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# FELLER RESOLVENTS

by

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# Feller Resolvents

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## Summary

In this paper we consider a sub-Markov resolvent of kernels $(V_\lambda/\lambda\geq 0)$ on a locally compact space $E$ with a countable base and assume that the following conditions are fulfilled:

$1^o$ $V_\lambda C_b(E)\subset C_b(E)$, $\lambda\geq 0$ ,

$2^o$ $\lim_{\lambda\to\infty} \lambda V_\lambda f(x)=f(x)$ , $f\in C_c(E)$, $x\in E$ ,

$3^o$ $V_o 1$ is a potential.

In Section 1, we present an improvement of a wellknown technical result on convex cones of lower semicontinuous functions. In Section 2, we associate a Hunt process to the above resolvent. Section 3, contains an excessiveness criterion. In Section 4, we show that the process associated to the resolvent $(V_\lambda/\lambda\geq 0)$ is continuous if and only if the following relation holds for each open set $U$,

$$C_c(U)\subset \overline{V_o(C_c(U))} .$$

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# FELLER RESOLVENTS

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## 1. Convex Cones of Lower Semicontinuous Functions

In this section we shall prove an improvement of a wellknown result (see for example [4] Proposition 1 p.226). The proof follows from the original idea and an idea of C.Constantinescu and A.Cornea [1] (Lemma from page 160).

Let C be a convex cone of lower semicontinuous nonnegative functions on a locally compact space, E, which has a countable base. Assume that for each $x\in E$, there exists a function $c\in C$ such that $0<c(x)<\infty$. Let us denote by $C^*$ the family of all numerical nonnegative universally measurable functions, f, such that $\mu(f)<f(x)$ for each $x\in E$ and each measure $\mu$ that satisfies $\mu(c)<c(x)$ for any $c\in C$.

Let $f:E\longrightarrow R$ be a function such that there exists $c_0\in C$ with $f\leq c_0$. We shall use the notation

$$Rf=\inf\{c\in C^*|f\leq c\}$$

It follows that $Rf\leq 0$ if the function f satisfies $f\leq 0$. We denote by D the family of all functions $f\in C(E)$ which have the following properties:

1° there exists $c\in C$ such that $|f|\leq c$

2° $\inf\{R(|f|\chi_{CK})\,|\,K \text{ compact set}\}=0$.

Obviously D is a vector lattice that contains $C_c(E)$ and $Rf<\infty$ for each $f\in D$.

1.1. Theorem

Let f be an upper semicontinuous function such that there is $g\in D$ with $f\leq g$. Then for each $x\in E$ there exists a nonnegative measure $\mu$ such that

a) $\mu(c)\leq c(x)$ for each $c\in C$,

b) $Rf(x)=\mu(f)$.

Proof

We define, for each $g\in D$, $p(g)=Rg(x)$. One easily verifies that p is sub-liniar on D.

Let $\mu: D \longrightarrow R$ be a linear functional such that $\mu(g) \leq p(g)$ for each $g \in D$. Then for $g \leq 0$ we have $\mu(g) \leq p(g) = 0$ and hence $\mu$ is nonnegative. The restriction $\mu|_{C_c(E)}$ define a nonnegative measure on E, which we shall denote by $\bar{\mu}$. Now let $g \in D$, $g \geq 0$ and choose a sequence $\{h_n\} \subset C_c(E)$ such that $0 \leq h_n \leq h_{n+1} \leq 1$ and $\bigcup_n \overset{\circ}{\overbrace{\{h_n = 1\}}} = E$. Then $gh_n, g(1-h_n) \in D$ and $R(g(1-h_n)) \to 0$, as $n \to \infty$. Therefore $\mu(g(1-h_n)) \leq R(g(1-h_n))(x) \to 0$, and hence

$$\bar{\mu}(g) = \lim_{n\to\infty} \bar{\mu}(gh_n) = \lim_{n\to\infty} \mu(gh_n) = \mu(g) - \lim_{n\to\infty} \mu(g(1-h_n)) = \mu(g)$$

We conclude that $\bar{\mu} = \mu$ on D. On the other hand for $c \in C$, let $g \in C_c(E)$ be such that $g \leq c$. Then

$$\mu(g) \leq p(g) \leq c(x) \quad ,$$

which leads to $\mu(c) \leq c(x)$.

Conversely let $\mu$ be a nonnegative measure on E such that $\mu(c) \leq c(x)$ for each $c \in C$. Then $\mu$ is finite on D and $\mu(g) \leq p(g)$ for each $g \in D$.

Now let us suppose that $f \in D$. Then the assertion of the theorem results from the Hahn-Banach theorem applied on the space D.

If f is upper semicontinuous, let us consider a sequence $\{f_n\} \subset D$ such that $f_{n+1} \leq f_n$ and $f = \inf_n f_n$. The set $B = \{\mu \in D' \mid \mu(g) \leq p(g)$ for each $g \in D\}$ is a compact set in the topology $\sigma(D', D)$, because $-p(-g) \leq \mu(g) \leq p(g)$ for each $g \in D$ and each $\mu \in B$. The functions $\bar{f}, \bar{f}_n : B \to R$ defined by $\bar{f}(\mu) = \mu(f)$, $\bar{f}_n(\mu) = \mu(f)$ for $\mu \in B$ satisfy $\bar{f} = \inf \bar{f}_n$ and $\bar{f}_n$, $n \in N$ are continuous on B. Therefore from Lemma 1.2 stated below it follows

$$\sup_B \bar{f} = \inf_n \left[\sup_B \bar{f}_n\right]$$

From the first part of the proof we know $Rf_n(x) = \sup_B \bar{f}_n$. Since $Rf(x) \leq Rf_n(x)$, for each $n \in N$, and $\bar{f}(\mu) = \mu(f) \leq Rf(x)$ for each $\mu \in B$ we get

$$\sup_B \bar{f} \leq Rf(x) \leq \inf_n \left[\sup_B \bar{f}_n\right]$$

Since $\bar{\bar{f}}$ is an upper semicontinuous function on a compact spac
there exists $\mu_o \in B$ such that $\mu_o(f)=\sup_{B} \bar{\bar{f}}$.

1.2. Lemma

Let K be a compact space and $(f_n)$ a lower bounded decreasing sequence of upper semicontinuous numerical functions. Then the following equality holds:

$$\sup_{x\in K} (\inf_n f_n(x))=\inf_n (\sup_{x\in K} f_n(x)) \quad .$$

## 2. Feller Resolvents

Let E be a locally compact space with a countable base. In this section we shall study a sub-Markov resolvent of kernels $\{V_\lambda/\lambda>0\}$ which has the following property of W.Feller:

$$V_\lambda C_b(E) \subset C_b(E)., \text{ for each } \lambda\geq 0 .$$

We also assume that for each $f\in C_c(E)$ and each $x\in E$,

$$(1) \qquad \lim_{\lambda\to\infty} \lambda V_\lambda f(x)=f(x) .$$

2.1. Remark. The following fact is wellknown; let $\{V_\lambda|\lambda>0\}$ be a sub-Markov resolvent of kernels that satisfies property (1). If f is a lower semicontinuous nonnegative function such that $\lambda V_\lambda f\leq f$ for each $\lambda>0$, then f is excessive. Endeed if $g\in C_c(E)$ is such that $g\leq f$ then $g=\lim_{\lambda\to\infty}\lambda V_\lambda g\leq \lim_{\lambda\to\infty}\lambda V_\lambda f\leq f$. Since $f=\sup\{g\in C_c(E)|g\leq f\}$ it follows $f=\lim_{\lambda\to\infty}\lambda V_\lambda f$.

We shall denote by $\mathcal{S}$ the family of all excessive functions on E and by $\mathcal{S}_c$ the subfamily of all continuous excessive functions. For each bounded function f we define

$$Rf=\inf\{g\in\mathcal{S}|f\leq g\}$$

$$^cRf=\inf\{g\in\mathcal{S}c|f\leq g\}$$

Obviously $Rf\leq {}^cRf$. G.Mokobodzki proved in [4] (see p.220-221) that the cone of all excessive functions associated to an arbitrary sub-Markov resolvent is a potential cone. In our situation this property is stated in the following theorem.

2.1'. Theorem

If $s,t\in\mathcal{S}$, then $R(s-t)\in\mathcal{S}$ and $s-R(s-t)\in\mathcal{S}$.

The following three results stated in the theorem from below are easy consequences of some results of G.Mokobodzki [4]. (See Theorem 6 p.212, p.221, Proposition 8, p.229, Theorem 12, p.236, Proposition

14, p.232 and Proposition 16, p.233 in [4]. The proof of $2^{o}$ results by using Theorem 1.1 instead of Proposition 1 from p.226 in [4]).

2.2. Theorem

$1^{o}$ If f is a bounded lower semicontinuous function, then Rf is also lower semicontinuous.

$2^{o}$ If g is an upper semicontinuous function and there exists a bounded continuous function f such that $g \leq f$ and

$$\inf \{R(f\chi_{CK}) \mid K \text{ compact set} \subset E\}=0 , \tag{2}$$

then $Rg={}^{c}Rg$. Particularly Rg is upper semicontinuous.

$3^{o}$ If f is a bounded continuous function which fulfils relation (2), then Rf is a continuous function.

2.3. Remark. $1^{o}$ If f is a bounded lower semicontinuous function, then from the above theorem and Remark 2.1 it follows that Rf is excessive.

$2^{o}$ If f is a continuous function with compact support, then condition $3^{o}$ of the above theorem is obviously fulfilled. Therefore one deduces that each lower semicontinuous excessive function is the limit of an increasing sequence of bounded continuous excessive functions.

$3^{o}$ Let f be a bounded continuous excessive function which fulfils relation (2). Then f fulfils the following condition:

$$\inf \{{}^{c}R(f\chi_{CK}) \mid K \text{ compact set}\}=0$$

Endeed, let $\{g_n\}$ be a sequence in $C_c(E)$ such that $0 \leq g_n \leq g_{n+1} \leq 1$ and $\bigcup_n \overset{\circ}{\overbrace{\{g_n=1\}}}=E$. Then $R(f(1-g_n))$, $n \in N$ are continuous and

$${}^{c}R(f\chi_{\{g_n=0\}}) \leq R(f(1-g_n)) \leq R(f\chi_{\{g_n<1\}})$$

Since each compact set K satisfies $K \subset \overset{\circ}{\overbrace{\{g_n=1\}}}$ for some $n \in N$ we deduce

$R(f\chi_{\{g_n<1\}}) \to 0$, which implies the assertion.

$4^o$ Now let $(\Omega, M, M_t, X_t, \theta_t, P^x)$ be a standard process with state space E. Since $E^x[f(X_t)] \to f(x)$, $(t \to 0)$ for each $x \in E$ and each $f \in C_c(E)$, it follows that the resolvent of the process satisfies relation (1). If the potential kernel of the process is finite, i.e.

$$G1(x)=E^x[\zeta]<\infty \quad , \text{ for each } x \in E,$$

then from Hunt's theorem (see [1] p.141) for each compact set K and each $x \in K$, it follows

$$\inf \{s(x) \mid G1<s \text{ on } CK, \ s \text{ excessive}\}=E^x[\zeta - T_{CK}] \ .$$

If $K_n$ is an increasing sequence of compact sets such that $K_n \subset \mathring{K}_{n+1}$, $T_{CK_n} \to \zeta$, and hence

$$\inf_n \inf \{s(k) \mid G1 \leq s \text{ on } CK_n, \ s \text{ excessive}\}=0$$

In the sequell we want to associate a Hunt process to the given resolvent $(V_\lambda)$. Therefore <u>from now on we assume that $V_o 1$ satisfies relation (2)</u>.

The family of all excessive functions that satisfy relation (2) will be denoted by $P$. Our assumption implies $V_o C_{b+}(E) \subset P$. We put

$$T=\{f \in C_c(E) \mid \text{there exist } s,t \in P \cap C_b(E) \text{ such that } f=s-t\}$$

Obviously T is a vector lattice. We assert that T linearly separates the points of E, i.e. for each $x,y \in E$ there exist $f,g \in T$ such that

$$f(x)g(y) \neq f(y)g(x) \ .$$

Since $C_c(E)$ has this property from condition (1) and the relation $V_\lambda f=V(f-\lambda V_\lambda f)$ we first deduce that $V(C_b(E))$ linearly separates the points of E. Then $P \cap C_b(E)$ has the same property, because $V(C_{b+}(E)) \subset P \cap C_b(E)$.

Now let $f$ $P \cap C_b(E)$. If $g \in S_c$ then $\min(f,g) \in P \cap C_b(E)$. If $f \leq g$

on CK for some compact set K then $f-\min(f,g)\in T$, and the assertion follows on account of Remark 2.3.3°.

The Stone-Weierstrass theorem implies $T=C_o(E)$. Further Theorem 3.4 of J.C. Taylor [7] implies the following result:

2.4. Theorem

There exists a standard process $(\Omega, F, F_t, X_t, \theta_t, P^x)$ with state space E such that for each $x\in E$, $\lambda\geq 0$ and $f\in B_b(E)$,

$$E^x\left[\int_0^\infty \exp(-\lambda t) f(X_t)\,dt\right]=V_\lambda f(x)$$

Let us denote by $\{P_t\}$ the transition function of the process given by the above theorem.

2.5. Proposition

For each $f\in C_o(E)$, $\lim_{t\to 0} P_t f=f$ uniform on each compact set.

Proof

Let $f\in P\cap C_b(E)$. The sequence $f_n=nV_n f$ is increasing and $\lim f_n=f$. Dini's theorem implies that the convergence is uniform on each compact set. Further since

$$P_t f_n=P_t V(f-nV_n f)=\int_t^\infty P_t(f-nV_n f)\,dt$$

we get $f_n-P_t f_n\leq t||f-nV_n f||$, and hence $P_t f_n \to f_n$, uniform. The inequality $P_t f_n\leq P_t f\leq f$ shows that $P_t f \to f$ uniform on each compact set. Then the same holds for each $f\in T$, and since T is dense in $C_o(E)$ the proposition follows.

In order to show that the semigroup of the process given by Theorem 2.4 is in fact a Hunt semigroup we first give the next two lemmas.

2.6. Lemma

Let $(\Omega, \mathcal{N}, \mathcal{M}_t, Y_t, \theta_t, P^x)$ be a standard process with state space E. Let $\Delta$ be the Alexandrov point if E is noncompact or an additional isolated point if E is compact and set $E_\Delta = E \cup \{\Delta\}$. Assume that for each pair $x, y \in E_\Delta$, $x \neq \Delta$ there exist two finite excessive functions s,t such that $s-t \geq 1$ on a neighbourhood of x and $s-t \leq 0$ on a neighbourhood of y. Then $\lim_{\substack{t\to\zeta \\ t<\zeta}} Y_t$ exists in $E_\Delta$ a.s.

Proof

Let $U_1$, $U_2$ be open sets in $E_\Delta$ and s,t finite excessive functions such that $s-t \geq 1$ on $\overline{U}_1$ and $s-t \leq 0$ on $U_2$. We are going to prove that the set

(4) $$M = \{\omega \in \Omega / \text{there exists two sequences } (t_n), (t_n') \text{ such that } t_n \to \zeta(\omega),\ t_n' \to \zeta(\omega),\ Y_{t_n}(\omega) \in U_1,\ Y_{t_n'}(\omega) \in U_2\}$$

is negligible.

Let us define $T_1 = T_{U1}$ and $T_{k+1} = T_k + T_{Ui} \circ \theta_{T_k}$, where i is taken such that i=1 if k is even and i=2 if k is odd. Then

$$M = \bigcap_{n \geq 1} \{T_n < T_{n+1}\} .$$

Since $Y_{T_{2k+1}} \in U_1$ and $Y_{T_{2k}} \in U_2$ on $\{T_{2k} < T_{2k+1} < \zeta\}$ we deduce

$$P^x(\{T_{2k} < T_{2k+1} < \zeta\}) \leq E^x[(s-t)(Y_{T_{2k+1}}) - (s-t)(Y_{T_{2k}})] .$$

Further we have

$$nP^x(M) \leq \sum_{k=1}^{n} E^x[(s-t)(Y_{T_{2k+1}}) - (s-t)(Y_{T_{2k}})] \leq$$

$$\leq \sum_{k=1}^{n} E^x[t(Y_{T_{2k+1}}) - t(Y_{T_{2k}})] \leq E^x[t(Y_{T_3})] \leq t(x) ,$$

because $\{s(Y_{T_n})\}, \{t(Y_{T_n})\}$ are supermartingales. Thus we deduce $P^x(M) = 0$ for each $x \in E$.

The condition from the statement allows as to choose a countable family $\{(U_1^n, U_2^n)\}$ of pairs of open sets in $E_\Delta$ and a family $\{(s^n,t^n)\}$ such that $s^n$, $t^n$, $n\in N$ are finite excessive functions $s^n-t^n\geq 1$ on $U_1^n$ and $s^n-t^n\leq 0$ on $U_2^n$ and for each pair $(x,y)\in E\times E_\Delta$ there exists $n\in N$ such that $x\in U_1$, $y\in U_2$. Then denoting by $M_n$ the set defined by (4) for $(U_1^n, U_2^n)$ we have

$$\{\omega\in\Omega \mid \lim_{\substack{t\to\zeta(\omega)\\ t<\zeta(\omega)}} Y_t(\omega) \text{ do not exists in } E_\Delta\}\subset \bigcup_n M_n .$$

and the desired conclusion follows.

2.7. Lemma

Let $(\Omega,F,F_t,Y_t,\theta_t,P^x)$ be a standard process with state space E such that $\lim_{\substack{t\to\zeta\\ t<\zeta}} Y_t$ exists in $E_\Delta$ a.s. Let $\{G_\lambda \mid \lambda>0\}$ be its resolvent and assume that for each $f\in C_c(E)$, $\lim_{\lambda\to\infty}\lambda G_\lambda f=f$ uniform on each compact subset of E. Then the process is a Hunt process.

Note. In this lemma and in the next corollary $F$ and $F_t$ denote the canonical $\sigma$-fields associated to a Markov process.

Proof.

Let $\{T_n\}$ be an increasing sequence of stopping times and $T=\lim_{n\to\infty} T_n$. Let $L=\lim_{n\to\infty} Y_{T_n}$ and put

$$M=\{T=\zeta<\infty \quad\text{and}\quad L\in E\}$$

We are going to prove that M is negligible. First we note that for each bounded nonnegative universally measurable function f, $\{e^{-t}V_1f(X_t)\}$ is a nonnegative supermartingale and

$$E^x[e^{-T_n}V_1f(X_{T_n})]=E^x[e^{-T_n}\int_{T_n}^{\infty} f(X_t)dt]\to E^x[e^{-T}\int_T^{\infty} f(X_t)dt] .$$

Therefore $\lim_{n\to\infty} e^{-T_n}V_1f(X_{T_n})=e^{-T}V_1f(X_T)$ a.s.

Now let $f \in C_c(E)$. From the relation $V_k f = V_1(f-(k-1)V_k f)$ we deduce that $f_k = kV_k f$ satisfies

$$\lim_{n\to\infty} e^{-T_n} f_k(Y_{T_n}) = e^{-T} f_k(Y_T) \quad \text{a.s.},$$

and hence $\lim_{n\to\infty} f_k(Y_{T_n}) = 0$ a.s. on M. On the other hand for each $\omega \in M$, the set $\{Y_{T_n}(\omega) \mid n \in N\} \cup \{L(\omega)\}$ is compact. Since $f_k \to f$ uniform on each compact set we deduce $\lim_{n\to\infty} f(Y_{T_n}) = 0$, a.s. on M. But $\lim_{n\to\infty} f(Y_{T_n}) = f(L)$ a.s. an account of the continuity of f, which implies $f(L)=0$ a.s. Since f is arbitrary choosen we conclude that M is negligible.

2.8. Corollary

The process $(\Omega, F, F_t, X_t, \theta_t, P^x)$ given by Theorem 2.4 is a Hunt process.

Proof

For each $f \in T$ we have $\lim_{\lambda\to\infty} \lambda V_\lambda f = f$ uniform on each compact set. Since T is dense in $C_o(E)$ we deduce $\lim_{\lambda\to\infty} \lambda V_\lambda f = f$ uniform un each compact set for each $f \in C_o(E)$. The corollary follows from the preceding two lemmas.

## 3. Excessive Functions for Feller Resolvents

Let $(\Omega, M, M_t, X_t, \theta_t, P^x)$ be a standard process with state space E and suppose that its resolvent $\{V_\lambda | \lambda>0\}$ has the following property: $V_\lambda C_b(E) \subset C_b(E)$, for each $\lambda>0$. In this section we are going to prove a criterion of excessiveness. The proof makes use of the Choquet boundary associated to a convex cone of lower semicontinuous functions on a compact topological space. Namely we use Bauer's minimum principle.

### 3.1. Theorem

Let f be a bounded continuous nonnegative function on E. Assume that for each $x \in E$ there exists a base of neighbourhoods of $x, \mathcal{U}(x)$, such that

$$P_{CW} f(x) \leq f(x) \quad \text{for each} \quad W \in \mathcal{U}(x).$$

Then f is an excessive function.

Proof

In order to simplify the exposition we first assume that the potential kernel $V_o$ has also the property $V_o C_b(E) \subset C_b(E)$. From Remark 2.3, $4^o$ one deduces that all results of Section 2 apply for our resolvent. Let us denote by $g=f-\lambda V_\lambda f$, $\varphi=\max(0,g)$, $\psi=\max(0,-g)$. Then $V_\lambda f = V_o(f-\lambda V_\lambda f) = V_o \varphi - V_o(\psi)$. From Remark 2.3, $3^o$ we know that

$$\inf \{t \in C(E) \,|\, t \text{ is excessive and } V_o\varphi \leq t \text{ on } CK, \text{ for some compact set } K\} = 0 .$$

Therefore, in order to show $g \geq 0$, it suffices to show $g+t \geq 0$, for each continuous excessive function t such that $V_o\varphi \leq t$ on CK for some compact set K. For such a function t let us suppose that $\alpha = \inf (t+g) < 0$. Then $K_o = \{x \in E / (t+g)(x) = \alpha\}$ is a compact set because $K_o$ must satisfy $K_o \subset K$. Also $K_o$ must satisfy $K_o \subset \{g<0\}$.

Now let $x \in K_o$. Choose $W \in \mathcal{U}(x)$ such that $W \subset \{g<0\}$. Since $\{g<0\} \cap \{\varphi>0\} = \emptyset$ we have

$$E^x[\int_0^{T_{CW}} (x_t)dt]=0,$$ which implies

$$P_{CW}(V_o\varphi)(x)=E^x[\int_{T_{CW}}^{\infty} \varphi(X_t)dt]=V_o\varphi(x) .$$

Since $g=f-\lambda V_o\varphi+\lambda V_o\psi$ we deduce $P_{CW}g(x)\leq g(x)$. Further on account of $\alpha\leq t+g$ and $P_{CW}(1)(x)\leq 1$ we get $\alpha\leq P_{CW}(\alpha)(x)\leq P_{CW}(t+g)(x)\leq t(x)+g(x)=\alpha$ . It follows $P_{CW}(t+g-\alpha)(x)=0$, which shows $P_{CW}(\chi_{E\setminus K_o})=0$. If we denote by $\mu_x$ the measure on $K_o$ defined by $\mu_x(f)=P_{CW}(f)(x)$ we see that $\mu_x(1)=1$, $\mu_x(s)<s(x)$ for each excessive function s, $\mu_x(t+g)\leq(t+g)(x)$ and $\mu_x(\overset{o}{w})=0$ because $X_{T_{CW}}\in\widetilde{CW}$, $P^x$-a.s.

Now we can apply Lemma 1.5 of [5] for the space $K_o$ the cone of all lower semicontinuous excessive functions and the function g+t. We get $g+t\geq 0$. Finally we conclude $f\leq\lambda V_\lambda f$ and from Remark 2.1 deduce that f is excessive.

Now let us treat the general case (where we allow the potential kernel to be nonfinite). For $\lambda>0$ we first deduce $P^\lambda_{CW}f(x)\leq P_{CW}f(x)\leq f(x)$ for any $W\in\mathcal{U}(x)$ and any $x\in E$. Then from the first part of the proof we deduce that f is $\lambda$-excessive. Since $\lambda$ is arbitrary it follows that f is excessive.

The above theorem can be stated in the following more general form:

3.2. Theorem

Let f be a continuous bounded function on E such that $\inf\{R(-f\chi_{CK}) \mid K \text{ compact set} \subset E\}=0$, where

$$R(-f\chi_{CK})=\inf\{t \mid t \text{ is excessive and } -f\chi_{CK}\leq t\}$$

Assume that for each $x\in E$ there exists a base $\mathcal{U}(x)$ of neighbourhoods of x such that

$$P_{CW}f(x)\leq f(x) \text{ for each } W\in\mathcal{U}(x) .$$

Then f is nonnegative and excessive.

Proof

Let K be a compact set. From Theorem 2.2, $1^o$ we know that $t=R(-f\chi_{CK})$ is lower semicontinuous. Let us suppose that $\inf (t+f)=\alpha<0$. Then put $K_o=\{x\in E \mid (t+f)(x)=\alpha\}$. It follows that $K_o$ is a compact subset of K. Further we apply Lemma 1.5 of [5] and deduce $t+f\geq 0$ just like in the preceding proof. The assumption from the statement implies $f\geq 0$. The theorem results from the preceding one.

3.3. Remark. T.Watanabe in [6] proved other excessiveness criteria for a resolvent satisfying the condition $V_\lambda C_b(E)\subset C_b(E)$, for $\lambda>0$. Our results do not follow from his because we let the family $U(x)$ to depend on x.

## 4. The Local Character

Let E be a locally compact space with a countable base and $(V_\lambda | \lambda \geq 0)$ a sub-Markov resolvent of kernels on E that satisfies the conditions assumed in Section 2. We shall use the notation from Section 2. Particularly $(\Omega, M, M_t, X_t, \theta_t, P^x)$ will be a Hunt process such that for each $f \in C_b(E)$, $\lambda \geq 0$, $x \in E$ the following relation holds:

$$V_\lambda f(x) = E^x[\int_0^\infty \exp(-\lambda t) f(X_t) dt] \quad .$$

In this section we shall characterise those resolvents $(V_\lambda)$ which are associated to continuous Markov processes. In the sequel we shall use the following consequence of a result of G.Mokobodzki.

### 4.1. Theorem

Let $t \in P \cap C(E)$. There exists a unique kernel, $G_t$, on E such that $G_t 1 = t$ and for each $f \in C_{b+}(E)$ ,

$$G_t f \in P \cap C(E) \quad \text{and}$$

$$R(\chi_A G_t f) = G_t f \, , \quad \text{where } A = \text{supp } f.$$

The proof follows from Theorem 3 of Ch.IV in [4] and the next lemma.

### 4.2. Lemma

Let t be in $P \cap C(E)$. Then there exists a sequence $\{t_n\}$ in $P \cap C(E)$ such that $t = \sum_n t_n$ and for each $n \in N$ there exists a compact set $K_n$ such that $R(t_n \chi_{K_n}) = t_n$ .

#### Proof

Let $\{g_n\}$ be a sequence in $C_c(E)$ such that $0 \leq g_n \leq g_{n+1}$ and $\overline{\bigcup_n \{g_n = 1\}}^c = E$. We define $t_o = 0$ and

$$t_{n+1} = R(t - \sum_{k \leq n} t_k - R((t - \sum_{k \leq n} t_k)(1 - g_{n+1}))) \, .$$

Next we are going to prove by induction that the sequence $\{t_n\}$ has the following properties:

$$t_n \in P \cap C(E) , \quad t - \sum_{k\le n} t_k \in P \cap C(E) .$$

Suppose that the above properties are true. Let us prove them with n+1 instead of $n$. First we note that

$$R((t-\sum_{k\le n} t_k)(1-g_{n+1})) = t - \sum_{k\le n} t_k \quad \text{on} \quad \{g_{n+1}=0\} .$$

From Theorem 2.2, $3^o$ and Remark 2.3, $1^o$ it follows that $R((t-\sum_{k\le n} t_k)(1-g_{n+1}))$ is a continuous excessive function. Then the same arguments imply that $t_{n+1}$ is also a continuous excessive function. From Theorem 2.1' it follows that $t-\sum_{k\le n} t_k - t_{n+1}$ is also excessive. Further the inequality $t_{n+1} \le t$ implies $t_{n+1} \in P$ , and similarly we deduce $t-\sum_{k\le n+1} t_k \in P$.

Now for each $n \in N$ we put $K_n = \text{supp } g_n$ and remark that $t_{n+1} = R(t_{n+1} \chi_{K_{n+1}})$.

From the definition of $t_{n+1}$ we deduce

$$t-\sum_{k\le n} t_k - t_{n+1} \le R((t-\sum_{k\le n} t_k)(1-g_{n+1})) .$$

Further $t-\sum_{k\le n+1} t_k \le R(t(1-g_{n+1})) \le R(t\chi_{CK_{n+1}})$ .

Since the last term tends to zero we get $t = \sum_{k=1}^{\infty} t_k$ .

4.3. Notation

If $t \in P \cap C_b(E)$ and $f \in \mathcal{B}_b(E)$ we shall use the notation

$$f.t = G_t f \quad ,$$

where $G_t$ is given by Theorem 4.1.

4.3′. Remark

The unicity of the kernel $G_{V_o 1}$ associated to $V_o 1$ shows that

$$f.(V_o 1)=V_o f \quad \text{for each} \quad f\in\mathcal{B}_b(E) \quad .$$

4.4. Lemma

Let U be an open set such that $P^x(X_{T_{CU}}\in E\setminus\overline{U})=0$ for each $x\in U$. Assume that $s,t\in P\cap C_b(E)$ are such that $s=t$ on $\overline{U}$. If $u\in P\cap C_b(E)$ is such that $s-u\in P$ and there exists a compact set, $K\subset U$, such that $P_K u=u$, then $t-u\in P$.

Proof

We are going to apply Theorem 3.2 for the function t-u. If $x\in U$ we put $\mathcal{U}(x)=\{W \text{ open } | \overline{W}\subset U,\ x\in W\}$. The condition from the statement implies $P^x(X_{T_{CW}}\in E\setminus\overline{U})=0$, and hence $P_{CW}t(x)=P_{CW}s(x)$ for each $W\in\mathcal{U}(x)$.

Then $(t-u)(x)-P_{CW}(t-u)(x)=(s-u)(x)-P_{CW}(s-u)(x)\geq 0$.

If $x\in E\setminus U$ we put $\mathcal{U}(x)=\{W \text{ open} | W\cap K=\emptyset,\ x\in W\}$. Since $P_K u(x)=P_{CW}u(x)$, we have

$$(t-u)(x)-P_{CW}(t-u)(x)=t(x)-P_{CW}t(x)\geq 0.$$

Then Theorem 3.2 implies that t-u is nonnegative and excessive. Since $t-u\leq t\in P$ we have $t-u\in P$ .

4.5. Proposition

If U, s, t satisfy the requirements of the preceding Lemma, then f.s=f.t for each $f\in\mathcal{B}_b(E)$ which satisfies f=0 on $E\setminus U$.

Proof

From the construction of the kernel $G_s$ (see [4] p.239) it follows that for each open set D,

$\chi_D.s=\sup\{u\in S\cap C(E)\,|\,s-u\in S$ and $R(u\chi_k)=u$ for some compact set $K\subset D\}$.

A similar relation holds for $\chi_D.t$, and the equality $\chi_D.s=$ $=\chi_D.t$ follows from the preceding lemma for $D\subset U$. Further the monotone class theorem implies the desired conclusion.

4.6. Lemma

Let U be an open and $x_o\in U$. Then there exist two functions $p,q\in P\cap C_b(E)$ such that $p\geq q$, $p-q\in C_c(U)$ and $p(x_o)>q(x_o)$.

Proof

Put $p=V_o1$ and choose a function $g\in C(E)$ such that $g=0$ on an open neighbourhood D of $x_o$, $0\leq g\leq 1$ and $g=1$ on $E\setminus U$. Then put $q=R(gp)$. From Theorem 2.2, $3^o$ we get $q\in C(E)$. On the other hand we have

$$P_{CD}p(x_o)=E^{x_o}[\zeta-T_{CD}]<E^{x_o}[\zeta]=p(x_o)$$

From Hunt's theorem (see [1] p.141) it follows $P_{CD}p(x_o)=R(q\chi_{CD})(x_o)$. Since $R(q\chi_{CD})=q$ we get $q(x_o)<p(x_o)$.

4.7. Lemma

Let u be a continuous excessive function and K a compact set such that $P_Ku=u$. If $\{u_n\}$ is an increasing sequence of continuous excessive functions which converges to u, then the convergence is uniform.

Proof

By Dini's theorem we deduce that for each $\varepsilon>0$ there exists $n\in N$ such that $u<u_n+\varepsilon$ on K. Then

$$u=P_Ku\leq P_K(u_n+\varepsilon)\leq u_n+\varepsilon \quad \text{on } E.$$

4.8. Proposition

Let u be a continuous excessive function and K a compact set such that $P_Ku=u$. Assume that $t\in P\cap C(E)$ and $\{f_n\}$ is a sequence of continuous functions such that the sequence $\{f_n.t\}$ is increasing and $\lim_{n\to\infty} f_n.t=u$. If $g\in C_c(E)$ is such that $0\leq g\leq 1$ and $g=1$ on an open set D with $K\subset D$, then $\lim_{n\to\infty}(gf_n).t=u$ uniform.

Proof

Using Lemma 4.6 we first choose two continuous bounded excessive functions $p,q\in P$ such that $p=q$ on $E\setminus D$ and $p-q\geq 1$ on K. Then we can apply the method from Proposition 3.1 in [5]. Thus for each $x\in E$ we have a positive measure $\mu_x$ such that

$$s(x)=\mu_x(s-P_K s) \text{ for each } s\in P \text{ which fulfils } P_{E\setminus D}s=s .$$

Furthermore $\mu_x(1)<||q||$, for each $x\in E$. Hence

$$||s||\leq c||s-P_K s|| \text{ for each } s\in P \text{ which fulfils } P_{E\setminus D}s=s .$$

From Lemma 4.7 we know that $f_n.t\to u$ uniform. Then $P_K(f_n.t)\to P_K(u)$ uniform. Since $P_K u=u$ we deduce $f_n.t-P_K(f_n.t)\to 0$. Further from the inequality

$$f_n.t-P_K(f_n.t)=(gf_n).t-P_K((gf_n).t)+((1-g)f_n).t-P_K(((1-g)f_n).t)$$

$$\geq((1-g)f_n).t-P_K(((1-g)f_n).t)\geq(1/c)((1-g)f_n).t$$

we get $((1-g)f_n).t\to 0$, which implies

$$(gf_n).t=f_n.t-((1-g)f_n).t\to u.$$

4.9. Remark. If in the preceding proposition we assume K is closed and CK is relatively compact instead of assuming K is compact, then the conclusion is still valid with uniform convergence on each compact subset of E instead of uniform convergence on the whole space E.

4.10. Theorem

Let U be a relatively compact open set such that for each $x\in CU$, $P^x(X_{T_{\bar U}}\in U)=0$. Then for each open set, A, such that $\bar U\subset A$, the following inclusion holds:

$$C_c(U) \subset \overline{V_o(C_c(A))}$$

Proof

Let us define

$T(U)=\{f \in C_c(U) \mid$ there exist $s,t \in P \cap C_b(E)$ such that $f=s-t\}$

From Lemma 4.6 and the Stone-Weierstrass theorem it follows

$$\overline{T(U)}=C_o(U) \; .$$

Therefore for each $f \in C_c(U)$ and each $\varepsilon>0$ there exist $s,t \in P \cap C_b(E)$ such that $s-t \in C_c(U)$ and $|s-t-f|<\varepsilon$. Let now $g \in C_c(A)$ be such that $0 \le g \le 1$ and $g=1$ on U. From Lemma 4.5 we deduce $(1-g).s=(1-g).t$ and hence $g.s-g.t=s-t$.

Since $g.s$ is excessive we have $\lim_{\lambda\to\infty} \lambda V_\lambda(g.s)=g.s$. If we put $p=V_o 1$ and $f_n=n(g.s-nV_n(g.s))$ we have $nV_n(g.s)=f_n.p$. Now we apply Proposition 4.8 for $K=\mathrm{supp}\,g$ and $u=g.s$. Let $g' \in C_c(A)$ be such that $0 \le g' \le 1$ and $g'=1$ on a neighbourhood of K. Then there exists $n \in N$ such that $|(g'f_n).p-g.s|<\varepsilon$. Putting $h=g'f_n$ we can writte $(g'f_n).p=h.p=V_o h$ and $|V_o h-g.s|<\varepsilon$.

Similarly we can find $h' \in C_c(A)$ such that $|V_o h'-g.t|<\varepsilon$. Therefore

$$|V_o(h-h')-f|<3\varepsilon \quad \text{and} \quad h-h' \in C_c(A) \; .$$

4.11. Lemma

Let A,U be two open sets such that $U \subset A$. If

$$C_c(U) \subset \overline{V_o(C_c(A))} \quad ,$$

then for each $x \in E \setminus A$ we have $P^x(X_{T_A} \in U)=0$.

Proof

If $f \in C_c(A)$ and $x \in E \setminus A$, then we have $E^x[\int_0^{T_A} f(X_t)\,dt]=0$, and hence

$$P_A V_o f(x) = E^x[\int_{T_A}^{\infty} f(X_t)\,dt] = V_o f(x) = 0 .$$

The density condition leads to $P_A g(x)=0$ for each $g \in C_c(U)$, which implies $P^x(X_{T_A} \in U)=0$.

4.12. Corollary

The process $(\Omega, M, M_t, X_t, \theta_t, P^x)$ is continuous if and only if for each open set, U, the following inclusion holds:

$$(*) \qquad C_c(U) \subset \overline{V_o(C_c(U))}$$

Proof

If the process is continuous one uses Theorem 4.10 and get relation (*) for each open set.

Now let us suppose that relation (*) is valid for each open set. Let W be an open set and put $U=E\setminus\overline{W}$. From Lemma 4.11 we get

$$P^x(X_{T_{E\setminus W}} \in E\setminus\overline{W})=0 \quad \text{for each} \quad x \in W .$$

Then from the result of Annexe 1 in [3] we deduce that the process is continuous.

# R E F E R E N C E S


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