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# A NULLSTELLENSATZ OVER ORDERED FIELDS.

by

Serban A. Basarab

Contents

## 1. Introduction. Statement of principal results.

The aim of the present paper is to prove a Nullstellensatz over ordered fields which generalizes some results of Dubois [5] and Stengle [14]. Our results and their proofs are presented in the spirit of the paper [6] of Jarden and Roquette on the Nullstellensatz over $\mathfrak{p}$-adically closed fields. The role of the Kochen ring from the theory of formally $p$-adic fields is played in the present situation by the Baer ring of a field extension of an ordered field.

This work was elaborated while the author was a Humboldt fellow at the University Heidelberg. He would like to thank Professor Peter Roquette for his warm stimulation.

We consider the following situation :

$V$ — an affine variety defined over the ordered field $(K,P)$, where $P$ denotes the semiring of non-negative elements

$x$ — $= (x_1, \ldots, x_n)$ a generic point of $V$ over $K$

$K[x]$ — its coordinate ring; the elements in $K[x]$ are regarded as polynomial functions defined on $V$

$F$ — $= K(x)$ the field of rational functions on $V$ over $K$

$u$ — $= (u_1, \ldots, u_m)$ a finite family of elements in $K[x] \setminus \{0\}$

$V(\tilde{K})$ — the space of $\tilde{K}$-rational points on $V$, where $(\tilde{K},\tilde{P})$ denotes the real closure of $(K,P)$

$V_u(\tilde{K})$ — the subset of $V(\tilde{K})$ consisting of those points $a \in V(\tilde{K})$ which satisfy the condition $u_i(a) \in \tilde{P}$ for $i=1,\ldots,m$.

$J_u$ — the semiring generated by $P \cup u \cup K[x]^2$, where $K[x]^2 = \{ z^2 \mid z \in K[x] \}$

$\underline{a}$ — an ideal in $K[x]$

$r_u(\underline{a})$ — the $J_u$-radical of $\underline{a}$ consisting of the elements $z \in K[x]$ subject to condition : $z^2 + b \in \underline{a}$ for some positive integer $\ell$ and some $b \in J_u$

$V_{u,\underline{a}}(\tilde{K})$ the subset of $V_u(\tilde{K})$ consisting of those points $b \in V_u(\tilde{K})$ with $f(b) = 0$ for every $f \in \underline{a}$

$I(V_{u,\underline{a}}(\tilde{K}))$ the ideal in $K[x]$ consisting of those elements $f \in K[x]$ which satisfy the condition $f(b) = 0$ for every $b \in V_{u,\underline{a}}(\tilde{K})$.

<u>Theorem 1.1.</u> If the variety $V$ is nonsingular then $I(V_{u,\underline{a}}(\tilde{K})) = r_u(\underline{a})$ for every ideal $\underline{a}$ in $K[x]$.

Clearly, the nonsingularity condition is satisfied if $V$ is the full affine space. In this particular case we obtain Stengle's Theorem 1 (semialgebraic Nullstellensatz) from [14]. The later has as an immediate consequence the real Nullstellensatz discovered by Dubois [5] (see also [14] Theorem 2 and [10] Theorem 5.12).

The Nullstellensatz will be supplemented by the following criterion for $V_u(\tilde{K})$ to contain a simple point. We assume in addition that the multiplicative monoid generated by the family u is a group. If this condition is satisfied then $V_u(\tilde{K})$ consists of the points $a \in V(\tilde{K})$ with $u_i(a) > 0$ for $i=1,\ldots,m$. As in the case where the base field is $p$-adically closed [6] Theorem 1.3., this criterion is of birational nature, reffering only to the function field F and not to the particular variety $V$.

<u>Definition.</u> The field extension $F|(K,P)$ is formally real over u if there exists an order T on F such that $P \cup u \subset T$.

<u>Theorem 1.2.</u> Suppose that the monoid generated by the family u is a subgroup of the multiplicative group $F^{\times}$. Then the necessary and sufficient condition for $V_u(\tilde{K})$ to contain a simple point is that the field extension $F|(K,P)$ is formally real over u.

## 2. The Baer ring of a field extension of an ordered field.

Let $K$ be a field equipped with an order $P$, i.e. $P+P\subset P$, $P.P\subset P$, $K = P\cup -P$ and $P\cap -P = \{0\}$. Let $F$ be an arbitrary field extension of $K$ and $u$ be a subset of $F$. Denote by $J_u(F)$ the semiring of $F$ generated by $P\cup u\cup F^2$. The following result is well known.

Proposition 2.1. The following assertions are equivalent ;

i) $F|(K,P)$ is formally real over $u$, i.e. there is an order $T$ on $F$ such that $P\cup u\subset T$.

ii) $-1\notin J_u(F)$.

iii) $J_u(F)\neq F$.

If $u$ is empty then we obtain the notion of a formally real field extension and the equivalence ; $F|(K,P)$ is formally real iff $-1\notin J(F)$ iff $J(F)\neq F$, where $J(F)$ denotes the semiring generated by $P\cup F^2$.

It was Baer [1] who for the first time established and investigated the connection between the orders of a field and related valuation rings.

If $T$ is an order on $F$ which extends the order $P$ of $K$ we denote by $B(T)$ the Baer ring of $T|P$ consisting of the finite elements of $F$ relative to $T|P$, i.e. $B(T)=\{a\in F \mid \bigvee_{b\in P^\times} b\pm a\in T\}$. Here $P^\times=P\setminus\{0\}$. Denote by $\underline{b}(T)$ the Baer ideal in $B(T)$ consisting of the infinitely small elements, i.e.

$\underline{b}(T)=\{a\in F \mid \bigwedge_{b\in P^\times} b\pm a\in T\}$. $B(T)$ is a valuation ring of $F$, $K\subset B(T)$ and $\underline{b}(T)$ is the maximal ideal of $B(T)$.

Similarly we define the Baer ring of the field extension $F|(K,P)$ to be the ring $B=B(F)=\{a\in F \mid \bigvee_{b\in P^\times} b\pm a\in J(F)\}$ of finite elements relative to $J(F)|P$ and the Baer ideal in $B$

to be the ideal $\underline{b} = \underline{b}(F) = \{a \in F \mid \bigwedge_{b \in P^\times} b \pm a \in J(F)\}$ of infinitely small elements relative to $J(F)|P$. By definition the Baer ring $B$ is an intermediate ring between $K$ and $F$. The following result is immediate.

Proposition 2.2. The field extension $F|(K,P)$ is formally real iff $\underline{b} \neq B$ iff $\underline{b} \cap K = \{0\}$ iff $\underline{b} \neq F$.

Remarks. i) If $F|(K,P)$ is not formally real then $B = F$. The converse is not generally true. For instance, let $F = \tilde{K}$ be the real closure of $(K,P)$ or more simple let $F = K$. Then $B(F) = F$ and $\underline{b}(F) = 0$.

ii) If $F|(K,P)$ is formally real then $K\left[\frac{1}{1+t} \mid t \in J(F)\right]$ is a subring of $B$. Indeed $1+t \neq 0$, $1 + \frac{1}{1+t} \in J(F)$, $1 - \frac{1}{1+t} = \frac{t}{1+t} \in J(F)$ for every $t \in J(F)$.

Proposition 2.3. Suppose that $F|(K,P)$ is formally real and let $Q$ be a place of $F|K$. Then the following assertions are equivalent:

i) $Q$ lies over $B$, i.e. $B$ is contained in the valuation ring $\mathcal{O}_Q$ of the place $Q$.

ii) The residue extension $F.Q|(K,P)$ is formally real ;

iii) There is an order $T$ on $F$ which extends $P$ and is compatible with the place $Q$, i.e. $1+\underline{m}_Q \subset T$.

Proof. i) $\rightarrow$ ii) Let $J(F.Q)$ be the semiring generated by $P \cup (F.Q)^2$. We have the equality $J(F.Q) = (J(F) \cap \mathcal{O}_Q).Q$. If $F.Q|(K,P)$ is not formally real, i.e. $-1 \in J(F.Q)$, then $1+t \in \underline{m}_Q$ for some $t \in J(F) \cap \mathcal{O}_Q$, i.e. $\frac{1}{1+t} \in B \setminus \mathcal{O}_Q$ and hence $B \not\subset \mathcal{O}_Q$.

ii) $\rightarrow$ iii) There is a bijection from the set of orders $T$ on $F$ which extend $P$ and are compatible with $Q$ onto the set of pairs $(\overline{T}, \chi)$ where $\overline{T}$ is an order on $F.Q$ which extends $P$ and $\chi$ is a group morphism from $F^\times/\mathcal{O}_Q^\times.(F^\times)^2$ to $\mathbb{Z}/2\mathbb{Z}$. [7] §12 [3] Satz 2.4.

iii) $\to$ i) Let $a \in B$, i.e. $b \pm a \in J(F)$ for some $b \in P^{\times}$. If $T$ is an order on $F$ which extends $P$ and is compatible with $Q$ then $b \pm a \in T$, i.e. $|a|_T \leq b$. If $a \notin \mathcal{O}_Q$ then $a^{-1}b \in \underline{m}_Q$ and $|a^{-1}b|_T < 1$, i.e. $|a|_T > b$, which is absurd. Q.E.D.

Theorem 2.4. Suppose that $F|(K,P)$ is formally real. Then the following hold :

i) $B = K\left[\frac{1}{1+t} \,\middle|\, t \in J(F)\right] = \bigcap B(T)$ where $T$ ranges over the set of all orders on $F$ which extend $P$.

ii) $\underline{b} = \sqrt{\underline{b}} = \bigcap \underline{b}(T)$ where $T$ ranges over the same set as in i).

iii) Every overring $R$ of $B$ in $F$ is a Prüfer ring with $F$ as its field of quotients and the ideal-class group $C(R)$ is a 2-group.

iv) $\text{Arch}(B) \cap B^{\times} = J(F) \cap B^{\times}$, where

$$\text{Arch}(B) = \left\{ a \in B \,\middle|\, \bigwedge_{b \in P^{\times}} b+a \in J(F) \right\}.$$

Proof. Let $A = K\left[\frac{1}{1+t} \,\middle|\, t \in J(F)\right]$. We have to show that $B \subset A$. First let us show that the field of quotients of $A$ is $F$. Observe that $J(F)$ is contained in the field of quotients of $A$. Indeed, let $t \in J(F)$. Then $\frac{1}{1+t} \in A$, $\frac{t}{1+t} = 1 - \frac{1}{1+t} \in A$ and hence $t \in \text{Quot}(A)$. On the other hand $F = J(F) - J(F) \subset \text{Quot}(A)$. Indeed, $a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 \in J(F) - J(F)$ for each $a \in F$. Now let us show that $B \subset A$. Let $a \in B$, i.e. $b \pm a \in J(F)$ for some $b \in P^{\times}$. Then $1+b+a \in B \cap J(F)$ and $\frac{1}{1+b+a} \in A \subset B$, i.e. $1+b+a \in B^{\times} \cap J(F)$ and hence there is $f \in P^{\times}$ such that $f-1-b-a \in J(F)$. It follows that $1+b+a = f(1+(f-1-b-a)(1+b+a)^{-1})^{-1} \in \frac{f}{1+J(F)} \subset A$ and hence $a = (1+b+a) - (1+b) \in A$. Thus we proved the equality $B = K\left[\frac{1}{1+t} \,\middle|\, t \in J(F)\right]$.

Now let us show that $B$ is a Prüfer ring. Let $\underline{m}$ be any maximal ideal in $B$. We have to prove that $B_{\underline{m}}$ is a valuation ring of $F$. First observe that either $t \in B_{\underline{m}}$ or $t^{-1} \in \underline{m}B_{\underline{m}}$ if $t \in J(F)$. Indeed, if $t \in J(F) \setminus B_{\underline{m}}$ then $\frac{1}{1+t} \in \underline{m}$ and $\frac{1}{1+t} = 1 - \frac{1}{1+t} \in B \setminus \underline{m}$, and hence $t^{-1} \in \underline{m}B_{\underline{m}}$. Let us show that the integral closure $B'_{\underline{m}}$ of $B_{\underline{m}}$ in $F$ is a valuation ring. Let $\underline{m}'$ be a maximal ideal of $B'_{\underline{m}}$ such that $B_{\underline{m}} \cap \underline{m}' = \underline{m}B_{\underline{m}}$. Given any $a \in F$, since $a^2 \in J(F)$, it follows that either $a^2 \in B_{\underline{m}}$ or $a^{-2} \in B_{\underline{m}}$. Thus either $a \in B'_{\underline{m}}$ or $a^{-1} \in B'_{\underline{m}}$ and hence $B'_{\underline{m}}$ is a valuation ring of $F$. Let us show that $B_{\underline{m}} = B'_{\underline{m}}$. If $a \in B'_{\underline{m}}$ and $a^2 \notin B_{\underline{m}}$ then $a^{-2} \in \underline{m}B_{\underline{m}} \subset \underline{m}'$ which is absurd. Hence $a^2 \in B_{\underline{m}}$ if $a \in B'_{\underline{m}}$. Since the residue field $L = B'_{\underline{m}}/\underline{m}'$ extends $K$, the characteristic of $L$ is zero. Let $a \in B'_{\underline{m}}$. We have to show that $a \in B_{\underline{m}}$. Since $a^2 \in B_{\underline{m}}$ and $(1+a)^2 \in B_{\underline{m}}$, we conclude that $a = \frac{(1+a)^2 - a^2 - 1}{2} \in B_{\underline{m}}$. Thus we proved that $B_{\underline{m}}$ is a valuation ring of $F$ for every maximal ideal $\underline{m}$ in $B$, i.e. $B$ is a Prüfer ring with $F$ as its field of quotients.

Now let us show that $B$ equals the intersection of the Baer rings $B(T)$ where $T$ ranges over the set of orders on $F$ which extend the order $P$ on $K$. The inclusion $B \subset \bigcap_{T \supset P} B(T)$ is immediate. Let $\underline{m}$ be a maximal ideal of $B$. Since $B$ is a Prüfer ring, $B_{\underline{m}}$ is a valuation ring. We show that there is an order $T$ on $F$ such that $P \subset T$ and $B(T) = B_{\underline{m}}$. Then

$$\bigcap_{T \supset P} B(T) \subset \bigcap_{\underline{m} \in \mathrm{Max}(B)} B_{\underline{m}} = B \quad \text{and hence} \quad B = \bigcap_{T \supset P} B(T).$$

First observe that, by Proposition 2.3., there exists an order $T$ on $F$ such that $P \subset T$ and $1 + \underline{m}B_{\underline{m}} \subset T$. It follows that $B(T) \subset B_{\underline{m}}$ and $\underline{m}B_{\underline{m}} \subset \underline{b}(T)$. Indeed, let $a \in \underline{m}B_{\underline{m}}$ and $b \in P^{\times}$. Then

$b \pm a = b(1 \pm ab^{-1}) \in P^{\times}.(1+\mathfrak{m}B_{\mathfrak{m}}) \subset T$ and hence $a \in \underline{b}(T)$. Thus $\mathfrak{m}B_{\mathfrak{m}} \subset \underline{b}(T)$ and $B(T) \subset B_{\mathfrak{m}}$. Since $B_{\mathfrak{m}}$ is minimal in the ordered set of valuation rings extending $B$ we conclude that $B_{\mathfrak{m}} = B(T)$. So we proved that $B = \bigcap_{T \supset P} B(T)$. By Prüfer criterion for holomorphy rings [13] Theorem 1 it follows that $C(B)$ is a 2-group. We denote by $C(B)$ the ideal class group of $B$, i.e. the factor group of the finitely generated fractional $B$-ideals modulo the principal ones.

The equalities $\underline{b} = \bigcap_{T \supset P} \underline{b}(T) = \sqrt{\underline{b}}$ are immediate from definitions.

It remains to show that $\mathrm{Arch}(B) \cap B^{\times} \subset J(F)$. Let $a \in \mathrm{Arch}(B) \cap B^{\times}$. In particular $a \in B(T) \setminus \underline{b}(T)$ for each order $T \supset P$ such that $-a \in T^{\times}$. On the other hand, for each $b \in P^{\times}$, $b+a \in J(F) \subset T$ and hence $a \in \underline{b}(T)$, which is absurd. We conclude that $a \in J(F)$. Q.E.D.

<u>Remark</u> The particular case $K = \mathbb{Q}$ was considered by Pejas [9] and Dress [4]. A more general situation is investigated by Becker [2] Theorem 3.7.

<u>Theorem 2.5</u>. There is a canonical bijection from the set of prime ideals of $B$ onto the set of places $Q$ of $F|K$ with the property that the residue extension $F.Q|(K,P)$ is formally real ;

$\underline{q} \in \mathrm{Spec}(B) \mapsto$ the place attached to the valuation ring $B_{\underline{q}}$

$Q \longmapsto \mathfrak{m}_Q \cap B$

This bijection induces an embedding of the set of maximal ideals of $B$ into the set of orders of $F$ extending $P$, up to the equivalence relation ; two such orders $T$ and $T'$ are equivalent iff $B(T) = B(T')$.

Proof. This theorem is an immediate consequence of the fact that B is a Prüfer ring and of Proposition 2.3. Q.E.D.

Remark. The embedding considered in the last part of Theorem 2.5. is not necessarily a bijection. Indeed, there exist situations of the following type ; (K,P) an ordered field, F a field extension of K, T an order on F extending P such that $\mathfrak{b}(T) \cap B = 0$, $B \neq F$, and hence B is not a field and $\mathfrak{b}(T) \cap B \notin \mathrm{Max}(B)$. For instance, let $K = \mathbb{Q}$, $F = \mathbb{Q}(x)$, the field of rational functions in an indeterminate x. Let ${}^*\mathbb{Q}$ be an enlargement of $\mathbb{Q}$ in Robinson' sense [12] and let $t' \in {}^*\mathbb{Q} \setminus \mathbb{Q}$ be such that t' is infinitely small with respect to the order T' induced on K(t') by the internal order ${}^*P$ on ${}^*\mathbb{Q}$. It is known that $B({}^*P)/\mathfrak{b}({}^*P) \cong \mathbb{R}$. It follows that B(T') is the valuation ring $\mathbb{Q}[t']_{\mathfrak{p}}$ where $\mathfrak{p}$ is the prime ideal in $\mathbb{Q}[t']$ generated by t'. In particular $B(T') \neq \mathbb{Q}(t')$ and hence $B(F) \neq F$. On the other hand, let $t \in {}^*\mathbb{Q} \setminus \mathbb{Q}$ be such that t is finite with respect to ${}^*P$, i.e. $t \in B({}^*P)$, and the standard part of t, i.e. t mod $\mathfrak{b}({}^*P)$, is a transcendental real number c. Let T denote the order induced by ${}^*P$ on $\mathbb{Q}(t)$. Then $B(T) = \mathbb{Q}(t)$ and $\mathfrak{b}(T) = 0$, in particular, identifying $\mathbb{Q}(t)$ with F, we obtain $\mathfrak{b}(T) \cap B(F) = 0$.

Corollary to Theorem 2.5. Let $\mathfrak{p}$ be a prime ideal in B. The set $X_{\mathfrak{p}} = \{\mathfrak{q} \in \mathrm{Spec}(B) \mid \mathfrak{q} \subset \mathfrak{p}\}$ is linearly ordered with respect to the inclusion and the cardinal number of $X_{\mathfrak{p}}$ equals the rank of the value group of the valuation ring $B_{\mathfrak{p}}$ attached to $\mathfrak{p}$.

Proof. We have $\mathfrak{q} \subset \mathfrak{p}$ iff $B_{\mathfrak{p}} \subset B_{\mathfrak{q}}$. The set of overrings of the valuation ring $B_{\mathfrak{p}}$ is linearly ordered and their number equals the rank of the value group of $B_{\mathfrak{p}}$. Q.E.D.

Remark. The previous result remains valid for an arbitrary overing of B.

## 3. The Riemann space of a field extension of an ordered field.

Let $(K,P)$ be an ordered field and $F$ be a field extension of $K$. The space of all places $Q$ of $F|K$, subject to the condition : the residue extension $F.Q|(K,P)$ is formally real, is called the Riemann space of $F|(K,P)$ and is denoted by $S(F)$. In order to simplify the notation we shall use the symbol $S$ instead of $S(F)$. It is well known that the necessary and sufficient condition for $F|(K,P)$ to admit a non-empty Riemann space $S$ is that $F|(K,P)$ is formally real.

We assume in the following that $F|(K,P)$ is formally real. Let $Q$ be an arbitrary place of $F|K$. We have shown in Section 2 that $Q\in S$ iff $\mathcal{O}_Q \supset B$ iff $\mathcal{O}_Q = B_{\mathfrak{q}}$ for some $\mathfrak{q}\in \mathrm{Spec}\,(B)$. In addition the Baer ring $B$ of $F|(K,P)$ is the holomorphy ring $\bigcap_{Q\in S} \mathcal{O}_Q$ of the Riemann space. $S$. We may identify the Riemann space $S$ with the prime spectrum $\mathrm{Spec}\,(B)$ of the Baer ring $B$ and consider on $S$ the Zariski topology admiting as basis of open sets the family $\{D_f\}_{f\in B}$ where $D_f = \{Q\in S \mid f.Q\neq 0\}$. Moreover $S$ has a natural structure of ringed space. The structural sheaf $\mathcal{F}$ is given by $\mathcal{F}(D_f) = B_{(f)} = \bigcap_{Q\in D_f} \mathcal{O}_Q$ = the holomorphy ring of $D_f$. Equipped with Zariski topology, the Riemann space $S$ is quasi-compact.

If $x$ is an arbitrary subset of $F$ we denote by $S^x$ the subset of $S$ consisting of all places $Q\in S$ which lie over $x$, i.e. $x\subset \mathcal{O}_Q$. It is easy to see that the family $\{S^x\}_x$ where $x$ ranges over the family of finite subsets of $F$ is a basis of open sets for the Zariski topology on $S$ and $\mathcal{F}(S^x)$ is the holomorphy ring $H^x = \bigcap_{Q\in S^x} \mathcal{O}_Q$ for each finite subset $x$ of $F$.

4. Holomorphic functions on the Riemann space.

Let $(K,P)$ be an ordered field and $F$ be a field extension of $K$. We suppose that the extension $F|(K,P)$ is formally real, i.e. the Riemann space $S$ is non-empty.

If $x$ and $u$ are arbitrary subsets of $F$ we denote by $S_u^x$ the subset of $S$ consisting of all places $Q$ of $F|K$ which lie over $x \cup u$ and satisfy the condition : the residue extension $F.Q|(K,P)$ is formally real over $u.Q$. In particular if $x \subset B$ and $u \subset P$, for instance if $x$ and $u$ are empty, $S_u^x$ coincides with the whole Riemann space $S$.

We denote by $H_u^x = \bigcap_{Q \in S_u^x} \mathcal{O}_Q$ the holomorphy ring of $S_u^x$. If either $x$ or $u$ is empty we denote the corresponding sets and holomorphy rings by $S^x$, $S_u$, $H^x$, $H_u$. For $x \subset B$ and $u \subset P$, $H_u^x$ is the Baer ring $B$ of $F|(K,P)$.

First let us observe that if $F|(K,P)$ is formally real over $u$ then $S_u^x$ is non-empty for every subset $x$ of $F$. Indeed, in this case the trivial place $1_F$ is contained in $S_u^x$ for each subset $x$ of $F$. The converse is not generally true. [For instance, let $K = \mathbb{Q}$ and $F = {}^*\mathbb{Q}$ be an enlargement of $\mathbb{Q}$ in Robinson'sense [12]. Let $a \neq 0$ be an infinitely small element of $F$ with respect to the internal order ${}^*P$ on $F$. Let $Q$ denote the place of $F|K$ whose valuation ring is $B({}^*P)$, the ring of finite elements of $F$ with respect to ${}^*P$. Then $Q \in S_u$ where $u = \{a, -a\}$ but $F|(\mathbb{Q}, P)$ is not formally real over $u$ ]. If the monoid generated by $u$ is a subgroup of the multiplicative group $F^\times$, it is a simple exercise to verify that $S_u$ is non-empty iff $F|(K,P)$ is formally real over $u$.

The aim of this section is to describe the holomorphy rings $H_u^x$ for arbitrary sets $x$ and $u$. As overrings of the Baer ring $B$, all this rings are Prüfer rings.

Proposition 4.1. Assume that $S_u^x$ is non-empty. Then $H_u^x$ is the smallest overring $A$ of $B[x,u]$ subject to $1 + J_u(A) \subset A^\times$, where $J_u(A)$ denotes the semiring generated by $P \cup u \cup A^2$.

Proof. First let us observe that the intersection $A = \bigcap_{i \in I} A_i$ of a family $\{A_i\}_{i\in I}$ of overrings of $B[x,u]$ subject to $1 + J_u(A_i) \subset A_i^\times$ satisfies the condition $1 + J_u(A) \subset A^\times$ too.

Let $A$ denote the smallest overring of $B[x,u]$ subject to $1 + J_u(A) \subset A^\times$. Observe that $A \subset H_u^x$. Indeed, let $Q \in S_u^x$ and $J_u(\mathcal{O}_Q)$ be the semiring generated by $P \cup u \cup \mathcal{O}_Q^2$. If $1 + J_u(\mathcal{O}_Q) \not\subset \mathcal{O}_Q^\times$ then $-1 \in J_{uQ}(F.Q) = J_u(\mathcal{O}_Q) \bmod \underline{m}_Q$, i.e. $F.Q|(K,P)$ is not formally real over $u.Q$, which is absurd.

On the other hand $A$ is a Prüfer ring with $F$ as its field of quotients and hence $A = \bigcap A_{\underline{p}}$ where $\underline{p}$ ranges over the set of maximal ideals of $A$. For every maximal ideal $\underline{p}$ of $A$ let $Q_{\underline{p}}$ be the corresponding place. If we show that $Q_{\underline{p}} \in S_u^x$ for each maximal ideal $\underline{p}$ then $H_u^x = \bigcap_{Q \in S_u^x} \mathcal{O}_Q \subset \bigcap_{\underline{p} \in \mathrm{Max}(A)} \mathcal{O}_{Q_{\underline{p}}} = A$ and hence $A = H_u^x$.

It remains to show that $Q_{\underline{p}} \in S_u^x$ for $\underline{p} \in \mathrm{Max}(A)$. We have $u \cup x \subset \mathcal{O}_{Q_{\underline{p}}} = A_{\underline{p}}$ and $F.Q_{\underline{p}} \cong A/\underline{p}$, $J_{uQ_{\underline{p}}}(F.Q_{\underline{p}}) = J_u(A) \bmod \underline{p}$ and hence $-1 \notin J_{u.Q_{\underline{p}}}(F.Q_{\underline{p}})$ because $(1 + J_u(A)) \cap \underline{p} = \emptyset$. We conclude that $F.Q_{\underline{p}}|(K,P)$ is formally real over $u.Q_{\underline{p}}$ and hence $Q_{\underline{p}} \in S_u^x$.

Q.E.D.

Corollary to Proposition 4.1. $H^x = B[x]$.

Remark. If $Q$ is a place of $F|K$ such that $\mathcal{O}_Q \supset H_u$, where $u \not\subset P$, this does not imply that $Q \in S_u$. For instance let $K = \mathbb{R}$, $F = {}^*\mathbb{R}$ be an enlargement of $\mathbb{R}$ in Robinson'sense and let $a \in \underline{b}(F) \cap F_{<0}$.

Then $H_{\{a\}} = B(F)$, the trivial place $Q = 1_F$ lies over $H_{\{a\}}$ but $Q \notin S_{\{a\}}$, i.e. $F|K$ is not formally real over $\{a\}$.

The following proposition describes the holomorphy ring $H_u^x$ as an inductive limit of certain overrings of $B[x,u]$.

Proposition 4.2. Assume that $S_u^x$ is non-empty. Then there exists an unique sequence $(A_n)_{n\in\mathbb{N}}$ of intermediate rings between $B[x,u]$ and $H_u^x$ satisfying the conditions :

i) $A_o = B[x,u]$

ii) $-1 \notin J_u(A_n)$

iii) $A_{n+1}$ is the ring of fractions of $A_n$ with respect to the monoid $1 + J_u(A_n)$.

In addition $H_u^x = \bigcup_{n\in\mathbb{N}} A_n$.

Proof. First we have to show by induction that $-1 \notin J_u(A_n)$ and $A_n \subset H_u^x$ for each $n \in \mathbb{N}$.

For $n = 0$, if $-1 \in J_u(A_o)$ then $F.Q$ is not formally real over $u.Q$ for every $Q \in S_u^x \neq \emptyset$, which is absurd. e conclude that $-1 \notin J_u(A_o)$. On the other hand the inclusion $A_o = B[x,u] \subset H_u^x$ is trivial.

Suppose that $A_n \subset H_u^x$ and $-1 \notin J_u(A_n)$. We have to show that $A_{n+1} \subset H_u^x$ and $-1 \notin J_u(A_{n+1})$. Since $1 + J_u(A_n) \subset H_u^x \subset \mathcal{O}_Q$ and $F.Q$ is formally real over $u.Q$ for every $Q \in S_u^x$ it follows that $1 + J_u(A_n) \subset \mathcal{O}_Q^\times$ for every $Q \in S_u^x$ and hence $A_{n+1} \subset H_u^x$. With the same argument we conclude that $1 + J_u(A_{n+1}) \subset (H_u^x)^\times$, therefore $-1 \notin J_u(A_{n+1})$.

Now we have to show that $H_u^x = A = \bigcup_{n\in\mathbb{N}} A_n$. By Proposition 4.1. it suffices to prove that $A$ is minimal with the properties :

$B[x,u] \subset A$ and $1 + J_u(A) \subset A^\times$. First let us show that $A$ satisfies the later condition. Let $t \in J_u(A)$. We have to show that $1 + t$ is invertible in $A$. By construction of $A$, there is $n \in \mathbb{N}$ such that $t \in J_u(A_n)$. Since $-1 \notin J_u(A_n)$ we conclude that $\frac{1}{1+t} \in A_{n+1} \subset A$.

Now let $C$ be an overring of $B[x,u]$ such that $1+J_u(C) \subset C^\times$. We have to show that $A_n \subset C$ for every $n \in \mathbb{N}$. For $n = 0$, the inclusion $A_o = B[x,u] \subset C$ is trivial. Assume that $A_n \subset C$ for some $n \in \mathbb{N}$. We must show that $A_{n+1} \subset C$. Since $A_n \subset C$ it follows that $1 + J_u(A_n) \subset 1 + J_u(C)$ and hence $1 + J_u(A_n) \subset C^\times$, therefore $A_{n+1} \subset C$. Q.E.D.

5. The Nullstellensatz for holomorphy rings - a weak form.

Let $(K,P)$ be an ordered field and $F$ be a field extension of $K$. Assume that $F|(K,P)$ is formally real. Let $x$ and $u$ be arbitrary subsets of $F$. Our goal in this section is to give a weak form of the Nullstellensatz for an arbitrary subring $A$ of $H_u^x$ which contains $B[x,u]$.

Definition Given a subset $X$ of $S_u^x$, let $I(X)$ be the ideal of $A$ consisting of the elements $z \in A$ which vanish on $X$, i.e. $zQ = 0$ for each $Q \in X$. Given a subset $M$ of $A$ let $W(M)$ be the set of common zeros $Q \in S_u^x$ of elements in $M$, i.e. $zQ = 0$ for each $z \in M$.

Definition. (Stengle [14]) Let $C$ be a commutative ring, $\underline{a}$ an ideal in $C$, and $\Gamma$ a subsemiring of $C$ containing all squares in $C$. Then the $\Gamma$- radical of $\underline{a}$ is the subset $r_\Gamma(\underline{a}) = \{z \in C \mid z^{2m} + b \in \underline{a}$ for some $m \geq 1$ and some $b \in \Gamma\}$.
An ideal is a $\Gamma$-radical ideal if it is own $\Gamma$-radical. According to Stengle [14] Proposition 2, $r_\Gamma(\underline{a})$ is a $\Gamma$-radical ideal and equals the intersection of all prime $\Gamma$-radical ideals containing $\underline{a}$.

Let $J_u(A)$ be the subsemiring of $A$ generated by $P \cup u \cup A^2$. If $\underline{a}$ is an ideal in $A$ we denote by $r_u(\underline{a}) = r_{J_u(A)}(\underline{a})$ the $J_u(A)$- radical of $\underline{a}$. It is easy to see that $\mathcal{W}(M) = \mathcal{W}(r_u(\underline{a}))$ where $\underline{a}$ is the ideal in $A$ generated by the subset $M$ of $A$, and $r_u(\underline{a}) \subset I(\mathcal{W}(M))$.

Proposition 5.1. Assume that $F|(K,P)$ is formally real. Let $A$ be a subring of $H_u^x$ containing $B[x,u]$, and $Q$ be a place of $F|K$. Then $Q \in S_u^x$ iff $Q$ lies over $A$ and the center $\underline{m}_Q \cap A$ of $Q$ on $A$ is a $J_u(A)$-radical ideal.

Proof. If $Q \in S_u^x$ then $\mathcal{O}_Q \supset H_u^x \supset A$ and $F.Q|(K,P)$ is formally real over $u.Q$. Let $z \in A$ be such that $z^{2m} + b \in \underline{m}_Q$ for some $m \geqslant 1$ and some $b \in J_u(A)$. Then $(z.Q)^{2m} + b.Q = 0$ and hence $z.Q = 0$, i.e. $z \in \underline{m}_Q \cap A$, since $FQ|(K,P)$ is formally real over $u.Q$.

Conversely, we have only to show that $F.Q|(K,P)$ is formally real over $u.Q$. Since $A$ is a Prüfer ring $\mathcal{O}_Q = A_{\underline{p}}$ where $\underline{p} = A \cap \underline{m}_Q$ and $F.Q$ is isomorphic to the field of quotients of $A/\underline{p}$. If $F.Q|(K,P)$ is not formally real over $u.Q$, i.e. $-1 \in J_{u.Q}(F.Q)$, then $z^{2m} + b \in \underline{p}$ for some $z \in A \setminus \underline{p}$, some $m \geqslant 1$ and some $b \in J_u(A)$, and hence $\underline{p}$ is not a $J_u(A)$-radical ideal, which contradicts the hypothesis. Q.E.D.

The following weak form of Nullstellensatz is an immediate consequence of Proposition 5.1. and of Stengle [14] Proposition 2.

Proposition 5.2. Suppose that $F|(K,P)$ is formally real. Let $A$ be a subring of $H_u^x$ containing $B[x,u]$, $M$ be an arbitrary subset of $A$ and $\underline{a}$ be the ideal in $A$ generated by $M$. Then $I(\mathcal{W}(M)) = r_u(\underline{a})$.

Corollary to Proposition 5.2. Suppose that $F|(K,P)$ is formally real. Let $M$ be a subset of $H^x = B[x]$ and $\underline{a}$ be the ideal in $H^x$ generated by M. Then $I(\mathcal{W}(M)) = \sqrt{\underline{a}}$. In particular $I(S^x) = \bigcap_{Q\in S^x} m_Q = 0$.

6. The restricted Riemann space.

Let (K,P) be an ordered field and F be a field extension of K. We denote by $\tilde{S} = \tilde{S}(F)$ the subspace of the Riemann space S consisting of the places Q of F|K which are rational over the real closure $(\tilde{K},\tilde{P})$ of (K,P), i.e. F.Q is a subextension of $\tilde{K}|K$. It is possible that $\tilde{S}$ is empty though S is non-empty, i.e. F|(K,P) is formally real. For instance let $K = \mathbb{Q}$ and $F = \mathbb{R}$. Then $B(F) = \mathbb{R}$, $S = \{1_{\mathbb{R}}\}$ and $\tilde{S}$ is empty. However, if F|K is finitely generated it follows by Lang [8] Theorem 5, p.278, that F|(K,P) is formally real iff $\tilde{S}(F)$ is non-empty.

We shall assume in the rest of this paper that F|K is finitely generated and F|(K,P) is formally real.

The Zariski topology on S induces a topology on $\tilde{S}$. A basis of open sets for this induced topology is given by the sets $\tilde{S}^x = \tilde{S} \cap S^x = \{Q\in\tilde{S} \mid x \subset \mathcal{O}_Q\}$ where x ranges over all finite subsets of F. There is also another topology on $\tilde{S}$ induced by the order $\tilde{P}$ on $\tilde{K}$. This topology admits as basis the sets $\tilde{S}_u = \{Q\in\tilde{S} \mid u.Q \subset \tilde{P}\}$ where u ranges over all finite subsets of F.

We also consider sets of the form $\tilde{S}_u^x = \tilde{S}_u \cap S^x$ where u and x are finite subsets of F. Any such set will be called a basic subset of $\tilde{S}$.

The following result establishes a non-trivial relation between the sets $S_u^x$ and $\tilde{S}_u^x$.

Proposition 6.1. Let u,x,u',x' be finite subsets of F. If $\tilde{S}_u^x \subset \tilde{S}_{u'}^{x'}$, then $S_u^x \subset S_{u'}^{x'}$.

Proof. Suppose that $S_u^x \not\subset S_{u'}^{x'}$, i.e. either the set

$\{Q \in S_u^x \mid u' \cup x' \not\subset \mathcal{O}_Q\}$ is non-empty or the set $\{Q \in S_u^x \mid u'Q \subset \mathcal{O}_Q$ and $F.Q \mid (K,P)$ is not formally real over $u'.Q\}$ is non-empty. The proposition is a consequence of the following two lemmata.

Lemma 6.2. Let $u$ and $x$ be finite subsets of $F$ and $z$ be an element of $F$. Then the following assertions are equivalent :

i) The set $\{Q \in S_u^x \mid z.Q = \infty\}$ is non-empty.

ii) The set $\{Q \in \tilde{S}_u^x \mid z.Q = \infty\}$ is non-empty.

Proof. Only the implication i) $\rightarrow$ ii) is non-trivial. Suppose that there is a place $R \in S_u^x$ such that $z.R = \infty$, i.e. $z^{-1}R = 0$. We have to show that the set $M_x = \{Q \in \tilde{S}_u^x \mid z^{-1}.Q = 0\}$ is non-empty. If $M_{x'}$ is non-empty for some finite subset $x'$ of $F$ containing $x$, then clearly $M_x$ is also non-empty. Hence we may enlarge $x$ if convenient by adding finitely many elements of $F$. After a suitable enlargement we may assume that $F = K(x)$, $u \cup \{z^{-1}\} \subset x \subset \mathcal{O}_R$ and, by Zariski's local uniformization theorem [15], $xR$ is a simple point on the affine model $V$ of $F \mid K$ whose generic point is $x$. After a renumbering of the elements $x_1, \ldots, x_n$ of $x$ we may assume that $u_i = x_i$ for $i = 1, \ldots, m$, $z^{-1} = x_{m+1}$ and $m+1 \leqslant n$. Thus there exists an order $T$ on the residue field $FR$ which extends the order $P$ and $x_i R \in T$ for $i = 1, \ldots, m$. In addition the point $x.R \in V(F.R)$ is simple and $x_{m+1}.R = 0$.

We see the affine variety $V$ in $n$-space as being defined by a finite system of polynormial equations over $K$. Let $f_1, \ldots, f_s \in K[X]$, where $X = (X_1, \ldots, X_n)$, be some polynomials defining the variety $V$. The condition for a point to be simple on $V$ is that at least one of the minors of order $n$-dim $V$ of the Jacobian matrix $(\frac{\partial f_i}{\partial X_j})$ does not vanish at this point. Let $h \in K[X]$ be a proper minor such that $h(x.R) \neq 0$. Thus the ordered field $(F.R,T)$ satisfies the following existential sentence in the language of ordered fields extending $(K,P,)$:

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$$\varphi, = (\exists X) \bigwedge_{i=1}^{s} f_i(X) = 0 \wedge h(X) \neq 0 \wedge X_{m+1} = 0 \wedge \bigwedge_{i=1}^{m} X_i \geqslant 0$$

By Robinson [11] Theorem 4.3.5., the real closure $(\widetilde{K},\widetilde{P})$ of $(K,P)$ satisfies the sentence $\varphi$ too, i.e. there exists a point $b = (b_1,\ldots,b_n)$ of $V$, which is rational over $\widetilde{K}$, such that $h(b) \neq 0$ (therefore $b$ is simple), $b_i \in \widetilde{P}$ for $i = 1,\ldots,m$ and $b_{m+1} = 0$. Since the point $b$ is simple on $V$ it follows, by [6] Corollary A 2, that the specialization $x \to b$ can be extended to a K-rational place $Q$ of $F|K$. We conclude that $Q \in \widetilde{S}^x_u$ and $z.Q = \infty$ and hence the set $M_x$ is non-empty. Q.E.D.

<u>Lemma 6.3</u>. Let $x,u$ and $u'$ be finite subsets of F. If there is a place $Q \in S^{x\cup u'}_u$ such that $F.Q|(K,P)$ is not formally real over $u'.Q$ then there is a place $R \in \widetilde{S}^{x\cup u'}_u$ such that $u'.R \not\subset \widetilde{P}$.

The proof of this lemma is similar with the proof of Lemma 6.2.

<u>Corollary to Proposition 6.1.</u> Let $x$ and $u$ be finite subsets of F. Then $S^x_u$ is non-empty iff $\widetilde{S}^x_u$ is non-empty. In particular the restricted Riemann space $\widetilde{S}$ is dense in the Riemann space $S$ with respect to the Zariski topology.

## 7. <u>The Nullstellensatz for holomorphy rings-a strong form.</u>

Let $(K,P)$ be anordered field and $F$ be a finitely generated extension of $K$ such that $F|(K,P)$ is formally real.

First we give a description of the holomorphy rings $H^x_u$ in terms of holomorphy rings of basic subsets of the restricted Riemann space $\widetilde{S}$.

<u>Theorem 7.1</u>. Let $x$ and $u$ be finite subsets of F. Then the holomorphy ring $H^x_u$ of the basic subset $S^x_u$ of $S$ equals

the holomorphy ring of the basic subset $\tilde{S}^x_u$ of $\tilde{S}$, i.e. $H^x_u = \bigcap_{Q\in\tilde{S}^x_u} \mathcal{O}_Q$

<u>Proof</u>. We have to show that $\bigcap_{Q\in\tilde{S}^x_u} \mathcal{O}_Q \subset H^x_u$. Let $z \in F \setminus H^x_u$, i.e. there is $Q \in S^x_u$ such that $z.Q = \infty$, By Lemma 6.2. there exists a place $R \in \tilde{S}^x_u$ such that $z.R = \infty$ and hence $z \notin \bigcap_{Q\in\tilde{S}^x_u} \mathcal{O}_Q$. Q.E.D.

Now let us consider an arbitrary subring of $H^x_u$ which contains $B[x,u]$. Given a subset $X$ of $\tilde{S}^x_u$, let $I(X)$ be the ideal of $A$ consisting of the elements $z \in A$ which vanish on $X$, i.e. $zQ = 0$ for each $Q \in X$. Given a subset $M$ of $A$ let $\tilde{W}(M)$ be the set of common zeros $Q \in \tilde{S}^x_u$ of elements in $M$, i.e. $z.Q = 0$ for each $z \in M$. Let $J_u(A)$ denote the subsemirnig of $A$ generated by $P \cup u \cup A^2$. If $\underline{a}$ is an ideal in $A$ let $r_u(\underline{a}) = r_{J_u(A)}(\underline{a})$ be the $J_u(A)$ - radical of $\underline{a}$.

<u>Theorem 7.2</u>. (<u>Nullstellensatz</u> - <u>a</u> <u>strong</u> <u>form</u>). Let $A$ be a subring of $H^x_u$ which contains $B[x,u]$. Let $M$ be an arbitrary <u>finite</u> subset of $A$ and $\underline{a}$ be the ideal in $A$ generated by $M$. Then $I(\tilde{W}(M)) = r_u(\underline{a})$.

<u>Proof.</u> By Proposition 5.2. , $r_u(\underline{a}) = I(W(M)) = \{ z \in A \mid z.\ Q=0$ for each $Q \in W(M)\}$, where $W(M) = \{ Q \in S^x_u \mid z.Q=0$ for each $z \in M\}$. Since $\tilde{W}(M) = W(M) \cap \tilde{S}^x_u$ we have to show that $I(\tilde{W}(M)) \subset I(W(M))$. Let $f \in A \setminus I(W(M))$, i.e. the set $\{ Q \in S^x_u \mid z.Q = 0$ for each $z \in M$ and $f.Q \neq 0\}$ is non-empty. Using a similar argument as in the proof of Lemma 6.2. we conclude that the set $\{ Q \in \tilde{S}^x_u \mid z.Q = 0$ for each $z \in M$ and $f.Q \neq 0\}$ is non-empty too and hence $f \notin I(\tilde{W}(M))$. Q.E.D.

<u>Corollary to Theorem 7.2</u>. Let $M$ be a finite subset of

$H^x = B[x]$ and $\underline{a}$ be the ideal in $H^x$ generated by M. Then $I(\widetilde{W}(M)) = \sqrt{\underline{a}}$, where $\widetilde{W}(M) = \{Q \in \widetilde{S}^x \mid z.Q=0$ for each $z \in M\}$ and $I(\widetilde{W}(M)) = \{z \in B[x] \mid z.Q = 0$ for each $Q \in \widetilde{W}(M)\}$. In particular $I(\widetilde{S}^x) = \bigcap_{Q \in \widetilde{S}^x} \underline{m}_Q = 0$.

8. Proof of the results from Section 1.

First let us observe that we have formulated Theorem 1.1. in such a way that it appears entirely similar to Theorem 7.2. We shall show in this section that the former can be easily deduced from the latter. So let us consider the situation of Theorem 1.1; we use the notations as introduced in Section 1. In particular $x = (x_1,\ldots,x_n)$ is a generic point over K of the affine variety $V$ and $F = K(x)$ is the function field of $V$, $u = (u_1,\ldots,u_m)$ is a finite family of elements in $K[x] \setminus \{0\}$ and $\underline{a}$ is an ideal in $K[x]$. We consider the restricted Riemann space $\widetilde{S}$ of $F|(K,P)$ and the basic subset $\widetilde{S}^x$. If $Q \in \widetilde{S}^x$ then $x.Q = (x_1Q,\ldots,x_nQ)$ is a specialization of $x$ over $\widetilde{K}$ and hence $x.Q$ is a $\widetilde{K}$-rational point of the variety $V$. If we attach to every $Q \in \widetilde{S}^x$ the point $x.Q \in V(\widetilde{K})$ then we obtain the projection map $\widetilde{S}^x \rightarrow V(\widetilde{K})$. If $Q \in \widetilde{S}^x_u$ then Q satisfies the non-negativity conditions $u_jQ \in \widetilde{P}$ for $j = 1,\ldots,m$. Considering $u_j = u_j(x)$ as a polynomial expression in $K[x]$ we observe that $u_j.Q = u_j(x.Q)$ and hence $x.Q \in V_u(\widetilde{K})$. Conversely if $x.Q \in V_u(\widetilde{K})$ it follows in the same way that $Q \in \widetilde{S}^x_u$. We obtained a lemma which is similar with Lemma 2.4. from [6].

Lemma 8.1. $\widetilde{S}^x_u$ is the inverse image of $V_u(\widetilde{K})$ with respect to the projection map $\widetilde{S}^x \rightarrow V(\widetilde{K})$.

By a well known result from algebraic geometry [6] Corollary A 2 we also have:

Lemma 8.2. The image of $\widetilde{S}^x_u$ with respect to the projection map $\widetilde{S}^x \to V(\widetilde{K})$ contains at least all simple points of $V_u(\widetilde{K})$. In particular, if the variety $V$ is nonsingular then the projection map $\widetilde{S}^x_u \to V_u(\widetilde{K})$ is surjective.

Proof of Theorem 1.1.

Let $M = \{f_1,\ldots,f_s\}$ be a system of generators of the ideal $\underline{a}$ in $K[x]$. We consider $f_i = f_i(x)$ as a polynomial expression in $K[x]$.

We have :

$$V_{u,\underline{a}}(\widetilde{K}) = \{ b \in V(\widetilde{K}) \mid u_j(b) \in \widetilde{P} \text{ for } j = 1,\ldots,m \text{ and } f_i(b) = 0 \text{ for } i = 1,\ldots,s \}$$

and

$$\widetilde{W}(M) = \{ \mathcal{Q} \in \widetilde{S}^x_u \mid f_i \cdot \mathcal{Q} = 0 \text{ for } i = 1,\ldots,s \}$$

Since $V$ is by hypothesis nonsingular we conclude by Lemma 8.2. that $\widetilde{W}(M)$ is the inverse image of $V_{u,\underline{a}}(\widetilde{K})$ and $V_{u,\underline{a}}(\widetilde{K})$ is the image of $\widetilde{W}(M)$ with respect to the projection map $\widetilde{S}^x_u \to V_u(\widetilde{K})$.

Let us consider the corresponding ideals in $K[x]$ and $B[x] = B[x,u]$ :

$$I(V_{u,\underline{a}}(\widetilde{K})) = \{ g \in K[x] \mid g(b) = 0 \text{ for every } b \in V_{u,\underline{a}}(\widetilde{K}) \} = \{ g \in K[x] \mid g.\mathcal{Q} = 0 \text{ for every } \mathcal{Q} \in \widetilde{W}(M) \} \text{ and}$$

$$I(\widetilde{W}(M)) = \{ g \in B[x] \mid g.\mathcal{Q} = 0 \text{ for every } \mathcal{Q} \in \widetilde{W}(M) \}.$$

It follows that $I(V_{u,\underline{a}}(\widetilde{K})) = K[x] \cap I(\widetilde{W}(M))$. By Theorem 7.2. $I(\widetilde{W}(M))$ is the $J_u(B[x])$ - radical of the ideal $\underline{a}B[x]$, denoted by $r_{J_u(B[x])}(\underline{a}B[x])$. It follows that $I(V_{u,\underline{a}}(\widetilde{K})) = K[x] \cap r_{J_u(B[x])}(\underline{a}B[x])$. It remains to show that the contraction of the ideal $r_{J_u(B[x])}(\underline{a}B[x])$ on $K[x]$ is the $J_u(K[x])$ - radical $r_u(\underline{a})$ of the ideal $\underline{a}$. The inclusion $r_u(\underline{a}) \subset K[x] \cap r_{J_u(B[x])}(\underline{a}B[x])$ follows easily from definitions.

Conversely, let $g \in K[x] \setminus r_u(\underline{a})$. We have to show that

$g \notin r_{J_u(B[x])}(\underline{a} B[x])$. By [14] Proposition 2, $r_u(\underline{a})$ is the intersection of all prime $J_u(K[x])$ - radical ideals in $K[x]$ containing $\underline{a}$. It follows that there is a prime $J_u(K[x])$ - radical ideal $\underline{p}$ containing $\underline{a}$ such that $g \notin \underline{p}$. Let $L = K(x \bmod \underline{p})$ be the field of quotients of the factor ring $K[x]/\underline{p}$. Then $L|(\tilde{K},P)$ is formally real over $u \bmod \underline{p}$ and $x \bmod \underline{p}$ is an $L$-rational point of the variety $V$. Since by hypothesis $V$ is nonsingular, the point $x \bmod \underline{p}$ is simple on $V$. By [6] Corollary A 2 the specialization $x \to x \bmod \underline{p}$ can be extended to an $L$-rational place $Q$ of $F|K$ such that $xQ = x \bmod \underline{p}$ and $F.Q = L$. It follows that the center $\underline{m}_Q \cap K[x]$ of the place $Q$ on $K[x]$ is the prime ideal $\underline{p}$ and hence $g \notin \underline{m}_Q$. On the other hand, $Q \in S_u^x$ and hence, by Proposition 5.1., $Q$ lies over $B[x]$ and the center $\underline{m}_Q \cap B[x]$ of $Q$ on $B[x]$ is a prime $J_u(B[x])$ - radical ideal in $B[x]$ containing the ideal $\underline{a} B[x]$. We conclude by [14] Proposition 2 that $g \notin r_{J_u(B[x])}(\underline{a} B[x])$ as contended. Q.E.D.

<u>Proof of Theorem 1.2</u>

By hypothesis of Theorem 1.2. the monoid generated by the family $u$ is a subgroup of the multiplicative group $F^\times$.

First, suppose that $F|(K,P)$ is not formally real over $u$. Then it is easy to see that $\tilde{S}_u^x$ is empty. We conclude from Lemma 8.2 that there are no simple points in $V_u(\tilde{K})$.

Conversely, assume that $F|(K,P)$ is formally real over $u$. Then the trivial place $1_F$ is contained in $S_u^y$ for every family $y$ of elements in $F$. It follows by Corollary to Proposition 6.1. that $\tilde{S}_u^y$ is non-empty for each finite family $y$ of elements in $F$. Now let us observe that the generic point $x$ is simple on $V$. Therefore, considering a system of defining equations for $x$

over K and its Jacobin matrix, it follows that there exists at least one minor $H \in K[X]$ of order n - dim (V) such that $H(x) \overset{say}{=} h \neq 0$. Let y be a finite family of elements in F which contains the elements $x_1, \ldots, x_n, h^{-1}$. The set $\tilde{S}_u^y$ is non-empty. Let Q be a place in $\tilde{S}_u^y$. The place Q is contained in $\tilde{S}_u^x$ and its projection $b = x.Q$ is simple on V since $H(b) = h.Q \neq 0$. We conclude by Lemma 8.1. that $V_u(\tilde{K})$ contains at least one simple point. Q.E.D.

<u>References.</u>

[1] R.Baer, Über nicht - Archimedisch geordnete Körper, Sitzungsberichte Heidelberger Akademie d.Wissenschaften, Math.- naturw. Klasse 1927, 8.Abhandlung, 3-13.

[2] E.Becker, Partial orders on a field and valuation rings, Preprint.

[3] E. Becker, Summen n-ter Potenzen in Körpern, J.reine angew. Math., 307/308 (1979), 8-30.

[4] A.Dress, Lotschnittebenen mit halbierbarem rechten Winkel, Archiv Math. 16 (1965), 388-392.

[5] D.W. Dubois, A nullstellensatz for ordered fields, Arkiv Mat. 8 (1969), 111-114.

[6] M.Jarden and P.Roquette, The Nullstellensatz over p -adically closed fields, to appear in J.Math.Soc.Japan.

[7] W.Krüll, Allgemeine Bewertungstheorie, J.reine angew. Math. 167(1932), 160-196.

[8] S.Lang, Algebra, Addison-Wesley, 1965.

[9] W.Pejas, Die Modelle des Hilbertschen Axiomensystems der absoluten Geometrie, Math.Ann.143 (1961), 212-235.

[10] A.Prestel, Lectures on formally real fields, IMPA Lecture Notes, Rio de Janeiro 1975.

[11] A.Robinson, Introduction to model theory and to the metamathematics of algebra, North Holland 1963.

[12] A.Robinson and P.Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, J. Number Theory 7(1975), 121-176.

[13] P.Roquette, Principal ideal theorems for holomorphy rings in fields, J.reine angew. Math. 262/263 (1973) 361-374.

[14] G.Stengle, A nullstellensatz and a positiv-stellensatz in semialgebraic geometry, Math. Ann.2o7 (1974),87-97.

[15] O.Zariski, Local uniformization on algebraic varieties, Ann. Math. 41 (194o), 852-896.