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by

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Introduction

In our previous paper [6], we dealt with some criteria for properness for the morphisms of schemes.

If $f:X \to Y$ is a separated morphism of algebraic schemes over a field k, then the following properties are equivalent:

i) f is proper.

ii) every closed integral 1-dimensional subscheme $C \subseteq X$ is proper over Y.

(cf. [6], Corollary 3).

In general, when f is a separated morphism of finite type of noetherian schemes, i) is not equivalent with ii), as shows the following

Example 1 - Let A be a discrete valuation ring, $t \in A$ a generator of the maximal ideal of A, A[T] the ring of the polynomials in one indeterminate and $x=(tT-1)$ the ideal of A[T] generated by tT-1. Then x is a maximal ideal since $A[T]/(tT-1)$ is isomorphic to the quotient field A_t. Denote $Y=\text{Spec } A[T]$, $X=Y-\{x\}$ and $f:X \hookrightarrow Y$ the canonical open immersion. Since x is a closed 1-codimensional point of Y, it is easy to see that f satisfies ii), f is not proper since it is not surjective.

In Example 1, X and Y are schemes of finite type over A.

Then it is natural to put the following problem: if S is a noetherian scheme, which conditions must satisfy S such that for every separated morphism $f:X \to Y$ of schems of finite type over S, the above conditions i) and ii) are equivalent ?

In [6] we have shown (see the proofs of Theorem 1 and Propositions 1 and 2, loc.cit.) that the obstruction for a separated morphism $f:X \rightarrow Y$ of finite type of noetherian schemes to be proper (under some "good" conditions) consists in the existence of some closed integral subschemes $X' \subseteq X$ of dimension >1, which have closed 1-codimensional points.

In connection with this remark, in §1 we introduce a class of noetherian rings, called universally 1-equicodimensional. Precisely, a ring A belongs to this class if it is noetherian and if every integral A-algebra of finite type, which has a maximal 1-height ideal, is 1-dimensional. In an obvious manner one defines the universally 1-equicodimensional schemes.

In §2, Theorem 2 a) $\Longleftrightarrow$ c), we show that a scheme S has the property required in the above problem iff it is universally 1-equicodimensional. Other characterizations of universally 1-equicodimensional schemes are given in Theorem 2, some of them being pure topological (see b)). Theorem 2 appears also in [4].

In the other theorems of this paper we establish some general properties of the above class of schemes. The main results are Theorem 1 and 3.

In §3, Theorem 3, we point out that the integral universally 1-equicodimensional schemes which have generically some good properties (more precisely, those which contain an open non-empty catenary and equicodimensional scheme), can be characterized by some stronger properties. From this theorem we may easily deduce that an integral 1-dimensional (resp.2-dimensional) ring is universally 1-equicodimensional iff it is noetherian and Jacobson (resp. Jacobson universally catenary and equicodimensional) (see also Corollaries 1 and 8).

In [5], we have proved that if a subalgebra of an algebra of finite type over a field is universally 1-equicodimensional then it is finitely generated (and conversely). This result is not contained in this paper, but it will appear in [7]. This result can be related in a natural manner with some new solutions for the known affirmative cases of Hilbert's 14th problem.

In §1, Theorem 1, we give a class of morphisms of finite type of schemes, by which the universally 1-equicodimensionality goes down. This class of morphisms, given by topological properties, includes the surjective proper morphisms, the surjective universally open (in particular, all faithfully flat) morphisms of finite type and also some morphisms which appear in connection with the problem of the finite generation of the subalgebras: the strongly submersive morphisms (of finite type) introduced in [18] by Nagata and in particular the morphisms of finite type of the form Spec A $\longrightarrow$ Apec A^G, where G is a (geometrically) reductive group acting rationally on the ring A and A^G is the subring of the invariants (for details, see the proofs of Corolaries 2-6 and Remark 2).

In connection with Theorem 1, we may mention that by a morphisms $f:X \longrightarrow Y$ of reduced schemes over a field k, which belongs to the class of morphisms described in Theorem 1, the property to be algebraic over k goes down (see [7]).

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Throught this paper we follow in general the terminology and the notations of EGAI-IV, except the term of "prescheme" (resp. "scheme") which is replaced by "scheme" (resp. "separated scheme").

1. Definition, examples and some general properties

We shall introduce the following:

Definition - A ring A is called universally 1-equicodimensional if it is noetherian and if every integral A-algebra of finite type which has a maximal 1-height ideal, is 1-dimensional.

A scheme X is called universally 1-equicodimensional if there exists a finite covering $(U_i)_{i\in I}$ of X with open affine subsets such that for every $i\in I$, the ring $\Gamma(U_i, \mathcal{O}_X)$ is universally 1-echicodimensional.

The following Example 2 gives a class of universally 1-equicodimensional rings. This Examples appears also in EGA IV, 10.6.:

Example 2 - An integral ring which is Jacobson, equicodimensional and universally catenary, is universally 1-equicodimensional.

Indeed, let A be such a ring. It suffices to prove that every polynomial algebra $A[T_1,\ldots,T_n]$ is equicodimensional. (Since it is also catenary, it follows that all maximal chains of prime ideals of $A[T_1,\ldots,T_n]$ have the same lenght and so every integral A-algebra B of finite type is equicodimensional; then, if B has a maximal 1-height ideal, it follows dim B=1).

Since $A[T_1,\ldots,T_n]$ is a Jacobson universally catenary ring (cf.[2], ch.V, 3, no.4, Th.3), by induction on n, we may suppose that n=1. Let $\underline{m}\subset A[T]$ be a maximal ideal. Since A is a Jacobson ring, it follows that $\underline{n}=\underline{m}\cap A$ is a maximal ideal of A (cf.[2], loc. cit.). The local ring $A[T]_{\underline{m}}$ is flat over $A_{\underline{n}}$, and then let $\mathrm{ht}\,\underline{m} =$
$=\dim A[T]_{\underline{m}}=\dim A_{\underline{n}}+\dim A[T]_{\underline{m}}/\underline{n}A[T]_{\underline{m}} =\mathrm{ht}\,\underline{n}+\dim(A[T]/_{\underline{n}}A[T])_{\underline{m}} =$
$=\dim A+\dim(A/\underline{n}[T])_{\underline{m}}=\dim A+1.$

Q.E.D.

We have the following consequences of Example 2:

Example 2a - Every artinian ring k is universally 1-equicodimensional. Every k-algebra of finite type (resp. every scheme of finite type) over such a ring k is universally 1-equicodimensional.

Indeed an integral k-algebra of finite type is a k/m-algebra, where m is a maximal ideal of k. Then Example 2a follows from the fact that every field is universally 1-equicodimensional.

Example 2b - Every algebra of finite type over an integral noetherian Jacobson 1-dimensional ring is universally 1-equicodimensional.

The ring $\mathbb{Z}$ of the integer numbers is universally 1-equicodimensional.

It follows from Example 2 and Definition, since every integral noetherian 1-dimensional ring is equicodimensional and universally catenary (cf. EGA IV, 5.6.3 and 6.3.7).

Example 2c - Every algebra of finite type over a noetherian normal Jacobson ring A, such that for every maximal ideal $\underline{m} \subset A$ ht $\underline{m}$ = =2, is universally 1-equicodimensional.

It follows from Example 2, since every normal 2-dimensional ring is universally catenary. Indeed, for every prime ideal $\underline{p} \subset A$, dim $A_{\underline{p}} \leq 2$ and $A_{\underline{p}}$ is a Cohen-Macaulay ring (cf. [22], ch.IV B, Ex.2) then $A_{\underline{p}}$ is universally catenary (cf. EGA IV, 6.3.7) and hence A is also universally catenary (cf. EGA IV, 5.6.3).

Example 2d - Let A be a local integral universally catenary ring and $m \subset A$ its maximal ideal. Then Spec A-{m} is an universally 1-equicodimensional scheme.

Indeed, cf. EGA IV, 10.5.9, it follows that X=Spec A-{$\underline{m}$} is a Jacobson scheme. Since for every closed point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is universally catenary and dim $\mathcal{O}_{X,x}$=dim A-1, the assertion follows from Example 2.

Example 2d appears also in EGA IV, 10.7.2.

We have the following

<u>Example 3</u> - <u>A noetherian semilocal ring of dimension > 0 is not universally 1-equicodimensional</u>.

In fact, let A be such a ring. Choosing a prime ideal $\underline{p} \subset A$, and replacing A by A/$\underline{p}$, we may suppose that A is an integral 1-dimensional semilocal ring. Let $f \in A$ be a non-zero element such that the quotient ring $A_f = A[1/f]$ is local and $t \in A[1/f]$ a non-zero element of the maximal ideal. Then the principal ideal (tT-1) of the polynomial algebra $A[1/f][T]$ is maximal and ht(tT-1)=1. It follows that $A[1/f,T]$ is an A-algebra of finite type of dimension 2, having a maximal 1-height ideal.

We shall give some general properties of the universally 1-equicodimensional schemes.

<u>Proposition 1</u> - a) <u>An universally 1-equicodimensional scheme is a Jacobson scheme</u>

b) <u>If X is an universally 1-equicodimensional scheme, for every open affine subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is universally 1-equicodimensional</u>.

c) <u>Every scheme of finite type over an universally 1-equicodimensional scheme is still universally 1-equicodimensional. In particular, every subscheme of such a scheme is universally 1-equicodimensional</u>.

d) <u>A noetherian scheme is universally 1-equicodimensional iff every irreducible component with reduced scheme structure is universally 1-equicodimensional</u>.

<u>Proof</u> - a) Let X be an universally 1-equicodimensional scheme and let us suppose that X is not a Jacobson scheme. Then there exists an open affine subset $U \subseteq X$ such that $A=\Gamma(U,\mathcal{O}_X)$ is an universally 1-equicodimensional ring, but it is not a Jacobson ring. From EGA IV 10.5.2, it follows that there exists a prime ideal $\underline{p} \subset A$, such that $A/\underline{p}$ is a semilocal 1-dimensional ring. Then $A/\underline{p}$ is not universally 1-equicodimensional, by Example 3. This contradicts the fact that A is universally 1-equicodimensional.

b) By definition, it follows that if A is an universally 1-equicodimensional ring, then for every $f \in A$, the quotient ring A_f is universally 1-equicodimensional. Hence an universally 1-equicodimensional scheme has a topological basis $(U_i)_{i\in I}$ such that for every $i \in I$, U_i is affine and $\Gamma(U_i,\mathcal{O}_X)$ is an universally 1-equicodimensional ring. Let $U \subseteq X$ be an open affine subset and $J \subseteq I$ such that $(U_j)_{j\in J}$ is a covering of U. If A is an integral $\Gamma(U,\mathcal{O}_X)$-algebra of finite type which has a maximal ideal $\underline{m}$ of height 1 and φ:Spec A $\longrightarrow$ U is the canonical morphism of affine schemes, then there exists $j \in J$ such that $\varphi(\underline{m})=\underline{n} \in U_j$. Since φ is an affine morphism of finite type, it follows that $\varphi^{-1}(U_j)$ is an affine scheme of finite type over U_j (cf. EGA II, 1.2.1) and $\underline{m} \in \varphi^{-1}(U_j)$ is a maximal 1-height ideal in $B=\Gamma(\varphi^{-1}(U_j),\mathcal{O}_{\mathrm{Spec}A})$. Since B is of finite type over the universally 1-equicodimensional ring $\Gamma(U_j,\mathcal{O}_X)$, it follows that dim $\varphi^{-1}(U_j)=1$. From a), it results that X is a Jacobson scheme. Then $\Gamma(U,\mathcal{O}_X)$ is a Jacobson ring (cf. EGA IV, 10.4.2) and then A is still Jacobson. Applying Corollary 2 of [6], to the scheme Spec A and to the 1-dimensional open subset $\rho^{-1}(U_j)$ it follows that dim Spec A=1. Hence dim A=1.

c) and d) are standard.

Corollary 1 - An integral 1-dimensional ring is universally 1-equicodimensional iff it is a noetherian Jacobson ring.

Indeed, an implication follows by Proposition 1 a) and another by Example 2b).

In the following Theorem 1 we give a class of morphisms, by which the property of universal 1-equicodimensionally, goes down.

Theorem 1 - Let $f:X \longrightarrow Y$ be a morphism of finite type of noetherian schemes, which satisfies the following property:

(P) For every integral Y-scheme Y′ the canonical morphism $f_{(Y')}: X \times_Y Y' \longrightarrow Y'$ restricted to the (finite) union of all irreducible components of $X \times_Y Y'$ which dominate Y′, is surjective.

Then, if X is an universally 1-equicodimensional scheme, Y is also universally 1-equicodimensional.

Proof - Let Y′ be an integral Y-scheme of finite type and $y \in Y'$ a closed point. By the property (P), there exists an integral scheme X′ of finite type over X and a dominant morphisms $f': X' \longrightarrow Y'$ of finite type such that $y \in f'(X')$. Let $x \in X'$ be a closed point such that $y=f'(x)$ and $f'^{*}: \mathcal{O}_{Y',y} \longrightarrow \mathcal{O}_{X',x}$ the canonical homomorphisms of local rings. Then f'^{*} is injective and if $\underline{m}_y$ is the maximal ideal of the ring $\mathcal{O}_{Y',y}$, we have $\underline{m}_y \mathcal{O}_{X',x} \neq 0$. It is easy to see that there exists a prime ideal $\underline{p} \subset \mathcal{O}_{X',x}$ such that cht $\underline{p}=1$ and $\underline{p} \not\supseteq \underline{m}_y \mathcal{O}_{X',x}$.

Let $C\subseteq X'$ be the integral closed subscheme which passes through x and which corresponds to $\underline{p}$. C is of finite type over X and $\dim \mathcal{O}_{C,x}=\dim \mathcal{O}_{X',x}/\underline{p}=1$. If $U\subseteq C$ is an open affine neighbourhood of x, $\Gamma(U,\mathcal{O}_C)$ is universally 1-equicodimensional since C is an universally 1-equicodimensional scheme and because of Proposition 1b). The maximal ideal $\underline{m}\subset\Gamma(U,\mathcal{O}_C)$ corresponding to $x\in C$ is of height 1. Then $\dim U=\dim \Gamma(U,\mathcal{O}_C)=1$. Since C is a Jacobson scheme, it follows that dim C=1, by Corollary 2 of [6] . We deduce that $f'(C)\supsetneqq\{y\}$, since $\underline{p}\neq\underline{m}_y\mathcal{O}_{X',x}$.

Therefore we have proved that there exists a closed integral 1-dimensional subscheme $C\subseteq X'$ such that $f'(C)\supset\{y\}$ and C is a X-scheme of finite type.

Using this remark, we may prove that Y is a Jacobson scheme. Indeed, let us suppose that Y is not Jacobson. Then there exists an open affine subset $U\subseteq X$ such that the ring $A=\Gamma(U,\mathcal{O}_X)$ is not Jacobson. By EGA IV, 10.5.2, it follows that there exists a prime ideal $\underline{p}\subset A$ such that $A/\underline{p}$ is a semilocal 1-dimensional ring. Let be $Y'=\operatorname{Spec} A/\underline{p}$ and let $y\in Y'$ be a closed point.

By above remark, there exists a morphism of finite type $\varphi:C\longrightarrow Y'$, such that C is an integral 1-dimensional scheme of finite type over X and $\varphi(C)\supset\{y\}$. Then φ is dominant, since $\overline{\varphi(C)}\supset\{y\}$ and $\operatorname{codim}_{Y'}\{y\}=1$. Let $\eta\in Y'$ be the generic point; $\{\eta\}$ is open in Y' and so $\varphi^{-1}(\eta)$ is an open nonempty subset of C. Since $\varphi^{-1}(y')$ is a finite set, by Zariski Main Theorem it follows that $U=\{x\in C\mid x \text{ is isolated in } \varphi^{-1}\varphi(x)\}$ is an open nonempty subset in C. Hence $C'=\varphi^{-1}(\eta)\cap U$ is nonempty subset of C.

If $x\in C'$, then $\varphi(x)=\eta$ and x is isolated in $\varphi^{-1}(\eta)$. But $\varphi^{-1}(\eta)$ is irreducible of dimension 1, since C is a Jacobson scheme, by Proposition 1a. This is a contradiction.

Now we shall prove that Y is a universally 1-equicodimensional scheme. Let $U \subseteq Y$ be an open affine subset and A an integral $\Gamma(U,\mathcal{O}_Y)$-algebra of finite type which has a maximal ideal of height 1. Then $Y' = \operatorname{Spec} A$ is an integral Y-scheme of finite type and has a closed 1-codimensional point $y \in Y'$. By the remark made above, there exists a closed integral 1-dimensional subscheme $C \subseteq X'$ and a morphism of finite type $\varphi : C \longrightarrow Y'$ such that $\overline{\{y\}} \subset \varphi(C)$. Since $\{y\}$ is closed of codimension 1 in Y', we have $\overline{\varphi(C)} = Y'$ and so φ is dominant. φ is then quasi-finite and hence it is easy to see that it is generically a finite morphism, via Zariski Main Theorem (EGA III, 4.4.5). Let $U \subseteq Y'$ be an affine nonempty open subset such that $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \longrightarrow U$ is finite. We have $\dim U = \dim \varphi^{-1}(U) = 1$, since C is an integral Jacobson 1-dimensional scheme. Y' being of finite type over the Jacobson scheme Y, it is also Jacobson. By Corollary 2 of [6], it follows $\dim Y' = 1$. Hence $\dim A = 1$ and Y is a universally 1-equicodimensional scheme.

Q.E.D.

Remark 1 - In Theorem 1, it suffices to assume that f is a morphism of finite type of noetherian schemes which satisfies the following weaker property:

(P') for every affine integral Y-scheme Y' of finite type and for every closed point $y \in Y'$, there exists an integral component X' of $X \times_Y Y'$ such that the canonical morphism $f' : X' \longrightarrow Y'$ is dominant and $y \in f'(X')$.

The proof remains unchanged.

We don't know whether in Theorem 1 Y is noetherian if X is universally 1-equicodimensional, even under some additional conditions. In some particular cases this fact is true.

For instance, in [7], one proves that the answer to the above question is affirmative if X is an algebraic scheme over a field (more precisely if $f:X \longrightarrow Y$ is a morphism of reduced k-schemes over a field k, where X is algebraic over k and f satisfies the property (P) of Theorem 1, then Y is also algebraic over k).

We shall present some particular cases of Theorem 1.

Corollary 2 - Let $f:X \rightarrow Y$ be a morphism of schemes, where X is universally 1-equicodimensional and Y is noetherian. If f is either proper surjective or a universally open surjective morphism of finite type, then Y is universally 1-equicodimensional.

Indeed, it is easy to see that a proper surjective morphism or a universally open surjective morphism of finite type has the property (P) of Theorem 1. Then Corollary 2 follows by Theorem 1.
Q.E.D.

Corollary 3 - Let $f:X \rightarrow Y$ be a faithfully flat morphism of finite type of schemes, where X is universally 1-equicodimensional. Then Y is universally 1-equicodimensional.

Indeed, by the hypothesis of Corollary 3, it follows that Y is a noetherian schemes and f is universally open and surjective. Corollary 3 is then a consequence of Corollary 2.

Corollary 4 - Let X be an affine universally 1-equicodimensional scheme over a field k, G a linearly reductive algebraic group over k and $\sigma:G \times X \longrightarrow X$ an action of G on X. Suppose that the canonical morphism $X \longrightarrow X/G$ of X to the categorical quotient X/G of X by G (which exists by Munford's Theorem (cf. [14], Ch.1 §2, Th.1.1)) is of finite type. Then X/G is a universally 1-equicodimensional scheme.

Proof - We shall follow the first part of the proof of Mumford's Theorem (cf. [14], loc.cit.).

Denote $A=\Gamma(X,\mathcal{O}_X)$ and by $\hat{\sigma}:A \longrightarrow \Gamma(G,\mathcal{O}_G) \otimes_k A$ the dual action induced by σ. Let $A_o=\{x \in A \mid \hat{\sigma}(x)=1 \otimes x\} \subseteq R$ be the k-subalgebra of invariants of $\hat{\sigma}$, Y=Spec A_o and $\varphi: X \longrightarrow Y$ the canonical morphism induced by the inclusion $A_o \subseteq A$.

In the proof of <u>Mumford</u>'s Theorem the following facts are proved:

1) if B_o is an A_o-algebra then the ring of invariants of the dual action induced by $\hat{\sigma}$ on $A \otimes_{A_o} B_o$ is B_o.

2) if $(\underline{a}_i)_{i\in I}$ is a set of invariant ideals of A then $(\sum_{i\in I} \underline{a}_i) \cap A_o = \sum_{i\in I} (\underline{a}_i \cap A_o)$.

3) Y=X/G and it is noetherian.

Corollary 4 follows from Theorem 1 if we prove that $\varphi: X \longrightarrow Y$ has the property (P) of Theorem 1.

Let Y' be an integral Y-scheme and $y \in Y'$. From 1) and 3), applied to the action of G on the scheme $X x_Y \operatorname{Spec} \mathcal{O}_{Y',y}$, it follows that Spec $\mathcal{O}_{Y',y}$ is a categorical quotient of this action. By 2), applied also to this action, it follows that for every set $(\underline{a}_i)_{i\in I}$ of invariant ideals of the ring $A \otimes_{A_o} \mathcal{O}_{Y',y}$ we have $(\sum_{i\in I} \underline{a}_i) \cap \mathcal{O}_{Y',y} = \sum_{i\in I} (\underline{a}_i \cap \mathcal{O}_{Y',y})$. If $\psi: X x_Y \operatorname{Spec} \mathcal{O}_{Y',y} \longrightarrow \operatorname{Spec} \mathcal{O}_{Y',y}$ is the canonical morphism, then this equality of ideals implies that for every family $(W_i)_{i\in I}$ of closed invariant subsets of $X x_Y \operatorname{Spec} \mathcal{O}_{Y',y}$ we have $\overline{\psi(\bigcap_{i\in I} W_i)} = \bigcap_{i\in I} \overline{\psi(W_i)}$.

Let us denote by $X''_1, \ldots, X''_n$ the irreducible components of the scheme $X x_Y \operatorname{Spec} \mathcal{O}_{Y',y}$ which dominate Spec $\mathcal{O}_{Y',y}$. Then $\bigcup_{i=1}^{n} X''_i$ is a closed invariant subset of $X x_Y \operatorname{Spec} \mathcal{O}_{Y',y}$. Since $\psi^{-1}(y)$ is also a closed invariant subset of $X x_Y \operatorname{Spec} \mathcal{O}_{Y',y}$ (y being indetified with the closed point of Spec $\mathcal{O}_{Y',y}$), we have:

$$\overline{\Psi(\Psi^{-1}(y)\cap(\bigcup_{i=1}^{n}X_i''))}=\{y\}\cap\overline{\Psi(\bigcup_{i=1}^{n}X_i'')}=\{y\}\cap Y'=\{y\}$$

If $y\notin\Psi(\bigcup_{i=1}^{n}X_i'')$, then $\Psi^{-1}(y)\cap(\bigcup_{i=1}^{n}X_i'')=\emptyset$, and so $\{y\}=\emptyset$, which is not possible.

Thus $y\in\Psi(\bigcup_{i=1}^{n}X_i'')=\bigcup_{i=1}^{n}\Psi(X_i'')$. There exists an irreducible component X_i'' of $X\times_Y \mathrm{Spec}\,\mathcal{O}_{Y',y}$, which dominates Y', such that $y\in\Psi(X_i'')$. Then the closure of the image of the natural map $X_i'\to X\times_Y Y'$ is a closed irreducible subset X' of $X\times_Y Y'$, which dominates Y' and such that y is in the image of the application $X'\to Y'$. Choosing an irreducible component $Z\subseteq X\times_Y Y'$ which contains X', it is clear that Z dominates Y' and y is in the image of the application $Z\to Y'$.

Corollary 5 - Let A be a universally 1-equicodimensional algebra over an algebraically closed field k and G a linearly reductive group over k which acts rationally on A. If A is of finite type over the subring A^G of invariants, then A^G is a universally 1-equicodimensional ring.

Indeed, Corollary 5 follows from Corollary 4, since it is known that the rational action of G on A corersponds to an action of G on X=Spec A, such that X/G=Spec A^G.

Remark 2 - The fact that in Corollary 4, the morphism $X\to X/G$ has the property (P), is more general: let A be an algebra over an algebraically closed field k and G a (geometrically) reductive group over k acting rationally on A. If A is of finite type over the subring of invariants A^G then the cannonical morphism Spec A $\to$ Spec A^G satisfies the condition (P) of Theorem 1.

We don't prove this fact here. The proof uses the known lemmas of *Nagata* (see [17] or [20] pag.54) in a more general version (see [7]).

Recall that a homomorphism $f:A \to A'$ of rings is called *strongly submersive* if for every minimal prime ideal $\underline{p} \subset A$ and for every valuation subring $V \supseteq A/\underline{p}$ of the quotient field of $A/\underline{p}$, there exist a valuation ring W dominating V and a homomorfism $A' \longrightarrow W$ such that the following diagram

$$\begin{array}{ccc} A' & \longrightarrow & W \\ f \uparrow & & \uparrow \\ A & \longrightarrow & V \end{array}$$

is commutative. (cf. [18], p.193-194).

In a natural manner, one defines the notion of strongly submersive morphism of affine schemes.

Corollary 6 - *Let $f:A \to A'$ be a strongly submersive homomorphism of rings, such that A is noetherian and A' is an A-algebra of finite type. Then A is universally 1-equicodimensional if A' is universally 1-equicodimensional.*

Proof - We shall prove that the morphism $\varphi : X = \mathrm{Spec}\ A' \to Y = \mathrm{Spec}\ A$, induced by f, has the property (P) of Theorem 1. Then Corollary 6 follows from Theorem 1.

Let Y' be an affine integral Y-scheme and $y \in Y'$. By [18], Theorem 1, it follows that $\varphi_{(Y')} : X \times_Y Y' \to Y'$ is strongly submersive. Then if V is a valuation subring of the field K(Y') of rational functions on Y', such that V dominates $\mathcal{O}_{Y',y}$, then there exist a valuation ring W dominating V and a morphism $\psi : \mathrm{Spec}\ W \to X \times_Y Y'$ such that the diagram

$$\begin{array}{ccc} X\times_Y Y' & \xleftarrow{\ \psi\ } & \operatorname{Spec} W \\ {\scriptstyle \varphi_{(Y')}}\downarrow & & \downarrow \\ Y' & \longleftarrow & \operatorname{Spec} V \end{array}$$

is commutative. Then $X''=\overline{\psi(\operatorname{Spec} W)} \subseteq X\times_Y Y'$ is an integral closed subscheme dominating Y' such that $y\in\varphi_{(Y')}(X'')$. If X' is an irreducible component of $X\times_Y Y'$ containing X'', then X' dominates Y' and $y\in\varphi_{(Y')}(X')$.

Now it is clear that the property (P) of Theorem 1 is fulfilled by φ.

Remark 3 - Let $f:A\longrightarrow A'$ be a homomorphism of rings. It is easy to show that the converse of the assertion made in the proof of Corollary 6 is true, i.e. if the morphism $\operatorname{Spec} A'\longrightarrow \operatorname{Spec} A$ has the property (P) of Theorem 1, then f is strongly submersive.

Q.E.D.

2. Some characterizations of universally 1-equicodimensional schemes

In the following Theorem 2, we shall prove that the universally 1-equicodimensional schemes are the solution of the problem put in Introduction (see a) $\Longleftrightarrow$ c)). The assertion b) of this Theorem gives a topological characterization for the universally 1-equicodimensional schemes.

Theorem 2 - Let S be a noetherian scheme. The following assertions are equivalent:

a) S is universally 1-equicodimensional.

b) for every irreducible scheme X of finite type over S of dimension > 0 and for every closed point $x\in X$, the set of all closed points $x'\in X$, such that there exists a closed irreducible (resp.connected)

1-dimensional subset of X passing through x and x', is dense in X.

c) for every separated morphism $f:X \longrightarrow Y$ of S-schemes X,Y of finite type over S, the following assertions are equivalent:

i) f is proper

ii) every closed integral 1-dimensional subscheme $C \subseteq X$ is proper over Y.

d) for every separated morphism $f:X \longrightarrow Y$ of S-schemes X, Y of finite type over S, the following assertions are equivalent:

i) f is finite

ii) every closed integral 1-dimensional subscheme $C \subseteq X$ is finite over Y.

e) S is a Jacobson scheme and every integral scheme X, which is finite over S and has a closed 1-codimensional point, is 1-dimensional.

f) S is Jacobson and every closed integral subscheme $S' \subseteq S$ such that its normalization has a closed 1-codimensional point, is 1-dimensional.

Proof - a) $\Longrightarrow$ b) We proceed by noetherian induction on X. Since for dim X=1, b) is clear, we may assume that dim X>1 and every closed irreducible subscheme $X' \subset X$ has the property given in b). If X does not satisfy b), then there exists a closed point $x \in X$ such that the subset $Y \subseteq X$ of the closed points $x' \in X$ which can be "joined" with x, by a closed irreducible 1-dimensional subset of X, is not dense in X. Let $U \subseteq X$ be an open nonempty subset, such that $U \cap Y = \emptyset$. It is clear that $x \notin U$ and every closed integral subscheme $X' \subset X$ passing through x does not meet U. As in the proof of Lemma 1 of [6], it follows that $\dim \mathcal{O}_{X,x}=1$. Hence x is a closed 1-codimensional point of X. Let $V \subseteq S$ and $V' \subseteq X$ be two open affine subsets, such that V' is a V-scheme and $x \in V'$. Then $\Gamma(V', \mathcal{O}_X)$ is an integral $\Gamma(V, \mathcal{O}_S)$-

-algebra of finite type, which has a maximal 1-height ideal. Aplying Proposition 1b) to $V' \subseteq S$, it follows that dim $V'=1$. By Corollary 2 of [6] and from the fact that S and X are Jacobson schemes (cf. Proposition 1a)), we deduce that dim X=1. This contradicts the assumption dim X>1.

b) $\Rightarrow$ a) Let $U \subseteq S$ be an open affine subset and A an integral $\Gamma(U, \mathcal{O}_S)$-algebra of finite type which has a maximal 1-height ideal.

Then X=Spec A is an integral S-scheme of finite type which has a closed 1-codimensional point $x \in X$. Since every closed connected subscheme passing through x is either $\{x\}$ or X, it follows that the subset $Y \subset X$ of all closed points of X which can be joined with x by a connected closed 1-dimensional subset of X is either $\{x\}$ or $X_{cl} = \{x' \mid x' \text{ closed in } X\}$. If $Y=\{x\}$, then from b) it follows that $\{x\}$ is dense in X and thus A is a field; but this is not possible. If $Y=X_{cl}$, then dim X=1; hence dim A=1, which proves that $\Gamma(U, \mathcal{O}_S)$ is a universally 1-equicodimensional ring.

b) $\Rightarrow$ c) We may assume that X and Y are integral.

If dim X=0, then $X=\{x\}$ and x is closed. Since $f:X \to Y$ is a morphism of finite type of Jacobson schemes, y=f(x) is closed in Y and the extension of the residue fields $k(y) \hookrightarrow k(x)$ is finite (cf. [2], Ch.V, §3, n°4, Th.3). Then it is easy to see that f is proper.

If dim X>0, let us suppose that f is not proper. Let $\bar{f}:\bar{X} \to Y$ be a (dense) compactification of f (cf. [16], [8])

$$\begin{array}{ccc} X & \hookrightarrow & \bar{X} \\ & f \searrow \quad \swarrow \bar{f} & \\ & Y & \end{array}$$

and $x \in \overline{X} - X$ a closed point. From b) it follows that there exists a connected closed 1-dimensional subscheme $\overline{C} \subseteq \overline{X}$ such that $x \in \overline{C}$ and $C = \overline{C} \cap X \neq \emptyset$. Since C is proper over Y, C is closed in $\overline{C}$. Hence $\overline{C} = C$ and so $x \in C \subseteq X$, which is not possible.

c) $\Longrightarrow$ d) From c) and ii) of d) we have that $f: X \longrightarrow Y$ is proper. Let $y \in Y$ be a closed point. If dim $f^{-1}(y) > 0$, then there exists a closed integral 1-dimensional subscheme $C \subseteq f^{-1}(y)$; hence C is a closed integral 1-dimensional subscheme of X, which is not finite over Y; this fact contradicts the hypothesis ii) of d).

Therefore, for every closed point $y \in f(X)$, dim $f^{-1}(y) = 0$.

For every closed point $x \in X$, x is then isolated in $f^{-1}f(x)$. By Zariski's Main Theorem, the subset $X' = \{x \in X \mid x \text{ isolated in } f^{-1}(x)\}$ is open in X.

Since X′ contains all closed points of X, it is easy to see that $X' = X$. Therefore every $x \in X$ is isolated in $f^{-1}f(x)$ and so, for every $y \in f(X)$, $f^{-1}(y)$ is finite. By EGA III, 4.4.2 it follows that f is finite.

d) $\Longrightarrow$ a) If S is not universally 1-equicodimensional then there exist an open affine subset $U \subseteq S$ and an integral $\Gamma(U, \mathcal{O}_S)$-algebra A of finite type of dimension > 1, which has a maximal ideal $\mathfrak{m}$ of height 1. Let be $Y = \operatorname{Spec} A$, $X = Y - \{m\}$ and $f: X \hookrightarrow Y$ the natural open immersion. Then X and Y are of finite type over S and it is easy to see that for every closed integral subscheme $C \subset X$ of dimension 1, $f(C)$ is closed in Y. Therefore f satisfies (ii) of d) but f is not finite, since it is not surjective.

a) $\Rightarrow$ e) From Proposition 1a), S is a Jacobson scheme. If $x \in X$ is a closed 1-codimensional point, then let $V \subseteq S$ and $U \subseteq X$ be open affine subsets such that $x \in U$ and U is a V-scheme. Hence $\Gamma(U, \mathcal{O}_X)$ is an integral $\Gamma(V, \mathcal{O}_S)$-algebra of finite type with maximal 1-height ideals. By Proposition 1b), applied to $U \subseteq S$, $\Gamma(V, \mathcal{O}_S)$ is an universally 1-equicodimensional ring and then $\dim \Gamma(U, \mathcal{O}_X) = 1$. Since X is a Jacobson scheme, by Corollary 2 of [6], we have $\dim X = 1$.

e) $\Rightarrow$ a) Let $U \subseteq S$ be an open affine subset, A an integral $\Gamma(U, \mathcal{O}_S)$-algebra of finite type with a maximal 1-height ideal $\underline{m} \subset A$ and $f: X = \operatorname{Spec} A \longrightarrow S$ the canonical morphism. It is clear that $\underline{m}$ is isolated in $f^{-1}f(\underline{m})$ and by Zariski's Main Theorem (cf. EGA III, 4.4.5), there exists an open neighbourhood U of $\underline{m}$ in X and an open dense immersion $U \hookrightarrow Y$, with Y a finite integral S-scheme. Since S is a Jacobson scheme, the closed point $\underline{m}$ of U is closed in Y (cf. [2], Ch. V, §3, no. 4, Th. 3). Hence Y contains a closed 1-codimensional point. By e), we deduce $\dim Y = 1$ and then $\dim U = 1$. Via Corollary 2 of [6], it follows $\dim X = \dim A = 1$, since X is a Jacobson scheme. Therefore $\Gamma(U, \mathcal{O}_S)$ is universally 1-equicodimensional and by definition it follows a).

e) $\Rightarrow$ f) Let S'^N be the normalization scheme of $S' \subseteq S$ and $x \in S'^N$ a closed 1-codimensional point. If $p: S'^N \longrightarrow S'$ is the normalization morphism and $U \subseteq S'$ is an open affine subset containing $p(x)$, then $p|_{p^{-1}(U)}$ factors in the following manner:

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & X \\ & {\scriptstyle p|_{p^{-1}(U)}} \searrow & \swarrow {\scriptstyle \pi} \\ & U & \end{array}$$

where π is a finite morphism and φ is a dominant integral morphism with the property that $\{x\}=\varphi^{-1}\varphi(x)$. Indeed, $p^{-1}(U)$ is an open affine subset and $B=\Gamma(p^{-1}(U), \mathcal{O}_{S',N})$ is the integral closure of the noetherian ring $A=\Gamma(U, \mathcal{O}_{S'})$. If $\underline{n}\subset B$ corresponds to $x\in S'^N$, then $\underline{n}$ is a maximal 1-height ideal. Let be $\underline{m}=\underline{n}\cap A$ and denote $\underline{n}_1=\underline{n}$, $\underline{n}_2,\ldots,\underline{n}_k$ the prime ideals of B lying over $\underline{m}$. For every i, $1\leq i\leq k$ we may choose $\alpha_i\in \underline{n}_i\setminus\bigcup_{j\neq i}\underline{n}_j$. Then $X=\operatorname{Spec} A[\alpha_1,\ldots,\alpha_k]$ is a finite U-scheme and $p^{-1}(U)$ is a scheme which is integral and dominant over X. Since for every $i\neq j$ we have $\underline{n}_i\cap A[\alpha_1,\ldots,\alpha_k]\neq\underline{n}_j\cap A[\alpha_1,\ldots,\alpha_k]$ it follows that $\underline{n}_i$ is the unique prime ideal of B lying over $\underline{n}_i\cap A[\alpha_1,\ldots,\alpha_k]$, for every i, $1\leq i\leq k$. Via <u>Cohen - Seidenberg</u> Theorem, we deduce $ht(\underline{n}\cap A[\alpha_1,\ldots,\alpha_k])=1$, and so $\varphi(x)$ is a closed 1-codimensional point in X. From e) $\Rightarrow$ a) and Proposition 1b), we get that U is universally 1-equicodimensional and so dim X=1. Then $\dim p^{-1}(U)=1$ and from Corollary 2 of [6]; we deduce $\dim S'^N=1$, since S'^N is a Jacobson scheme (cf. [2], Ch.V, § 3, no.4, Prop.5). Therefore $\dim S'=1$.

f) $\Rightarrow$ e) If X is an integral finite S-scheme and $S'\subseteq S$ is the closed integral image of X in S, consider the following commutative diagram:

$$\begin{array}{ccc} X^N & \xrightarrow{p_X} & X \\ {\scriptstyle f^N}\downarrow & & \downarrow{\scriptstyle f} \\ S'^N & \xrightarrow{p_{S'}} & S' \end{array}$$

where p_X and $p_{S'}$ are the normalization morphisms. If $x\in X$ is a closed 1-codimensional point, $p_X^{-1}(x)$ is a set of closed 1-codimensional points and hence the points of $f^N p_X^{-1}(x)$ are closed of codimension 1 in S'^N since S'^N is normal and f^N is integral.

Q.E.D.

Corollary 7 - Let $f: X \longrightarrow Y$ be an integral morphism of noetherian schemes. If Y is universally 1-equicodimensional, then Y is universally 1-equicodimensional. Conversely, if f is surjective and X is universally 1-equicodimensional, then Y is universally 1-equicodimensional.

Proof - Suppose that Y is universally 1-equicodimensional. Then Y is a Jacobson scheme, by Proposition 1b). Hence X is Jacobson, via [2], ch.V, §3, no.4, Th.3. Let X' be a closed integral subscheme of X and $Y'=f(X')$ with the reduced scheme structure. We have the commutative diagram:

$$\begin{array}{ccc} X'^N & \xrightarrow{p_{X'}} & X' \\ \varphi \downarrow & & \downarrow f|_{X'} \\ Y'^N & \xrightarrow{p_{Y'}} & Y' \end{array}$$

where $p_{X'}$ and $p_{Y'}$ are the normalization morphisms and φ is integral and surjective. If $x \in X'^N$ is a closed 1-codimensional point, then $\varphi(x)$ is closed in Y'^N and of codimension one, by Cohen-Seidenberg Theorem. By Theorem 2 (a) $\Rightarrow$ f)) dim $Y'=1$. Hence dim $X'=1$. Via Theorem 2 (f) $\Rightarrow$ a)) it follows that X is universally 1-equicodimensional.

Conversely, suppose that f is surjective and X is universally 1-equicodimensional. Then Y is Jacobson, via Cohen-Seidenberg Theorem. Let Y' be an integral scheme, which is finite over Y and has a closed 1-codimensional point y. Since the canonical morphism $X \times_Y Y' \longrightarrow Y'$ is integral and surjective, there exists an integral component $X' \subseteq X \times_Y Y'$ such that $f': X' \longrightarrow Y'$ is integral and surjective.

Let be $x \in X'$ such that $f'(x)=y$ and $\underline{p} \subset \mathcal{O}_{X',x}$ a prime ideal such that $\mathrm{cht}\ \underline{p}=1$ and $\underline{p} \not\supset \underline{m}_y \mathcal{O}_{X',x} \neq 0$ ($\underline{m}_y$ is the maximal ideal of the local ring $\mathcal{O}_{Y',y}$). If $C \subseteq X'$ is the closed integral subscheme passing through x and corresponding to $\underline{p}$, then $\dim \mathcal{O}_{C,x} = \dim \mathcal{O}_{X',x}/\underline{p} = 1$.

Therefore dim C=1, since x is a closed 1-codimensional point of C and C is of finite type over X. Moreover, $f'(C) \supsetneqq \{y\}$ and so $f'|_C : C \longrightarrow Y'$ is dominant. Since $f'|_C$ is integral, it follows that dim Y'=dim C=1. By Theorem 2, e) $\Rightarrow$ a), Y is universally 1-equicodimensional.

<u>Corollary 8</u> - <u>An integral 2-dimensional ring is universally 1-equicodimensional iff it is a Jacobson, universally catenary and equicodimensional ring</u>.

<u>Proof</u> - An implication follows from Example 2. Conversely, let A be an integral 2-dimensional ring, which is universally 1-equicodimensional and $\underline{m} \subset A$ a maximal ideal. By a <u>Theorem of Mori--Nagata</u> (cf. [15], Th.33.12), the integral closure A^N of A is noetherian, since dim A=2. By Corollary 7 it follows that A^N is universally 1-equicodimensional. Since $\dim A^N=2$, we have that A^N is equicodimensional. Therefore every maximal ideal of the integral closure $(A_{\underline{m}})^N$ of $A_{\underline{m}}$ is of height 2. By a result of <u>Ratliff</u> (cf. [21], Cor. 3.4 (i)) it follows that $A_{\underline{m}}$ is universally catenary. Thus A is universally catenary and equicodimensional. A is Jacobson by Proposition 1 a).

3. Universally 1- equicodimensional schemes, generically catenary and equicodimensional

In Example 2 we have shown that every integral Jacobson, universally catenary and equicodimensional ring is universally 1-equicodimensional. Corollaries 1 and 8 show that for the integral rings of dimension $\leqslant 2$, the converse of the above assertion is still true. We don't know if this converse is true for the integral rings of dimension $\geqslant 3$.

In Theorem 3 below, we shall give a class of integral universally 1-equicodimensional rings, which are Jacobson, universally catenary and equicodimensional. This class contains all integral universally 1-equicodimensional rings of dimension $\leqslant 2$ and thus Theorem 3 is a generalization of Corollaries 1 and 8.

Theorem 3 - Let S be an integral scheme. The following assertions are equivalent:

i) S is a universally 1-equicodimensional scheme, which generically is catenary and equicodimensional.

ii) S is a Jacobson, universally catenary and equicodimensional scheme.

iii) S is noetherian and for every integral S-scheme X of finite type, all maximal chains of closed irreductible subsets have the same length.

Proof - i) $\Longrightarrow$ ii) By Proposition 1a), it follows that S is Jacobson.

First we shall prove that S is catenary and equicodimensional. It is easy to see that this is equivalent with the fact that all maximal chains of closed irreducible subsets of S have the same length.

Let $\{x\}=S_o \subset S_1 \subset \ldots \subset S_n = S$ be such a chain and $U \subseteq S$ an open nonempty subset, which is catenary and equicodimensional. We shall prove by induction on dim U that n=dim U.

If dim $U \leq 1$, Corollaries 1 and 2 of [6] imply that dim $S \leq 1$. Then S is catenary and equicodimensional.

Suppose that dim $U > 1$. Then $n \geq 2$, since otherwise, x would be a closed 1-codimensional point in S; then from i) would follow dim S=1, which is not possible. The local ring $\mathcal{O}_{S,S_{n-2}}$ is noetherian of dimension ≥ 2. Let $\underline{m} \supset \underline{p} \supset 0$ the saturated chain of prime ideals of $\mathcal{O}_{S,S_{n-2}}$ corresponding to the saturated chain of closed irreducible subsets $S_{n-2} \subset S_{n-1} \subset S_n$, and $\underline{p}_1, \ldots, \underline{p}_\ell$ the prime ideals of $\mathcal{O}_{S,S_{n-2}}$ corresponding to the irreducible components of S-U, containing S_{n-2}. From a result of Ratliff-McAdam (cf.[13], Prop.1), there exist infinitely many maximal chains of prime ideals in $\mathcal{O}_{S,S_{n-2}}$ of length 2. Then we may find a saturated chain $\underline{m} \supset \underline{p}' \supset 0$ such that $\underline{p}' \neq \underline{p}_i$, for every i, $1 \leq i \leq \ell$. Let S'_{n-1} be the closed irreducible subset of S, which contain S_{n-2}, corresponding to the prime ideal $\underline{p}' \subset \mathcal{O}_{S,S_{n-2}}$. Then the chain $\{x\}=S_o \subset S_1 \subset \ldots \subset S_{n-2} \subset S'_{n-1} \subset S_n$ is maximal and $S'_{n-1} \cap U \neq \emptyset$. The scheme S'_{n-1} is universally 1-equicodimensional and $U \cap S'_{n-1}$ is an open nonvoid catenary and equicodimensional subscheme. Since dim $(U \cap S'_{n-1}) <$ dim U, we may apply the inductive hypothesis to S'_{n-1} and to the maximal chain $S_o \subset S_1 \subset \ldots \subset S_{n-2} \subset S'_{n-1}$. We have n-1=dim $(U \cap S'_{n-1})$. Since U is catenary and equicodimensional and S'_{n-1} is 1-codimensional in S, it follows that dim $(U \cap S'_{n-1})=$ =dim U-1. Hence n-1=dim U-1 and so n=dim U.

It remains to show that S is universally catenary. This fact follows from the following.

Lemma 1 - An universally 1-equicodimensional scheme which is catenary is universally catenary.

First let us do the following.

Remark 4 - In [21], p.517, Ratliff proves the following Theorem:

Let R be an integral noetherian local ring. The following assertions are equivalent:

(1) R is universally catenary.

(2) R is quasi-unmixed (i.e. the completion $\hat{R}$ of R in the radical topology is equidimensional).

In [21], loc.cit., one proves that (2) $\Rightarrow$ (1) and Non (2) $\Rightarrow$ Non (1). In the proof of Non (2) $\Rightarrow$ Non (1), one shows in [21], that Non (2) $\Rightarrow$ either R is not catenary, or R is catenary and there exists an integral finite R-algebra D of dimension > 1, which has a maximal ideal of height 1.

We may complete the theorem of Ratliff by adding the following equivalent property.

(3) R is catenary and every integral R-algebra which is finite and has a maximal ideal of height 1 is of dimension 1.

Indeed, by the above remark we have Non (2) $\Rightarrow$ Non (3) and so (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3) Let A be an integral R-algebra which is finite and which has a maximal ideal $\underline{m} \subset A$ of height 1. Denote $\underline{p} \subset R$, the prime ideal such that $A \supseteq R/\underline{p} = R'$ and $\underline{n} = \underline{m} \cap R'$. Then R' is universally catenary, $A \supseteq R'$ is a finite extension of rings and $\underline{n}$ is the maximal, ideal of R'. By EGA IV, 5.6.10, applied to $A \supseteq R'$, it results $1 = \dim A_{\underline{m}} = \dim R'$. Then $\dim A = \dim R = 1$.

Proof of Lemma 1 - Let X be a catenary universally 1-equicodimensional scheme. We may assume that X is integral and it suffices to prove that for every closed point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is universally catenary. Since $\mathcal{O}_{X,x}$ is catenary, by the above Theorem of Ratliff it is sufficient to prove that every integral $\mathcal{O}_{X,x}$-algebra A which is finite and has a maximal ideal $\underline{m} \subset A$ of height 1, is 1-dimensional. In fact, let $\underline{p} \subset \mathcal{O}_{X,x}$ be a prime ideal such that $A \supset \mathcal{O}_{X,x}/\underline{p}$ and $\{f_1,\dots,f_n\}$ a set of generators of the $\mathcal{O}_{X,x}$-module A. For every i, $1 \leq i \leq n$, f_i satisfies an equation $f_i^{n_i}+a_{1i}f_i^{n_i-1}+\dots+a_{n_i-1,i}f+a_{n_i,i}=0$, with $a_{ji} \in \mathcal{O}_{X,x}/\underline{p}$ If $X' \subseteq X$ is the closed integral subscheme corresponding to $\underline{p}$, we have $x \in X'$ and $\mathcal{O}_{X,x}/\underline{p}=\mathcal{O}_{X',x}$. Let $U \subseteq X'$ be an open affine neighbourhood of x such that $a_{ji} \in B=\Gamma(U,\mathcal{O}_{X'})$, for every i,j. Then $B[f_1,\dots,f_n]$ is a subring of A finite over B. If $\underline{n} \subset B$ is the maximal ideal corresponding to x, we have $B_{\underline{n}}=\mathcal{O}_{X,x}/\underline{p}$ and so $B[f_1,\dots,f_n]_{\underline{n}}=A$. The ideal $\underline{q}=\underline{m} \cap B[f_1,\dots,f_n]$ is maximal in $B[f_1,\dots,f_n]$ and $B[f_1,\dots,f_n]_{\underline{q}}=A_{\underline{m}}$. Since $\dim A_{\underline{m}}=1$, we have $ht\underline{q}=1$. By Proposition 1c) and 1b), B is universally 1-equicodimensional. Thus $\dim B[f_1,\dots,f_n]=$

$$=1 \text{ and then } 1=\dim B=\dim B_{\underline{n}}=\dim \mathcal{O}_{X,x}/\underline{p}=\dim A.$$

Q.E.D.

Proof of Theorem 3

ii) $\Rightarrow$ iii) Let $V \subseteq S$ and $U \subseteq X$ be two open affine subsets such that U is a V-scheme. Then $\Gamma(V,\mathcal{O}_S)$ is Jacobson, universally catenary and equicodimensional and $\Gamma(U,\mathcal{O}_X)$ in an integral $\Gamma(V,\mathcal{O}_S)$-algebra of finite type. As in Example 2, one shows that $\Gamma(U,\mathcal{O}_S)$ is catenary and equicodimensional.

Therefore there exists a covering $(U_i)_{i \in I}$ of X with open affine subsets, which are catenary and equicodimensional. Then it is easy to see that X is catenary and equicodimensional.

iii) $\Rightarrow$ i) Let $V \subseteq S$ be an open affine subset and A an integral $\Gamma(V,\mathcal{O}_S)$-algebra of finite type. Then X=Spec A is equicodimensional. Hence, if A has a maximal 1-height ideal then dim A=1. Therefore $\Gamma(V,\mathcal{O}_S)$ is universally 1-equicodimensional.

Q.E.D.

Corollary 9 - Let $f:X \to Y$ be a dominant morphism of finite type of integral schemes, such that X is regular and equicodimensional. Then Y is universally 1-equicodimensional iff it is a Jacobson, universally catenary and equicodimensional scheme.

Proof - We shall prove that if Y is universally 1-equicodimensional, there it is Jacobson, universally catenary and equicodimensional. Via Theorem 3, it suffices to prove that Y is generically catenary and equicodimensional.

Since f(X) is constructible and dense in Y, we may choose an open nonempty subset $U \subseteq Y$, such that $U \subseteq \operatorname{Im} f$. By restricting U, we may assume that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat and surjective.

Let $y \in U$ be a closed point and $x \in f^{-1}(U)$ a closed point such that $f(x)=y$. Since $\mathcal{O}_{X,x}$ is regular, it follows that $\mathcal{O}_{Y,y}$ is regular (cf. EGA IV, 6.5.2) We have $\dim \mathcal{O}_{Y,y} = \dim \mathcal{O}_{X,x} - \dim(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y))$ (cf.EGA IV, 6.1.2). Since Y is a Jacobson scheme, by Proposition 1a), X is Jacobson and hence x is closed in X. Thus $\dim \mathcal{O}_{X,x} = \dim X$. On the other hand, $\dim(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} k(y)) = \dim_x f^{-1} f(x) = \dim f^{-1}(\eta)$, where η is the generic point of Y (cf. EGA IV, 13.2.2 and 13.2.3). Therefore $\dim \mathcal{O}_{Y,y} = \dim X - \dim f^{-1}(y)$

Thus for all closed points $y \in U$, the local rings $\mathcal{O}_{Y,y}$ are regular of the same dimension. Since every regular ring is (universally) catenary, it follows that U is a catenary equicodimensional scheme.

Q.E.D.

As a consequence of Corollary 9 we have

Corollary 10 - Let A be a subring of an integral generically regular and equicodimensional ring B such that B is an A-algebra of finite type. Then A is universally 1-equicodimensional iff it is Jacobson, universally catenary and equicodimensioanal ring.

Indeed, if $f \in B$ is a non-zero element such that B_f is regular and equicodimensional, then Corollary 10 results from Corollary 9 applied to the natural morphism Spec $B_f \longrightarrow$ Spec A.

Remark 5 - In [5], Proposition 1, we have proved that an integral subalgebra B of an algebra A of finite type over a field k is finitely generated iff B is an universally 1-equicodimensional ring. (It is easy to show that we may suppose in this assertion that B is an arbitrary subalgebra of an algebra A of finite type over a field). A proof for this result can be obtained using Corollary 10 in connection with some properties of k-schemes generically algebraic over k (see [7]).

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