term $p\binom{p+\nu}{p-1}$ is a bound for the embedding dimension of the $\mathcal{O}_p$-module $P = T(\mathfrak{E}^{\nu},\varphi)/m_n^{\nu+1}\Delta(\varphi)$ and by proposition 1 (applied to $P \subset M = J/m_n^{\nu+1}\Delta(\varphi)$ and $M' = m_n^{q_j(\nu)}. M$), if

$$\ell(J/T(\mathfrak{E}^{\nu},\varphi) + m_n^{q_j(\nu)}J) = \ell(J/T(\mathfrak{E}^{\nu},\varphi) + m_n^{q_{j+1}(\nu)}J),$$

then $m_n^{q_j(\nu)} J \subseteq T(\mathfrak{E}^{\nu},\varphi)$.

Because of $\ell(J/T(\mathfrak{E}^{\nu},\varphi)) \leq \ell(\nu)$ we obtain therefore, if we put

$$a(k,\nu) = q_{\ell(\nu)}(\nu),$$

the inclusion

$$m_n^{a(k,\nu)} J \subset T(\varphi, \mathfrak{E}^{\nu})$$

Now we want to prove (3). We will apply again Proposition 1 to $M = J/m_n \Delta(\varphi')$, $M' = m_n^{\ k} M$ and $P = T(\varphi', \mathfrak{E})/m_n \Delta(\varphi')$, thus we get: If $c = c(k) = k+1+p^2 \binom{k+p}{p}$ and $j_c(\varphi') = j_c(\varphi)$, then $m_n^{\ k} J \subset T(\varphi', \mathfrak{E})$ q.e.d.

<u>Proposition 4</u> C-finite mapping germs are the following ones

(i) finite mapping germs

(ii) mapping germs φ such that $X(\varphi)$ is a complete intersection of dimension n-p, which has only an isolated singularity

Proof: C-finiteness is obviously equivalent to the property, that the jacobian matrix of φ defines a linear map $Q(\varphi)^n \to Q(\varphi)^p$, whose cokernel is concentrated on the maximal ideal, which is equivalent to (i) or (ii).

<u>Remark</u> If $\varphi \in J(n,p)$ is $\mathfrak{E}$-finite, then it must be C-finite, hence φ is finite or $X(\varphi)$ is a complete intersection with an isolated singularity.

If φ is finite and $rg(\varphi) < p$, then necessarily $\deg(\varphi) \leq \frac{p}{p - rg\varphi}$.

If the fibre $X(\varphi)$ is a complete intersection then φ is necessarily represented as a germ of a morphism $U \xrightarrow{\varphi} V$ of smooth algebraic varieties at points $O \in U$, $O' = \varphi(O) \in V$ such that, if $C(\varphi) \subset U$ denotes the critical locus, then $C(\varphi) - \{O\} \to V - \{O\}$ is a closed embedding.

<u>Proof</u> Assume φ is finite of rank $r < p$, that means if L is the quotient field of $\mathcal{O}_n$, then the vector space $\Delta(\varphi) \otimes \mathcal{O}_n L$ has dimension r over L. Let us denote by $K \subset L$ the

quotient field of $\varphi^*\mathcal{O}_p \subset \mathcal{O}_n$, then $\deg(\varphi) = [L : K]$ and $\mathcal{O}_n \otimes_{\varphi^*\mathcal{O}_p} K = L$. Because of the $\mathbb{C}$-finiteness of φ we have $L^p = K^p + \Delta(\varphi)\otimes_{\mathcal{O}_n} L$, hence $p \deg\varphi \leq p + r\cdot\deg\varphi$, i.e. $\deg\varphi \leq \frac{p}{p-r}$.

Proposition 5 If two mapping germs $\varphi, \varphi' \in J(n,p)$ are formally C-equivalent, then they are C-equivalent.

Proof: $\hat{C}$-equivalence means that there exists an isomorphism $\bar{\sigma} : \hat{Q}(\varphi') \xrightarrow{\sim} \hat{Q}(\varphi)$ hence we get a $Q(\varphi)$-homomorphism

$$\bar{u} : Q(\varphi)\langle X\rangle / I(\varphi')Q(\varphi)\langle X\rangle \to \hat{Q}(\varphi)$$

by $\bar{u}(X_j) = \bar{\sigma}(X_j)$. By the approximation theorem for algebraic equations, for any integer c we can find a $Q(\varphi)$-homomorphism

$$u_c : Q(\varphi)\langle X\rangle / I(\varphi')Q(\varphi)\langle X\rangle \to Q(\varphi)$$

such that $u_c \equiv \bar{u} \bmod m_{\hat{Q}(\varphi)}^{c+1}$ and u_c composed with the canonical map $i : Q(\varphi') \to Q(\varphi)\langle X\rangle / I(\varphi')Q(\varphi)\langle X\rangle$, $i(X_j) = X_j \bmod I(\varphi')Q(\varphi)\langle X\rangle$ yields a homomorphism $\sigma_c = u_c \circ i : Q(\varphi') \to Q(\varphi)$ such that $\sigma_c = \bar{\sigma} \bmod m_{\hat{Q}(\varphi)}^{c+1}$. For $c \geq 1$ the homomorphism σ_c is therefore an isomorphism, since, in the same way, we can approximate the inverse isomorphism $\bar{\tau} = \bar{\sigma}^{-1}$ to get a homomorphism τ_c and $\tau_c \circ \sigma_c$ is a homomorphism of $Q(\varphi)$ into itself, which coincides up to order c with the identity and therefore it must be an automorphism by [11] as well as $\sigma_c \circ \tau_c$.

Proposition 6 If $\varphi \in J(n,p)$ is a finite mapping germ and if the mapping germ $\varphi' \in J(n,p)$ is formally equivalent to φ, then φ' is equivalent to φ.

Proof Consider $\varphi^* : \mathcal{O}_p \to \mathcal{O}_n$, which makes $\mathcal{O}_n$ to a finite $\mathcal{O}_p$-algebra and the following functor on the category of Henselian-K-Algebras: For a K-algebra R the elements of $F(R)$

are tripels $(\varepsilon,\sigma,\tau)$, where $\varepsilon : \mathcal{O}_p \to R$ is a K-morphism of local rings $\sigma : \mathcal{O}_p \otimes \mathcal{O}_p \to R$ is a $\mathcal{O}_p$-morphism of local rings ($\otimes$ denotes the Henselian tensor product) i.e. $\sigma(f \otimes g) = \varepsilon(f)\sigma(1 \otimes g)$

$\tau : \mathcal{O}_p \otimes \mathcal{O}_n \to R \otimes_{\mathcal{O}_p} \mathcal{O}_n$ a. $\mathcal{O}_p$-morphism of local rings, i.e.

$$\tau(f\otimes g) = \varphi^*\varepsilon(f)\,\tau(1\otimes g)$$

such that

$$\tau(1\otimes\varphi'^*(g)) = \sigma(1\otimes g) \otimes 1$$

Then F commutes with filtered limits, and if $(\bar{\alpha},\bar{\beta})$ is a formal equivalence of φ' and φ, i.e. $\bar{\alpha}^* \circ \hat{\varphi}'^* = \hat{\varphi}^* \circ \bar{\beta}^*$, (where $\hat{\varphi}^* : \hat{\mathcal{O}}_p \to \hat{\mathcal{O}}_n$ denotes the prolongation of φ^* to the completion), we get an element

$$(\bar{\varepsilon},\bar{\sigma},\bar{\tau}) \in F(\hat{\mathcal{O}}_p)\ ,$$

where $\bar{\varepsilon} : \mathcal{O}_p \to \hat{\mathcal{O}}_p$ is the canonical embedding,

$$\bar{\sigma}(f \otimes g) = f\bar{\beta}^*(g)$$

$$\bar{\tau}(f \otimes g) = \varphi^*(f)\bar{\alpha}^*(g)$$

(here we use the canonical isomorphism $\hat{\mathcal{O}}_p \otimes_{\mathcal{O}_p} \mathcal{O}_n \cong \hat{\mathcal{O}}_n$, which follows from the finiteness of φ)

Hence, by the approximation theorem for algebraic equations, for any integer $c \geq 0$, we can find an element $(\varepsilon_c,\sigma_c,\tau_c) \in F(\mathcal{O}_p)$ which coincides up to order c with $(\bar{\varepsilon},\bar{\sigma},\bar{\tau})$. If $c > 0$, ε_c is an automorphism $\mathcal{O}_p \xrightarrow{\sim} \mathcal{O}_p$ (see [11]) and if we define $\beta_c \in \mathrm{Aut}(\mathbb{A}^p,0))$, $\alpha_c \in \mathrm{Aut}((\mathbb{A}^n,0))$ by

$$\beta^*_c(f) = \sigma_c(1 \otimes f)$$

$$\alpha^*_c(f) = \tau_c(1 \otimes f) \in \mathcal{O}_p \otimes_{\mathcal{O}_p} \mathcal{O}_n = \mathcal{O}_n\ ,$$

then α_c, β_c coincide up to order c with $\bar{\alpha}$, $\bar{\beta}$, hence they are automorphisms. Moreover

$$\alpha^*_C(\varphi'^*(f)) = \varphi^*(\beta^*_C(f)),$$

i.e. $\varphi' \circ \alpha_C = \beta_C \circ \varphi$

<u>Proposition 7</u> If φ is a germ of a function, i.e. $\varphi \in J(n,1)$, which has only an isolated critical point, and if $\varphi' \in J(n,1)$ is a function germ which is formally equivalent to φ, then φ' is equivalent to φ.

We only will sketch the proof: One considers the group $\mathrm{Aut}((\mathbb{A}^n,0)) = R \subset \mathfrak{C}$. acting on $J(n,1)$, and its "tangent space to the orbits" $T(\varphi,R) = m_n\Delta(\varphi)$ similar as for the group $\mathfrak{C}$ or C. If φ has only an isolated critical point, then $T(\varphi,R)$ is open in the Krull topology, i.e. there is a k such that $m_n^{k+1} \subseteq m_n\Delta(\varphi)$.

In the same way as for C (see the next section) one shows that this implies, that φ is formally $(2k+1)$-R-determined, i.e. $\varphi + m^{2k+1}J(n,1) \subset \hat{R}\varphi$ ($\hat{R}$-orbit of φ in $\hat{J}(m,1)$). Now if φ' is formally equivalent to φ, we can replace φ' by an equivalent function φ'' such that $j_{2k+1}(\varphi'') = j_{2k+1}(\varphi)$, therefore the function φ'' is formally R-equivalent to φ, i.e. the

$$\varphi''(\alpha(x)) = \varphi(x)$$

has a formal solution $\bar{\alpha}(x)$. Therefore, by the approximation theorem for algebraic equations it has an algebraic solution $\alpha(x)$ such that $j_1(\alpha) = j_1(\bar{\alpha})$, which implies that $\alpha(x)$ is an automorphism q.e.d.

3. Finitely determined mapping germs

We say that a mapping germ $\varphi \in J(n,p)$ is c-S-determined resp. c-$\hat{S}$-determined (where S is one of the groups $\mathfrak{C}$ or C) if any mapping germ φ' such that $j_c(\varphi') = j_c(\varphi)$ is S - resp. $\hat{S}$ equivalent to φ, i.e. $m_n^c J(n,p) + \varphi \subseteq S_\varphi$ resp. $\subseteq \hat{S}_\varphi$.

Proposition 8 For any integer k there holds:

If a mapping germ $\varphi \in J(n,p)$ is k-C-finite then it is (2k+1)-C-determined.

If the characteristic of K is zero, then "k-C-finite" implies "(k+1)-C-determined)" and "k-$\mathfrak{C}$-finite" implies "c(k)-$\hat{\mathfrak{C}}$-determined" where $c(k) = p^2 \binom{k+p}{p} + k + 1$.

Proof: For $r \geq \nu$ we denote the unipotent algebraic group $\mathrm{Ker}\,(C_r \to C_\nu)$ by C_r^ν.

If $r \leq 2\nu$ and $\gamma = (\alpha(x), \beta(x,y)), \alpha(x) = x + \alpha_o(x), \beta(x,y) = y + \beta_o(x,y)$, the action of C_r^ν on $J_r(n,p)$ is given by

$$\gamma P(x) = P(x) + \frac{\partial P}{\partial x}(x) \cdot \alpha_o(x) + \beta_o(x, P(x)) \qquad (P \in J_r(n,P))$$

where $\frac{\partial P}{\partial x}(x)$ denotes the matrix with columns $\frac{\partial P}{\partial x_i}$ and $\frac{\partial P}{\partial x_i}$ is defined by considering P as a polynomial in x.

Therefore $C_r^\nu P = P + m_n^{\nu+1}\Delta(P) + m_n^\nu I(P) = P + m_n^\nu T_P(CP)$

$$C_\varphi^\nu + m_m^r J = \varphi + m_n^\nu T(\varphi, C) + m_n^r J$$

and if φ is k-finite, this implies

$$C_\varphi^\nu + m_n^r J \supset \varphi + m_n^{\nu+k} J$$

Assume $\varphi' \in J$ and $j_{2k+1}(\varphi') = j_{2k+1}(\varphi)$, i.e. $\varphi' \in \varphi + m_n^{2k+1} J$.

Therefore, by induction we find a sequence $\gamma^1 \in C^{k+1}$, $\gamma^2 \in C^{k+2}$, ..

and

$$j_{2k+a}(\gamma^a \circ \gamma^{a-1} \circ \ldots \circ \gamma^1 \varphi') = j_{2k+1}(\varphi)$$

If $\bar{\gamma} = \lim_{a \to \infty} (\gamma^a \circ \gamma^{a-1} \ldots \gamma^1) \in \hat{C}$, then $\bar{\gamma}\varphi'$ and φ have the same jets for any order, hence $\bar{\gamma}\varphi' = \varphi$.

By the preceding proposition φ' is C-equivalent to φ.

Now, for $r \geq c$ we consider the affin subspace $N \subset J_r$ of all jets P having the c-jet $j_c(\varphi)$. If φ is C-finite and $c \geqslant k+1$, all elements $P \in N$ represent C-finite mapping germs. The same is true for the group $\mathfrak{C}$, if $c \geqslant c(k)$

If G is the group C_{2k+1}, $c = k+1$, then k-C-finiteness of φ implies, for $P \in N$,

$$T_p(GP) \supset m_n^k J_r \geqslant m_n^c J_r = T_p(N)$$

If $G = \mathfrak{C}_r^{\nu}$ $(r \geqslant \nu)$ and $c = \max(c(k), a(k,\nu))$, the k-finiteness of φ implies

$$T_p(GP) \supseteq m_n^c J_r = T_p(N)$$

We will show that this implies $N \subset GP$ for any $P \in N$, provided the ground field K is of characteristic 0.

Hence, for the group C we get : If the mapping germ is k-C-finite and φ' is a mapping germ with $j_{k+1}(\varphi') = j_{k+1}(\varphi)$, then there is a transformation $\gamma \in C$ and $j_{2k+1}(\gamma\varphi') = j_{2k+1}(\varphi)$. Hence, by the first part of the proof, φ' is C-equivalent to φ. By repeated application of the argument we get, for the group $\mathfrak{C}$, chosing

$$c_\nu = \max(c(k), a(k,\nu)) \; r_\nu = c_{\nu+1} \; :$$

If $\varphi \in J(n,p)$ is a k-$\mathcal{G}$-finite mapping germ and $\varphi' \in J(n,p)$ is a mapping germ with $j_{c(k)}(\varphi') = j_{c(k)}(\varphi)$, there exist a sequence of transformations $\gamma^{\nu} \in \mathcal{G}^{\nu}$ and

$$j_{c_{\nu+1}}(\gamma^{\nu} \circ \gamma^{\nu-1} \circ \dots \circ \gamma^{\circ} \varphi') = j_{c_{\nu+1}}(\varphi)$$

If $\bar{\gamma} = \lim_{\nu\to\infty} (\gamma^{\nu} \circ \gamma^{\nu-1} \circ \dots \circ \gamma^{\circ})$, then $\bar{\gamma}\varphi' = \varphi$.

To prove $N \subseteq GP$ we first need the following lemma

<u>Lemma 1</u> If G is an algebraic group acting on smooth varieties X and Y and if $j : X \to Y$ is a smooth equivariant morphism and $N = j^{-1}(Q)$ $(Q \in Y)$, then $T_P(N) \subseteq T_P(GP)$ for all points $P \in N$ implies $\dim(GP) = \dim(GQ) + \dim N$, i.e. all orbits through N have the same dimension.

<u>Proof</u> $T_Q(GQ) = T_P(GP)/T_P(GP) \cap T_P(N) = T_P(GP)/T_P(N)$ if $T_P(N) \subseteq T_P(GP)$

hence $\dim(GP) = \dim T_P(GP) = \dim T_P(N) + \dim T_Q(GQ)$

$= \dim(N) + \dim(GQ)$ q.e.d.

In our case $X = J_r$, $Y = J_c$ and therefore all orbits through N have the same dimension. Now the result $N \subset GP$ follows from

<u>Lemma 2</u> Let V be a quasi-projective algebraic variety, W an algebraic subvariety and G a connected algebraic group acting on V. Assume that for any $Q \in W$ the following conditions are satisfied

(i) $T_Q(W) \subseteq T_Q(GQ)$

(ii) $\dim GQ = d$ independent of Q

Then W is contained in an orbit GQ, provided the ground field K has characteristic 0.

Proof: The proof is reduced to the case where V is a nonsingular variety and W a smooth curve. It is sufficient to prove that any curve on W is contained in an orbit, because the intersection of W with an arbitrary orbit is closed by (ii), and if W is not contained in an orbit, there would also exist a curve on W not contained in an orbit.
Hence we may assume that W is a curve. Now we can replace V by the smallest subvariety $V_1 \subset V$ containing W which is G-stable. Therefore, any G-stable open subvariety $U_1 \subset V_1$ has a nonempty intersection with W, and if $W \cap U_1$ is contained in an orbit then W is contained in the same orbit, too.
Hence we can replace V_1 by its nonsingular locus, moreover, we can assume that any orbit meets W (the union of orbits intersecting W contains a G-stable open subvariety), and only finite many orbits intersect W in a singular point (because of dim W = 1). Taking the complementary set of this orbits, we are in the case described above.
If W is not contained in an orbit, we will show that condition (i) cannot be satisfied (assuming V, W nonsingular, dim W = 1). Any orbit meets W in a finite set of points, hence dim V = d+1 if d is the dimension of the orbits. The variety V is contained in some projective space and if B is a sufficiently general section of V with a linear subspace of codimension d, then B will be a nonsingular curve on V, and almost all orbits GQ, $Q \in W$, will transversally intersect B (by Bertinis theorem).

If $\tilde{V} \subset V \times B$ denotes the closure of the image of $G \times B \to V \times B$, $(g,Q) \to (gQ,Q)$ and $p : \tilde{V} \to B$, $q : \tilde{V} \to V$ the projections, then q will be etal over a nonempty open set $U \subset V$ and almost all fibres of p have the form $GQ \times \{Q\}, Q \in B$. If $\tilde{W} \subset \tilde{V}$ is the inverse image of W, then property (i) would imply that the restriction $\pi = p/\tilde{W}$ would have a zero tangent map, hence it would be a constant map and therefore W would be contained in an orbit.

<u>Remark</u> This lemma is not true in positive characteristic as one can see by the following example: $V = \mathbb{A}^2$, $G = \mathbb{G}_a$ acting on V by $(t,(x,y)) \to (x+t, y+t^p)$.
The orbits are the curves $x^p - y = \text{const}$, hence the line $W : y = 0$ has property (i) and (ii), but is not contained in an orbit.

4. <u>Unfoldings</u>

If φ is a mapping germ $(\mathbb{A}^n, 0) \to (\mathbb{A}^p, 0)$, by an unfolding of φ over a germ T, we understand a mapping germ

$$\Phi : (\mathbb{A}^n \times T, 0) \to (\mathbb{A}^p \times T, 0)$$

$$\Phi(x,t) = (\Phi_1(x,t), t)$$

such that $\Phi_1(x,0) = \varphi(x)$

Two unfoldings Φ, Φ' over $(T,0)$ are called equivalent, if

$$\Phi' = L \circ \Phi \circ R^{-1}$$

where R is an unfolding of the idendity of $(\mathbb{A}^n, 0)$ and L an unfolding of the idendity of $(\mathbb{A}^p, 0)$ (i.e. $L(y,t) = (L_1(y,t), t)$, $L_1(y,0) = y$ and $R(x,t) = (R_1(x,t), t)$, $R_1(x,0) = x$), equivalently

$$\Phi'_1(R_1(x,t),t) = L_1(\Phi_1(x,t),t)$$

If $T = \mathrm{Spec}(A)$, where A is a local (Henselian) K-algebra with residua field K, an unfolding of φ is given by a p-tupel of functions

$$\Phi \in A\langle X\rangle^p \qquad (X = (X_1,\ldots,X_n))$$

which reduces $\bmod\ m_A\langle X\rangle^p$ to the p-tupel φ.

<u>Example</u> Any mapping germ $\varphi \in J(n,p)$ of rank r (i.e. the rank of the linear part of φ) is equivalent to an unfolding of a mapping germ $\varphi_o \in J(n-r,p-r)$ of rank 0.

If the components $\varphi_{p-r+1},\ldots,\varphi_p$ are such that their differentials $d\varphi_{p-r+1},\ldots,d\varphi_p$ are linearly independent, we can find coordinates $(x_1,\ldots,x_{n-r},t_1,\ldots,t_r)$ on $(\mathbb{A}^n,0)$ in such a way that $\varphi_{p-r+1} = t_1,\ldots,\varphi_p = t_r$ and on $(\mathbb{A}^p,0)$ in such a way that

$$\varphi_i = \varphi_{oi}(x_1,\ldots,x_{n-r}) + \sum_{j=r+1}^{n} \psi_{ij}(x,t)t_j$$

for $i = 1,\ldots,p-r$, hence after this coordinate transformations φ becomes an unfolding of the mapping germ

$$\varphi_o(x_1,\ldots,x_{n-r}) = \begin{pmatrix} \varphi_{o1}(x_1,\ldots,x_{n-r}) \\ \varphi_{op-r}(x_1,\ldots,x_{n-r}) \end{pmatrix}$$

Consider an unfolding Φ over T of a mapping germ φ.
If $f : S \to T$ is a morphism, we get an unfolding over S induced by f as follows:

$$f^*\Phi(x,S) = (\Phi_1(x,f(s)),s)$$

The unfolding Φ is called <u>versal</u> if to any unfolding of φ,

$$\psi : (\mathbb{A}^n \times S,0) \to (\mathbb{A}^p \times S,0)$$

and to any mapping germ $\bar{f} : \bar{S} \to T$, where $\bar{S} \subset S$ is a zero-dimensional closed subgerm, such that $\bar{f}^*\Phi$ is equivalent to the restriction $\Psi|\mathbb{A}^n\times\bar{S}$, there exists a prolongation $f : S \to T$ of $\bar{f}$, such that Ψ is equivalent to $f^*\Phi$.

If, moreover, the tangent map

$$T_o(f) : T_o(S) \to T_o(T)$$

is uniquely determined by Ψ, then Φ is called <u>semiuniversal</u> or <u>miniversal</u>.

If this property holds only for arbitrary 0-dimensional germs S, Φ is called <u>formally versal</u> resp. <u>formally semiuniversal</u>.

Assume the unfolding Φ is (formally) semiuniversal and consider the germ $I_1 = \mathrm{Spec}(K(t)/(t^2))$. If Ψ is an unfolding over I_1, it corresponds to a linear map $T_o(I_1) = K \to T_o(T)$, hence to a tangent vector τ of T at 0. Conversely the tangent vectors of T at 0 are in (1,1)-correspondence with morphisms $I_1 \xrightarrow{f} T$ because any f is given by a linear map

$$f^* : \mathrm{Hom}(m_{T,O}/m_{T,O}^2, K)$$

$$f^*(b_1t_1+\ldots+b_st_s+\ldots) = b_1f^*(t_1)+\ldots+b_sf^*(t_s),\quad f^*(t_i) = c_it$$

Hence $T_o(T)$ is in (1,1)-correspondence with the set of equivalence classes of unfoldings of φ over I_1.

An unfolding of φ over I_1 has the form

$$\Phi_1(x,t) = \varphi(x) + tg(x)$$

where $g(x)$ is an arbitrary vector of (algebraic) power series.

Two such vectors g, g' determine equivalent unfoldings if and only if

$$\varphi(x) + tg'(x) = \varphi(x) + t\left(\sum_{i=1}^{n} r_i(x)\frac{\partial\varphi}{\partial x_i} + g(x) + \lambda(\varphi(x))\right)$$

where $r(x)$ is a vector of power series with components $r_i(x)$,

$\lambda(y)$ a vector of power series with p components

(i.e. $\Phi' = L\circ\Phi\circ R$

$L(y,t) = y + t\lambda(y), R(x,t) = x + tr(x)$)

Hence g and g' define equivalent unfoldings if and only if

$$g - g' \epsilon \sum_{i=1}^{n} \mathcal{O}_n \frac{\partial\varphi}{\partial x_i} + \varphi^*(\mathcal{O}_p)^p = \Delta(\varphi) + \varphi^*(\mathcal{O}_p)^p$$

Consequently we get the following isomorphism

$$T_O(T) \cong \mathcal{O}_n^p/\Delta(\varphi) + \varphi^*\mathcal{O}_p^p \cong J(n,p)/(\Delta(\varphi) \cap J(n,p)) + \varphi^*J(p,p)$$

Remark If $\dim T_O(T) < \infty$, then

$T_O(T) \cong \hat{\mathcal{O}}_n^p/\hat{\mathcal{O}}_n\Delta(\varphi) + \varphi^*\hat{\mathcal{O}}_p^p$ (by Prop. 1), hence there is

no difference between the formal and the algebraic case.

Assume that this space is of finite dimension; if

$g^1,\ldots,g^\tau \epsilon\, m_n \mathcal{O}^p$ represent a base of this space, then

$$\Phi(x,t) = (\varphi(x) + t_1g^1(x)+\ldots+t_\tau g^\tau(x), t_1,\ldots,t_\tau)$$

should be a candidate for a semiuniversal unfolding.

Examples 1) $\varphi : (A^1,O) \to (A^1,O)$

$$\varphi(x) = x^{n+1}$$

$$K\langle x\rangle/K\langle x\rangle x^n + K\langle x^{n+1}\rangle =: Kx+\ldots+Kx^{n-1}$$

hence

$$\Phi(x,t) = (x^{n+1} + t_1x^{n-1}+\ldots+t_{n-2}x, t_1,\ldots,t_{n-1})$$

is the semiuniversal unfolding of φ

2) $\varphi : (A^2,O) \to (A^1,O)$ $\quad \varphi(x_1,x_2) = x_1x_2$

$$K\langle x_1,x_2\rangle/K\langle x_1,x_2\rangle x_1 + K\langle x_1,x_2\rangle x_2 + K\langle x_1,x_2\rangle = O$$

hence the semiuniversal unfolding is trivial.

3) $\varphi : (\mathbb{A}^2,0) \to (\mathbb{A}^1,0) \quad \varphi(x_1,x_2) = x_1^2 - x_2^3$

$$K\langle x_1,x_2\rangle / K\langle x_1,x_2\rangle x_1 + K\langle x_1,x_2\rangle x_2^2 + K\langle x_1^2 - x_2^3\rangle = Kx_2$$

$$\Phi(x_1,x_2,t) = x_1^2 - x_2^3 + tx_2$$

<u>Proposition 9</u> Let $g^1(x),\ldots, g^\tau(x) \in \mathcal{J}(n,p)$ represent a base of the vector space $T_o(\varphi) = \mathcal{O}_n^p / \Delta(\varphi) + \varphi^* \mathcal{O}_p^p$.
The unfolding $\phi(x,t) = (\varphi(x) + t_1 g^1(x) + \ldots, \ldots t_\tau g^\tau(x))$ is formally semiuniversal.

<u>Proof</u> The proof is deduced from Schlessinger's criterium [15].
Consider the functor (R local Artin-k-algebra with residual field K)

$$D(R) = \left\{ \begin{array}{l} \text{set of all unfoldings of} \\ \varphi \quad \text{over} \quad \mathrm{Spec}(R) \end{array} \right\} \Big/ \{\text{equivalence}\}$$

It satisfies Schlessinger's conditions, hence there exists a formal semiuniversal unfolding, i.e. a complete local K-algebra $\bar{A}$ with residual field K and for any ν on unfolding Φ_ν over $\mathrm{Spec}((\bar{A})/m_A^{\nu+1})$ such that $\Phi_{\nu+1} \bmod m_{\bar{A}}^{\nu+1} = \Phi_\nu$ and any infinitesimal unfolding is induced from some Φ_ν with a unique tangent map.

Moreover: By construction of Schlessinger

$$\Phi_1 = \Phi(z,t) \bmod (t_1,\ldots,t_\tau)^2$$

and $\bar{A} = K[\![t_1,\ldots,t_\tau]\!]/\text{some ideal } I$

$$I \subseteq (t_1,\ldots,t_\tau)^2$$

But because for $\Phi(x,t)$ there are no relations among the t and Φ is $\bmod (t)^{\nu+1}$ induced by a morphism

$f_\nu : \bar{A} \to K[\![t_1,\ldots,t_\tau]\!]/(t_1,\ldots,t_\tau)^{\nu+1}$ from some Φ_μ with uniquely determined tangent map $T_o(f_\nu) = 1$, we infer $\bar{A} \simeq K[\![t_1,\ldots,t_\tau]\!]$ and $\Phi \bmod (t)^{\nu+1}$ is equivalent to Φ_ν, hence Φ is formally semiuniversal.

Thus it is clear that the condition

(*) $\dim_K(J(n,p)/(\Delta(\varphi)\cap J(n,p)) + \varphi^*J(p,p)) < \infty$

is necessary for the existence of a semiuniversal unfolding, and if there exists a semiuniversal unfolding of φ, then

$$\Phi(x,t) = (\varphi(x) + \sum_{\nu=1}^{\tau} t_\nu g^\nu(x),\ t_1,\dots,t_\tau)$$

represents one (if $g^1(x),\dots,g^\tau(x) \in J(n,p)$ represents a base mod$((\Delta(\varphi)\cap J(n,p)) + \varphi^*J(p,p))$), and any semiuniversal unfolding is (in a non-canonical way) equivalent to $\Phi(x,t)$. The condition (*) is obviously equivalent to the property

(**) φ is $\mathfrak{E}$-finite.

Before we consider the question of existence of semiuniversal unfoldings, we will consider deformations of germs of schemes.

5. Deformations

Let V be a variety or more general an algebraic scheme or a germ of an algebraic scheme. By a deformation of V over a germ T we understand, as usual, a flat morphism $X \to T$ of germs of algebraic schemes together with an isomorphism $V \xrightarrow{\sim} X_o$ = special fibre of X over T.

Equivalence of deformations over T is defined in an obvious way : X,X' are defined to be equivalent if there exists a T-isomorphism $X \xrightarrow{\sim} X'$ (of germs) which moreover is compatible with the isomorphisms $V \simeq X_o$ and $V \simeq X_o'$. The pull back with respect to a map $f : S \to T$ induces a deformation $f^*X \to S$ of V over S. The deformation is called semiuniversal if, to any other deformation $Y \to S$ of V and to any mapping germ $\bar{f} : \bar{S} \to T$

of a zero-dimensional closed subgerm $\bar{S} \subset S$ such that $\bar{f}^*X \cong Y/\bar{S}$ (as deformations), there exists a prolongation $f : S \to T$ such that $f^*X \cong Y$ (an isomorphism of deformations) and such that the tangent map $T_o(f)$ is uniquely determined by $(Y \to S, V \cong Y_o)$.

Now, concerning the existence of semiuniversal deformations, the following facts which are consequences of the flatness should be observed:

We can think of V as the special fibre of a mapping germ

$$\varphi : (\mathbb{A}^n,0) \to (\mathbb{A}^p,0)$$

(if V is embedded into $(\mathbb{A}^n,0)$ and defined by p equations).

Then

(i) Any deformation $X \to T$ is embedded into $(\mathbb{A}^n,0) \times T$ and defined by a mapping germ

$$\Phi_1 : (\mathbb{A}^n,0)\times T \to (\mathbb{A}^p,0)$$

such that $\Phi_1(x,o) = \varphi(x)$, $X = \Phi_1^{-1}(0)$ and $V \cong X_o$ is induced by the embedding into $\mathbb{A}^n \times T$.

(ii) If $\varrho(x) = (\varrho_1(x),\dots,\varrho_p(x))$ is a relation of φ i.e. $\varrho \cdot \varphi \equiv 0$, then $\varrho(x)$ can be prolongated to a relation $P(x,t)$ of $\Phi_1(x,t)$:

$$P(x,t)\cdot\Phi_1(x,t) \equiv 0, \quad P(x,0) = \varrho(x)$$

(which is, in fact, equivalent to the property of $\Phi_1^{-1}(0)$ to be flat over T).

If $\Phi_1' : (\mathbb{A}^n,0)\times T \to (\mathbb{A}^p,0)$ is an other map with the property $\Phi_1'(x,0) = \varphi(x)$, then $X' = \Phi_1'^{-1}(0)$ defines a deformation equivalent to X if and only if there is a T-isomorphism $\sigma : X' \cong X$ inducing the idendity on $X'_o = X_o \subseteq (\mathbb{A}^n,0)$. The isomorphism σ can be prolongated to an T-isomorphism $R : (\mathbb{A}^n,0)\times T \simeq (\mathbb{A}^n,0)\times T$, such that $R(x,0) = (x,0)$ and $\Phi_1 \circ R$ generates the same ideal as Φ'. Using property (ii) we see that this is equivalent to the existence of a map $L(x,y,t)$ such that

(1) $L(x,0,t) = 0$

(2) $L(x,y,0) = y$

(3) $L(x,\Phi_1'(x,t),t) = \Phi_1(R(x,t))$

Furthermore, the property (ii) (flatness) is equivalent to

(4) If $\Phi_1(x,t) = \varphi(x) + F_1(x,t)$, where $F_1(x,0) = 0$, then for any relation $\varrho\cdot\varphi = 0$ there holds $\varrho . F_1 \in m_T I(\Phi_1)$

(because the elements of $m_T I(\Phi_1)$ have the form $Q(x,t)\cdot\Phi_1(x,t)$, $Q(x,t) = (Q_1(x,t),\dots,Q_p(x,t)), Q(x,0)=0$; but $\varrho . F_1 = Q(x,t)\cdot\Phi_1(x,t)$ is equivalent to $(\varrho(x) + Q(x,t))\cdot\Phi_1(x,t) = 0$ because of $\varrho\cdot\varphi = 0$, thus $P(x,t) = \varrho(x) + Q(x,t)$ is a lifting of the relation $\varrho(x)$).

If $T = \operatorname{Spec}(K[t]/(t^2))$, we get, by (1) ... (4), a description of the tangent space to a semiuniversal deformation (if it exists), it must be isomorphic to

$$T_o \cong \operatorname{Hom}(I(\varphi),\mathcal{O}_V)/\Delta(\varphi;V) \subseteq \mathcal{O}_n{}^p/I(\varphi)^{\oplus p} + \Delta(\varphi)$$

To $\Phi_1(x,t) = \varphi(x) + tF(x)$ we associate the map $I(\varphi) \to \mathcal{O}_V$ given by $\sum_{\nu=1}^{p} f_\nu\varphi_\nu \to \sum_{\nu=1}^{p} f_\nu F_\nu|V$ (which is well defined by (4)), from (3) we infer that two such maps define equivalent deformations if and only if their difference is contained in $\Delta(\varphi,V) = \Delta(\varphi)|V$. There are different possibilities to prove the existence of a semiuniversal deformation, provided $\dim T_0 < \infty$. I will describe one of them (which was suggested by a conversation with B. Tessier).

By a result of Gruson and Raynaud [10], to any algebraic scheme $\tilde{X}$ over a germ T there exists a closed subgerm $T \subset T$ which represents the functor

$$F(S) = \{f : S \to T;\ f^*\tilde{X} \to S \text{ flat}\}$$

This result we will apply to our situation.

Proposition 10 Let $\varphi \in \mathcal{J}(n,p)$ be a mapping germ, $V = X(\varphi) \subseteq (\mathbb{A}^n, 0)$, and assume

$$\dim_K(\mathcal{O}_n^{\,p}/\Delta(\varphi) + I(\varphi)^{\oplus p}) < \infty .$$

Let $g^1(x), \ldots, g^r(x) \in \mathcal{O}_n^{\,p}$ represent a base of the vector space $\mathcal{O}_n^{\,p}/\Delta(\varphi) + I(\varphi)^{\oplus p}$ and $\tilde{X} \subset (\mathbb{A}^n \times \mathbb{A}^r, 0)$ the germ defined by

$$\varphi(x) + \sum_{\varrho=1}^{r} t_\varrho g^\varrho(x) = 0 .$$

If $T \subset (\mathbb{A}^r, 0)$ is the closed subgerm which represents the functor

$$S \mapsto \{f : S \to (\mathbb{A}^r, 0);\ f^*\tilde{X} \to S \text{ flat}\} ,$$

then $X = \tilde{X} \times_{(\mathbb{A}^r,0)} T \to T$ together with the canonical isomorphism $V \simeq X_0$ is a deformation of V, which is formally semiuniversal.

Proof Consider unfoldings $\phi : (\mathbb{A}^n \times S, 0) \to (\mathbb{A}^p \times S, 0)$ of φ and define equivalence of unfoldings in the following sense: Unfoldings $\phi, \phi' : (\mathbb{A}^n \times S, 0) \to (\mathbb{A}^p \times S, 0)$ are called equivalent if there exists an unfolding $R(x,s)$ of the identity of $(\mathbb{A}^n, 0)$ and an unfolding $L(x,y,s)$ of the identity of $(\mathbb{A}^n \times \mathbb{A}^p, 0)$ such that $L(x,y,s) = (x, L_1(x,y,s))$ and $L_1(x,0,s) = 0$.

By $F(S)$ we will denote the set of all equivalence classes of unfoldings of φ.

Then $S \mapsto F(S)$ is a cofunctor on the category of germs.
By the properties (1) - (4) above, if D denotes the cofunctor $D(S)$ = set of equivalence classes of deformations of V over S, then D is a subfunctor. It is easy to check that F satisfies the Schlessinger's criterion and
$\Phi(x,t) = \varphi(x) + \sum_{\varrho=1}^{r} t_\varrho g^\varrho(x)$ represents a formally semiuniversal element of F.
Since $D \subset F$ and by the definition of $T \subseteq (A^r,0)$ it is evident that $X \to T$ (defined above) is formally semiuniversal.

6. Semiuniversality of deformations and unfoldings

In both cases, unfoldings and deformations, we have a cofunctor on the category of germs,

$D : S \to$ {classes of unfoldings of φ over S} resp.

$S \to$ {classes of deformations of V over S}, and if the vector-space $J(n,p)/\Delta(\varphi) \cap J(n,p) + \varphi^*J(p,p) = T_\varphi$ resp.
$Hom(I(\varphi),\mathcal{O}_V)/\Delta(\varphi),V) = T_V$ (if $V = X(\varphi)$) is of finite dimension, a germ T and a class $\xi \in D(T)$ were constructed such that the corresponding natural transformation

$\hat{\xi} : T \to D \quad (f : S \to T \mapsto f^*(\xi) \in D(S))$

has the following properties:

(a) $\hat{\xi}$ is formally smooth

(b) $\hat{\xi}$ induces an isomorphism

$T_o(T) = Hom(Spec(K[t]/(t^2)),T) \to D(Spec(K[t]/(t^2)))$

(and $T_o(T) \simeq T_\varphi$ resp. $\simeq T_V$)

Recall that a natural transformation $v : F \to F'$ of cofunctors F,F' on the category of germs is called formally smooth if for any zero-dimensional germ S and any closed subgerm $\bar{S} \subset S$ the canonical map

$$F(S) \to F'(S) \times_{F'(\bar{S})} F(\bar{S}), \quad \eta \mapsto (v(\eta), \eta(\bar{S}))$$

is bijective. This property is stable under pull back along any natural transformation $G' \to F'$ and if F, F' are representable, it is equivalent to the fact, that the Jacobian of v at 0 has the rank equal to the dimension of the target germ F'.

The cofunctor D can also be considered as a functor on the category H_K on all local Henselian Noetherian K-algebras with residual field K.

It is easy to prove

Proposition 11 Assume D is a functor on the category H_K into the category of sets, such that there exist a finitely generated Henselian K-algebra $A \in H_K$ and an $\xi \in D(A)$ such that

(i) $\hat{\xi} : T = \mathrm{Spec}(A) \to D$ is formally smooth

(ii) The canonical map $\varinjlim D(B_\alpha) \to D(\varinjlim B_\alpha)$ is bijective for filtered limits $B = \varinjlim B_\alpha$

(iii) The canonical map $D(B) \to \varprojlim D(B/m^{\nu+1})$ is injective for algebras $B \in H_K$ of finite type.

Then, if B is an algebra from H_K, $\bar{f} : \mathrm{Spec}(A) \to \bar{S}$ is a morphism into a zero-dimensional closed subscheme $\bar{S}$ of $S = \mathrm{Spec}(B)$ and $\eta \in D(S)$ such that $\eta|\bar{S} = \bar{f}^* \xi$, there exists a prolongation $f : S \to T$ of $\bar{f}$ such that $\eta = f^* \xi$

Proof Because of (ii) we can assume that B is of finite type.

Let ν be big enough such that $\bar{S} \subset S_\nu = \mathrm{Spec}(B/m_B^{\nu+1}) \subset S_{\nu+1} = \mathrm{Spec}(B/m_B^{\nu+2}) \subset \ldots$ By (i) the morphism $\bar{f}$ extends to a map $\hat{f} : \hat{S} = \mathrm{Spec}(\hat{B}) \to T$ such that $\hat{f}^*\xi | S_\nu = \eta | S_\nu$. By (iii), therefore $\hat{f}^*\xi = i^*\eta$, if $i : \hat{S} \to S$ is the canonical morphism. Therefore $\hat{\sigma} = (\hat{f}, i) : S \to T \times_D S$ is a natural transformation (we identify schemes with the corresponding functors by the Yoneda embedding) $\hat{\sigma}/\bar{S} = (\bar{f}, \bar{i}) \underset{\text{def.}}{=} \bar{\sigma}$ and the diagram

$$\begin{array}{ccc} T \times_D S & \xrightarrow{\;q\;} & S \quad \text{(projection)} \\ & {\scriptstyle \hat{\sigma}} \nwarrow & \uparrow {\scriptstyle i} \\ & & \hat{S} \end{array}$$

is commutative. By (ii) the functor $T \times_D S$ commutes with filtered inductive limits, hence we can apply the approximation property for algebraic equations: for any integer c there exists a natural transformation $\sigma_c : S \to T \times_D S$ such that $\sigma_c | S_c = \hat{\sigma} | S_c$. We choose $c \geqslant 1$ and such that $S_c \supset \bar{S}$. Then the morphism $\tau_c = q \circ \sigma_c : S \to S$ induces the identity on S_c. By [11], proposition 3.4.5., page 92, the morphism τ_c is then an automorphism and $\sigma = \sigma_c \circ \tau_c^{-1}$ is a section of q, prolongating $\bar{\sigma}$. Thus the first component of σ is a morphism $f : S \to T$, prolongating $\bar{f}$ and such that $f^*\xi = \eta$ q.e.d.

In the cases of unfoldings and deformations (assuming T_φ resp. T_V are of finite dimension) the conditions (i) and (ii) are satisfied, the crucial property is (iii), which in this case means obviously the following: If unfoldings (resp. deformations) are formally equivalent (resp. formally isomorphic), then they are equivalent (resp. isomorphic).

<u>Proposition 12</u> Let V be an algebraic germ over K such that T_V has finite dimension. If V has only an isolated singularity or if K is of characteristic 0, then it has a semiuniversal deformation.

Proof: We have to show that two deformations X, X' over $S = \mathrm{Spec}(A)$ (A a Henselian Noetherian K-algebra), which are formally isomorphic, are also isomorphic.

We can assume that X and X' are given by unfoldings $\Phi, \Phi' \in A\langle x_1,\dots,x_n\rangle^p$, formal equivalence means: If

$$\hat{A}\{x\} = \hat{A}\{x_1,\dots,x_n\} = \varprojlim_{\nu} (A\langle x_1,\dots,x_n\rangle / m^{\nu} A\langle x_1,\dots,x_n\rangle)$$

then

$$\Phi'(\bar{\alpha}(x)) = \bar{\beta}(x, \Phi(x))$$

where $\bar{\alpha}(x) \in x + m_A \bar{A}\{x\}^n$

$\bar{\beta}(x,y) \in y + m_A \hat{A}\{x,y\}^p, \quad y = (y_1,\dots,y_p)$

$\bar{\beta}(x,0) = 0$

This equation is equivalent to an equation

$$\bar{\beta}(x,y) = \Phi'(\bar{\alpha}(x)) + \bar{G}(x,y)(y - \Phi(x))$$

and the formal solution $\bar{\alpha}(x), \bar{\beta}(x,y), \bar{G}(x,y)$ (with components in $\hat{A}\{x,y\} \subset \hat{A}[\![x,y]\!] = K[\![t,x,y]\!]$ if $A = K\langle t\rangle$) can be approximated by algebraic solutions up to arbitrary order by [12]; if K has characteristic 0.

The case of isolated singularities goes back to R. Elkik. One uses the following result of Elkik: If R is a Noetherian ring, which is Henselian along a closed subset $V = V(I) \subset \mathrm{Spec}(R)$ and if $Y \to \mathrm{Spec}(R)$ is a quasiprojective R-scheme which is smooth over $\mathrm{Spec}(R)$ outside a closed subset $W \subset Y$, then any R-morphism $\bar{\varepsilon} : \mathrm{Spec}(\hat{R}) \to Y$ such that $\bar{\varepsilon}(\mathrm{Spec}(\hat{R}) - V) \subset Y - W$

($\hat{R}$ denotes the I-adic completion), then for any integer $c \geq 0$ there exists a section $\varepsilon_c : \mathrm{Spec}(R) \to Y$ such that $\varepsilon_c = \bar{\varepsilon}$ on $\mathrm{Spec}\,(R/I^{c+1})$. (see [7] or [12]).

It can be applied to $R = \hat{A}\langle x\rangle/I(\Phi)$ and $Y = \mathrm{Spec}(R\langle x'\rangle/I(\Phi')R\langle x'\rangle)$ (which is a filtered colimit of affine schemes of finite type over B). The formal isomorphy of the deformations means that there is an isomorphism

$$\bar{\sigma}\colon \hat{A}\{x'\}/I(\Phi')\hat{A}\{x'\} \simeq \hat{A}\{x\}/I(\Phi)\hat{A}\{x\}$$

inducing the identity mod m_A).

If $H' \subset R' = \hat{A}\langle x'\rangle I(\Phi')\hat{A}\langle x'\rangle$ is an ideal defining the critical locus of $\mathrm{Spec}(R')$ over $\mathrm{Spec}(\hat{A})$, then r'/H'' is finite over $\hat{A}$, because it is quasi-finite by the assumption that $V = \mathrm{Spec}(R'/m_A R')$ has only an isolated singularity. (For Henselian rings, quasi-finite implies finite, by Zariski's main theorem, see for example [11]).

Therefore the quotient $\hat{R}'/H'\hat{R}$ is isomorphic to R'/H' (because it is the completion of R'/H', but R'/H' is already complete); if $H = R \cap \bar{\sigma}(H'\hat{R}')$, then $R/H \subset \hat{R}/\bar{\sigma}(H'\hat{R}') \simeq R'/H'$ is finite over $\hat{A}$, hence complete, hence $R/H \simeq \hat{R}/\sigma(H'\hat{R}')$, and H defines the critical locus of $\mathrm{Spec}(R)$ over $\hat{A}$.

We apply Elkik's theorem to the ideal $I = m_A H$, the ring R is Henselian with respect to $V(I)$ and because all quotients R/I^{ν} are finite over $\hat{A}$, hence complete, the I-adic completion is $\hat{R} = \hat{A}\{x\}/I(\Phi)\hat{A}\{x\}$.

The ideal $H'\mathcal{O}_Y$ defines the critical locus W of Y over $\mathrm{Spec}(R)$ (because Y is defined by the same equations over R

as $\mathrm{Spec}(R')$ over $\hat{A}$). By $\bar{\varepsilon}(x') = \bar{\sigma}(x')$ we get a R-morphism $\mathrm{Spec}(\hat{R}) \to Y$ and $\bar{\varepsilon}^*(H'\mathcal{O}_Y) = H\hat{R} > m_A H\hat{R} = I\hat{R}$, hence $\bar{\varepsilon}(\mathrm{Spec}(\hat{R}) - V) \subset Y - W$. Let $\varepsilon : \mathrm{Spec}(R) \to Y$ be a section which coincides on $\mathrm{Spec}(R/I^2)$ with $\bar{\varepsilon}$. If $q : Y \to \mathrm{Spec}(R')$ denotes the canonical projection (given by the embedding $R' \subset R\langle x'\rangle/I(\Phi')R\langle x'\rangle$), then $q \circ \varepsilon : \mathrm{Spec}(R) \to \mathrm{Spec}(R')$ is an $\hat{A}$-morphism which is mod I^2 an isomorphism, hence by [11] it is an $\hat{A}$-isomorphism $\sigma : \hat{A}\langle x'\rangle/I(\Phi')\hat{A}\langle x'\rangle \simeq \hat{A}\langle x\rangle/I(\Phi)\hat{A}\langle x\rangle$. Now by the approximation theorem it can be approximated by an A-isomorphism $A\langle x'\rangle/I(\Phi') \simeq A\langle x\rangle/I(\Phi)$, which induces the identity mod m_A, hence by an isomorphism of deformations q.e.d.

7. Stability and minimal stable unfoldings

A mapping germ $\varphi \in J(n,p)$ is called stable, if any unfolding of φ is equivalent to the constant unfolding over the same parameter germ (at least formally, later we will see that this implies also algebraic equivalence). Using the (formally) semiuniversal unfolding, we see that stability of a mapping germ is characterized by $T^1_\varphi = 0$, i.e. by

$$\mathcal{O}^p_n = \Delta(\varphi) + \varphi^*\mathcal{O}^p_p$$

or also by

$$J(n,p) \subseteq \Delta(\varphi) + \varphi^*J(p,p) + K^p$$

Here are some properties of stable mappings

Proposition 13 Let $\varphi \in J(n,p)$ be a mapping germ, $q \geq p$

(1) φ stable $\Longleftrightarrow$ $J(n,p) \subset \Delta(\varphi) + I(\varphi)^{\oplus p} + m^q_n J(n,p) + K^p$ (hence stability depends only on the (p+1) jet of a mapping germ)

(2) If $n \geq p$ and φ stable, then φ is flat with finite critical locus over $(\mathbb{A}^p, 0)$

(3) If $n < p$ and φ is stable, then φ is finite and $\dim_K Q(\varphi) < \frac{p}{p-n}$ or φ is a closed embedding and $n=0$.

Proof (1) If $M = \mathcal{O}^p_n/\Delta(\varphi) \supset P = \varphi^*\mathcal{O}^p_p + \Delta(\varphi)/\Delta(\varphi)$, then $P = M$ if and only if the canonical map $\bar{P} = P/m_p P \to M/m_p M = \bar{M}$ is surjective, and because of $\dim_K \bar{P} \leq p$ this is equivalent to the surjectivity of

$$\bar{P} \to \bar{M}/m_n^{q+1}\bar{M} = \mathcal{O}_n^p/\Delta(\varphi) + I(\varphi)^{\oplus p} + m_n^q J(n,p),$$

hence stability is equivalent to

$$\mathcal{O}_n^p = \varphi^*\mathcal{O}_p^p + \Delta(\varphi) + I(\varphi)^{\oplus p} + m_n^q J(n,p) = K^p + \Delta(\varphi) + I(\varphi)^{\oplus p} + m_n^q J(n,p)$$

Property (2) is obvious, because stability implies that the image of the Jacobian $Q(\varphi)^n \to Q(\varphi)^p$ contains $m_n Q(\varphi)^p$, hence the Jacobian of φ is surjective except at the origin, hence if $n > p$, then $\dim X(\varphi) = n-p$ and it has only an isolated singularity, which means that φ is flat with finite critical locus.

If $n = p$, then φ must be finite, hence flat (because both local rings are regular).

To prove (3) we consider again the cokernel, say C, of the Jacobian $Q(\varphi)^n \to Q(\varphi)^p$. Because of $n < p$ and $\dim_K(C) \leqslant p$ we see that $Q(\varphi)$ must be finite and $\dim(C) \geqslant (p-n)\dim_K Q(\varphi)$.

In case of equality we must have $\dim(C) = p$ and the Jacobian must be injective, hence φ must be of rank n and therefore a closed embedding, $\dim_K(Q) = 1 = \frac{p}{p-n}$, $n = 0$.

<u>Corollary</u> If $p \geqslant 2n$ and $\varphi \in J(n,p)$ is stable, then φ is a closed embedding.

This is clear because of $\dim_K(Q(\varphi)) < \frac{p}{p-n} = 1 + \frac{n}{p-n} \leqslant 2$ or $n = 0$.

<u>Proposition 14</u> Let $\varphi \in J(n,p)$ be a mapping germ of rank 0 and $(\Phi(x,t),t) \in J(n+r,\ p+r)$ be an unfolding of φ, then the following statements are equivalent

(i) $(\Phi(x,t),t)$ is stable

(ii) For some $c \geqslant r$ the vector space

$$J(n,p)/\Delta(\varphi) + I(\varphi)^{\oplus p} + m^{p+c} J(n,p)$$

is generated by the classes of the vectors

$$\frac{\partial\Phi}{\partial t_1}(x,0) - \frac{\partial\Phi}{\partial t_1}(0,0), \dots, \frac{\partial\Phi}{\partial t_r}(x,0) - \frac{\partial\Phi}{\partial t_r}(0,0)$$

Proof: (i) $\rightarrow$ (ii): If $(\Phi,t) = \Psi$ is stable, then $\mathcal{O}^{p+r}_{n+r} = \Psi^* \mathcal{O}^{p+r}_{p+r} + \Delta(\Psi)$, hence for any germ $\psi(x) \in J(n,p)$ there exists a relation

$$\begin{pmatrix} \psi(x) \\ 0 \end{pmatrix} = \begin{pmatrix} u(\Phi(x,t),t) \\ v(\Phi(x,t),t) \end{pmatrix} + \sum_{i=1}^{n} f_i(x,t) \frac{\partial\Psi}{\partial x_i}(x,t) + \sum_{\varrho=1}^{r} w_\varrho(x,t) \frac{\partial\Psi}{\partial t_\varrho}(x,t)$$

and for $t = 0$ we get

$$\Phi(x,0) = \varphi(x), \quad \frac{\partial\Psi}{\partial x_i}(x,0) = \begin{pmatrix} \frac{\partial\varphi}{\partial x_i} \\ 0 \end{pmatrix} \text{ and}$$

$$\frac{\partial\Psi}{\partial t}(x,0) = \begin{pmatrix} \partial\Phi/\partial t_\varrho(x,0) \\ e_\varrho \end{pmatrix}$$ (where e_ϱ denotes the ϱ-th canonical unit vector of K^r), hence

$$\Psi(x) = u(\varphi(x),0) + \sum_{i=1}^{n} f_i(x,0) \frac{\partial\varphi}{\partial x_i}(x) - \sum_{\varrho=1}^{r} v_\varrho(\varphi(x),0) \frac{\partial\Phi}{\partial t_\varrho}(x,0)$$

$$\equiv - \sum_{\varrho=1}^{r} v_\varrho(0,0)\left[\frac{\partial\Phi}{\partial t_\varrho}(x,0) - \frac{\partial\Phi}{\partial t_\varrho}(0,0)\right] \bmod(\Delta(\varphi) + I(\varphi)^{\oplus p})$$

(because $\psi(0) = 0$ and $rk(\varphi) = 0$ implies

$$u(0,0) - \sum_{\varrho=1}^{r} v_\varrho(0,0)\frac{\partial\Phi}{\partial t_\varrho}(0,0) = 0)$$

(ii) → (i) : Because of $(\Phi - \sum_{\nu=1}^{r} c_\nu t_\nu, t) \sim (\Phi, t)$ we can assume $\frac{\partial\Phi}{\partial t_\nu}(0,0) = 0$ for $\nu = 1,\ldots,r$ (replacing Φ by $\Phi - \sum_{\nu=1}^{r} c_\nu t_\nu$, $c_\nu = \frac{\partial\Phi}{\partial t_\nu}(0,0)$). Because of $t_1,\ldots,t_r \in I(\Psi)$, from (ii) we infer

$$J(n+r,p) \subseteq I(\Psi)^{\oplus p} + m_{n+r}^{p+r} J(n+r,p) + \sum_{\nu=1}^{n} \mathcal{O}_{n+r}\frac{\partial\Phi}{\partial x_\nu} + \sum_{\varrho=1}^{r} K\frac{\partial\Phi}{\partial t_\varrho}$$

We can consider $J(n+r,p)$ as a direct summand of $J(n+r,p+r)$, and therefore as a submodule of $\mathcal{O}_{n+r}^{p+r}$. In $\mathcal{O}_{n+r}^{p+r}$ there holds

$$\Delta(\Psi) + J(n+r,p+r) = \mathcal{O}_{n+r}^{r} + J(n+r,p)$$

which implies

$$J(n+r,p+r) \subseteq I(\Psi)^{\oplus p+r} + m_{n+p}^{p+r} J(n+r,p+r) + \Delta(\Psi),$$

hence stability of Ψ by the preceeding proposition.

<u>Example</u> If $\varphi(x,y) = \begin{pmatrix} xy \\ x^a+y^b \end{pmatrix}$ and if the characteristic 1 of K does not divide $a+b$, then the smallest stable unfolding of φ can be determined as follows:

Consider the vector space $E = J(2,2)/\Delta(\varphi) + I(\varphi)^{\oplus 2}$ and determine a base of this vector space. In our case, $\Delta(\varphi)$ is generated by $\begin{pmatrix} y \\ ax^{a-1} \end{pmatrix}$ and $\begin{pmatrix} x \\ by^{b-1} \end{pmatrix}$ and it contains $\delta\mathcal{O}_2^{\,2}$, where $\delta = by^b - ax^a$ is the determinant of the Jacobian of φ.

Because of $\delta + a\varphi_2 = (a+b)y^b$, $b\varphi_2 - \delta = (a+b)x^a$ $(\varphi_2 = x^a+y^b)$,

the vector space E is the quotient space

$$[xK[x]/(x^a) \oplus yK[y]/(y^b)]^{\oplus 2}/[\begin{pmatrix} x \\ by^{b-1}\end{pmatrix},\begin{pmatrix} y \\ ax^{a-1}\end{pmatrix}]K[x,y]/(xy,x^a,y^b)$$

$$\cong K\binom{o}{x} \oplus \dots \oplus K\binom{o}{x^{a-1}} \oplus K\binom{o}{y} \oplus \dots \oplus K\binom{o}{y^{b-1}}$$

By $\Psi(x,y,s,t) = (\Phi(x,y,x,t),s,t)$, where

$$\Phi(x,y,s,t) = \begin{pmatrix} x\,y \\ x^a+s_{a-1}x^{a-1}+\dots+s_1x+y^b+t_{b-1}y^{b-1}+\dots+t_1y \end{pmatrix}$$

we get therefore a stable unfolding of $\varphi(x,y)$.

This example shows that a stable unfolding is not necessarily a versal unfolding of φ.

Now we consider the following functor on the category H_K of local Noetherian K-algebras with residual class field K.

Let $\varphi \in J(n,p)$ be a mapping germ, and consider $\Phi \in A\langle x_1,\dots,x_n\rangle^p$ such that $\Phi \bmod m_A A\langle X\rangle^p = \varphi$ and $\Phi(0) = 0$.

We define $\Phi \sim \Phi'$ $(\Phi' \in A\langle X\rangle^p)$ if there exists an $R \in A\langle X\rangle^n$ such that $R \bmod mA\langle X\rangle = X$ and an $L \in A\langle X,Y\rangle^p$ $(Y = (Y_1,\dots,Y_p))$ such that $L(X,0) = 0$ and $L \bmod mA\langle X,Y\rangle = Y$ and

$$L(X,\Phi'(X)) = \Phi(R(X)).$$

By $D(A)$ we denote the set of all equivalence classes $[\Phi]$ of such unfoldings Φ with respect to the equivalence relation $\sim$.

It is easy to check that

$$D(A) \to D(A') \times_{D(A'/tA')} D(A'') \quad \text{is surjective if}$$

$A = A' \times_{A'/tA'} A''$, $tA' \cong K$ and $A'' \to A'/tA'$ is a morphism in H_K: If $[\Phi''] \in D(A'')$, $[\Phi'] \in D(A')$ and if $\bar{\Phi}'$, $\bar{\Phi}''$ denotes the image of Φ', Φ'' in $(A'/tA')\langle X\rangle$, then $\bar{\Phi}' \sim \bar{\Phi}''$ means that there are transformations

$\bar{L}(X,Y) \in (A'/tA')\langle X,Y\rangle^p$, $\bar{R}(X) \in (A'/tA')\langle X\rangle^p$ as above and

$$\bar{\Phi}'' = L(X,\bar{\Phi}'(\bar{R}(X)))$$

We can lift $\bar{L}$, $\bar{R}$ to $L, \in A'\langle X,Y\rangle^p$ and $R \in A\langle X\rangle^n$ such that

$$L(X,0) = 0,\ L(X,Y) \equiv Y \bmod m_{A'}A'\langle X,Y\rangle$$

$$R(X) \equiv X \bmod m_{A'}A'\langle X\rangle$$

and we can replace Φ' by $L(X,\Phi'(R(X)))$, hence we can assume

$$\bar{\Phi}'' = \bar{\Phi}'$$

and if $\Phi = (\Phi',\Phi'') \in A\langle X\rangle^p$, then $[\Phi]$ is mapped on $[\Phi'],[\Phi'']$.

Hence the first condition of Schlessinger's criterion is fulfilled.

If $A' = K[t]/(t^2)$, then $A = A'' \oplus K\bar{t}$ where $\bar{t}^2 = m_{A''}\bar{t} = 0$ and the lifting Φ is uniquely determined, and

$$D(K[t]/(t^2)) \cong J(n,p)/[J(n,p) \cap \Delta(\varphi) + I(\varphi)^{\oplus p}]$$

This is a finite dimensional vector space if φ is finite or if $X(\varphi)$ is a complete intersection with an isolated critical point.

<u>Proposition 15</u> Assume φ is a finite mapping germ or $X(\varphi)$ is a complete intersection with an isolaed critical point. Let $\psi_1,\dots,\psi_q \in J(n,p)$ be mapping germs which represent a base of the vector space

$$J(n,p)/[J(n,p)\cap\Delta(\varphi) + I(\varphi)^{\oplus p}] \text{ and}$$

$$\Phi(x,t) = \varphi(x) + \sum_{\nu=1}^{q} t_\nu\psi_\nu(x)$$

Then $(\Phi(x,t),t)$ is a stable unfolding of φ and if $(\Psi(x,s),s)$ is any stable unfolding of φ over a germ S, $\bar{f} : \bar{S} \to T$ a mapping germ of a 0-dimensional subgerm $\bar{S} \subset S$ into T, such that $(\Phi(x,\bar{f}(\bar{s})),\bar{s})$ is equivalent to $(\Psi(x,s),\bar{s})$, there exists a prolongation $f : S \to T$ of $\bar{f}$, such that $(\Psi(x,s),s)$

is equivalent to $(\Phi(x,f(s)),s)$.

Moreover, the tangent map $T_o(f)$ is uniquely determined by $(\Psi(x,s),s)$.

The proof is completely analogous with the proof of the existence of semiuniversal deformations in the preceeding section. We call $(\Phi(x,t),t)$ a <u>minimal stable unfolding</u> of φ (it is uniquely determined up to equivalence. As a corollary we get

<u>Corollary 1</u> For a finite mapping germ $\varphi \in J(n,p)$ or a mapping germ which defines a complete intersection with isolated singularity, the properties

- infinitesimal stable
- formal stable
- stable

are equivalent.

Using the result of Gruson and Raynaud quoted in the last section, we also get the following

<u>Corollary 2</u> If $(\Phi(x,t),t) : (\mathbb{A}^{n+q},0) \to (\mathbb{A}^{p+q},0)$ is a minimal stable unfolding of φ and $T \subset (\mathbb{A}^{p+q},0)$ the maximal closed subspace over which $(\Phi(x,t),t)$ is flat and $\psi : X \to T$ the mapping germ induced from $(\Phi(x,t),t)$, then ψ is semiuniversal (i.e. $\psi : X \to T$ is a semiuniversal deformation of $X(\varphi)$).

8. Stable equivalence classes

Proposition 16 The set of stable jets in $J_{p+1}(n,p)$ is Zariski-open in $J_{p+1}(n,p)$.

Proof: The points of $J_{p+1}(n,p)$ are p-tupels $Z(x)$ of polynomials of degree $\leq p+1$ such that $Z(0) = 0$. Let $Z^{(\nu)}(X)$ be the components of $Z(X)$ and $Z_\alpha^{(\nu)}$ the coefficients of $Z^{(\nu)}(X)$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $0 < |\alpha| = \alpha_1+\ldots+\alpha_n \leq p+1$. Then the $Z_\alpha^{(\nu)}$ are affine coordinates on $J_{p+1}(n,p)$, and the coordinate ring of $J_{k+1}(n,p)$ is $K[Z] = K[(Z_\alpha^{(\nu)})]$.

Consider the modules

$$S_{p+1} = (x_1,\ldots,x_n)^{p+1}K[Z,X]^p + K[Z^{(1)}(X),\ldots,Z^{(p)}(X)]^p + \sum_{\nu=1}^{n} \frac{\partial Z}{\partial x_\nu} K[Z,X]$$

and

$$T_{p+1} = K[Z,X]^p/S_{p+1}$$

The latter one is a finitely generated $K[Z]$-module and from proposition 13 we infer that $\operatorname{supp}(T_{p+1}) = \Pi_{p+1}(n,p) \subset J_{p+1}(n,p)$ is the set of non-stable jets.

Corollary For each $q > p$ the set of stable jets in $J_q(n,p)$ is a Zariski-open set. Let us denote the set of non-stable mapping germs by $\Pi(n,p) \subset J(n,p)$ and its image in $J_k(n,p)$ by $\Pi_k(n,p)$.

Proposition 17 If $\varphi \in J(n,p)$ is a stable mapping germ, then $C\varphi \setminus \Pi(n,p) \subseteq \hat{\mathfrak{E}}\varphi$.

We show: $T(\varphi,\mathfrak{E}) = T(\varphi,C)$, if φ is stable, i.e.

$I(\varphi)^{\oplus p} \subseteq T(\varphi,\mathfrak{E}) = m_n\Delta(\varphi) + \varphi^* m_p^{\oplus p}$, which follows immediately from

$$\mathcal{O}_n^p = \Delta(\varphi) + \varphi^*\mathcal{O}_p^p$$

by multiplication with $\varphi^* m_p$.

Therefore for any $q \geqslant p+1$ we can conclude

$$\mathfrak{C}_q j_q(\varphi) = C_q j_q(\varphi) \smallsetminus \Pi_q(n,p)$$

(because the $\mathfrak{C}_q$-orbits are open in the C_q-orbits and the C_q-orbit is covered by $\mathfrak{C}_q$-orbits), hence C-equivalence of stable mapping germs implies formal $\mathfrak{C}$-equivalence.

<u>Corollary</u> Stable mapping germs $\varphi, \varphi' \in J(n,p)$ are formally equivalent if and only if

$$Q(\varphi)/m_{Q(\varphi)}^{2p+2} \simeq Q(\varphi')/m_{Q(\varphi')}^{2p+2} .$$

If the characteristic of K is zero, then already

$$Q(\varphi)/m_{Q(\varphi)}^{p+2} \simeq Q(\varphi')/m_{Q(\varphi')}^{p+2}$$

implies formal equivalence of φ and φ'.

If $p = 1$ or if $n \leqslant p$, then formal equivalence implies equivalence.

<u>Proof</u> An isomorphism $Q(\varphi)/m_{Q(\varphi)}^{2p+2} \simeq Q(\varphi')/m_{Q(\varphi')}^{2p+2}$ implies

$\varphi' \in C\varphi + m^{2p+1}J(n,p) = C\varphi$ by proposition 8.

For characteristic zero the proof is analogue.

For $p = 1$ or $n \leqslant p$ the equivalence follows from proposition 6 resp. proposition 7.

<u>Remark</u> Stability is not invariant with respect to contact equivalence. Example: $\varphi = \binom{x^3}{y^3+xy}$ and $\varphi' = \binom{x^3}{y^3}$ are contact equivalent, φ is stable, but φ' is not.

9. Normal forms of stable mapping germs

Let Q be a local Artinian K-algebra. We want to determine all stable mapping germs φ such that $Q(\varphi) \cong Q$.

If Q is of the embedding dimension $m = \dim(m_Q/m_Q^2)$, then there exists a surjection

$$\mathcal{O}_m = K\langle x_1, \ldots, x_m\rangle \to Q$$

We denote its kernel by I and define (according to Mather)

$$i(Q) = \dim(I/m_m I) - m$$

which is an invariant of the algebra Q.

If $\varphi \in J(n,p)$ is a mapping germ and $Q(\varphi) \cong Q$ then necessarily

$$p - n \geqslant i(Q)$$

The integer $\mu = \dim(Q^p/\overline{\Delta(\varphi)})$ (where $\overline{\Delta(\varphi)}$ denotes the image of $\Delta(\varphi)$ in $Q^p = \mathcal{O}_n^{\,p}/I(\varphi)^{\oplus p}$) has also an invariant meaning, it depends only on Q and $p-n$ and we denote it, according to Mather, by

$$\mu_{p-n}(Q) = \dim(Q^p/\Delta(\varphi))$$

The invariance follows from

<u>Lemma 1</u> There exists a constant $g(Q) \geqslant e(Q)$ and

$$\mu_{p-n}(Q) = (p-n)\dim Q + g(Q)$$

$$(g(Q) = \mu_i(Q) - i(Q)\dim Q, \text{ if } i = i(Q))$$

<u>Proof</u> Consider a presentation $Q = \mathcal{O}_e/I$ where $e = \dim(m_Q/m_Q^2)$, and choose a minimal set of generators $\psi_1, \ldots, \psi_q$ of I, then $\psi = \begin{pmatrix}\psi_1\\ \vdots \\ \psi_q\end{pmatrix} \in J(e,q)$ and $Q = Q(\psi)$. We call ψ a minimal

presentation of Q, we have $i(Q) = q - e$ and if

$\mu = \dim(Q^q/\overline{\Delta(\psi)})$ $(= \mu_{i(Q)}(Q))$, then $\mu - q$ is the number of parameters of a minimal stable unfolding of ψ, i.e. the smallest integers p,n, such that there exists a stable mapping germ $\Phi \in J(n,p)$ with fibre Q, are

$p = \mu$ and $n = e + \mu - q = i(Q) + \mu$ (by proposition 14).

Now we consider the mapping germ

$$\varphi' = \begin{pmatrix} \psi \\ x_{e+1} \\ \vdots \\ x_n \\ 0 \end{pmatrix} \in J(n,p) \quad (p-n-i(Q) \text{ zeros})$$

then $Q(\varphi') \cong Q(\psi) = Q$, hence φ' and φ are contact equivalent, which implies $\dim Q^p/\overline{\Delta(\varphi')} = \dim Q^p/\overline{\Delta(\varphi)}$ by using the definition of contact equivalence. On the other hand, we can calculate $\dim Q/\overline{\Delta(\varphi')}$ very easily because of the particular simple form of φ':

$\overline{\Delta(\varphi')} = \overline{\Delta(\psi)} \times 0 \oplus Qe_{e+1} + \ldots + Qe_n$ (where $e_1, \ldots, e_p$ denotes the canonical base of Q^p), so we get

$$\dim Q^p/\overline{\Delta(\varphi)} = \dim(Q^q/\overline{\Delta(\psi)}) + (p-n)\dim Q - i(Q)\dim Q$$

hence
$$\begin{aligned} g(Q) &= \dim(Q^q/\overline{\Delta(\psi)}) - i(Q)\dim Q \\ &= q\dim Q - \dim \overline{\Delta(\psi)} - (q-e)\dim Q \\ &= e\dim Q - \dim \overline{\Delta(\psi)} \geq e \end{aligned}$$

because $\overline{\Delta(\psi)}$ has at most e generators, each of which is contained in $m_Q^{\oplus q}$.

Assume now that p, n are positive integers such that

(1) $p-n \geq i(Q)$

(2) $p \geq \mu_{p-n}(Q)$

and let $\psi \in J(e,q)$ be a minimal presentation of Q.

We want to construct a stable germ $\varphi \in J(n,p)$ such that $Q \simeq Q(\varphi)$.

We consider $\psi' = \binom{\psi}{0} \in \mathcal{O}_e^{p-n+e}$ (observe that $p-n+e \geq q$ by (1)), then

$$0 \leq (p-n)\dim(m_Q) + (g(Q) - e) = \mu_{p-n}(Q) - (p-n+e) \leq n-e$$

by (2) and therefore there exist an integer d, $0 < d \leq n-e$ and germs $v_1,\ldots,v_d \in J(e,p-n+e)$ which represent a base of $J(e,p-n+e)/\Delta(\psi') + I(\psi)^{\oplus p-n+e}$. Consider coordinates $x_1,\ldots,x_e$, $t_1,\ldots,t_d$ and $z_1,\ldots,z_s$ (if $s = n-e-d > 0$) on $(\mathbb{A}^n,0)$ and the germ

$$(3) \quad \varphi = \begin{pmatrix} \psi'+t_1v_1+\ldots+t_dv_d \\ t \\ z \end{pmatrix} \in J(n,p) \; .$$

It is stable by proposition 14, and $Q(\varphi) \simeq Q$ by construction.

This proves the "if-part" of the following

<u>Theorem</u> If Q is a local Artinian K-algebra, there exists a stable mapping germ $\varphi \in J(n,p)$ with $Q(\varphi) \simeq Q$ if and only if (1) and (2) (see above) are satisfied.

Any such mapping germ is equivalent to one which has the "normal form" (3), and

$$d = \mu_{p-n}(Q) - (p-n+e) = (p-n)\dim(m_Q) + g(Q) - e \; ,$$

$$s = p - \mu_{p-n}(Q)$$

<u>Proof</u> We have to show that for stable mapping germs $\varphi \in J(n,p)$ it holds that $p \geq \mu_{p-n}(Q)$ (the inequality $p-n \geq i(Q)$ holds for any mapping germ φ such that $Q(\varphi) \simeq Q$). But this is obvious

because of $\dim(Q^p/\overline{\Delta(\varphi)}) \leq p$ for stable germs φ, hence $\mu_{p-n}(Q) = \dim(Q^p/\overline{\Delta(\varphi)}) \leq p$.

<u>Example</u> $Q = K[x]/(x^{c+1})$

If $c+1$ is not a multiple of $\operatorname{char}(K)$, then $\mathfrak{g}(Q) = c$, $\dim(Q) = c+1$, $i(Q) = 0$, hence stable mapping germs $\varphi \in J(n,p)$ exist if and only if $p \geq n$ and $p \geq (p-n)(c+1) + c$. In this case, they can be put into the form

$$\varphi = \begin{pmatrix} x^{c+1} + t_{c-1}x^{c-1} + \ldots + t_1 x \\ t_{1c}x^c + t_{1c-1}x^{c-1} + \ldots + t_{11}x \\ t_{p-n,c}x^c + t_{p-n,c-1}x^{c-1} + \ldots + t_{p-n,1}x \\ t \\ z \end{pmatrix}$$

where $z = \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix}$, $s = n - c(p-n+1)$.

If $c+1$ is a multiple of $\operatorname{char}(K)$, we get a similar result, but with $g(Q) = c+1$, hence $p \geq n$, $p \geq (p-n+1)(c+1)$.

<u>Corollary 1</u> If $\varphi \in J(n,p)$ is stable and $p > n$, then $\dim(Q(\varphi)) \leq \frac{p-e}{p-n}$, where $e = \dim(m_Q/m_Q^2)$, and equality holds if and only if $p = \mu_{p-n}(Q)$ and $e = \mathfrak{g}(Q)$.

<u>Corollary 2</u> If $\varphi \in J(n,p)$ is stable and $p \geq n + \operatorname{rk}(\varphi)$, then either φ is a closed embedding or $\operatorname{rk}(\varphi) = n-1$, $p = 2n-1$, and (up to equivalence)

$$\varphi = \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ \vdots \\ x_1 x_n \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

<u>Proof</u> If $p > n + rk(\varphi)$, then $\dim(Q(\varphi)) \leqslant 1 + \frac{rk(\varphi)}{p-n} < 2$ (by the preceding corollary), hence φ is a closed embedding. If $p = n + rk(\varphi)$, then $\dim(Q(\varphi)) \leqq 2$, hence φ is a closed embedding or $\dim(Q(\varphi)) = 2$, $rk(\varphi) = n-1 = p-n$, and the normal form of φ yields the assertion.

Remark: If Q is local Artinian K-algebra, $\dim(Q) = 1$, then $i(Q) \leqslant \binom{l-1}{2}$.

Proof: If $q_1,\dots,q_{l-1} \in m_Q$ is a K-base of m_Q, then Q is determined by at most $\binom{l}{2}$ equations $q_i q_j = \sum_k a_{ij} q_k$ $i \leqslant j$, hence $\binom{l}{2} - (l-1) = \binom{l-1}{2} \geqslant i(Q)$ q.e.d.

<u>Example</u> Stable mapping germs $\varphi \in J(n,p)$ such that $4p \geqslant 4n + rk(\varphi)$

For the corresponding local algebra $Q = Q(\varphi)$ there holds $\dim(Q) \leqslant 5$.

If $\dim(Q) = 5$, then necessarily $e = g(Q)$, which implies by definition of $g(Q)$: $\dim(\overline{\Delta(\varphi)}) = e \dim(m_Q)$. Therefore, if $Q = Q(\psi)$, where $\psi \in J(e,q)$ is a minimal presentation of Q, the vectors $\frac{\partial\psi}{\partial x_1},\dots,\frac{\partial\psi}{\partial x_e}$ must be linearly independent over $Q/\mathrm{ann}(m_Q)$ and $\mathrm{ann}(m_Q)$ must be of dimension 1. If $m_Q^{c+1} = 0$, $m_Q^c \neq 0$, then necessarily $m_Q^c = \mathrm{ann}(m_Q)$, $\dim(m_Q^2/m_Q^c) = 3-e \geqslant 1$ or $c = 2$ and $e = 3$. If $e = 1$, then $c = 4$ and $g(Q) = 4$, which is not possible. If $e = 2$, then $c = 3$ and by multiplication we get a non-zero quadratic form

$$(m_Q/m_Q^2) \otimes (m_Q/m_Q^2) \to m_Q^2/m_Q^3 .$$

Hence there are generators x,y of m_Q, such that $m_Q^2 = x^2K + x^3K$, $x^4 = 0$ and $xy = 0$, and up to isomorphism there are the following possibilities for a minimal presentation of Q:

$\psi_1 = (x^4, y_2^2, xy)$, $g(Q) = 6$

$\psi_2 = (x^3 + y^2, xy)$, $g(Q) = 5$

These algebras have no stable representative in the range $4p \geq 4n + rk(\varphi)$.

If $e = 3$, we get, in the same way, up to isomorphism, the following possibilities for a minimal presentation of Q

$\psi_1 = (xy, xz, yz, y^2, z^2, x^3)$ $g(Q) = 10$

$\psi_2 = (xy, xz, yz, y^2 + x^2, z^2)$ $g(Q) = 8$

$\psi_3 = (xy, xz, yz, y^2 + x^2, z^2 + x^2)$ $g(Q) = 7$

These algebras have again no stable representatives in the range $4p \geq 4n + rk(\varphi)$.

Therefore it remains the case $\dim(Q) \leq 4$. We describe the corresponding algebras by minimal presentations.

(i) $e = 3$

$\psi = (x^2, y^2, z^2, xy, xz, yz)$, $i(Q) = 3$, $g(Q) = 6$, $\dim(Q) = 4$

Range of existence: $p - n \geq 3$, $p \geq 4(p-n) + 6$

(ii) $e = 2$

$\psi_1 = (x^3, y^2, xy)$ $i(Q) = 1$, $g(Q) = 5$, $\dim(Q) = 4$

Range of existence: $p - n \geq 3$, $p \geq 4(p-n) + 5$

$\psi_2 = (x^2 + y^2, xy)$ $i(Q) = 0$ $g(Q) = 4$, $\dim(Q) = 4$

Range of existence: $p - n \geq 2$, $p \geq 4(p - n) + 4$

$\psi_3 = (x^2, y^2, xy)$ $i(Q) = 1$, $g(Q) = 4$, $\dim(Q) = 3$

Range of existence: $p - n \geq 1$, $p \geq 3(p - n) + 4$

(iii) $e = 1$

$\psi_s = (x^{s+1})$, $i(Q_s) = 0$, $g(Q_s) = s$, $\dim(Q_s) = s + 1$

($s = 1, 2, 3$)

Range of existence: $p \geq (s + 1)(p - n) + s$ and $p - n \geq 2$ or $s \leq 2$ and $p - n \geq 1$.

Appendix: Multi-germs and multi-jets

For later use we need some slight generalizations of the results about germs (jets) to multi-germs (multi-jets).

By a multi-germ (resp. multi-jet) we mean an m-tupel of germs, $\varphi = (\varphi^{(1)},\ldots,\varphi^{(m)}) \in J(n,p)^m$, i.e. a mapping $\coprod_m (\mathbb{A}^n,0) \to (\mathbb{A}^p,0)$ (resp. jets $z = (z^{(1)},\ldots,z^{(m)}) \in J_k(n,p)^m$).

The group C^m (resp. C_k^m) of contact equivalences on $J(n,p)^m$ (resp. on $J_k(n,p)^m$) is defined as the semidirect product of the symmetric group S_m with the m-fold direct product of the group of contact equivalences on $J(n,p)$ (resp. on $J_k(n,p)$).

The group of equivalences $\mathfrak{E}^m$ (resp. $\mathfrak{E}_k^m$) is the subgroup of the semidirect product of S_m with the m-fold direct product of the group $\mathfrak{E}$ of equivalences, consisting of all elements $((\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m),\pi)$, $(\alpha_i,\beta_i) \in \mathfrak{E}$ such that $\beta_1 = \ldots = \beta_m$.

This group acts on $J(n,p)^m$ resp. on $J_k(n,p)^m$ (algebraically), for example

$$(\gamma^{(1)},\ldots,\gamma^{(m)},\pi)(\varphi^{(1)},\ldots,\varphi^{(m)}) = (\gamma^{(1)}\varphi^{(\pi(1))},\ldots,\gamma^{(m)}\varphi^{(\pi(m))})$$

$\mathfrak{E}_k^m$ resp. C_k^m are linear algebraic groups having m! connected components and $(\mathfrak{E}_k^m)^o = (C_k^m)^o \cap \mathfrak{E}_k^m$ (where ^{o} indicates the connected component of the unit).

We put
$$Q(\varphi) = Q(\varphi^{(1)}) \times \ldots \times Q(\varphi^{(m)})$$
$$Q(z) = Q(z^{(1)}) \times \ldots \times Q(z^{(m)}).$$

The tangent spaces to the orbits are

$$T(C^m\varphi) = (m_n^{(1)}\Delta(\varphi^{(1)}) \times \ldots \times m_n^{(m)}\Delta(\varphi^{(m)})) + I(\varphi)^{\oplus p} \subset J(n,p)^m$$

$$T(C_k^m z) = (m_n^{(1)}\Delta(z^{(1)}) \times \ldots \times m_n^{(m)}\Delta(z^{(m)})) + I(z)^{\oplus p} \subset J_k(n,p)^m$$

$$T(\mathfrak{E}^m\varphi) = (m_n^{(1)}\Delta(\varphi^{(1)}) \times \ldots \times m_n^{(m)}\Delta(\varphi^{(m)})) + \varphi^* m_p^{\oplus p} \subset J(n,p)^m$$

$$T(\mathfrak{E}_k^m z) = (m_n^{(1)}\Delta(z^{(1)}) \times \ldots \times m_n^{(m)}\Delta(z^{(m)})) + z^* m_p^{\oplus p} \subset J_k(n,p)^m$$

<u>Proposition A 1</u> Multi-germs $\varphi, \psi \in J(n,p)^m$ are contact equivalent if and only if $Q(\varphi) \cong Q(\psi)$. The same holds for multi-jets.

Proof: Obviously, contact equivalence implies isomorphy of the corresponding algebras. Conversely, assume $Q(\varphi) \cong Q(\psi)$. By a permutation we can assume $Q(\varphi^{(j)}) \cong Q(\psi^{(j)})$. Then there are (pxp)-matrices $A^{(j)}$, $B^{(j)}$ with coefficients in $\mathcal{O}_n$ and automorphisms α_j of $(\mathbb{A}^n, 0)$ and $\varphi^{(j)} = A^{(j)}\psi^{(j)} \circ \alpha_j$, $\psi^{(j)} = B^{(j)}\varphi^{(j)} \circ \alpha_j^{-1}$. We can assume $A^{(j)}B^{(j)} = I_p$ because of $\psi^{(j)} = (C(I_p - A^{(j)}B^{(j)}) + B^{(j)})\varphi^{(j)} \circ \alpha_j^{-1}$, by the following

<u>Lemma</u> If A,B are (p x p)-matrices over a local ring (with infinite residual class field), there exists a (pxp)-matrix C over this ring such that $C(I_p - AB) + B$ is invertible.

Proof of the lemma: Reducing modulo the maximal ideal we can consider matrices over a field K. Let $E \subset K^p$ be a subspace complementary to Ker(B) (consider B as an endomorphism of K^p) and let $F \subset K^p$ be a subspace complementary to Im(B), and let C_o be an isomorphism $\mathrm{Ker}(B) \xrightarrow{\sim} F$.
We define

$$Cv = \begin{cases} C_o v & \text{if } v \in \mathrm{Ker}(B) \\ \lambda v & \text{if } v \in E \end{cases}$$

where $\lambda \in K$ is a scalar such that $\lambda(I_p - AB) + B$ induces an isomorphism $E \xrightarrow{\sim} K^p/F$ (observe that B induces an isomorphism $E \xrightarrow{\sim} K^p/F$). Then $C(I_p - AB) + B$ is an isomorphism of K^p q.e.d.

Now the proposition follows by choosing the contact transformation

$$\gamma^{(j)}(x,y) = (\alpha_j(x), B^{(j)}(\alpha_j(x))y), j = 1,\dots,m.$$

Unfoldings of multi-germs are defined in the same way as for germs. If T is a parameter germ, an unfolding of the multi-germ φ over T is given by an m-tupel $(\Phi^{(1)}(x^{(1)},t),\dots,\Phi^{(m)}(x^{(m)},t))$, where $(\Phi^{(j)}(x,t),t)$ is an unfolding of $\varphi^{(j)}$.

The multi-germ φ is called stable if and only if any unfolding of φ is equivalent to a trivial unfolding. As for the case of germs there holds

<u>Proposition A 2</u> If $n \leq p$, then the multi-germ $\varphi \in J(n,p)^m$ is stable if and only if $Q_{p+1}(\varphi)^p = K^p + \overline{\Delta(\varphi)}$.
$(Q_k(\varphi) = Q(\varphi)/m^{k+1}$, $\overline{\Delta(\varphi)}$ = image of $\Delta(\varphi)$ in $Q_{p+1}(\varphi)^p$.)

<u>Proposition A 3</u> If $n \leq p$ and if $\varphi, \psi \in J(n,p)^m$ are stable, then φ and ψ are equivalent if and only if they are contact equivalent.

The proof of proposition A 2 is exactly the same as for the analogous proposition about germs. For proposition A 3 the proof is also the same as for germs, provided we know that the set $\Pi_k^m(n,p) \subset J_k(n,p)^m$ of non-stable multi-jets is Zariski-closed. This will be proved in more general context in § 21.

Stability of multi-germs (resp. multi-jets) can be characterized as follows: We say that branches of the multi-germ φ are ~~transversal~~, if the linear subspaces
normally crossing

T_i = image of $(\Theta_{\mathbb{A}^p,0} \cap (T(\varphi^{(i)})(\Theta_{\mathbb{A}^n,0}) + m_p \varphi^{(i)*} \Theta_{\mathbb{A}^p,0}))$ in $T_0(\mathbb{A}^p)$ are in general position, i.e. if

$$\operatorname{codim}(T_1 \cap \ldots \cap T_m) = \operatorname{codim} T_1 + \ldots + \operatorname{codim} T_m.$$

In other terms

$$T_1 = \mathcal{O}_p^p \cap (\Delta(\varphi^{(i)}) + I(\varphi^{(i)})^{\oplus p})/m_p^{\oplus p} + K^p + (\Delta(\varphi^{(i)}) + I(\varphi^{(i)})^{\oplus p}.$$

<u>Proposition A 4</u> The multi-germ φ is stable if and only if each component $\varphi^{(j)}$ is stable and the branches of φ are normally crossing.

If φ is stable, then each $\varphi^{(j)}$ is stable. Now assume that each $\varphi^{(j)}$ is stable, i.e. $\mathcal{O}_n^p = K^p + \Delta(\varphi^{(j)}) + I(\varphi^{(j)})^{\oplus p}$. Consider the diagonal map

$$d : K^p \to \bigtimes_{j=1}^{m} (K^p + \Delta(\varphi^{(j)}) + I(\varphi^{(j)})^{\oplus p})/\Delta(\varphi^{(j)}) + I(\varphi^{(j)})^{\oplus p}.$$

Then φ is stable if and only if this map is surjective. On the other hand, the linear subspaces $T_1,\ldots,T_m$ are in general position if and only if d is surjective q.e.d.

An analogous proposition holds for k-multi-jets, $k \geq p + 1$, Further properties of multi-germs are derived in § 22.

§ 10 The nice range

To apply the local considerations for the classification of ordinary singularities, i.e. the singularities which appear as generic projections of smooth projective varieties into some lower dimensional projective space, we have to clarify the meaning of 'generic' in this case. One condition which we impose for 'generic' projections will be that they have stable germs everywhere. As J. Mather found out, there is a 'nice range' of dimensions (n,p), $n = \dim(V) < p = \dim \mathbb{P}^p$, such that the set of projections $V \to \mathbb{P}^p$ (fixing an embedding of V into some $\mathbb{P}^N$) which are locally stable everywhere, is not empty; outside the 'nice range' there are always examples of varieties V and embeddings $V \subset \mathbb{P}^N$ such that there are no projections $V \to \mathbb{P}^p$ which are stable everywhere.

We will derive here the results of Mather over arbitrary algebraically closed ground fields K, but for simplicity we exclude char(K) = 2 or 3. Most of the ideas are roughly the same as in Mather's work.

In § 20 we will show that the nice domain is characterized by

$$\operatorname{codim} \Pi_k (n,p) > n$$

In § 12 we will show that, if $W_k(n,p) \subset J_k(n,p)$ denotes the Zariski-closed set of contact classes depending on moduli (for precise definition see § 12), then $\operatorname{codim} W_k(n,p) \leq n$ implies $\operatorname{codim} \Pi_k(n,p) \leq n$, and $\operatorname{codim} W_k(n,p) > n$ implies that there are only finite many contact classes $C_1,\dots,C_S$ of codimension being at most n.

If

$$\dim (\Delta(z) / m\Delta(z)) = n$$

holds for $z \in C_i$, this implies $\operatorname{codim} \Pi_k(n,p) > n$.

Hence we have to determine

a) pairs (n,p), $n < p$, such that $\operatorname{codim} W_k(n,p) > n$

b) the (finite many) contact classes of codimension $c < n$, and we have to check for them that $\dim(\Delta(z) / m\Delta(z)) = n$.

The result will be the following:

The 'nice range' of pairs $(p, p-n)$, for which $n < \operatorname{codim} \Pi_k(n,p)$, is characterized by the inequalities

$$p \leq 7(p-n) + 7 \quad \text{or} \quad p \leq 7(p-n) + 8, \; p-n \leq 3$$

If $(p,p-n)$ is in the nice range and $\varphi \in J(n,p)$ is a stable mapping germ, the algebra $Q(\varphi)$, which determines φ up to equivalence, is one of the following 48 local Artin algebras:

Table 1:

Stable equivalence classes in the nice range (classified by the corresponding algebra $Q = Q(\varphi) = \mathcal{O}_e / I$)

Nr.	Hilbert-sequence of Q	Generators of I	dim(Q)	g(Q)	i(Q)
	$e(Q)=h_1=1$				
1	10	x^2	2	1	0
2	110	x^3	3	2	0
3	1110	x^4	4	3	0
4	11110	x^5	5	4	0
5	111110	x^6	6	5	0
6	1111110	x^7	7	6	0
7	11111110	x^8	8	7	0
	$e(Q)=h_1=2$				
8	20	x^2,xy,y^2	3	4	1
9	210	y^2+x^2,xy	4	4	0
10		x^3,xy,y^2	4	5	1
11	2110	y^2+x^3,xy	5	5	0
12		x^4,xy,y^2	5	6	1
13	21110	y^2+x^4,xy	6	6	0
14		x^5,xy,y^2	6	7	1
15	21111	y^2+x^5,xy	7	7	0
16		x^6,xy,y^2	7	8	1

Nr.	Hilbert-sequence of Q	Generators of I	dim(Q)	g(Q)	i(Q)
17	220	xy, x^3, y^3	5	6	1
18		y^2, x^2y, x^3	5	7	1
19	2210	xy, x^3+y^3	6	6	0
20		y^2, x^3	6	7	0
21		xy, x^4, y^3	6	7	1
22		y^2-x^3, x^2y, x^4	6	8	1
23		y^2, x^2y, x^4	6	9	1
24	22110	xy, x^4+y^3	7	7	0
25		y^2-x^3, x^2y	7	8	0
26		xy, x^5, y^3	7	8	1
27	2220	xy, x^4, y^4	7	8	1
28	230	x^3, x^2y, xy^2, y^3	6	10	2
	$e(Q) = 3$				
29	30	$(x,y,z)^2$	4	9	3
30	310	$y^2+x^2, z^2+x^2, xy, xz, yz$	5	7	2
31		y^2+x^2, z^2, xy, xz, yz	5	8	2
32		$x^3, y^2, z^2, xy, xz, yz$	5	10	3
33	3110	$y^2+x^3, z^2+x^3, xy, xz, yz$	6	8	2

Nr.	Hilbert-sequence of Q	Generators of I	dim(Q)	g(Q)	i(Q)
34		y^2+x^3, z^2, xy, xz, yz	6	9	2
35		$x^4, y^2, z^2, xy, xz, yz$	6	11	3
36	320	$x^2+y^2+z^2 xy, xz, yz$	6	7	1
37		$x^2, y^2, z^2, xz+yz$	6	8	1
38		x^2+yz, xz, y^2, z^2	6	9	1
39		x^2, yz, xz, y^2, z^3	6	9	2
40		x^2+yz, xz, xy, y^2, z^3	6	10	2
41		x^2, y^2, z^2, yz	6	11	1
42	3210	x^2+z^3, y^2+z^3, xz, yz	7	8	1
	$e(Q)=h_1=4$				
43	40	$(x_1 \ldots, x_4)^2$	4	16	6
44	41	$(x_2^2+x_1^2, x_3^2+x_1^2, x_4^2+x_1^2, x_i x_j; i>j)$	6	11	5
45	41	$(x_2^2+x_1^2, x_3^2+x_1^2, x_4^2, x_i x_j; i>j)$	6	12	5
46	41	$(x_2^2+x_1^2, x_3^2, x_4^2, x_i x_j; i>j)$	6	14	5
47	41	$(x_1^3, x_2^2, x_3^2, x_4^2, x_i x_j; i>j)$	6	17	6
	$e(Q)=h_1=5$				
48	50	$(x_1, \ldots, x_5)^2$	6	25	10

§ 11 The codimension of nonstable jets in a contact class

We fix $k \geq p + 1$ and consider $J_k(n,p)$. For k-jets $z \in J_k(n,p)$ we consider the algebra $Q(z) = \mathcal{O}_n/m_n^{k+1} + I(z)$ and the contact class $C(z)$ of z in $J_k(n,p)$. The modul $\Delta(z) = \sum_{i=1}^{n} \frac{\partial z}{\partial x} \cdot Q(z)$ is only determined modulo $(m_{Q(z)}{}^k) \oplus p$ therefore $m_{Q(z)} \overline{\Delta(z)}$ is uniquely determined and

$$c(z) = \dim(J_k(n,p) / m_n \Delta(z) + I(z) \oplus p$$

is uniquely determined and is the codimension of $C(z) \subset J_k(n,p)$.

We define

$$\mu(z) = \dim(Q_k^p \; / \overline{\Delta(z)} + (m_{Q(z)}{}^k) \oplus p \;)$$

then

$$c(z) \geq \mu(z) - p + \dim(\overline{\Delta(z)} + (m^k) \oplus p \; / \; m\overline{\Delta(z)} + (m^k) \oplus p \;)$$

(where $m = m_{Q(z)}$).

We have equality if $m^k = 0$, so for instance for stable jets (because of $k \geq p + 1$, but $\dim(Q(z)^p \; / \; \overline{\Delta(z)} + (m^k) \oplus p \;) \leq p$).

By $V_k(n,p) \subset J_k(n,p)$ we denote the closed subset of contact classes of codimension $> n$.

Proposition 18 Let C be the contact class of a jet z in $J_k(n,p)$.

(1) $V_k(n,p) \subseteq \Pi_k(n,p)$

(2) If $c(z) \leq p - n$, then z is a closed embedding, hence $C \cap \Pi_k(n,p) = \phi$

(3) If $p = n$ and $c(z) = 1$, then $C \cap \Pi_k(n,p) = \phi$ and z is equivalent to $(x_1^2, x_2, \ldots, x_n)$

(4) If $c(z) \leq n$, then

$$\operatorname{codim}_C (C \cap \Pi_k \; n,p)) = \max(0, p - \mu(z) + 1)$$

Proof: If z is stable, then

$p \geq \mu(z)$ and $c(z) = \mu(z) - p + \dim \overline{\Delta(z)} / m \overline{\Delta(z)}$

$\leq n$, this proves (1).

If $c(z) \leq p - n$, then, because of

$c(z) \geq \dim(J_k(n,p) / m_n\Delta(z) + I(z) \oplus p + (m_n^2) \oplus p) = (p-r)(n-r)$

(if $r = rk(z)$), we get

$$p - n \geq (p-r)(n-r) \geq (p-n)(n-r),$$

hence $n - r \leq 1$, hence $r = n$, this proves (2).

If $c(z) = 1$ and $p = n$, we get by the same argument $r = n - 1$, hence

z is equivalent to

$$z = (x_1^2 + \sum_{i=2}^{n} x_i w_i, x_2, \ldots, x_n)$$

and $2x_1 \neq 0$, hence z is stable, hence equivalent to $(x_1^2, x_2, \ldots, x_n)$.

Now we will introduce the following notation:

If $e \leq n$, we denote by

$$\Lambda_k(n,p)_e \subset J_k(n,p)$$

the subspace of all jets of the form

$$z = (z_1, \ldots, z_{p-n+e}, x_{e+1}, \ldots, x_n)$$

where $z_i = f_i(x_1, \ldots, x_e) + \sum_{j=e+1}^{n} x_j w_{ij}(x)$, ord $(f_i) \geq 2$ and

ord $(w_{ij}) \geq 1$ (unfoldings of jets from $J_k(e, p - n + e)$).

Each contact class meets exactly one $\Lambda_k(n,p)_e$ (e is the embedding dimension of the corresponding algebra).

By $J_k(n,p)_e$ I denote the locally closed subspace of $J_k(n,p)$ of jets of rank $n - e$ and by A_k the group of k-jets of automorphisms of the germ $(A^n, 0)$. Then A_k acts on $J_k(n,p)_e$ (transformation of the coordinates), $\Lambda_k(n,p)_e$ is a closed subspace of $J_k(n,p)_e$ and the action of A_k defines a smooth morphism

$$\Lambda_k(n,p)_e \times A_k \longrightarrow J_k(n,p)_e$$

which has local sections.

If X is any $\mathcal{C}_k(n,p)$-stable subset of $J_k(n,p)_e$, then

X open (closed, irreducible) $\Longrightarrow$ $X \cap \Lambda_k(n,p)_e$ open (closed, irreducible) in $\Lambda_k(n,p)_e$

$$\operatorname{codim}_{J_k(n,p)_e}(X) = \operatorname{codim}_{\Lambda_k(n,p)_e}(X \cap \Lambda_k(n,p)_e)$$

holds true.

If Π denotes the morphism

$$\Lambda_k(n,p)_e \longrightarrow J_k(e,p-n+e)_0 \times J_{k-1}(e,p-n+e)^{n-e} \underset{\text{def.}}{=} V$$

$$z \longmapsto \left(z'(x_1,\dots,x_e,\underline{0}),\ \frac{\partial z'}{\partial x_{e+1}}(x_1,\dots,x_e,\underline{0}),\dots,\ \frac{\partial z'}{\partial x_n}(x_1,\dots,x_e,\underline{0})\right)$$

(where $z = (z', x_{e+1},\dots,x_n)$),

then by proposition 14 we get that the closed subset $\Pi_k(n,p) \cap \Lambda_k(n,p)_e$ is the preimage of the closed subset of all points $(z'_0, w_1,\dots,w_{n-e}) \in V$ such that the projection

$$Kw_1 + \dots + Kw_{n-e} \longrightarrow J_{k-1}(e,p-n+e)/\Delta(z'_0) + I(z'_0)^{\oplus p-n+e} = T_{z'_0}$$

is not surjective.

Fixing z, hence z'_0, the codimension of this set (in $\{z'_0\} \times J_{k-1}(e,p-n+e)^{\oplus n-e}$) is then equal to the codimension of $C \cap \Pi_k(n,p)$ in C. So the result follows from the following

<u>Lemma</u>: Let $E \to T$ be an epimorphism of vector spaces and r a positive integer. Then the closed subset $\Delta \subset E^{\oplus r}$ of all r-tupels whose image in T does not generate T has codim equal to $\max(0, r - \dim T + 1)$.

To prove the lemma we can divide by the kernel F of $E \to T$ (Δ is stable under translations from $F^{\oplus r}$), hence we may assume $E = T$.

In this case the lemma is easy.

§ 12 Simple contact classes

If a connected algebraic group G acts algebraically on an algebraic variety X, there may be G-stable non-empty open sets which contain only finite many orbits. The orbits of the largest of such open sets are called the simple orbits of G in X.

By $W_k(n,p)$ we will denote the closed subset of $J_k(n,p)$ of non-simple $\mathcal{C}_k(n,p)$-orbits.

Proposition 19 (1) If codim $W_k(n,p) > n$, then $W_k(n,p) \subset V_k(n,p)$, hence $W_k(n,p) \subset \Pi_k(n,p)$.

If for the finite many simple contact classes $C(z_1),\ldots,C(z_s)$ which meet $\Pi_k(n,p) - V_k(n,p)$ $\dim(\overline{\Delta(z)} / m\overline{\Delta(z)}) = n$ holds, then codim $\Pi_k(n,p) > n$, too.

(2) If codim $W_k(n,p) \leq n$, then also codim $\Pi_k(n,p) \leq n$.

Proof: Let V be a component of $W_k(n,p)$, then V contains no simple orbits, hence no orbits which are open in V, hence codim$C >$ codim$V$ for any orbit C in V. Thus we have $V \subset V_k(n,p)$ if codim $W_k(n,p) > n$, i.e. $W_k(n,p) \subset V_k(n,p)$.

If $C = C(z)$ is a simple orbit meeting $\Pi_k(n,p) \setminus V_k(n,p)$ and if $\dim(\overline{\Delta(z)} / m\overline{\Delta(z)}) = n$, then

$$\operatorname{codim}(C \cap \Pi_k(n,p)) = \max(c(z),\ c(z) + p - \mu(z) + 1)$$
$$\geq \max(c(z),\ n + 1)$$

hence codim $\Pi_k(n,p) \geq n + 1$, which proves (1).

If a component of $W_k(n,p)$ of codimension $\leq n$ is contained in $\Pi_k(n,p)$, then codim $\Pi_k(n,p) \leq n$.

If there is no such component and if codim $W_k(n,p) \leq n$, there is a component V of $W_k(n,p)$ of codimension $m \leq n$ and there are contact classes $C = C(z)$ of codimension $c \leq n$ (otherwise $V \subset V_k(n,p) \subseteq \Pi_k(n,p)$) and $c > m$ (since V contains no open orbits) in V.

Then $\operatorname{codim}(\Pi_k(n,p)) \leq (V \cap \Pi_k(n,p))$

$$= \operatorname{codim} V + \operatorname{codim}_V (V \cap \Pi_k(n,p))$$

$$\leq \operatorname{codim} V + \operatorname{codim}_C (C \cap \Pi_k(n,p))$$

$$= \max(m,\ m + p - \mu(z) + 1)$$

If $C \notin \Pi_k(n,p)$, then

$$m < c = c(z) = \mu(z) - p + \dim(\overline{\Delta(z)} \,/\, m\overline{\Delta(z)})$$

$$\leq \mu(z) - p + n$$

hence $m + p - \mu(z) < n$, hence $\operatorname{codim} \Pi_k(n,p) \leq n$ in this case.

If $C \subseteq \Pi_k(n,p)$, then $\operatorname{codim}_C(C \cap \Pi_k(n,p)) = 0$, hence $\operatorname{codim}(\Pi_k(n,p)) \leq m < c \leq n$ q.e.d.

That is why we have to determine the codimension of $W_k(n,p)$, the simple contact classes meeting $\Pi_k(n,p) \setminus V_k(n,p)$ if $\operatorname{codim} W_k(n,p) > n$ and to check $\dim(\overline{\Delta(z)} \,/\, m\overline{\Delta(z)}) = n$ in this case.

We add some general remarks before we determine the simple contact classes.

<u>Lemma 1</u> If G: is a connected algebraic group and $j : X \to Y$ is an open morphism such that G acts equivariantly on X and Y, then $j^{-1}(W(Y)) \subset W(X)$, where W(X) and W(Y) denote the closed subspaces of non-simple orbits of X and Y.

<u>Proof:</u> The set $X \setminus W(X)$ is open, G-stable, and contains only finite many orbits, hence $j(X - W(X))$ is open, G-stable, and contains only finite many orbits, hence $j(X - W(X)) \subseteq Y - W(Y)$.

By this lemma we have $W_k(n,p) \subseteq j^{-1}(W_l(n,p))$ if $l < k$ and $j : J_k(n,p) \to J_l(n,p)$ is the truncation map.

<u>Lemma 2</u> Let $h = (h(1),\dots,h(l))$, $l \leq k$, be a sequence of positive integers and let $J_k(n,p)_h$ be the subset of all jets z such that, for

the corresponding local algebra Q, $\dim(m(z)^v / m(z)^{v+1} = h(v)$ holds.

Then $J_k(n,p)_h$ is locally closed, $\mathcal{C}_k(n,p)$ stable, and for $k \geq 1$, $e \leq n$, we have

$$\operatorname{codim} J_k(n,p)_e = e(p - n + e).$$

For $k \geq 2$ we have

$$J_k(n,p)_{(e_1,e_2)} \neq \emptyset \quad \text{iff} \quad \max\left\{0, \binom{e_1}{2} - (p-n)\right\} \leq e_2 \leq \binom{e_1+1}{2}$$

and $\operatorname{codim} J_k(n,p)_{(e_1,e_2)} = (e_1+e_2)(p-n)+e_1^2+e_2\left(e_2-\binom{e_1}{2}\right)$

if it is not empty.

<u>Proof:</u> Only the last assertion may not be obvious, so we will give a proof for it. Let Λ be the subspace $\Lambda_2(n,p)_{e_1}$, which was considered in the last section. Then

$$\operatorname{codim} J_k(n,p)_{(e_1,e_2)} \triangleq \operatorname{codim} J_2(n,p)_{e_1} + \operatorname{codim}_\Lambda (\Lambda \cap J_2(n,p)_{(e_1,e_2)}).$$

The jets of $\Lambda \cap J_2(n,p)_{(e_1,e_2)}$ are determined by

i) $p - n + e_1$ quadratic forms $f_i(x_1,\dots,x_{e_1})$ generating a subspace of codimension e_2 in the space of all quadratic forms

ii) $(p - n + e_1)(n - e_1)$ arbitrary linear forms $w_{ij}(x)$.

If S is a vector space of dimension N, $U \subset S^{\oplus r}$ the space of all r-tupels spanning a subspace of codimension e_2, then $\operatorname{codim} U = e_2(r + e_2 - N)$. In our case $r = p - n + e_1$, $N = \binom{e_1+1}{2}$, which gives $\operatorname{codim}_\Lambda (\Lambda \cap J_2(n,p)_{(e_1,e_2)}) = e_2\left(p - n + e_1 + e_2 - \binom{e_1+1}{2}\right)$

Therefore

$$\operatorname{codim} J_k(n,p)_{(e_1,e_2)} = e_1(p-n+e_1) + e_2\left(p-n+e_1+e_2-\binom{e_1+1}{2}\right)$$

$$= (e_1+e_2)(p-n) + e_1^2 + e_2\left[e_2-\binom{e_1}{2}\right].$$

§ 13 Local algebras with $\dim(m^2/m^3) = 1$

If e, s, t are integers, $s \geq 2$, $2 \leq t \leq e$, $e \geq 2$, we denote by $Q_{s,t}(e)$ the local K-algebra

$$K[[x]]/I,$$

where $x = (x_1,\dots,x_e)$ and I is the ideal generated by

$$x_1^{s+1},\ x_2^2 + x_1^s,\ \dots,\ x_t^2 + x_1^s,\ x_{t+1}^2,\dots,x_e^2 \quad \text{and}$$

$$x_i x_j \qquad 1 \leq i \neq j \leq e.$$

By $Q_{s,1}(e)$ we denote the algebra defined by $x_1^{s+1} = x_2^2 = \dots = x_e^2 = x_i x_j = 0$ $(i<j)$.

Proposition 20 If Q is a local Artinian K-algebra such that $\dim(m/m^2) = e > 1$ and $\dim(m^2/m^3) = 1$, then Q is isomorphic to exactly one of the algebras $Q_{s,t}(e)$. We have

$$\dim Q_{s,t}(e) = e + s$$

$$i(Q_{s,t}(e)) = \begin{cases} \binom{e}{2} & \text{if } t = 1 \\ \binom{e}{2} - 1 & \text{if } t > 1 \end{cases}$$

$$\dim(\Delta/\Delta m) = e$$

$$g(Q_{s,t}(e)) = \begin{cases} e^2 + s - 1 & \text{if } t = 1 \\ e(e - t + 1) + \binom{t}{2} + s - 1 & \text{if } t > 1 \end{cases}.$$

Proof: The calculation of the invariants is easy. Now assume $\dim(m/m^2) = e > 1$ and $\dim(m^2/m^3) = 1$.

If $x^2 \in m^3$ for all $x \in m$, then $m^2 = 0$, which contradicts $\dim(m^2/m^3) = 1$. Hence there is an $x_1 \in m$ and $x_1^2 \notin m^3$, therefore $m^v = x_1^v Q$, $v \geq 2$, by Nakayamas lemma.

If $m^s \neq 0$, $m^{s+1} = 0$, then $x_1^s \neq 0$, $x_1^{s+1} = 0$.

Furthermore

$$\dim(0:x_1) = \dim Q - \dim x_1 Q$$

$$= \dim (Q/x_1 Q)$$

$= \dim (Q / m^2 + x_1 Q)$ (since $m^2 \in x_1{}^2 Q \subseteq x_1 Q$)

$= e.$

Let $x_1{}^s, x_2, \ldots, x_e$ be a base of $(0 : x_1)$ and consider $\bar{Q} = Q / x_2 Q + \ldots + x_e Q$, then a multiplication by x_1 yields a surjection

$$Q \longrightarrow Qx_1$$

and $x_1 \bar{Q}$ is mapped onto $Qx_1{}^2 = m^2$, hence $\bar{Q} / x_1 \bar{Q} \cong Qx_1 / m^2 \cong K$, i.e. $m = x_1 Q + x_2 Q + \ldots + x_e Q$.

If there is a $\nu \geq 2$ and $x_i x_j \in m^\nu$, but not all of them are in $m^{\nu+1}$ for $i,j \geq 2$, then we get a non-zero quadratic form on the vector space $Kx_2 + \ldots + Kx_e$ by

$$(x,y) \longrightarrow x\, y \bmod m^{\nu+1}.$$

We can diagonalize it, hence we can assume $x_i x_j \in m^{\nu+1}$ for $i \neq j$. If $x_2{}^2 = n\, x_1{}^\nu$, $n \in Q^*$, then multiplying by x_1 we get $x_1{}^{\nu+1} = 0$, hence $\nu = s$ and normalizing the x_ν, $\nu = 2, \ldots, e$ we get $t \geq 2$ (the rank of the quadratic form) such that

$$x_2{}^2 = x_1{}^s, \ldots, x_t{}^2 = x_1{}^s, \quad x_\nu{}^2 = 0 \quad (\nu > t) \text{ and}$$

$x_i x_j = 0$ $(i \neq j)$, hence Q is isomorphic to $Q_{s,t}(e)$.

If all $x_i x_j = 0$, $i,j \geq 2$, then Q is isomorphic to $Q_{s,1}(e)$.

Remark: If $C_{s,t}(e)$ denotes the contact class of $Q_{s,t}(e)$ in $J_k(n,p)_e$, the boundary of $C_{s,t}(e)$ consists of all contact classes $C_{\alpha\,\beta}(e)$, where $\alpha > s$ or $\alpha = s$ and $\beta < t$.

Proof: The jet $(x_1{}^{s+1}, x_2{}^2 + x_1{}^s, \ldots, x_\beta{}^2 + x_1{}^s, x_{\beta+1}{}^2 + \lambda^2 x_1{}^s, \ldots$
$\ldots, x_t{}^2 + \lambda^2 x_1{}^s, x_{e+1}{}^2, \ldots, x_e{}^2, x_i x_j, i<j)$ is contained in $C_{s,t}(e)$ for $\lambda \neq 0$ and specializes to $C_{s,\beta}$ for $\lambda = 0$.

If $\alpha > s$, $\beta > t$, consider the jet

$$(x_1{}^{\alpha+1}, x_2{}^2 + \lambda^2 x_1{}^s + x_1{}^\alpha, \ldots, x_t{}^2 + \lambda^2 x_1{}^s + x_1{}^\alpha, x_{t+1}{}^2 + x_1{}^\alpha, \ldots$$
$$\ldots, x_\beta{}^2 + x_1{}^\alpha, x_{\beta+1}{}^2, \ldots, x_e{}^2, x_i x_j, i<j).$$

It is contained in $C_{s,t}(e)$ if $\lambda \neq 0$ and specializes to $C_{\alpha,\beta}(e)$ for $\lambda = 0$.

If $C_{\alpha,\beta}$ (e) is a specialization of $C_{s,t}(e)$, then, necessarily, $\dim Q_{s,t}(e) \leq e + s \leq \dim Q_{\alpha,\beta}(e) = e + \alpha$, hence $s \leq \alpha$. If $s = \alpha$, then because of $\mu_{p-n}(Q_{s,t}(e)) \leq \mu_{p-n}(Q_{\alpha,\beta}(e))$ we must have $g(Q_{s,t}(e)) \leq \dim g(Q_{\alpha,\beta}(e))$, hence $t \geq \beta$ q.e.d.

Corollary 1 Assume Q is a local Henselian K-algebra and $\dim(m / m^2) > 1$, $\dim(m^2 / m^3) = 1$, $\dim (Q) = \infty$. Then Q contains an algebra isomorphic to the algebra $K\langle x_1, \ldots, x_e\rangle / (x_2^2, \ldots, x_e^2, x_1x_2, \ldots, x_1x_e, x_2x_3, \ldots$ $\ldots, x_2x_e, \ldots, x_{e-1}x_e)$ and both have the same completion.

Proof: Q / m^{s+1} is isomorphic to $Q_{s,t}(e)$, hence for any integer s the system of equations

$$x_1U_2 = \ldots = x_1U_e = 0$$

has a solution modulo m^{s+1} which together with x_1 forms a base of m.

For $s = 2e - 1$ we can apply Newton's lemma, hence there exists a base $x_1, \ldots, x_e$ of m such that $x_1x_j = 0$ for $j \geq 2$.

If $x_ix_j \neq 0$ for some $i,j \geq 2$, then we would have $x_ix_j = ux_1^{\nu}$, $u \in Q^*$, which would imply $x_1^{\nu+1} = 0$, i.e. $m^{\nu+1} = 0$, hence a contradiction.

§ 14 Non-simple contact classes of $J_k(n,p)_e$, $e \geq 4$

Proposition 21 For $k \geq 2$ and $e \geq 4$ we have

$$J_k(n,p)_e \cap W_k(n,p) \supseteq \begin{cases} J_k(n,p)_e & \text{if } p-n \leq \binom{e}{2} - 2 \\ \overline{J_k(n,p)_{(e,2)}} & \text{if } p-n \geq \binom{e}{2} - 1 \end{cases}$$

($\overline{J_k(n,p)_{(e,2)}}$ means the Zariske closure in $J_k(n,p)_e$).

Proof: It is sufficient to proof this for $k = 2$.

If $p-n \leq \binom{e}{2} - 2$, then $J_2(n,p)_{(e,q)} = \phi$ for $q = 0$ and 1 by lemma 2 of § 12.

Therefore we have to show (by proposition 20) that

$$J_2(n,p)_{(e,q)} \subseteq W_2(n,p) \quad \text{for} \quad q \geq 2.$$

Consider the map

$$\Lambda_2(n,p)_e \cap J_2(n,p)_{(e,q)} \longrightarrow \text{Grass}(q, S^2(Kx_1 +\ldots+ Kx_e)) = G$$

(subspaces of codimension q) which associates, to z, the kernel of the multiplication

$$S^2(Kx_1 +\ldots+ Kx_e) \longrightarrow m(z)^2$$

(S^2 means the second symmetric power).

This is a morphism and the contact classes correspond to the $\mathbb{P}Gl(e)$-orbits of G.

But
$$\dim G = q(\binom{e+1}{2} - q),$$
$$\dim \mathbb{P}Gl(e) = e^2 - 1$$

and the smallest value of q, for which $J_2(n,p)_{(e,q)} \neq \phi$, is $q = \max(0, \binom{e}{2} - (p-n))$, in this case $J_2(n,p)_{(e,q)}$ is dense in $J_2(n,p)_e$.

So it is sufficient to prove:

If $q_o = \max(2, \binom{e}{2} - (p-n))$, then

(*) $$e^2 - 1 < q_o(\binom{e+1}{2} - q_o),$$

because then there are no open contact classes, hence no simple contact classes. If $q_o = 2$, then (*) is true because of $e \geq 4$.

Assume $q_o = \binom{e}{2} - (p-n) > 2$, then we have to prove, by putting $p-n = r$, that

$$f(r) = [\binom{e}{2} - r][\binom{e+1}{2} - \binom{e}{2} + r] - e^2$$
$$= [\binom{e}{2} - r][e + r] - e^2 \geq 0$$

for $r = 0, \ldots, \binom{e}{2} - 3$.

But $f(0) = \binom{e}{2} e - e^2 > \frac{e^2(e-3)}{2} > 0$

$$f(\binom{e}{2} - 3) = 3[e + \binom{e}{2} - 3] - e^2$$
$$= \frac{e^2}{2} + \frac{3e}{2} - 9 > 0 \quad \text{if} \quad e \geq 4,$$

hence $f(r) > 0$ q.e.d.

For the following sections take into consideration that $\bigcup_{e \geq 4} J_k(n,p)_e$ is closed in $J_k(n,p)$, hence contact classes which are simple in $J_k(n,p)_0 \cup J_k(n,p)_1 \cup J_k(n,p)_2 \cup J_k(n,p)_3$ are simple in $J_k(n,p)$, too.

§ 15 Non-simple contact classes of $J_k(n,p)_3$

The same calculation as in § 14 with $e = 3$ yields

$$J_k(n,p)_{(3,3)} \cup J_k(n,p)_{(3,4)} \cup J_k(n,p)_{(3,5)} \cup J_k(n,p)_{(3,6)} \subseteq W_k(n,p).$$

Hence the only case left is that of $J_k(n,p)_{(3,2)}$, $p > n$.

There must be 4 equations with linear independent quadratic initial forms in x,y,z, hence the first step is to classify 3-dimensional linear systems of conics in $\mathbb{P}^2$. If the conics are given by 4 linear independent symmetric (3×3)-matrices S_1,S_2,S_3,S_4 , we can consider the space of all symmetric (3×3)-matrices S satisfying

$$Tr(\mathfrak{S}_1 S) = Tr(\mathfrak{S}_2 S) = Tr(\mathfrak{S}_3 S) = Tr(\mathfrak{S}_4 S) = 0$$

which gives a duality between 3-dimensional systems of conics and pencils of conics in $\mathbb{P}^2$. Choosing a suitable base in the pencil we have 4 possibilities for the discriminant

$$\det(\lambda S_1 + \mu S_2) = \Delta(\lambda,\mu)$$

of the generic member of the conic

(I) $\Delta(\lambda,\mu) \equiv 0$

(II) $\Delta(\lambda,\mu) = \lambda^3$

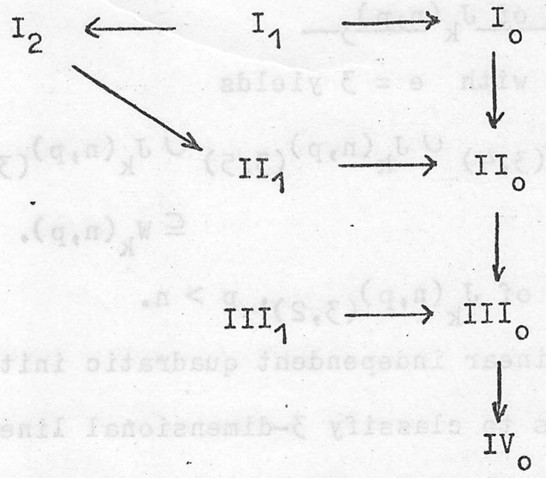
(III) $\Delta(\lambda,\mu) = \lambda^2\mu$

(IV) $\Delta(\lambda,\mu) = \lambda\mu(\lambda+\mu)$

To each root of $\Delta(\lambda,\mu)$ a degenerate conic corresponds, i.e. a line pair or a double line. To a simple root there never corresponds a double line because if we consider a double line given by $S_1 = \begin{pmatrix}100\\000\\000\end{pmatrix}$, then

$$\det(S_1+\lambda S_2)(e_1\wedge e_2\wedge e_3) = \lambda^2((e_1+\lambda S_2 e_1)\wedge S_2 e_2 \wedge S_2 e_3)$$

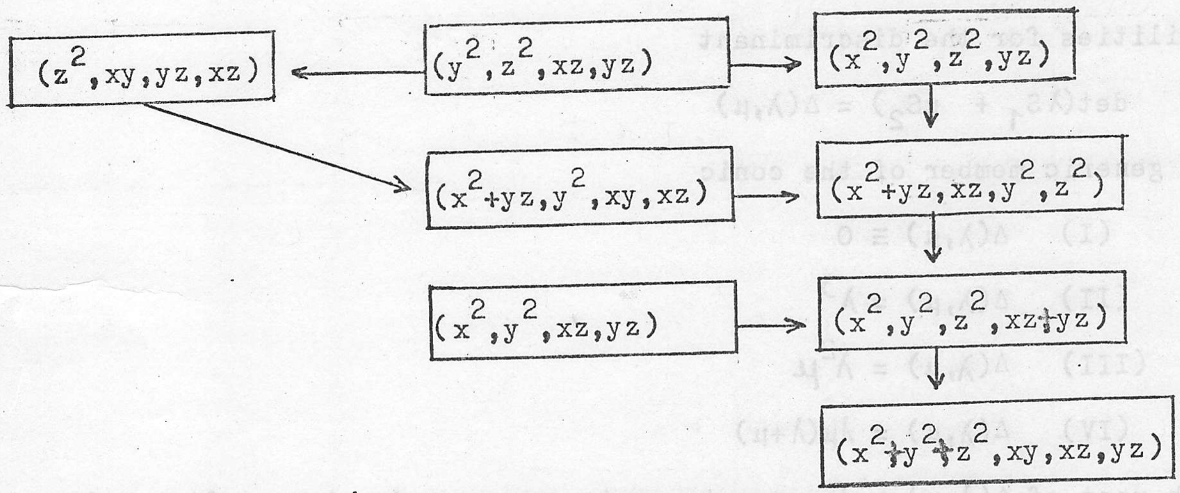
(e_1,e_2,e_3) denotes the columns of the unit matrix), hence λ is a double root. Indicating the number of double lines corresponding to $\lambda = 0$ we get the following possibilities



(The arrow A → B indicates that A lies on the boundary of B)

(In case I we consider (2×2)-minors, too.)

If we write out the matrices and change over to dual systems, we get the following possibilities for the quadratic initial forms:



The contact classes corresponding to I_o, II_o, III_o, IV_o (the last column) do not split in $J_k(n,p)$, hence we found 4 simple contact classes, and 4 remain to be investigated.

Let $C_{I_2} = C$ be the contact class in $J_2(n,p)$ corresponding to I_2; if $z \in j^{-1}(C_{I_2})$ $(z \in J_k(n,p),\ k \geq 3)$ and if $Q = Q(z)$ is the corresponding algebra, then $m^\nu = x^\nu Q + y^\nu Q$ $(\nu \geq 2)$ and we can choose $x \bmod m^2$, $y \bmod m^2$ in such a way that $xy = 0$ (consider the equation

$XY = 0$ in Q and use Newton's lemma).

If $k = 3$, then Q is a specialization of

$K[[x,y,z]] / (xy, z^2 - t_o x^3 - t_1 y^3, xz - t_2 y^3, yz - t_3 x^3) + (x,y,z)^4$.

If $t_o, t_2, t_3 = 0$, this algebra is isomorphic to

$Q_t = K[[x,y,z]] / (xy, z^2 - x^3 - ty^3, xz - y^3, yz - x^3)$. One easily checks that

$Q_{t_1} \cong Q_{t_2}$ iff $t_1 = t_2$, this means:

$$j^{-1}(C_{I_2}) \subseteq W_k(n,p)$$

and therefore also

$$j^{-1}(C_{I_1}) \subseteq W_k(n,p)$$

(because C_{I_1} lies on the boundary of C_{I_2}).

So only the cases II_1 and III_1 are left.

If $z \in j^{-1}(C_{II_1})$, $Q = Q(z)$, then $m^\nu = z^\nu Q$ for $\nu \geq 3$, and using Newton's lemma we can choose x, y such that

$$x^2 + yz = xz = 0$$

(consider the equation $zX = 0$ and then the equation $zY + x^2 = 0$).

Hence Q is defined by $x^2 + yz = xz = 0$ and by at least two further equations with the initial forms y^2 and xy.

If $xy \in m^a \setminus m^{a+1}$, then the multiplication by z yields $m^{a+1} = z^{a+1}Q = 0$, hence

$$xy = t_o z^a.$$

If $y^2 \in m^b \setminus m^{b+1}$, then, because of $y^2 z = -x^2 y = -xt_o z^a = 0$, we get $z^{b+1} = 0$, hence $b = a$, $y^2 = t_1 z^a$. We can always normalize in such a way that $t_o = 0$ or 1 and $t_1 = 0$ or 1. Thus we get that Q is isomorphic to one of the following algebras $K[[x,y,z]] / I$

$Q_{a,o,o}$: $I = (x^2+yz, xz, xy, y^2, z^{a+1})$, $\quad a \geq 2$

$Q_{a,o,1}$: $I = (x^2+yz, xz, y^2+z^a, xy)$, $\quad a \geq 3$

$Q_{a,1,o}$: $I = (x^2+yz, xz, y^2, xy+z^a)$, $\quad a \geq 3$

$Q_{a,1,1}$: $I = (x^2+yz, xz, xy+z^a, y^2+z^a)$, $\quad a \geq 3$

But $Q_{a,1,1}$ is isomorphic to $Q_{a,0,1}$.

They are all of dimension $a + 4$ and

$$g(Q_{a,0,0}) = a + 8$$

$$g(Q_{a,0,1}) = a + 6$$

$$g(Q_{a,1,0}) = a + 7$$

(by direct calculation).

Similarly we treat the case III_1. By using Newton's lemma we may assume that

$$xz = yz = 0$$

and if $m^a \neq 0$, $m^{a+1} = 0$, we get 4 cases

$$x^2 = t_o z^a,\ y^2 = t_1 z^a,\ z^{a+1} = 0,\ t_o, t_1 \in \{0,1\},$$

but because of symmetry with respect to x,y we get for each a 3 algebras $Q' = K[[x,y,z]] / I$ of dimension $a + 4$

$$Q'_{a,0} : I = (xy, yz, x^2, y^2, z^{a+1}) \qquad a \geq 2,\ g(Q) = a + 7$$

$$Q'_{a,1} : I = (xz, yz, x^2, y^2 + z^a) \qquad a \geq 3,\ g(Q) = a + 6$$

$$Q'_{a,2} : I = (xz, yz, x^2 + z^a,\ y^2 + z^a) \qquad a \geq 3,\ g(Q) = a + 5.$$

Hence we got

<u>Proposition 22:</u> $J_k(n,p)_3 \cap W_t(n,p) \supseteq \overline{j^{-1}(C_{I_2})} \cup \overline{J_k(n,p)_{(3,3)}}$ (closure in $J_k(n,p)_3$) for $k \geq 3$ and the contact classes of $J_k(n,p)_3 \setminus (\overline{j^{-1}(C_{I_2})} \cup \overline{J_k(n,p)_{(3,3)}})$ are given by the following list

Table 2:

	Generators of I	dim(Q)	g(Q)	i(Q)	
$C_{s,1}(3)$	$x^{s+1}, y^2, z^2, yz, xz, xy$	s+3	s+8	3	s=1,...,k
$C_{s,2}(3)$	y^2+x^s, z^2, yz, xz, xy	s+3	s+6	2	s=2,...,k
$C_{s,3}(3)$	$y^2+x^s, z^2+x^s, yz, xz, xy$	s+3	s+5	2	

	Generators of I	dim(Q)	g(Q)	i(Q)	
C_{I_o}	x^2,y^2,z^2,yz	6	11	1	
C_{II_o}	x^2+yz,xz,y^2,z^2	6	9	1	
C_{III_o}	$x^2,y^2,z^2,xz+yz$	6	8	1	
C_{IV_o}	$x^2+y^2+z^2,xy,xz,yz$	6	7	1	
$Q_{a,0,0}$	x^2+yz,xz,xy,y^2,z^{a+1}	a+4	a+8	2	a=2,...,k
$Q_{a,1,0}$	$x^2+yz,xz,xy+z^a,y^2$	a+4	a+7	1	a=3,...,k
$Q_{a,0,1}$	x^2+yz,xz,xy,y^2+z^a	a+4	a+6	1	
$Q'_{a,0}$	xz,yz,x^2,y^2,z^{a+1}	a+4	a+7	2	a=2,...,k
$Q'_{a,1}$	xz,yz,x^2,y^2+z^a	a+4	a+6	1	a=3,...,k
$Q'_{a,2}$	xz,yz,x^2+z^a,y^2+z^a	a+4	a+5	1	

§ 16 Non-simple contact classes in $J_k(n,p)_e$, $e \leq 2$

$J_k(n,p)_0$ is an open orbit and $J_k(n,p)_1$ consists of finite many simple orbits. It remains to investigate $J_k(n,p)_{(2,2)}$ and $J_k(n,p)_{(2,3)}$.

If $z \in J_3(n,p)_{(2,3,\nu)} \cap \Lambda_3(n,p)_2$, then it is, up to contact equivalence, uniquely determined by the multiplication $S^3(Kx + Ky) \longrightarrow m(z)^3$. Hence associating the kernel of this multiplication to z we get a morphism

$$J_k(n,p)_{(2,3,\nu)} \cap \Lambda_3(n,p)_2 \longrightarrow \mathrm{Grass}(\nu, S^3(Kx + Ky)),$$

and the contact classes correspond to the $\mathrm{PGl}(2)$-orbits.

Because of $\dim \mathrm{PGl}(2) = 3$, $\dim \mathrm{Grass}(\nu, S^3(Kx + Ky)) = \nu(4 - \nu)$ there are no open contact classes and hence no simple contact classes in $J_3(n,p)_{(2,3,2)}$, i.e. $W_k(n,p) \supseteq J_k(n,p)_{(2,3,2)} \cup J_k(n,p)_{(2,3,3)} \cup J_k(n,p)_{(2,3,4)}$ $(k \geq 3)$. $J_3(n,p)_{(2,3,0)}$ consists of exactly one orbit corresponding to the algebra $Q = K[[x,y]] / (x,y)^3$, which does not split in $J_k(n,p)$, $k \geq 3$, hence it is a simple orbit in $J_k(n,p)_{(2,3)}$.

$J_3(n,p)_{(2,3,1)}$ consists of exactly 3 orbits corresponding to 2-dimensional linear systems of degree 3 on $\mathbb{P}^1$, i.e. to morphisms $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ given by 3 cubic forms.

Case 1 (fixed component)

$$C(0) : Q = K[[x,y]] / (x^2y, xy^2, y^3) + (x^4)$$

Case 2 (no fixed component, the image of $\mathbb{P}^1$ in $\mathbb{P}^2$ has a cusp)

$$C'(0) : Q = K[[x,y]] / (x^3, y^3, xy^2),\ \dim(Q) = 7,\ g(Q) = 10$$

Case 3 (no fixed component, the image of $\mathbb{P}^1$ in $\mathbb{P}^2$ has a node)

$$C''(0) : Q = K[[x,y]] / (xy^2, x^2y, x^3+y^3),\ \dim(Q) = 7,\ g(Q) = 8.$$

The classes corresponding to case 2 and 3 do not split in $J_k(n,p)$ and the contact class corresponding to case 1 splits in $J_k(n,p)$, $k \geq 3$, into finite many classes corresponding to $C(0)_a$: $Q = K[[x,y]] / (x^{a+1}, x^2y, xy^2, y^3)$, $a = 3,\dots,k$, $\dim Q = a + 4$, $g(Q) = a + 8$.

Now we consider contact classes in $J_k(n,p)_{(2,2)}$. If $k = 2$, then one quadratic form in x,y must vanish, i. e. for a suitable choice of coordinates x,y: if $Q = Q(z)$, $z \in J_k(n,p)_{(2,2)}$

either $xy \in m^3$ or $y^2 \in m^3$.

In the first case we can use Newton's lemma (cf. [11]) for the equation $XY = 0$ and we infere, that x,y can be choosen in such a way that $xy = 0$. Hence we get up to isomorphism the following algebras:

$C(1)_{a,c}$: $Q = K[[x,y]]/(xy,x^a-y^c)$ $c \geq a > 2$

$C(2)_{a,c}$: $Q = K[[x,y]]/(xy,x^{a+1},y^{c+1})$ $c \geq a \geq 2$

Now if $y^2 \in m_Q^3$, then $m^{\nu+1} = xm^{\nu}$ and $\dim(O:x) = \dim(Q/xQ) = 2$.
Let $m_Q^{c+1} \neq 0$, but $m_Q^c = 0$, then Q is a quotient of

$Q(a,c) = K[[x,y]]/(y^2-x^a,x^cy,x^{c+1})$ $3 \leq a \leq c+1$

We consider the following cases:

case (i): $h = (2,\dots,2)$ and $c \geq 2$; i.e. $\dim(m^c) = 2$.

Then we have

$C(3)_{a,c}$: $Q = Q(a,c)$ $3 \leq a \leq c+1$

case (ii): $h = (2,\dots,2,1)$ and $c \geq 3$; i.e. $\dim(m^c) = 1$,

but $\dim(m^{c-1}/m^c) = 2$.

Of course we have $Q = Q(a,c)/(f)$, $f \in m_{a,c}^c = Kx^c + Kx^{c-1}y$.
Suppose $f = sx^c + tx^{c-1}y$ and $s \neq o$ and $t \neq o$.
Using the following substitutions we get $t = o$ or $s = o$:

1st step: $x := x - \frac{t}{c-1}y$, then $Q = K[[x,y]]/(y^2-x^a\varepsilon_1+x^{a-1}y\varepsilon_2,x^c)$

ε_i are unites from $K[[x]]$.

2nd step: $y := y - \frac{1}{2}x^{a-1}\varepsilon_2$, then $Q = K[[x,y]]/(y^2-x^a\varepsilon,x^c)$,

ε a unite.

3rd step: $x := x\eta$, η a suitable unite.

Hence we have the following algebras:

$C(4)_{a,c}$: $Q = K[[x,y]]/(y^2-x^a,x^c)$ $3 \leq a \leq c$

$C(5)_{a,c}$: $Q = K[[x,y]]/(y^2-x^a,x^{c-1}y,x^{c+1})$ $3 \leq a,c$; $a \leq c+1$

case (iii): $h = (2,\dots,2,1,1)$ and $c \geq 4$; i.e.

$\dim(m^{c-2}/m^{c-1}) = 2$ and $\dim(m^{c-1}/m^{c}) = \dim(m^{c}) = 1$.

We have again $Q = Q(a,c)/(f)$, but $f \in m_{a,c}^{c-1} - m_{a,c}^{c}$,
$m_{a,c}^{c-1} = Kx^{c-1} + Kx^{c} + Kx^{c-2}y + Kx^{c-1}y$. If we assume

$$x^{c-1} = sx^{c} + tx^{c-2}y + ux^{c-1}y \text{ , we get}$$

$$x^{c-1}(1 - sx - uy) = tx^{c-2}y$$

$$x^{c-1} = tx^{c-2}y(1 + sx + uy) = tx^{c-2}y + stx^{c-1}y$$

$$x^{c-1}(1 - sty) = tx^{c-2}y$$

$$x^{c-1} = (1 + sty)tx^{c-2}y = tx^{c-2}y.$$

But then we obtain the relation

$$x^{c} = tx^{c-1}y = t^2x^{c-2}y^2 = t^2x^{c-2+a} \text{ , hence}$$

$$x^{c} = 0 \text{ and } x^{c-1}y = 0 \text{ , which is a contradiction}$$

because of $m^{c} \neq 0$.

Therefore there remains only the case

$$x^{c-2}y = sx^{c} + tx^{c-1}y$$

$$x^{c-2}y(1 - tx) = sx^{c}$$

$$x^{c-2}y = sx^{c}(1 + tx) = sx^{c}$$

If $s = 0$ we get

$$C(6)_{a,c}: \quad Q = K[[x,y]]/(y^2-x^a, x^{c-2}y, x^{c+1})$$

$$3 \leq a,\ 4 \leq c,\ a+1 \leq c+1$$

(for $a = 3$ $x^{c+1} \in (y^2-x^3, x^{c-2}y)$.)

If $s \neq 0$ and $a = 3$ +) or $c = 4$ the following transformation reduces Q to the case $s = 0$:

$y := y + sx^2$; $x := x\varepsilon$, ε a unite.

If $s \neq 0$ and $4 = a < c$ the contact classes

$$C_c^*(t): \quad Q_t = K[[x,y]]/(y^2-x^4, x^{c-2}y-tx^{c}, x^{c+1}) \qquad c \geq 5$$

are not simple, because $Q_t \cong Q_{t'}$ iff $t^2 = t'^2$.

All other contact classes are contained in the closure of

+) for $a = 3$ the transformation has the form
$x := x+\lambda y$; $y := y+\varepsilon x^2 + \varepsilon' xy$, $\lambda \in K$, $\varepsilon, \varepsilon' \in K[[x]]^*$

$\bigcup_t C_c^*(t)$:

$$J_k(n,p)_{(2,\ldots,2,1,1)} = C(6)_{3,c} \cup \left(\overline{\bigcup_t C_c^*(t)}\right) ; \quad c \geq 5.$$

Since $g(Q_t) = 2c + 2$ we get $\operatorname{codim}(\bigcup_t C_5^*(t)) = 8(n-p)+11$.

Proposition 23:

$$J_k(n,p)_2 \cap W_k(n,p) = \overline{J_k(n,p)_{(2,3,2)} \cup j^{-1}(\bigcup_t C_5^*(t))}$$

(closure in $J_k(n,p)_2$) for $k \geq 5$.

The contact classes of the complement of $W_k(n,p)$ in $J_k(n,p)_2$ are given by the following list.

$\operatorname{codim}(J_k(n,p)_2 \cap W_k(n,p)) = \operatorname{codim} J_k(n,p)_{(2,3,2)} = 7(p-n)+10$.

Table 3:

	Generators of I	dim(Q)	g(Q)	i(Q)	
$C_{s,2}(2)$	xy, y^2+x^s	s+2	s+2	0	$s \geq 2$
$C_{s,1}(2)$	xy, y^2, x^{s+1}	s+2	s+3	1	$s \geq 1$
$C(0)_a$	x^{a+1}, x^2y, xy^2, y^3	a+4	a+8	2	$a \geq 2$
$C'(0)$	x^3, y^3, xy^2	7	10	1	
$C''(0)$	x^3+y^3, xy^2, x^2y	7	9	1	
$C(1)_{a,c}$	xy, x^a-y^c	a+c	a+c*)	0	$c \geq a > 2$
$C(2)_{a,c}$	xy, x^{a+1}, y^{c+1}	a+c+1	a+c+2	1	$c \geq a > 2$
$C(3)_{a,c}$	y^2-x^a, x^cy, x^{c+1}	2c+1	2c+a	1	$3 \leq a \leq c+1$
$C(4)_{a,c}$	y^2-x^a, x^c	2c	2c+a-2*)	0	$3 \leq a \leq c$
$C(5)_{a,c}$	$y^2-x^a, x^{c-1}y, x^{c+1}$	2c	2c+a-1	1	$3 \leq a \leq c+1$ $3 \leq c$
$C(6)_{3,c}$	$y^2-x^3, x^{c-2}y$	2c-1	2c	0	$c \geq 4$
$C(6)_{a,4}$	$y^2-x^a.x^2y.x^5$	7	a+5	1	a = 4,5

*) not for all values of char K

§ 17 The codimension of $W_k(n,p)$

In each $J_k(n,p)_e$ we have now determined a $\mathcal{C}_k(n,p)$-stable subset which consists of non-simple orbits only and whose complement contains only finite many orbits. Therefore the union of these sets is dense in $W_k(n,p)$ and we can use it for computing the codimension of $W_k(n,p)$.

For $e \leq 1$ we got the empty set, for $e = 2$ we got a set of codimension $7(p-n)+10$, for $e = 3$ we got the sets $j^{-1}(C_{I_2}) \cup J_k(n,p)_{(3,3)}$ (where C_{I_2} was the contact class in $J_2(n,p)$ given by $Q = K[[x,y]] / (z^2, xy, yz, xz, x^3, y^3)$) which are of codimension $6(p-n)+11$ and $6(p-n)+9$ respectively, and for $e \geq 4$ we got the set $J_k(n,p)_e$ if $\binom{e}{2} \geq p-n+2$ and $J_k(n,p)_{(e,2)}$ if $\binom{e}{2} \leq p-n+1$, hence the codimension of this set is

$$e(p-n+e) \qquad \text{if } p-n \leq \binom{e}{2} - 2$$

$$(e+2)(p-n) + e^2 + 2\left[2-\binom{e}{2}\right] = (e+2)(p-n) + e + 4 \qquad \text{if } p-n \geq \binom{e}{2} - 2.$$

The codimension of $W_k(n,p)$ is the minimum of all these integers, that is why the result is

Proposition 24

$$\operatorname{codim} W_k(n,p) = 6(p-n)+9, \qquad p-n \leq 3$$

$$\operatorname{codim} W_k(n,p) = 6(p-n)+8, \qquad p-n \geq 4.$$

§ 18 Stable equivalence classes in the nice range

The nice range is characterized by $n \leq 6(p-n)+8$ or $n \leq 6(p-n)+7$, $p-n \geq 4$. We have $W_k(n,p) \subseteq V_k(n,p) \subseteq \Pi_k(n,p)$ in this range by proposition 19. Thus, a contact class not contained in $\Pi_k(n,p)$ must be simple. If it is represented by an algebra Q, we must have

(*) $\quad 7(p-n)+8 \geq p \geq \mu_{p-n}(Q) = \dim(Q)(p-n) + g(Q)$

and $i(Q) \leq p-n$.

If we want to determine all of these algebras, we can use the results of the preceeding sections.

For $\dim(Q) \leq 6$ the inequality (*) yields no restriction, but we must have $e \leq 5$ and we can use proposition 20 and 22 to determine all of these contact classes. For e = 2 the Hilbert-function of the corresponding algebra must be one of the following sequences h = (h(1),h(2),...)
(2,3,0), (2,2,1), (2,1,1,1), (2,2,0), (2,1,1), (2,1,0), and (2,0),
for e = 3 we have the possibilities (3,2,0), (3,1,1), (3,1,0), (3,0,0) , and for e = 4 we have the possibilities (4,1) and (4,0).

For dim (Q) = 7 the inequality (*) yields the restriction $g(Q) \leq 8$ and by proposition 20 we infere that in this case we have $e \leq 3$ as a necessary condition. For the Hilbert-function we have, if e = 2, the possibilities (2,3,1), (2,2,2), (2,2,1,1), and (2,1,1,1,1), and if e = 3, (3,2,1) and (3,1,1,1).

For dim (Q) > 7 we get the restriction

$$(\dim(Q) - 7)i(Q) + g(Q) \leq 8.$$

Checking the list in proposition 20, 22, and 23 we get as a necessary condition $e \leq 2$. By inspecting the list of the algebras of proposition 23 with Hilbert-function (2,2,1,1,1,...) and (2,2,2,1,1,...) we see that none of these algebras satisfies this restriction . Hence for e = 2 the Hilbert-function must be one of the following

(2,2,2,1,0) or (2,1,1,1,1,1,...).

The resulting contact classes are given in the following theorem.

Theorem: In the nice range, $n \leq 6(p-n)+8$ or $n \leq 6(p-n)+7$, $p-n \geq 4$, the following algebras described by the ideal of relations can appear as fibres of stable mapping germs $(\mathbb{A}^n,0) \longrightarrow (\mathbb{A}^p,0)$:

(1) If $p > n$, all algebras of table 1

(2) If $p = n$, there can appear those 15 algebras of table 1 for which $i(Q) = 0$ and still the following 4 algebras:

Hilbert-sequence	Generators of I	dim(Q)	g(Q)	i(Q)
(1,...,1,0)	x^9	9	8	0
(2,1,1,1,1,1,0)	y^2+x^6, xy	8	8	0
(2,2,1,1,1,0)	xy, x^5+y^3	8	8	0
(2,2,2,1,0)	xy, x^4+y^4	8	8	0

§ 18 Application to generic projections

We consider smooth projective varieties $V \subset \mathbb{P}^N$ of dimension n and p + 1 forms of the same degree d, $\varphi_o, \dots, \varphi_p$, then we get a rational map

$$\varphi = (\varphi_o : \dots : \varphi_p) : V \to \mathbb{P}^p.$$

If $p > n$ and $\varphi_o, \dots, \varphi_p$ are sufficiently general, then φ is a finite birational morphism, hence $V \to \varphi(V)$ is a normalization map.

We want to show

Theorem. If $(p, p-n)$ is in the nice range, there exists a nonempty Zariski open subset U of $\mathrm{Grass}\,(p, |\mathcal{O}_V(d)|)$, $d > p$, such that for each $\Lambda \in U$ the corresponding map $\varphi_\Lambda : V \to \mathbb{P}^p$ has the following properties

(1) φ_Λ is a finite birational morphism.

(2) For each $x \in V$ the germ $\varphi_\Lambda : (V,x) \to (\mathbb{P}^p, \varphi(x))$ is stable (hence up to equivalence there are only 48 of such germs).

(3) The branches of φ are crossing normally.

By condition (3) we mean the following property:

Let $V \xrightarrow{\varphi} W$ be a morphism of smooth varieties, then $x_1, \dots, x_r \in \varphi^{-1}(y)$ and $\varphi_i : (V, x_i) \to (W, y)$ the germ induced by φ.

Consider the map $T(\varphi_i) : \Theta_{V,x_i} \to (\varphi_i^* O_W)_{x_i}$ and the natural map

$$j : \Theta_{W,y} \to (\varphi_i^* \Theta_W)_{x_i} .$$

Then let $T_i \subset T_y(W)$ be the subspace

$$j^{-1}(T(\varphi_i)(\Theta_{V,x_i}) + m_y(\varphi^* \Theta_W)_{x_i}) / m_y \Theta_{W,y}$$

$$T_i \subset T_{x_i}(\varphi)(T_{x_i}(V)) \subset T_y(W) .$$

We say that the branches of φ are crossing normally if for any finite set $\{x_1, \dots, x_r\} \subset \varphi^{-1}(y)$ $\quad \mathrm{codim}\,(\bigcap_{i=1}^{r} T_i) = \sum_{i=1}^{r} \mathrm{codim}\, T_i$ (the codimension in $T_y(W)$) holds. If $p = \dim(W) > n = \dim(V)$, this implies $\#\varphi^{-1}(y) \leq \frac{p}{p-n}$ and moreover, if all φ_i are stable, this implies

$$\dim Q(\varphi) \leq \frac{p}{p-n}$$

where $Q(\varphi)$ is the semilocal algebra

$$(\varphi_* \mathcal{O}_V)_y \,/\, m_y(\varphi_* \mathcal{O}_V)_y \,.$$

For the proof observe $\dim T_i \leq n$, hence $p \geq \operatorname{codim}(\bigcap_{i=1}^{r} T_i) = \sum_{i=1}^{r} \operatorname{codim} T_i \geq r(p-n)$, hence $r \leq \frac{p}{p-n}$. Furthermore, if each φ_i is stable, then one shows by using the characterization of stable mapping germs and property (3) and by replacing V by $\operatorname{Spec}(\mathcal{O}^h_{V,y})$, W by $W \times_V \operatorname{Spec}(\mathcal{O}^h_{V,y}) = \coprod \operatorname{Spec}(\mathcal{O}^h_{W,x_i})$ that:

$$\varphi^* \Theta_W = K^p + T(\varphi)(\Theta_V) + m_y(\varphi^* \Theta_W).$$

As in the case of mapping germs we get from this : $(p-n)\dim Q(\varphi) \leq p$, hence

$$\dim Q(\varphi) \leq \frac{p}{p-n}\,.$$

As in the case of mapping germs one also gets that contact equivalence coincides with equivalence, hence we have

<u>Corollary</u>. If $(p,p-n)$ is in the nice range and $\Lambda \in U$, then the singularities of $\varphi_\Lambda(V)$ are up to etal equivalence uniquely determined by the Artinian algebras

$$Q_y(\varphi_\Lambda) = (\varphi_{\Lambda *} \mathcal{O}_V)_y \,/\, m_y(\varphi_* \mathcal{O}_V)\,.$$

If $p > 2n$, φ_Λ is a closed embedding.

§ 19 The jet bundle

We want to give an algebraic construction of a bundle $J_k(V,W) \to V \times W$, where V, W are smooth algebraic varieties such that the fibre over $(x,y) \in V \times W$ is the variety of k-jets $J_k((V,x), (W,y))$ and such that any morphism $\varphi : V \to W$ induces a V-morphism

$$j_k(\varphi) : V \to J_k(V,W)$$

which associates the k-jet of $\varphi : (V,x) \to (W, \varphi(x))$ to $x \in V$. More generally, let T be a scheme and $V \to T$, $W \to T$ be T-schemes. We want to show that the cofunctor on the category of Vx_TW-schemes which associates to each $\psi_o : U \to Vx_TW$ the set

$$J_k(U) = \{\psi \in Hom_V(\Delta_k(V/T)x_VU, Vx_TW) \mid \psi / U = \psi_o\}$$

is representable.

Here $\Delta_k(V/T)$ denotes the k^{th} infinitesimal neighbourhood of the diagonal $\Delta \in Vx_TV$, i.e. if Δ is defined by the sheaf of ideals $\mathfrak{J} \subset Q_{Vx_TV}$, then $\Delta_k(V/T)$ is defined by the sheaf $\mathfrak{J}^{k+1}$. We identify V with $\Delta \subset \Delta_k(V/T)$, then $\Delta_k(V/T)x_VU \supset \Delta x_VU = U$. $\Delta_k x_V \mathcal{U}$ is the pull back with resp. to the first projection $\Delta_k \to V$. If especially $U \to Vx_TW$ factors through a point $(x,y) \in Vx_TW$ over $t \in T$, then we get, because of $\Delta_k(V/T)x_V \{(x,y)\} = (V_t,x)_k \times \{y\}$ (where $(V_t,x)_k = Spec(Q_{V_t,x}/m_x^{k+1})$)

$$(J_k)_{(x,y)}(U) = \{\psi \in Hom((X_t,x)_k \times U, Y_t) \mid \psi(\{x\} \times U) = \{y\}\}$$

for the fibre $(J_k)_{(x,y)}$ over (x,y), hence $(J_k)_{(x,y)} = J_k((X_t,x)_k, (Y_t,y)_k)$.

If $\varphi : V \to W$ is a T-morphism, we get a V-morphism

$$j_k(\varphi) = \varphi \circ pr_2 : \Delta_k(V/T)x_VV = \Delta_k(V/T) \xrightarrow{pr_2} V \xrightarrow{\varphi} W$$

(where we consider V as a Vx_TW-scheme by $V \xrightarrow{(id,\varphi)} Vx_TW$) which gives for each $x \in V$ the k-jet $j_k(\varphi) \in J_k((V_t,x),(W_t, \varphi(x)))$. For $k' > k$ we have $\Delta_k(V/T)\ \Delta_{k'}(V/T)$, hence by restriction we also get a natural transformation which associates to each k'-jet the corresponding k-jet.

The representability of J_k is a consequence of the following lemma applied to $W' = W x_T V$, $V' = \Delta_k(V/T)$ under the condition that V is smooth over T (hence $\Delta_k(V/T) \xrightarrow{pr_1} V$ is flat and projective) and $W \to T$ is smooth and quasi-projective.

Lemma. Let V be a noetherian scheme, $V' \to V$ a flat, projective morphism, $W' \to V$ a flat quasi-projective morphism. Then the cofunctor on the category of V-schemes,

$$U \longmapsto \mathrm{Hom}_V(V' x_V U,\ W'),$$

is representable by a V-scheme $J \to V$ and a universal V-morphism $V' x_V T \xrightarrow{\psi} Y'$. The connected components of M are quasi-projective over V; if V' is finite over V, then J is quasi-projective over V. Furthermore, (J, ψ) has the following properties:

(1) For each morphism $\tilde{V} \to V$ the couple $(\tilde{V} x_V J,\ \tilde{V} x_V \psi)$ is a representation of the cofunctor $(\tilde{U} \to \tilde{V}) \longmapsto \mathrm{Hom}_V(\tilde{V}' x_V \tilde{U}, \tilde{V})$ $(\tilde{V}' = V' x_V \tilde{V}, \tilde{W}' = W' x_V \tilde{V})$.

(2) If $W'' \subset W'$ is an open subscheme and $J'' = J \setminus pr_{\mathfrak{Z}}(\psi^{-1}(W' \setminus W''))$, then $(J'', \psi / V' x_V J'')$ is a representation of cofunctor $(U \to V) \longmapsto \mathrm{Hom}_V(V' x_V U, W'')$.

(3) If $V' \to V$ has a section σ, then J has a natural structure as a W'-scheme described by μ and given as follows:

$$\begin{array}{ccc} V' x_V J & \xrightarrow{\ \psi\ } & W' \\ (\sigma \circ \mu, id) \uparrow & \nearrow \mu' & \\ J & & \end{array}$$

If furthermore $V' \to V$ is finite, then $J \xrightarrow{\mu'} W'$ is affine.

Remark: In our application the section σ will be the diagonal embedding.

Proof of the lemma: Let $W' \subsetneq \overline{W}'$ be a projective closure of W' over T, then $\mathrm{Hom}_V(V' x_V U, W') \subset \mathrm{Hom}_U(V' x_V U, \overline{W}' x_V U)$. It is well known that this functor $(U \to V) \longmapsto \mathrm{Hom}_U(V' x_V U, \overline{W}' x_V U)$ is representable by a V-scheme

$\underline{Hom}_V(V',\overline{W}')$ (a subscheme of the Hilbert scheme of $V'x_V W'$ over V) (see[FGA]) and a universal morphism $V'x_V\underline{Hom}_V(V',\overline{W}') \xrightarrow{\overline{\psi}} \overline{W}'x_V\underline{Hom}_V(V',\overline{W}')$.

If $F = \overline{W}' - W'$ and if $M \subset \underline{Hom}_V(V',\overline{W}')$ denotes the image of

$$\overline{\psi}^{-1}(Fx_V\underline{Hom}_V(V',\overline{W}')) \subseteq V'x_V\underline{Hom}_V(V',\overline{W}') \xrightarrow{pr} \underline{Hom}_V(V',\overline{W}'),$$

then M is closed in $\underline{Hom}_V(V',\overline{W}')$, hence $J = \underline{Hom}_V(V',\overline{W}') \setminus M$ is an open subscheme of $\underline{Hom}_V(V',\overline{W}')$, and $\overline{\psi} / V'x_V J$ factors through $W'x_V J$, hence yields a J-morphism

$$\psi : V'x_V J \longrightarrow W'x_V J$$

such that (J, ψ) represents the functor which we consider. The remaining assertion of the lemma immediately follows from the properties of the Hilbert scheme or directly by the construction , except perhaps the last assertion.

To prove that $J \rightarrow Vx_T W$ is an affine morphism if $V' \rightarrow V$ is finite and has a section σ, we will describe the construction explicitely in the following two remarks:

(i) Assume V, V', W' are affine with coordinate rings $\Gamma(V) \subset \Gamma(V')$, $\Gamma(W')$, assume $V' \rightarrow V$ is finite and flat and has a section corresponding to a decomposition $\Gamma(V') = \Gamma(V) \oplus I$, where we assume that the ideal I considered as a $\Gamma(V)$-module is free with generators $w_1,\dots,w_N$ (which is always true after localization).

Let W' be defined by polynomials over $\Gamma(V)$:

$$\Gamma(W') \cong \Gamma(V)[T_1,\dots,T_r] / (F_1,\dots,F_s).$$

We choose new indeterminates $Z_{\alpha,\rho}$, $\alpha = 1,\dots,N$, $\rho = 1,\dots,r$ and consider

$$F_i(T_1 + \sum_\alpha Z_{\alpha 1}w_\alpha,\dots,T_r + \sum_\alpha Z_{\alpha r}w_\alpha) = \sum_{\beta=0}^{N} g_{i\beta}(Z)w_\beta$$

(where we put $w_o = 1$), where $g_{i\beta}(Z)$ are polynomials over $\Gamma(W')$.

Let $J \subset W' \times \mathbb{A}^{rN}$ be the closed subscheme defined by

$$g_{i\beta}(Z) = 0 \quad , \quad i = 1,\ldots,s; \quad \beta = 0,\ldots,N.$$

Then the universal morphism ψ is described by

$$T_i \longmapsto T_i + \sum_{\alpha=0}^{N} Z_{\alpha i} w_\alpha \,.$$

(ii) If $f : \tilde{V} \to V$ is a morphism, there is a canonical $(\tilde{V} x_T W)$-morphism

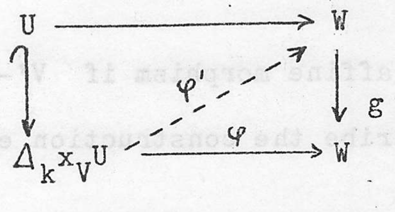
$$\tilde{V} x_V J_k(V/T, W/T) \longrightarrow J_k(\tilde{V}/T, W/T) \,.$$

If f is etal, this is an isomorphism (because of $\Delta_k(\tilde{V}/T) \simeq \tilde{V} x_V \Delta_k(V/T)$.

(iii) If $g : \tilde{W} \longrightarrow W$ is a morphism, there is a canonical $(Vx_T\tilde{W})$-morphism

$$J_k(V/T, \tilde{W}/T) \longrightarrow J_k(V/T, W/T) x_W \tilde{W} \,.$$

If f is etal, this is again an isomorphism, because we can lift any φ of the diagram below to a morphism φ'



(U a $Vx_T\tilde{W}$-scheme)

Using these remarks we get a rather explicite description of $J_k(V/T, W/T)$.

§ 20 Global description of everywhere stable morphisms

Consider $p \leq \ell < k$ and $J_k = J_k(V/T, W/T)$, where V, W are smooth quasi-projective T-schemes. We construct a coherent sheaf $\mathcal{T}_\ell$ on J_k which gives the coherent sheaf T on the fibres $(J_k)_{(x,y)}$ such that supp (T_ℓ) is the set of nonstable jets.

Then $\Pi_k(V/T,W/T) = \text{supp}(\mathcal{T}_\ell)$ is a closed subset of J_k consisting precisely of those jets which are not stable. More precisely, for each (Vx_TW)-morphism $\varphi : U \to J_k(V/T,W/T)$, i.e. for each U-morphism $\psi : \Delta_k(V/T)x_VU \longrightarrow Wx_TU$ (corresponsing to φ by the functorial description of J_k) we will construct $\varphi^*\mathcal{T}_\ell$ on U, where $U \xrightarrow{(s_1,s_2)} Vx_TW$ is a Vx_TW-scheme. Consider the embedding $i : U \to Wx_TU$, $u \to (s_2(u),u)$, the projection $p : \Delta_k(V/T)x_VU \to U$ and the closed subschemes

$$Z = \text{Ker}(\psi, i\circ p) \subset \Delta_k(V/T)x_VU,\quad Z_\ell = Z \cap (\Delta_\ell(V/T)x_VU).$$

We define

$$\mathcal{C} = \text{coker}(\mathcal{H}om(\Omega^1_{\Delta_k x_V U/U}, \mathcal{O}_{Z_\ell}) \xrightarrow{\psi^*} \mathcal{H}om(\psi^*\Omega_{Wx_TU/U}, \mathcal{O}_{Z_\ell}))$$

and $\mathcal{D} = p_*\mathcal{C}$. Then $\mathcal{D}$ is a coherent sheaf on U and the construction commutes, with base change, $U' \to U$ (since p is affine and $\Omega_{Wx_TU/U}$ is locally free).

Since ψ and $i\circ p$ coincide on Z_ℓ, there is a canonical morphism

$$p_*(i\circ p)^{-1}\,\Theta_{Wx_TU/U} = i^{-1}\Theta_{Wx_TU/U} \to \mathcal{D}$$

(induced by $\psi^{-1}\Theta_{Wx_TU/U} \to \mathcal{H}om(\psi^*\Omega_{Wx_TU/U}, \mathcal{O}_{Z_\ell}) = \psi^*\Theta_{Wx_TU/U} \otimes_{\mathcal{O}_{\Delta_k x_V U}} \mathcal{O}_Z$).

Since i is a closed embedding, this morphism factors through a morphism

$$i^*\Theta_{Wx_TU/U} \longrightarrow \mathcal{D}$$

and we define $\varphi^*\mathcal{T}_\ell$ as the cokernel of this morphism.

This construction commutes with base change $U' \to U$, taking $U = \{(x,y)\}$

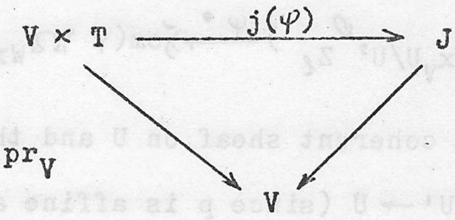
we get the sheaf we considered in § 8. Hence $\mathrm{supp}(\mathcal{F}_\ell) = \Pi_k(V/T, W/T)$ is the set of nonstable jets.

<u>Proposition 25.</u> If V, W are projective algebraic varieties, if T is a smooth algebraic variety and $\varphi : V \times T \to W$ a family of morphisms of V in W, then

(i) The set $T' = \{t \in T;\ \varphi_t : V \to W$ is stable everywhere$\}$ is open.

(ii) Let $j(\varphi) : V \times T \to J = J_k(V,W)$ be the corresponding morphism in the jet bundle, $k > p = \dim W$ (projection of $j((\varphi, pr_T))$: $V \times T \to J_k(V \times T/T,\ W \times T/T) = J_k(V,W) \times T$ onto J). Then $T' \neq \emptyset$ if for each $x \in V$ the induced morphism $t \mapsto j(\varphi)(x,t) : T \to J_x$ is smooth and $\mathrm{codim}(\Pi_k(n,p), J_k(n,p)) > n = \dim V$.

<u>Proof.</u> Obviously $T' = T \setminus pr_T(j(\varphi)^{-1}\ \Pi_k(n,p))$, hence T' is open. Now consider the commutative diagram



The morphisms pr_V and $J \to V$ are smooth, hence if $J(\varphi)$ induces smooth maps on the fibres $\{x\} \times T \to J_x$, then $j(\varphi)$ is smooth. This implies

$$\dim V + \dim T - \dim j(\varphi)^{-1}(\Pi) = \mathrm{codim}\,(\Pi, J) =$$
$$\mathrm{codim}\,(\Pi_k(n,p),\ J_k(n,p)) > n = \dim (V),$$

hence $\dim(T) > \dim j(\varphi)^{-1}(\Pi)$ and therefore $pr_T(j(\varphi)^{-1}(\Pi)) \subsetneq T$.

We consider now a smooth projective variety V, an embedding $V \subset \mathbb{P}^N$, and the subspace $T \subset \mathrm{Grass}\,(p, |O_V(d)|)$ of subspaces spanned by d-forms $\varphi_0, \ldots, \varphi_p$ which have no common zero on V.

Then we get a family of morphisms

$$\varphi : V \times T \to \mathbb{P}^p$$

$\varphi(x,\Lambda) = \varphi_\Lambda(x)$.

If $x \in V$, consider $j(\varphi)(x,-) : T \to J_x$ (where J denotes the bundle of k-jets over $V \times \mathbb{P}^p$, $k > p$).

<u>Proposition 26.</u> If $d \geq k$, then $j(\varphi)(x,-) : T \to J_x$ is smooth.

<u>Corollary:</u> If $(p,p-n)$ is in the nice range, there exists a nonempty Zariski-open set $T' \subset T$ such that for $\Lambda \in T'$ the morphism $\varphi_\Lambda : V \to \mathbb{P}^p$ is stable everywhere.

<u>Proof of the proposition:</u> Let x be the point $(1:0:\ldots:0)$ and let $x_1,\ldots,x_N$ be inhomogenous coordinates on $\mathbb{P}^N$ such that $dx_1,\ldots,dx_n$ are linear independent at x. Let Λ be spanned by forms $\varphi_o,\ldots,\varphi_p$ such that $\varphi_o(1,0,\ldots,0) = 1$, $\varphi_i(1,0,\ldots,) = 0$ for $i > 0$. Then $j(\varphi)(x,-)$ is (locally) given by

$$[\psi_o,\ldots,\psi_p] \longmapsto \left(\psi_o(x) : \ldots : \psi_m(x), \left(j_x\left(\frac{\psi_1}{\psi_o} - \frac{\psi_1(x)}{\psi_o(x)}\right), \ldots, j_x\left(\frac{\psi_p}{\psi_o} - \frac{\psi_p(x)}{\psi_o(x)}\right)\right) \in U_o \times J_k(n,p) \subset J_x$$

where $U_o \subset \mathbb{P}^p$ is the open set of all points $(1 : y_2 : \ldots : y_p)$ and j_x denotes the k-jet of a function at x. We can also write this map as

$$[\psi_o,\ldots,\psi_p] \longmapsto \left(j_x\left(\frac{\psi_1}{\psi_o}\right), \ldots, j_x\left(\frac{\psi_p}{\psi_o}\right)\right) .$$

If t is a parameter and if $\lambda_o,\ldots,\lambda_p$ are arbitrary d-forms, and if we identify d-forms ψ with the function $\frac{\psi}{x_o d}$ on $\mathbb{P}^N$, then

$$j(\varphi)(x,\varphi_o + t\lambda_o,\ldots,\varphi_p + t\lambda_p) \equiv \left(\frac{j_x(\varphi_1)}{j_x(\varphi_o)} + t\,\frac{j_x(\lambda_1)j_x(\varphi_o) - j_x(\varphi_1)j_x(\lambda_o)}{j_x(\varphi_o)^2}, \ldots\right.$$

$$\left.\ldots\right) \bmod t^2 \quad (\text{in } \mathcal{O}_{V,x} / m_{V,x}^{k+1}) .$$

Smoothness of $j(\varphi)(x,-)$ means that to each tupel of k-jets $(z_1,\ldots,z_p)$ $(z_i \in \mathcal{O}_{V,x} / m_{V,x}^{k+1})$ there exist d-forms $\lambda_o,\ldots,\lambda_p$ such that

$$z_i = \frac{j_x(\lambda_i)j_x(\varphi_o) - j_x(\varphi_i)j_x(\lambda_o)}{j_x(\varphi_o)^2} .$$

If $d \geq k$, these equations have solutions. By putting $\lambda_o = 0$ and if

$$j_x(\varphi_o)\, z_i = \sum_{|\alpha| \leq k} c_{i\alpha} X_1^{\alpha_1} \dots X_n^{\alpha_n}$$

then the d-forms

$$\lambda_i = \sum_{|\alpha| \leq k} c_{i\alpha} X_o^{d-|\alpha|} X_1^{\alpha_1} \dots X_n^{\alpha_n} , \qquad i = 1,\dots,p$$

satisfy the equations.

§ 21 Normal crossing of branches

For positive integers $d > p$ and for a smooth projective variety $V \subset \mathbb{P}^N$ of dimension $n < p$, such that $(p - n, p)$ is in the nice range, there is a nonempty Zariski-open subset T in $\mathrm{Grass}(p, |\mathcal{O}_{\mathbb{P}^N}(d)|)$ such that for $\Lambda \in T$ the projection $\varphi_\Lambda : V \to \mathbb{P}^p$ is defined everywhere, finite, birational onto its image and has stable germs $(V,x) \to (\mathbb{P}^p, \varphi_\Lambda(x))$ everywhere. Now we want to show that T contains a nonempty Zariski-open subset such that φ_Λ has nc- branches everywhere; with other words, for any point $y \in \varphi_\Lambda(V)$ and any finite subset $\{x_1, \ldots, x_m\} \subset \varphi_\Lambda^{-1}(y)$ the multi-germ induced by φ,

$$(V,x_1) \amalg \ldots \amalg (V,x_m) \longrightarrow (\mathbb{P}^p, y)$$

is stable (i.e. has only trivial unfoldings).

We will generalize the construction of the last section to multi-jets.

Consider positive integers $p \leq 1 < k$, a family of morphisms

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

where $V|S$ is a smooth family of varieties of dimension n, $W|S$ is a smooth family of varieties of dimension $p > n$, and consider the bundle $J = J_k(V|S, W|S)$. By $(J^m)_W$ I will denote the m-fold fibre product of J over W. If V^m denotes the m-fold product of V over $\mathcal{S}$, then $(J^m)_W$ is a bundle over $V^m \times_S W$, and for any $(V^m \times_S W)$-scheme $U \xrightarrow{(s_1, \ldots, s_m, s)} V^m \times_S W$, a $(V^m \times_S W)$-morphism $\varphi : U \to (J^m)_W$ is given by an m-tupel $(\psi_1, \ldots, \psi_m)$ of U-morphisms,

$$\psi_\mu : \Delta_k(V|T) \times_{s_\mu} U \longrightarrow W \times_{\mathcal{S}} U, \qquad \psi_\mu|U = (s, id_U).$$

For each ψ_μ we use the construction of the last section to get a coherent sheaf $\mathcal{D}_\mu$ on U and a morphism $i^*\partial_{W \times_{\mathcal{S}} U|U} \longrightarrow \mathcal{D}_\mu$, where

$i : U \to W \times_{\mathfrak{T}} U$ is the embedding $u \to (s(u), u)$.

By $\varphi^*\mathcal{T}_1$ we denote the cokernel of the diagonal embedding

$$i^*\Theta_{W \times_{\mathfrak{T}} U|U} \longrightarrow \mathcal{D}_1 \times \dots \times \mathcal{D}_m .$$

This construction is compatible with base change $U' \to U$, and thus we get a coherent sheaf $\mathcal{T}_1$ on $(J^m)_W$ such that $\operatorname{supp} \tilde{\mathcal{T}}_1 \underset{\text{def.}}{=} \Pi_k^m\ (V|\mathfrak{T}, W|\mathfrak{T})$ is the set of m-tuples of k-jets which are not stable or not nc .

If J^m denotes the m-fold product of J over $\mathfrak{T}$, then $(J^m)_W \subset J^m$, and codim $(J^m)_W = (m-1)p$, hence $\Pi_k^m\ (V|\mathfrak{T}, W|\mathfrak{T}) \subset J^m$ has the codimension $(m-1)p + \operatorname{codim}\ \Pi_k^m\ (n,p)$ in J^m.

We will prove (in the following section) that $\operatorname{codim} \Pi_k^m\ (n,p) > mn - (m-1)p$ if $(p-n,p)$ is in the nice range, hence

$$\operatorname{codim}\ \Pi_k^m\ (V|\mathfrak{T}, W|\mathfrak{T}) > m\,n.$$

As in the latter section we get the following analogue of proposition 25.

<u>Proposition 27.</u> If V, W are smooth projective varieties of dimension n and p, $n < p$ and if $\varphi : V \times T \to W$ is a family of morphisms of V in W such that the corresponding morphism $j(\varphi) : V \times T \to J_k(V,W) = J$ has the property that for any $x \in V$ the morphism $T \to J_x$, $t \longmapsto j(\varphi_t)(x)$, is smooth, then there exists a nonempty Zariski-open set $T' \subseteq T$ such that the morphism $\varphi_{t'}$, $t \in T'$ is stable everywhere and has crossing normally branches.

As a corollary we get the theorem stated in § 18.

For the proof of proposition 27 we consider, for all positive integers $m \leq \frac{p}{p-n} + 1$, the morphisms

$$\begin{array}{ccc} V^m \times T & \xrightarrow{\ j(\varphi) \times \dots \times j(\varphi)\ } & J^m \\ & \searrow^{pr} & \downarrow \\ & & V^m \end{array}$$

The morphism $j(\varphi) \times \ldots \times j(\varphi)$ is smooth, hence

$$\operatorname{codim} (j(\varphi) \times \ldots \times j(\varphi))^{-1} (\Pi_k^m (V,W)) =$$

$$m\,n + \dim T - \dim(j(\varphi) \times \ldots \times j(\varphi))^{-1} (\Pi_k^m (V,W)) > m\,n,$$

hence $\dim T > \dim(j(\varphi) \times \ldots \times j(\varphi))^{-1} (\Pi_k^m (V,W))$ q.e.d.

§ 22 Calculation of codim $\Pi_k^m(n,p)$

We say that subspaces $T_1,\ldots,T_m$ of a vector space T are in general position if

$$\operatorname{codim}(T_1 \cap \ldots \cap T_m) = \operatorname{codim} T_1 + \ldots + \operatorname{codim} T_m.$$

This condition is equivalent to the property that for any $\mu < m$

$$(T_1 \cap \ldots \cap T_\mu) + T_{\mu+1} = T.$$

If $\Sigma_k^m(n,p) \subset J_k(n,p)^m$ is the set of m-tupels of k-jets such that the images of the tangent spaces $T_\mu = T_o(z^{(\mu)})(T_o(\mathbb{A}^n)) \subset T = T_o(\mathbb{A}^p)$ (where $z^{(\mu)}$ denotes the μ-th component of m-tupel z) are not in general position, then $\Sigma_k^m(n,p) \subset \Pi_k^m(n,p)$, and we estimate codim $\Sigma_k^m(n,p)$ and codim$(\Pi_k^m(n,p) - \Sigma_k^m(n,p))$.

Furthermore, we will derive 'normal forms' for multi-jets of

$$J_k(n,p)^m - \Sigma_k^m(n,p) \quad \text{and} \quad J_k(n,p)^m - \Pi_k^m(n,p) .$$

<u>Lemma 1</u>. The subset $\Sigma_k^m(n,p)$ is closed and of codimension $m\,n - (m-1)p + 1$.

For the proof we only have to consider 1-jets, i.e. m-tupels of (pxn)-matrices $(z^{(1)},\ldots,z^{(m)})$. The space $J_1(n,p)^m$ is the union of the locally closed subsets

$$J_1(n,p)^{(r(1),\ldots,r(m))} = \{ z \,;\, rk(z^{(\mu)}) = r(\mu), \quad \mu = 1,\ldots,m \} .$$

By $T_\mu = T_\mu(z)$ we denote the subspace of $T_o(\mathbb{A}^p) = K^p$ generated by the columns of the matrix $z^{(\mu)}$, then T_μ is the image of the tangent space $T_o(\mathbb{A}^n)$ under $z^{(\mu)}$. By $z \to (T_1(z),\ldots,T_m(z))$ we get a surjective morphism $\pi : J_1(n,p)^{(r(1),\ldots,r(m))} \to \mathrm{Grass}(r(1),p) \times\ldots\times \mathrm{Grass}(r(m),p)$ whose fibres are of dimension $n(r(1)+\ldots+r(m))$. If

$$Z^{(r(1),\ldots,r(m))} = \left\{ \begin{array}{l} (T_1,\ldots,T_m)\ \mathrm{Grass}(r(1),p)\times\ldots\times \mathrm{Grass}(r(m),p) \\ T_1,\ldots,T_m \text{ are not in general position} \end{array} \right\}$$

then $Z^{(r(1),\ldots,r(m))}$ is closed and

$$\Sigma_k^m(n,p) \cap J_1(n,p)^{(r(1),\ldots,r(m))} = \pi^{-1}(Z^{(r(1),\ldots,r(m))}).$$

If $T_1,\dots,T_m$ are in general position, then $mp - \sum_{\mu=1}^{m} r(\mu) = \operatorname{codim}(T_1 \cap \dots \cap T_m) \leq p$. Therefore $Z^{(r(1),\dots,r(m))} = \operatorname{Grass}(r(1),p) \times\dots\times \operatorname{Grass}(r(m),p)$ if $(m-1)p > \sum_{\mu=1}^{m} r(\mu)$, and $\dim Z^{(r(1),\dots,r(m))} = \sum_{\mu=1}^{m} r(\mu)(p-r(\mu))$.

Now assume $(m-1)p \leq \sum_{\mu=1}^{m} r(\mu)$, then $\sum_{\mu=1}^{k} r(\mu) \geq (k-1)p$ for $k = 2,\dots,m$. If the subspaces $T_1,\dots,T_m$ are not in general position, there exists an integer $k \in [1,m-1]$ such that $T_1,\dots,T_k$ are in general position but $T_1,\dots,T_k, T_{k+1}$ are not.

If $S = \{ T_{k+1} \in \operatorname{Grass}(r(k+1),p) \; ; T_1,\dots,T_{k+1} \text{ not in general position} \}$,

$S_\nu = \{ T_{k+1} \in \operatorname{Grass}(r(k+1),p) \; ; \operatorname{codim}((T_1 \cap \dots \cap T_k) + T_{k+1}) = \nu \}$,

then $S = \bigcup_{\nu \geq 1} S_\nu$, and by

$$T_{k+1} \to T_{k+1} + (T_1 \cap \dots \cap T_k) / (T_1 \cap \dots \cap T_k)$$

we get a morphism $S_\nu \to \operatorname{Grass}(d,\bar{T})$, where $\bar{T} = T / (T_1 \cap \dots \cap T_k)$ and $d = k\,p - (r(1) + \dots + r(k) + \nu)$. The fibres are open sets in $\operatorname{Grass}(r(k+1),p-\nu)$ and not empty if and only if $\nu \geq \max(k\,p - (r(1) + \dots + r(k+1)),1)$. Hence

$$\dim S_\nu = \nu(kp - (r(1) + \dots + r(k) + \nu)) + r(k+1)(p-r(k+1) - \nu)$$
$$= r(k+1)(p-r(k+1)) - \nu(\nu - k\,p + r(1) + \dots + r(k+1)).$$

Because of $k\,p \leq r(1) + \dots + r(k+1)$ we have $\dim S_1 > \dim S_2 > \dots$ hence

$$\dim S = r(k+1)(p - r(k+1)) - (1 + r(1) + \dots + r(k+1) - k\,p)$$

and

$$\dim Z^{(r(1),\dots,r(m))} = \sum_{\mu=1}^{m} r(\mu)\,(p - r(\mu)) - (1 + r(1) + \dots + r(m) - (m-1)p)$$

$$\dim(\Sigma_k^m(n,p) \cap J_1(n,p)^{(r(1),\dots,r(m))}) =$$
$$= (n-1)(r(1) + \dots + r(m)) + \sum_{\mu=1}^{m} r(\mu)(p-r(\mu)) + (m-1)p-1$$
$$= m(n-1)n + mn(p-n) + (m-1)p-1$$
$$= mn(p-1) + (m-1)p-1$$

$$= \dim (\Sigma_k^m(n,p) \cap J_1(n,p)^{(n,\dots,n)})$$

$$\operatorname{codim} \Sigma_k^m(n,p) = m\,n - (m-1)p + 1 \qquad \text{q.e.d.}$$

Now we will describe normal forms of multi-jets. Assume $T_1,\dots,T_m$ are subspaces in general position of a vector space T, we define

$r(k) = \dim T_k$, $a(0) = p = \dim T$, $a(1) = r(1)$, $a(2) = r(1) + r(2) - p, \dots, a(k) = r(1) + r(2) + \dots + r(k) - (k-1)p$, $k = 1,\dots,m$.

Then we can find a base $e_1,\dots,e_p$ of T such that for $k = 1,\dots,m-1$

(i) the vectors $e_1,\dots,e_{a(k)}$ form a base of $T_1 \cap \dots \cap T_k$

(ii) the vectors $e_{a(k)+1},\dots,e_{a(k-1)}$ are contained in the subspace $(T_1 \cap \dots \cap T_{k-1}) \cap (T_{k+1} \cap \dots \cap T_m)$ and they form a base of $(T_1 \cap \dots \cap T_{k-1}) \bmod (T_1 \cap \dots \cap T_k)$

(observe that $(T_1 \cap \dots \cap T_{k-1}) \cap (T_{k+1} \cap \dots \cap T_m) + T_k = T$, hence $T_1 \cap \dots \cap T_{k-1} = (T_1 \cap \dots \cap T_{k-1}) \cap (T_{k+1} \cap \dots \cap T_m) + (T_1 \cap \dots \cap T_k)$).

Consequently, any multi-jet of multi-rank $(r(1),\dots,r(m))$ which is not contained in $\Sigma_k^m(n,p)$ is equivalent to a multi-jet such that (if we write the components $z^{(\mu)}$ as a row vector at the moment)

$$z^{(\mu)} = (x_1^{(\mu)},\dots,x_a^{(\mu)}, w_1^{(\mu)},\dots, w_{a(\mu-1)-a(\mu)}^{(\mu)}, x_{a(\mu)+1}^{(\mu)} \dots x_{r(\mu)}^{(\mu)}),$$

where $j_1(w_\alpha^{(\mu)}) = 0$. We write $w^{(\mu)}$ for the vector with the components $w^{(\mu)}$. We refer to such multi-jets as multi-jets in 'normal form'. The image of the module $\Delta(z^{(\mu)})$ in $Q_{k-1}^{(\mu)p}$ contains the vectors

$$e_j + \begin{pmatrix} 0 \\ \partial_j w^{(\mu)}{}_o \\ 0 \end{pmatrix}, \quad j \leq a(\mu) \text{ and } \quad e_{j+p-r(\mu)} + \begin{pmatrix} 0 \\ \partial_j w^{(\mu)}{}_o \\ 0 \end{pmatrix}, \quad a(\mu) < j \leq r(\mu)$$

where $\partial_j = \dfrac{\partial}{\partial x_j^{(\mu)}}$ and the index o indicates that we substitute

$x_1^{(\mu)} = x_2^{(\mu)} = \dots = x_{r(\mu)}^{(\mu)} = 0$. Thus we get

$$Q_{k-1}^{p}/\overline{\Delta(z)} \simeq \bigtimes_{\mu=1}^{m} [(Q_{k-1}^{(\mu)\, p-r(\mu)}/\overline{\Delta(w_o^{(\mu)})}].$$

If $\varepsilon_1,\ldots,\varepsilon_m$ denote the idempotents of Q_{k-1}, then the base vector $e_j = \sum_{\mu=1}^{m} \varepsilon_\mu e_j$ is mapped under this isomorphism to the image of the vector

$$(*)\quad \begin{cases} V_j = -\sum_{\mu=1}^{m} v_j^{(\mu)}\varepsilon_\mu & \text{if } j \leq a(m) \\ V_j^{(\mu)} e_j \varepsilon_i - \sum_{\mu=1}^{i-1} v_j^{(\mu)}\varepsilon_\mu - \sum_{\mu=i+1}^{m} v_{j-p+r(\mu)}^{(\mu)}\varepsilon_\mu & \text{if } a(i) < j \leq a(i-1), \end{cases}$$

where $v_j^{(\mu)} = \partial_j w_o^{(\mu)}$.

The following lemma is an immediate consequence of the definition.

Lemma 2. The multi-germ z is stable if and only if the vectors $V_1,\ldots,V_p$ (defined by (*)) generate the vector space

$$\bigtimes_{\mu=1}^{m}(Q_{k-1}^{(\mu)\, p-r(\mu)}/\overline{\Delta(w_o^{(\mu)})}).$$

If the jet $w'^{(\mu)}_o \in J_k(n-r(\mu),\, p-r(\mu))$ is contained in the contact class of $w_o^{(\mu)}$ for $\mu = 1,\ldots,m$ and if we choose vectors $V_1,\ldots,V_p$ in the form (*), where $v_j^{(\mu)} \in J_{k-1}(n-r(\mu),\, p-r(\mu))$, we get a multi-jet in normal form of the same size as z by putting $w'^{(\mu)} = w'^{(\mu)}_o + \sum_{j=1}^{r(\mu)} x_j^{(\mu)} v_j^{(\mu)}$ and z' is contact equivalent with z. Conversely, if the multi-jet z' is contact equivalent with z and has normal form, then the corresponding jets $w'^{(\mu)}_o$ are contact equivalent with $w_o^{(\mu)}$.

If C( ) denotes the contact class of a jet or multi-jet, we consequently get a surjective morphism

$$C(z) \cap \{\text{normal forms}\} \longrightarrow C(w_o^{(1)}) \times\ldots\times C(w_o^{(m)}) \times \bigtimes_{\mu=1}^{m} J_{k-1}(n-r(\mu),\, p-r(\mu))^{r(\mu)}.$$

As an abbreviation we shall denote the variety on the right hand side by X.

Lemma 3. If z is a multi-jet and if C(z) denotes its contact class, then:

(i) $C(z) \subset \Pi_k^m(n,p)$ if $p < \mu_{p-n}(Q_k(z))$

(ii) $\operatorname{codim}_{C(z)} (C(z) \cap \Pi_k^m(n,p) - \Sigma_k^m(n,p)) = \max(0, p-\mu_{p-n}(Q) + 1)$

where $Q = Q_k(z)$ is the corresponding Artinian algebra and

$$\mu_{p-n}(Q) = \dim (Q_{k-1}{}^p / \overline{\Delta(z)}).$$

The proof is analogous to that of proposition 18. It is sufficient to consider multi-jets in normal form. We denote this space by $Y^{(r(1),\ldots r(m))}$. Then $C(z) \cap \Pi_k^m(n,p) \cap Y^{(r(1),\ldots r(m))}$ is the inverse image of all elements of X for which the vectors $\bar{V}_1,\ldots,\bar{V}_p$ in $\bigtimes_{\mu=1}^m (Q_{k-1}{}^{(\mu),p-r(\mu)} / \Delta(w_o^{(\mu)})$ do not generate this space. Observe that $\bar{V}_{a(m)+1},\ldots,\bar{V}_p$ are always linearly independent and $(K\bar{V}_1 + \ldots + K\bar{V}_{a(m)}) \cap (K\bar{V}_{a(m)+1} + \ldots + K\bar{V}_p) = 0$. If $b = \mu_{p-n}(Q) - p + a(m) - 1$, we can choose b of the vectors $V_1,\ldots,V_{a(m)}$ arbitrarily and the remaining $\mu_{p-n}(Q) + p + 1$ vectors in the subspace generated by these b vectors and by $\Delta(w_o)$. Hence we get for the codimension c of non-stable jets in a contact class:

$$c = a(m) \dim[\bigtimes_{\mu=1}^m J_{k-1}(n-r(\mu), p-r(\mu))] + \sum_{j=1}^m (a(j-1) - a(j)) \cdot$$

$$\cdot \dim[\bigtimes_{\mu\neq j} J_{k-1}(n-r(\mu), p-r(\mu))]$$

$$- b \dim[\bigtimes_{\mu=1}^m J_{k-1}(n-r(\mu), p-r(\mu))] - (a(m) - b)(b + \dim \overline{\Delta(w_o)})$$

$$- \sum_{j=1}^m (a(j-1) - a(j)) \dim[\bigtimes_{\mu\neq j} J_{k-1}(n-r(\mu), p-r(\mu))]$$

$= p + 1 - \mu_{p-n}(Q)$, if $p \geq \mu_{p-n}(Q)$. Otherwise $C(z) \subseteq \Pi_k^m(n,p)$, q.e.d.

Exactly as in § 12 one derives from this result

<u>Lemma 4</u> If $W_k^m(n,p) \subset J_k(n,p) - \Sigma_k^m(n,p)$ denotes the set of non-simple contact classes, then $\operatorname{codim} W_k^m(n,p) > p - m(p-n)$ if and only if $\operatorname{codim} (\Pi_k^m(n,p) - \Sigma_k^m(n,p)) > p - m(p-n)$.

But the contact class z is determined by the isomorphism class of the corresponding Artinian algebra $Q(z) = Q(z^{(1)}) \times \ldots \times Q(z^{(m)})$.

If $(p-n,p)$ is in the nice range, we have $\operatorname{codim} W_k^m(n,p) > n \geq p - (p-n)m$ then, hence $\operatorname{codim} (\Pi_k^m(n,p) - \Sigma_k^m(n,p)) > p - m(p-n)$. Together with lemma 1 we get

Lemma 5: If $(p-n,p)$ is in the nice range, then

$$\operatorname{codim} \Pi_k^m(n,p) > mn - (m-1)p.$$

Immediately from lemma 2 we get the following result about normal forms:

Proposition 28: Given an Artinian algebra $Q = Q^{(1)} \times \ldots \times Q^{(m)}$ where $Q^{(1)}, \ldots, Q^{(m)}$ are the local factors, there exists a stable multi-germ $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(m)})$ in $J(n,p)^m$ such that $Q \cong Q(\varphi)$ if and only if

(1) $p-n \geq i(Q^{(k)})$, $k = 1, \ldots, m$

(2) $p \geq \mu_{p-n}(Q) = \sum_{k=1}^{m} (p-n)\dim Q^{(k)} + g(Q^{(k)})$.

We define $p_k = (p-n)\dim Q^{(k)} + g(Q^{(k)})$, $n_k = p_k - (p-n)$ and $s = p - \sum p_k = p - \mu_{p-n}(Q)$. Let $\psi_k : (\mathbb{A}^{n_k},0) \longrightarrow (\mathbb{A}^{p_k},0)$ be a stable germ to $Q^{(k)}$ in normal form, and let $\varphi^{(k)}$ denote the germ

$$(\mathbb{A}^n,0) = ((\mathop{\times}_{j=1}^{k-1} \mathbb{A}^{p_j}) \times \mathbb{A}^{n_k} \times (\mathop{\times}_{j=k+1}^{m} \mathbb{A}^{p_j}) \times \mathbb{A}^s,0)$$

$$\longrightarrow (\mathop{\times}_{j=1}^{m} \mathbb{A}^{p_j} \times \mathbb{A}^s,0) = (\mathbb{A}^p,0)$$

which is the direct product of the identical maps on the factors $\mathbb{A}^{p_j}$ $(j \neq k)$ and $\mathbb{A}^s$ with the germ ψ_k on the k-th factor. Then $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(m)})$ is a stable multi-germ with $Q \cong Q(\varphi)$.

Example: $Q = Q^{(1)} \times Q^{(2)}$, $Q^{(1)} = K[[x]]/(x^2)$, $Q^{(2)} = K[[y,z]]/(yz, y^2+z^2)$. The smallest integer n such that Q has a stable representative in dimension n is $n = 10$ and $p = 11$. A stable representation for $Q^{(1)}$ is given by the germ

$$(u, x^2, ux),$$

and a stable representation for $Q^{(2)}$ by

$$(v_1,\ldots,v_5, yz, y^2+z^2+v_1y+v_2z, v_3y+v_4z+v_5y^2).$$

Hence a stable representation for Q is given by $\varphi = (\varphi^{(1)}, \varphi^{(2)})$,

$$\varphi^{(1)} = (u_1,\ldots,u_6,u_7,u_8,u_9,x^2,u_1x)$$
$$\varphi^{(2)} = (v_1,\ldots,v_6,yz,y^2+z^2+v_2y+v_3z,v_4y+v_5z+v_6y^2,v_7,v_8).$$

Finelly proposition 28 immediatly yields

Proposition 29: If $(p-n,p)$ is in the nice range and $f\colon V^n \longrightarrow P^p$ is a generic projection of a non-singular projective n-dimensional variety into the p-dimensional projective space, the locus $S(Q) = \left\{ y \in f(V) \,\middle|\, Q_y(f) = Q \right\}$ is locally closed and of dimension $p - \mu_{p-n}$ or empty.

References

[1] W.I. Arnold, Singularities of smooth mappings, Uspechi math. nauk. XXIII (1968), p. 3-44

[2] -''- Critical functions and their normal forms, Uspechi math. nauk. XXX (1975), p.

[3] -''- Normal forms of functions near degenerate critical points, Functional anal. 6 (1972), p. 3-25

[4] -''- Classification of unimodal critical points of functions, Functional anal. 7 (1973), p. 75-76

[5] -''- Classification of bimodal critical points of functions, Functional anal. 9 (1975), p. 49-50

[6] A. Borel, Linear Algebraic groups, New York 1969

[7] R. Elkik, Solutions d'equations a coefficients dans un anneau henselien, Ann. Sc. Ec. Norm. Sup. 6 (1975), p. 533-604

[8] M. Golubitzky, M. Guillemin, Stable Mappings and their Singularities, Springer 1973

[9] A. Grothendieck, Elements de Geometrie Algebrique, Publ. Math. IHES 4, 8, 11, 17, 20, 24, 28, 32 and Fondaments de la Geometrie Algebrique, Extraite du Sem. Bourbaki 1957-1962

[10] L. Gruson, M. Raynaud, Platitude au geometrie algebrique, Inv. math. (1971)

[11] H.Kurke, G. Pfister, M. Roczen, Henselsche Ringe und algebraische Geometrie

[12] H.Kurke and others, Die Approximationseigenschaft lokaler Ringe, LN in Math. 634, Springer 1978

[13] B. Martin, Stabile Henselsche Abbildungskeime, Diss., Humboldt-Universitaet, Berlin 1977

[14] J. Mather, Stability of C^∞-Mappings:
1, Annals of Math. 87 (1968), p. 89-104
2, Annals of Math. 89 (1969), p. 254-291
3, Inst. Hautes Etudes Scien., Publ. Math. 35 (1969), p. 127-156
4, Inst. Hautes Etudes Scien., Publ. Math. 37 (1970), p. 223-248
5, Advances in Math. 4 (1970), p. 301-335
6, Proceedings of the Liverpool Symposium in Singularities of Manifolds and Mappings, LNM 192, (1971), p. 207-253

[15] J. Mather, Stable map-germs and algebraic geometry, LNM 197, (1971), p. 176-193

[16] -''- Generic Projections, Ann. of Math. 98(2), (1973) p. 226-245

[17] -''- On Thom-Boardman-Singularities, in Dynamical systems, New York, (197)

[18] J. Roberts, Generic Projections of Algebraic Varieties, Amer. Jour. of Math. 93, (1971), p. 191-213

[19] M. Schlessinger, Functors of Artin Rings, Trans. Amer. Math. Soc. 130 (1968), p. 208-222

[20] D. Siersma, The singularities of C^∞-functions of right-codimension smaller or equal eight, Indag. Math. 35/1, (1973), p. 31 -37

[21] D. Vohmann, Einige Eigenschaften der kritischen Menge der Diskriminante verseller Deformationen vollstaendiger Durchschnitte mit isolierter Singularitaet, Diss., Bonn, 1973

[22] Wasserman, Stability of Unfoldings, LNM 393, (1974)