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Introduction

In this paper we are dealing with the following problem: determine all normal (or smooth) projective varieties X over an algebraically closed field k supporting a given variety Y as an ample Cartier divisor. In §1 we assume $Y = P^{n-1}$ with $n \geq 3$ and show that such a normal variety X is isomorphic to the projective cone over $v_s(Y)$, where $s > o$ is the integer determined by the equality $O_X(Y) \otimes O_Y = O(s)$ and $v_s : P^{n-1} \hookrightarrow P^{N-1}$ $(N = \binom{n+s-1}{n-1})$ is the s^{th} Veronese embedding of P^{n-1}. A similar result is valid for $Y = P^s \times P^t$ with $s, t \geq 2$. In the second section we prove the following generalization of a result of Sommese ([13]). If $Y = H(d)$ is a hypersurface of prime degree d in P^{n+1} such that either $n \geq 3$, or else $char(k) = o$ and $H(d)$ is a generic surface in P^3 with $d \geq 5$, then Y can be contained in a smooth projective variety X as an ample divisor only in one of the following two cases: i) X is P^{n+1} and the inclusion $Y \subset X$ is just the inclusion $H(d) \subset P^{n+1}$, or ii) X is a smooth hypersurface of degree d in P^{n+2} and Y is the intersection of X with a hyperplane. In the last section we determine all smooth projective threefolds X with P^2 (resp. $P^1 \times P^1$) as an ample divisor. Note that if $char(k) = o$ the proofs are not so complicated (in the case of $Y = P^2$ the result being (well known and) contained in §1) because one applies the result of [2]. However, by the method of lifting to characteristic zero we show that in our situation we can apply [2] in positive characteristic as well.

The proofs of these results require Lefschetz type theorems in Grothendieck's form ([7],[8],[2]). Throughout this paper k will be an algebraically closed field of arbitrary characteristic and the notations and terminology will be standard, unless otherwise specified.

§1. Normal projective varieties containing P^{n-1} $(n\geqslant 3)$ or $P^s\times P^t$ $(s+t\geqslant 3)$ as an ample Cartier divisor

Let Y be an arbitrary connected smooth projective variety over k and choose a projectively normal embedding $i:Y\hookrightarrow P^m$ of Y (by a theorem of Serre such an embedding always exists). Denote by C(Y,i) the projective cone in P^{m+1} over i(Y). Then C(Y,i) is a normal projective variety containing i(Y) as an ample Cartier divisor.

Examples. 1) Take $Y = P^{n-1}$ with $n\geqslant 2$ and for every $s>o$ consider the s^{th} Veronese embedding $v_s:P^{n-1}\hookrightarrow P^{N-1}$ with $N = \binom{n+s-1}{n-1}$. Then v_s is projectively normal and hence P^{n-1} is an ample Cartier divisor in the normal variety $X^n_s = C(P^{n-1},v_s)$ such that the normal sheaf N_{P^{n-1},X^n_s} is $O(s) = O_{P^{n-1}}(s)$. Moreover, $X^n_1 = P^n$.

2) Take $Y = P^s\times P^t$ with $s>o$, $t>o$, $s+t\geqslant 3$ and for every $a>o$, $b>o$ consider the Segre-Veronese embedding $i_{a,b}:P^s\times P^t\hookrightarrow P^{N-1}$ with $N = \binom{s+a}{s}\binom{t+b}{b}$. Then $i_{a,b}$ is projectively normal and hence Y is an ample Cartier divisor on the cone $C(P^s\times P^t,i_{a,b}) = X^{s,t}_{a,b}$ such that the respective normal sheaf if $O(a,b) = p_1^*(O_{P^s}(a))\otimes p_2^*(O_{P^t}(b))$, p_1 and p_2 being the canonical projections of $P^s\times P^t$.

Theorem 1. Assume that $n\geqslant 4$ and that $Y = P^{n-1}$ is an ample Cartier divisor on the normal projective variety X. Then if the normal sheaf $N_{Y,X}$ is isomorphic to O(s) (necessarily $s>o$), X is isomorphic to X^n_s and Y is contained in X as in example 1 above. If $n = 3$ the same conclusion holds provided that char(k) = o. In particular, X is smooth if and only if $s = 1$, i.e. $X = P^n$.

Proof. Let Sing(X) be the singular locus of X and set $U = X - \mathrm{Sing}(X)$. Since Y is a smooth Cartier divisor on X, $Y \subset U$, and since Y is ample, $\dim(\mathrm{Sing}(X)) \leqslant \leqslant o$, i.e. Sing(X) consists of at most a finite set of closed points $\{x_1, \ldots, x_h\}$.

By [7], exposé X, example 2.2 the pair (X,Y) satisfies the effective Lefschetz condition, Leff(X,Y). Since this condition is local along Y we have also Leff(U,Y). If $n \geqslant 4$ we have $H^1(O_X(-mY)/O_X(-(m+1)Y)) = H^1(O(-ms)) = o$ for $i = 1,2$ and for every $m > o$. Hence by [7], exposé XI, théorème 3.12 the natural map of restriction $\alpha : \mathrm{Pic}(U) \longrightarrow \mathrm{Pic}(Y) \cong \mathbb{Z}$ is an isomorphism. If instead $n = 3$ and char(k) = o we have $H^1(O_X(-mY)/O_X(-(m+1)Y)) = H^1(O(-ms)) = o$ for every $m > o$, and then apply the theorem of [2] (in a slightly modified form) to deduce that α is injective and Coker(α) is torsion-free. Since $\mathrm{Pic}(U) \neq o$ ($O_X(Y)/U \not\cong O_U$) and $\mathrm{Pic}(Y) \cong \mathbb{Z}$ this yields that α is also an isomorphism.

Therefore in both cases there is an invertible O_U-module L such that $L \otimes O_Y = O(1)$. For every $m \in \mathbb{Z}$ put $F^{(m)} = j_*(L^{\otimes m})$, where $j : U \hookrightarrow X$ is the canonical open immersion. The following statements hold:

a) $F^{(m)}$ is a coherent O_X-module and $\mathrm{depth}_{O_{x_i}}((F^{(m)})_{x_i}) \geqslant 2$ for every $m \in \mathbb{Z}$.

Indeed, the coherence of $F^{(m)}$ comes from [7], exposé VIII, corollary VIII-II-3. On the other hand, the canonical map $F^{(m)} \longrightarrow j_*j^*(F^{(m)})$ is (by the very definition of $F^{(m)}$) an isomorphism, and the second affirmation follows from the exact sequence

$$o \longrightarrow \bigoplus_{i=1}^{h} H^o_{x_i}((F^{(m)})_{x_i}) \longrightarrow F^{(m)} \longrightarrow j_*j^*(F^{(m)}) \longrightarrow \bigoplus_{i=1}^{h} H^1_{x_i}((F^{(m)})_{x_i}) \longrightarrow o.$$

b) $F^{(ms)} \simeq O_X(mY)$ for every $m \in \mathbb{Z}$.

Indeed, $L^{\otimes ms} \simeq O_X(mY)/U$ because $O_X(mY) \otimes O_Y = O(ms)$ and the map α is injective. Applying j_* to this isomorphism and taking into account that $\mathrm{depth}(O_{X_i}) \geqslant 2$ (O_{X_i} is normal of dimension $\geqslant 2$) we get the conclusion.

c) $H^1(F^{(m)}) = o$ for every $m \ll o$.

First choose t big enough so that $O_X(tY)$ is very ample and consider the embedding $i: X \longrightarrow P = P(\Gamma(X, O_X(tY))$ such that $i^* O_P(1) \simeq O_X(tY)$.

Claim. For every coherent O_X-module G such that $\mathrm{depth}_{O_x}(G_x) \geqslant 2$ for every closed point $x \in X$, $H^1(G \otimes O_X(qY)) = o$ for every $q \ll o$.

Proof of the claim. Set $G' = i_*(G)$. For every closed point $y \in P - i(X)$ we have clearly $H^1_y(G'_y) = o$. If $y \in i(X)$ is a closed point, by [5], corollary 5.6 we have $H^1_y(G'_y) = H^1_y(G_y)$, and recalling the hypothesis the last group is zero. Thus we may apply [7], éxposé XII, corollary 1.3 and deduce that $H^1(X, G \otimes O_X(q'tY)) = H^1(P, G' \otimes O_P(q')) = o$ for every $q' \ll o$. Also, denoting by $G_r = G \otimes O_X(rY)$, $r = o, 1, \ldots, t-1$ ($G_o = G$), then $H^1(X, G_r \otimes O_X(q'tY)) = o$ for $q' \ll o$ (because for every closed point $x \in X$ $\mathrm{depth}((G_r)_x) \geqslant 2$). Now let q be arbitrary and divide $q = q't + r$, with $o \leqslant r \leqslant t-1$. The equality $G \otimes O_X(qY) = G_r \otimes O_X(q'tY)$ and the above discussion proves the claim.

Now in order to prove c) write $m = qs + r$, with $o \leqslant r \leqslant s-1$. Since $O_X(Y)$ is invertible on X, b) and projection formula yield

$$F^{(m)} = j_*(L^{\otimes r} \otimes L^{\otimes qs}) = j_*(L^{\otimes r} \otimes j^*(O_X(qY))) \simeq j_*(L^r) \otimes O_X(qY) = F^{(r)} \otimes O_X(qY).$$

The statement of c) follows applying the claim to $G = F^{(r)}$, $r = o, 1, \ldots, s-1$ and taking into account a).

d) Let $\sigma \in \Gamma(X, F^{(s)}) \simeq \Gamma(X, O_X(Y))$ be such that $\mathrm{div}_X(\sigma) = Y$. Then for every $m \in \mathbb{Z}$ there is the exact sequence on X

$$(1) \qquad o \longrightarrow F^{(m-s)} \xrightarrow{\ \sigma\ } F^{(m)} \longrightarrow O(m) \longrightarrow o\ ,$$

where the first map is multiplication by σ.

Indeed, the exact sequence $0 \longrightarrow O_X(-Y) \xrightarrow{\sigma} O_X \longrightarrow O_Y \longrightarrow 0$ tensorized by $F^{(m)}$ yields the exact sequence

$$F^{(m)} \otimes O_X(-Y) \simeq F^{(m-s)} \xrightarrow{\sigma} F^{(m)} \longrightarrow O(m) \longrightarrow 0.$$

Since $F^{(m)}$ is invertible on U the map σ/U is injective, and since $\sigma(x_i) \neq 0$ for every $i = 1,\dots,h$, σ is injective everywhere.

Now (1) yields the exact sequence of cohomology $(m \in \mathbb{Z})$

$$0 \longrightarrow \Gamma(X,F^{(m-s)}) \xrightarrow{\sigma} \Gamma(X,F^{(m)}) \longrightarrow \Gamma(Y,O(m)) \longrightarrow$$
$$\longrightarrow H^1(X,F^{(m-s)}) \xrightarrow{\psi_m} H^1(X,F^{(m)}) \longrightarrow H^1(Y,O(m)) = 0.$$

Thus for every $m \in \mathbb{Z}$ the map ψ_m is surjective. Thus from c) and induction on m it follows that $H^1(X,F^{(m)}) = 0$ for every $m \in \mathbb{Z}$. Thus for every m one gets the exact sequence

$$(2)\quad 0 \longrightarrow \Gamma(X,F^{(m-s)}) \xrightarrow{\sigma} \Gamma(X,F^{(m)}) \longrightarrow \Gamma(Y,O(m)) \longrightarrow 0\,.$$

Set $S = \bigoplus_{m=0}^{\infty} \Gamma(X,F^{(m)}) = \bigoplus_{m=0}^{\infty} \Gamma(U,L^{\otimes m})$. Then S is a graded k-algebra, $\sigma \in S_s$ and (2) yields the isomorphism of graded k-algebras

$$S/\sigma S \simeq \bigoplus_{m=0}^{\infty} \Gamma(Y,O(m)) \simeq k[T_1,\dots,T_n] \text{ (polynomial ring in n variables).}$$

Set $S' = S^{(s)}$, where $S'_t = S_{st}$ for every $t \in \mathbb{Z}$. Then $\sigma \in S'_1$ and

$$S'/\sigma S' = k[T_1,\dots,T_n]^{(s)}.$$

Choose $t_i \in S_1$ such that $t_i \bmod \sigma S = T_i$ and set $\sigma_{i_1,\dots,i_n} = t_1^{i_1}\dots t_n^{i_n} \in S_s = S'_1$, where $i_1+\dots+i_n = s$ and $i_m \geq 0$. Then $\sigma_{i_1,\dots,i_n}$ satisfy the well known Veronese equations

$$(3)\quad \sigma_{i_1,\dots,i_n}\cdot\sigma_{j_1,\dots,j_n} - \sigma_{e_1,\dots,e_n}\cdot\sigma_{f_1,\dots,f_n} = 0,$$

where $i_m + j_m = e_m + f_m$, $m = 1,\dots,n$.

Furthermore the images of $\{\sigma_{i_1,\dots,i_n}\}$ in $S'/\sigma S'$ generate the graded k-algebra $S'/\sigma S'$, and since $\sigma \in S'_1$, it follows that σ and $\{\sigma_{i_1,\dots,i_n}\}$ generate S' as a graded k-algebra.

In particular, $S' = \bigoplus_{m=0}^{\infty}\Gamma(X,O_X(mY))$ is generated by its part of degree one. Since Y is ample on X, $O_X(Y)$ results then very ample. Thus the canonical map $\varphi_Y: X \longrightarrow P(\Gamma(X,O_X(Y)))$ (such that $\varphi_Y^*(O(1)) \cong O_X(Y)$) is a closed immersion. If in (2) we take m = s we get $\dim\Gamma(X,O_X(Y)) = \dim\Gamma(X,O_X) + \dim\Gamma(Y,O(s)) = N+1$, where $N = \binom{n+s-1}{n-1}$. Thus $\varphi_Y(X)\subset P^N$ and φ_Y restricted to Y is precisely the Veronese embedding v_s. In particular, Y is the intersection of X with the hyperplane P^{N-1}. It remaines to be proved that $\varphi_Y(X)$ is isomorphic to the cone X_s^n.

Set $S'' = k[T_1,\dots,T_n]^{(s)}$, grade the polynomial k-algebra $S''[T]$ so that if $a\in S''$ is an arbitrary homogeneous element then $\deg(aT^m) = \deg(a) + m$, and define the homomorphism of graded k-algebras $\psi: S''[T] \longrightarrow S'$ by $\psi(T) = \sigma$ and $\psi(T_1^{i_1}\dots T_n^{i_n}) = \sigma_{i_1,\dots,i_n}$, where $i_m \geq 0$ and $i_1+\dots+i_n = s$. The equations (3) ensure us that this definition is correct. Since $\sigma_{i_1,\dots,i_n}$ and σ generate S' as a k-algebra, ψ is surjective. Also, the dimension of $S''[T]$ and S' are the same (namely n+1) and these graded algebras are integral domains. Therefore ψ is an isomorphism, which proves that $\varphi_Y(X) \cong X_s^n$. Q.E.D.

Exactly in the same way one can prove the following theorem.

Theorem 2. Assume that $Y = P^s \times P^t$ (with $s\geq 2$, $t\geq 2$ ~~and s+t≥3~~) is an ample divisor on the normal projective variety X. Then if the normal sheaf $N_{Y,X}$ is isomorphic to O(a,b) (necessarily a > o and b > o), X is isomorphic to the cone $X_{a,b}^{s,t}$ (from example 2 above). In particular, $P^s\times P^t$ ($s\geq 2$ and $t\geq 2$) cannot be contained in a smooth projective variety as an ample divisor.

Remark. The assumption about the normality of X in theorem 1 or theorem 2 cannot be dropped. Indeed, consider the Veronese embedding $v_2: P^2 \hookrightarrow P^5$ and take the generic projection Y' of $v_2(P^2)$ into P^4, i.e. the Veronese surface in P^4. Then Y' is isomorphic to P^2, Y' is an ample Cartier divisor on the cone $C(Y')\subset P^5$ over Y', but since Y' is the projection of $v_2(P^2)$ into P^4, the vertex of $C(Y')$ is not a normal point. Thus $C(Y')$ cannot be isomorphic to any X_s^3.

Corollary 1. i) *Assume that $Y = P^{n-1}$ is an effective Cartier divisor on the normal complete variety X such that $N_{Y,X} = O(s)$ with $s > o$, and assume moreover that either $n \geq 4$, or else $n = 3$ and $char(k) = o$. Then there is a birational morphism $f: X \longrightarrow X_s^n$ such that f is an isomorphism in a neighbourhood of Y and $f(Y) = v_s(P^{n-1})$.*

ii) *Assume that $Y = P^s \times P^t$ ($s \geq 2$, $t \geq 2$ [redacted]) is a effective Cartier divisor on the normal complete variety X such that $N_{Y,X} = O(a,b)$ with $a > o$ and $b > o$. Then there is a birational morphism $f: X \longrightarrow X_{a,b}^{s,t}$ such that f is an isomorphism in a neighbourhood of Y and $f(Y) = i_{a,b}(P^s \times P^t)$.*

Proof. Let us prove for example i). By [8], chapter III, theorem 4.2 there is a birational morphism $f: X \longrightarrow X'$ such that f is an isomorphism in a neighbourhood of Y and $Y' = f(Y)$ is an ample Cartier divisor on X'. Since X is normal, we may assume that X' is also normal. Then by theorem 1 $X' \cong X_s^n$ such that Y' corresponds to $v_s(P^{n-1})$. Q.E.D.

Corollary 2. *Assume that Y is as in corollary 1 i) or ii), and let $Y \hookrightarrow X_i$ ($i = 1,2$) two closed immersions such that X_1 and X_2 are smooth varieties of dimension equal to $\dim(Y) + 1$ and $N_{Y,X_1} \cong N_{Y,X_2}$ is ample. Then there is a birational map $u: X_1 \longrightarrow X_2$ which is an isomorphism on an open neighbourhood of Y in X and induces identity on Y.*

§2. A generalization of a result of Sommese

First we need the following extension to arbitrary characteristic of a result of Kobayashi-Ochiai (see [11]). For the intersection theory of line bundles needed in this section we send to [10].

Theorem 3. (Kobayashi-Ochiai) Let V be a complete Cohen-Macaulay algebraic scheme of pure dimension $t > o$ over k and L an ample invertible O_V-module such that $(L^{\cdot t})_V = 1$ and $\dim \Gamma(V,L) \geq t+1$. Then $\dim \Gamma(V,L) = t+1$ and the canonical map $\varphi_L : V \longrightarrow P(\Gamma(V,L)) \cong P^t$ is a biregular isomorphism.

Proof. First we prove that V is integral. Let $V_1, \ldots, V_n$ be the irreducible components of V naturally regarded as closed subschemes of V (see [10], page 298). Then by loc. cit. proposition 5 and corollary 1 one has

$$(L^{\cdot t})_V = (L_1^{\cdot t})_{V_1} + \ldots + (L_n^{\cdot t})_{V_n}, \text{ where } L_i = L \otimes O_{V_i}.$$

Since every V_i has dimension t and L_i is ample on V_i, $(L_i^{\cdot t})_{V_i} > o$ for every $i = 1, \ldots, n$. Thus if V were reducible the above equality would imply $(L^{\cdot t})_V \geq 2$, a contradiction.

Thus V is irreducible. By loc. cit. proposition 5 and corollary 2 (page 298) one has

$$(L^{\cdot t})_V = \text{length}(O_{V,\xi}) \cdot (M^{\cdot t})_{V_{red}},$$

where $M = L \otimes O_{V_{red}}$ and ξ is the generic point of V. Thus $\text{length}(O_{V,\xi}) = 1$, i.e. V is generically reduced. Now since V is Cohen-Macaulay and generically reduced, [1], chap. VII, proposition 2.2 shows that V is reduced everywhere. Thus V is integral.

Let now $s_1, \ldots, s_{t+1}$ be t+1 linearly independent sections (over k) from $\Gamma(V,L)$ and $D_i = \text{div}_V(s_i)$. Define the sequence of closed subsets of V

$$V = V_t \supseteq V_{t-1} \supseteq \ldots \supseteq V_o \supseteq V_{-1}$$

by $V_{t-i} = D_1 \cap \ldots \cap D_t$ for $i = 1, \ldots, t+1$. V_{t-i} can be naturally endowed with a structure of closed subscheme of V, $i = 1, \ldots, t+1$. Then one can easily prove as before that each V_{t-i} is an integral Cohen-Macaulay scheme of dimension t-i and that there is a natural exact sequence

$$o \longrightarrow (s_1, \ldots, s_i) \longrightarrow \Gamma(V,L) \longrightarrow \Gamma(V_{t-i}, L \otimes O_{V_{t-i}}),$$

where $(s_1,\dots,s_i)$ is the subspace of $\Gamma(V,L)$ generated by $s_1,\dots,s_i$ (see [11] for details). From this point one gets the conclusion exactly as in [11]. Q.E.D.

Theorem 4. Let $Y = H(d)$ be a hypersurface of P^{n+1} (i.e. a complete intersection of codimension one in P^{n+1}, not necessarily smooth) of degree d with d prime. Assume that one of the following conditions holds:

a) $n \geqslant 3$, or

b) $\mathrm{char}(k) = 0$ and Y is a generic surface in P^3 with $d \geqslant 5$.

Assume further that Y is embedded as an ample divisor in the projective smooth variety X. Then one has one of the following possibilities:

i) X is isomorphic to P^{n+1} and the inclusion $Y \subset X$ is just $H(d) \subset P^{n+1}$.

ii) X is isomorphic to a smooth hypersurface of degree d in P^{n+2} and Y is the intersection of X with a hyperplane.

Proof. In case a) by Lefschetz's theorem we have $\mathrm{Pic}(Y) = \mathbb{Z}[O_Y(1)]$. Also, since $Y = H(d)$ and $\dim(Y) = n \geqslant 3$, $H^i(O_Y(s)) = 0$ for $i = 1,2$ and for every $s \in \mathbb{Z}$; in particular, $H^i(O_X(-mY)/O_X(-(m+1)Y)) = 0$ for $i = 1,2$ and for every $m \geqslant 1$. Thus we may apply Lefschetz's theorem to (X,Y) and get that the map $\alpha: \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is an isomorphism.

In case b) we may apply Noether's theorem (see [8], page 182) and also deduce that $\mathrm{Pic}(Y) = \mathbb{Z}[O_Y(1)]$. By [2] α is injective and $\mathrm{Coker}(\alpha)$ is torsion-free. Hence α turns out to be also an isomorphism.

Therefore in both cases there is an invertible O_X-module L such that $L \otimes O_Y = O_Y(1)$. Further there is an integer $r > 0$ such that $O_X(Y) \cong L^{\otimes r}$. Let $\sigma \in \Gamma(X, O_X(Y)) \cong \Gamma(X, L^{\otimes r})$ be a section such that $\mathrm{div}_X(\sigma) = Y$. We have

$$(4) \quad (L^{\cdot(n+1)})_X = 1/r.(L^{\cdot n}.L^{\otimes r})_X = 1/r.(L^{\cdot n}.Y)_X = 1/r.(L_Y^{\cdot n})_Y =$$

$$= 1/r.(O_Y(1)^{\cdot n})_Y = d/r, \quad \text{where } L_Y = L \otimes O_Y$$

In particular r divides d, and since d is prime one has two possibilities.

1) $r = d$, i.e. $O_X(Y) = L^{\otimes d}$.

Then (4) gives $(L^{\cdot(n+1)})_X = 1$. On the other hand, exactly as in the proof of theorem 1 one shows that the sequence

$$o \longrightarrow \Gamma(L^{\otimes(1-d)}) \xrightarrow{\ \sigma\ } \Gamma(L) \longrightarrow \Gamma(O_Y(1)) \longrightarrow o$$

is exact. Since $d > 1$ and L is ample $\Gamma(L^{\otimes(1-d)}) = o$. Thus $\dim\Gamma(L) = n+2$. Now theorem 3 applied to $V = X$ leads to case i).

2) $r = 1$, i.e. $L \cong O_X(Y)$.

Again one deduces the exact sequence (for every $m \in \mathbb{Z}$)

$$(5) \qquad o \longrightarrow \Gamma(L^{\otimes(m-1)}) \longrightarrow \Gamma(L^{\otimes m}) \longrightarrow \Gamma(O_Y(m)) \longrightarrow o .$$

Denoting by S the graded k-algebra $\bigoplus_{m=o}^{\infty}\Gamma(X,L^{\otimes m})$, $\sigma \in S_1$ and by (5) $S/\sigma S \cong \bigoplus_{m=o}^{\infty}\Gamma(Y,O_Y(m))$. Recalling that Y is a hypersurface in P^{n+1}, this last algebra is generated by its homogeneous part of degree one. Hence S itself is generated by $S_1 = \Gamma(L)$, and in particular L is very ample on X.

If in (5) we take $m = 1$ we get $\dim\Gamma(L) = \dim\Gamma(O_X) + \dim\Gamma(O_Y(1)) = n+3$. Therefore the canonical map $\varphi = \varphi_L : X \longrightarrow P(\Gamma(L)) = P^{n+2}$ is a closed immersion. Since $\varphi^*(O(1)) \cong L$ (taking into account that $r = 1$ and (4))

$$\deg\varphi(X) = (O(1)^{\cdot(n+1)}\cdot\varphi(X))_{P^{n+2}} = (L^{\cdot(n+1)})_X = d .$$

The fact that $\varphi(Y)$ is the intersection of $\varphi(X)$ with a hyperplane of P^{n+2} is now clear. Thus case 2) leads to case ii). Q.E.D.

Corollary. Let Y be a hyperquadric in P^{n+1} with $n \geq 3$. Then Y can be an ample divisor on the smooth projective variety X if and only if either X is isomorphic to P^{n+1}, or to a smooth hyperquadric in P^{n+2}.

Remark. If $k = \mathbb{C}$ the above corollary has been previously obtained by Sommese in [13].

§3. Lifting to characteristic zero

Let k be an algebraically closed field of characteristic $p>o$ and $A = W(k)$ the ring of Witt vectors on k, which is a complete discrete valuation ring of characteristic zero, with residue field k and such that p generates its maximal ideal. Let X be a projective smooth variety over k. One says that X has a lifting to characteristic zero if there is a projective smooth morphism $f: \mathfrak{X} \longrightarrow \mathrm{Spec}(A)$ whose closed fibre is isomorphic to X. Then the generic fibre X' of f is a projective smooth variety over the quotient field k' of A.

Grothendieck proved in [6], exposé III, théorème 7.3 that a sufficient condition for the existence of a lifting to characteristic zero of X is the following "$H^2(T_X) = H^2(O_X) = o$", where $T_X = (\Omega^1_{X/k})^{\vee}$ is the tangent sheaf of X.

Let now k be a field (not necessarily algebraically closed) and X a smooth projective variety of dimension 3 over k. Let L be an ample invertible O_X-module and $\sigma \in \Gamma(X,L)$ a section such that $Y = \mathrm{div}_X(\sigma)$ is smooth over k. Then we have the following result which follows from [2].

<u>Proposition 1</u>. <u>Assume char(k) = o. Then the natural map $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is injective and its cokernel is torsion-free.</u>

<u>Lemma 1</u>. <u>Let k be an algebraically closed field of characteristic $p>o$ and X, L, σ, and Y as above. Assume moreover:</u>

i) <u>X has a lifting to characteristic zero.</u>

ii) $H^i(O_X) = o$ <u>for i = 1,2.</u>

ii) $H^i(O_Y) = o$ <u>for i = 1,2.</u>

iv) <u>$H^1(L) = o$.</u>

<u>Then the map of restriction $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$ is injective with cokernel a torsion-free group.</u>

Proof. Let $f: \mathcal{X} \longrightarrow \mathrm{Spec}(A)$ be a lifting to characteristic zero of X. First we prove that the natural map of restriction $\mathrm{Pic}(\mathcal{X}) \longrightarrow \mathrm{Pic}(X)$ is an isomorphism. Indeed, let $\hat{\mathcal{X}}$ be the formal completion of $\mathcal{X}$ along X. Then by Grothendieck's existence theorem (see [3], chap. III 5.4.1) the natural map $\mathrm{Pic}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\hat{\mathcal{X}})$ is an isomrophism. It will be therefore sufficient to show that the map of restriction $\mathrm{Pic}(\hat{\mathcal{X}}) \longrightarrow \mathrm{Pic}(X)$ is also an isomorphism. Let $\mathcal{X}_n$ be the closed subscheme of $\mathcal{X}$ defined by the sheaf of ideals $p^n O_{\mathcal{X}}$. In particular $\mathcal{X}_1 = X$. An invertible $O_{\hat{\mathcal{X}}}$-module is nothing but a sequence $(L_n)_{n \geqslant 1}$, where L_n in an invertible $O_{\mathcal{X}_n}$-module, plus isomorphisms $L_{n+1} \otimes O_{\mathcal{X}_n} \cong L_n$. Then the map $\mathrm{Pic}(\hat{\mathcal{X}}) \longrightarrow \mathrm{Pic}(X)$ is precisely $(L_n)_{n \geqslant 1} \rightsquigarrow L_1$. In order to see that this map is an isomorphism it will be sufficient to show that for each $n \geqslant 1$ the map of restriction $\mathrm{Pic}(\mathcal{X}_{n+1}) \longrightarrow \mathrm{Pic}(\mathcal{X}_n)$ is an isomorphism. But this follows from the standard exact sequence

$$o \longrightarrow p^n O_{\mathcal{X}}/p^{n+1} O_{\mathcal{X}} \cong O_X \longrightarrow O^*_{\mathcal{X}_{n+1}} \longrightarrow O^*_{\mathcal{X}_n} \longrightarrow 1 ,$$

which together with hypothesis ii) yields the assertion.

In particular there exists an invertible $O_{\mathcal{X}}$-module $\mathcal{L}$ such that $\mathcal{L} \otimes O_X \cong L$, and by [3], chap. III 4.7.1 $\mathcal{L}$ is ample. Moreover, from the exact sequence

$$\Gamma(\mathcal{X},\mathcal{L}) \longrightarrow \Gamma(X,L) \longrightarrow H^1(\mathcal{X},\mathcal{L}) \xrightarrow{\ p\ } H^1(\mathcal{X},\mathcal{L}) \longrightarrow H^1(X,L) = o$$

and Nakayama's lemma we deduce that the first map is surjective. In particular, σ lifts to a section $\tau \in \Gamma(\mathcal{X},\mathcal{L})$. Set $\mathcal{Y} = \mathrm{div}_{\mathcal{X}}(\tau)$ and $g = f/\mathcal{Y} : \mathcal{Y} \longrightarrow \mathrm{Spec}(A)$. Then the closed fibre of g is Y, and hence g is a smooth morphism. If Y' is the generic fibre of g, then $Y' = \mathrm{div}_{X'}(\tau/X')$ is a smooth surface in X'. By proposition 1 the map $\mathrm{Pic}(X') \longrightarrow \mathrm{Pic}(Y')$ is injective with cokernel a torsion-free group. In order to complete the proof of lemma 1 it will be therefore sufficient to show that there are isomorphisms $\mathrm{Pic}(X') \xrightarrow{\sim} \mathrm{Pic}(X)$ and

$Pic(Y') \xrightarrow{\sim} Pic(Y)$ making commutative the following diagram

$$\begin{array}{ccc} Pic(X') & \longrightarrow & Pic(Y') \\ \wr\downarrow & & \downarrow\wr \\ Pic(X) & \longrightarrow & Pic(Y) \end{array}$$

This fact is well known. For example we have firstly the isomorphism $Pic(Y') \xrightarrow{\sim} Pic(\mathcal{Y})$ defined by $[M] \rightsquigarrow [M']$, where M' is an invertible $O_{\mathcal{Y}}$-module such that $M'/Y' \cong M$. Such a M' always exists because $\mathcal{Y}$ is a regular scheme and Y' is an open subset in $\mathcal{Y}$. This definition is correct since the complement of Y' is Y and Y is defined as a closed subscheme of $\mathcal{Y}$ by the ideal $pO_{\mathcal{Y}}$, which is isomorphic as an $O_{\mathcal{Y}}$-module to $O_{\mathcal{Y}}$. Secondly, by the first part of the proof the natural map $Pic(\mathcal{Y}) \longrightarrow Pic(Y)$ is an isomorphism. Q.E.D.

<u>Lemma 2.</u> <u>Assume that $Y = P^2$ (resp. $Y = P^1 \times P^1$) is contained in the smooth projective variety X as an ample divisor, where k is an algebraically closed field of arbitrary characteristic. Then the map of restriction $Pic(X) \longrightarrow Pic(Y)$ is an isomorphism (resp. is injective and its cokernel is a torsion-free group).</u>

<u>Proof.</u> If char(k) = o this follows directly from proposition 1 taking into account (if $Y = P^2$) that $Pic(P^2) \cong \mathbb{Z}$. Assume therefore char(k) > o. Then the conclusion will follow from lemma 1 if we show that conditions i)-iv) are satisfied by $(X, L = O_X(Y), \sigma, div_X(\sigma) = Y)$. The verification of conditions ii), iii) and iv) are not difficult (using the explicit computation of the cohomology of P^2 and $P^1 \times P^1$ and the cohomological characterization of ampleness) and are left to the reader.

In order to verify condition i) it will be sufficient (using [6], exposé III, théorème 7.3) to show that $H^2(T_X) = o$ (the condition $H^2(O_X) = o$ being contained in ii)). Consider the exact sequence $(m \in \mathbb{Z})$

$$H^1(T_X\otimes O_X(mY)\otimes O_Y) \longrightarrow H^2(T_X\otimes O_X((m-1)Y)) \longrightarrow H^2(T_X\otimes O_X(mY)).$$

Since Y is ample on X, $H^2(T_X\otimes O_X(mY)) = o$ for $m \gg o$. Therefore in order to prove that $H^2(T_X) = o$ it will be sufficient (via descending induction on m) to see that

$$(6) \qquad H^1(T_X\otimes O_X(mY)\otimes O_Y) = o \quad \text{for every } m \geq 1.$$

Consider the exact sequence

$$(7) \qquad H^1(T_Y\otimes O_X(mY)) \longrightarrow H^1(T_X\otimes O_X(mY)\otimes O_Y) \longrightarrow H^1(O_X((m+1)Y)\otimes O_Y)$$

(induced by $o \longrightarrow T_Y \longrightarrow T_X\otimes O_Y \longrightarrow O_X(Y)\otimes O_Y \longrightarrow o$).

If $Y = P^2$ then $O_X(Y)\otimes O_Y = O(s)$ with $s > o$ (since Y is ample on X). Then $H^1(O_X((m+1)Y)\otimes O_Y) = H^1(P^2, O((m+1)s)) = o$ for every $m \in \mathbb{Z}$. On the other hand, the standard exact sequence on $Y = P^2$

$$o \longrightarrow O_Y \longrightarrow O(1)^{\oplus 3} \longrightarrow T_Y \longrightarrow o$$

yields the exact sequence of cohomology

$$o = H^1(O(ms+1)^{\oplus 3}) \longrightarrow H^1(T_Y\otimes O(ms)) \longrightarrow H^2(O(ms)) = o .$$

Therefore $H^1(T_Y\otimes O_X(mY)) = H^1(T_Y\otimes O(ms)) = o$. Now the exact sequence (7) proves (6) if $Y = P^2$.

If $Y = P^1\times P^1$ then $O_X(Y)\otimes O_Y = O(a,b)$ with $a > o$ and $b > o$. Then $H^1(O_X((m+1)Y)\otimes O_Y) = H^1(P^1\times P^1, O((m+1)a, (m+1)b)) = o$ for every $m \geq o$.

On the other hand, $T_Y \cong O(2,o)\oplus O(o,2)$, and therefore $H^1(T_Y\otimes O_X(mY)) = H^1(O(ma+2,mb))\oplus H^1(O(ma,mb+2)) = o$. Again the exact sequence (7) proves (6) if $Y = P^1\times P^1$. Q.E.D.

<u>Proposition 2.</u> <u>Assume that $Y = P^2$ is embedded as an ample divisor in the smooth projective variety X. Then X is isomorphic to P^3 and Y is contained in X as a hyperplane.</u>

<u>Proof.</u> If char(k) = o this result is contained in theorem 1. Thus we may

assume char(k) > o. Since $Pic(P^2) = \mathbb{Z}$ we may apply lemma 2 and deduce that the map $Pic(X) \longrightarrow Pic(Y)$ is an isomorphism. Now the argument is contained in the proof of theorem 1. Q.E.D.

Theorem 5. Assume that $Y = P^1 \times P^1$ is embedded in the smooth projective variety X as an ample divisor. Then we have one of the following possibilities:

i) $X \cong P^3$ and Y is a quadric in X.

ii) X is isomorphic to a hyperquadric in P^4 and Y is a hyperplane section.

iii) There are $a > o$, $b > o$, $c > o$ and $s > o$ positive integers such that $a+b+c = 2s$ and the exact sequence of O_{P^1}-modules

$$o \longrightarrow O_{P^1} \longrightarrow O(a) \oplus O(b) \oplus O(c) = E \xrightarrow{\ \varphi\ } O(s) \oplus O(s) = F \longrightarrow o$$

such that X is isomorphic to P(E) and $Y \cong P(F)$ is embedded in X via the surjection φ.

Proof. From lemma 2 we deduce that the map $Pic(X) \xrightarrow{\ \alpha\ } Pic(Y) \cong \mathbb{Z} \times \mathbb{Z}$ is injective and its cokernel is torsion-free. Thus we have two possibilities:

a) $Pic(X) \cong \mathbb{Z}$. Let L be an invertible O_X-module which is ample and generates Pic(X). Then $L \otimes O_Y \cong O(s,t)$ with $s > o$ and $t > o$. Since Coker(α) is torsion-free s and t are relatively prime integers. Writing $O_X(Y) \cong L^{\otimes r}$ and $\omega_X \cong L^{\otimes d}$, we get easily from the adjunction formula that $s(d+r) = t(d+r) = -2$, and thus $s = t = 1$.

Let $\sigma \in \Gamma(X, O_X(Y)) \cong \Gamma(X, L^{\otimes r})$ be such that $div_X(\sigma) = Y$. The exact sequence

$$o \longrightarrow L^{\otimes(m-r)} \xrightarrow{\ \sigma\ } L^{\otimes m} \longrightarrow O(m,m) \longrightarrow o$$

yields the exact sequence $(m \in \mathbb{Z})$

$$(8) \quad o \longrightarrow \Gamma(L^{\otimes(m-r)}) \xrightarrow{\ \sigma\ } \Gamma(L^{\otimes m}) \longrightarrow \Gamma(O(m,m)) \longrightarrow H^1(L^{\otimes(m-r)}) = o.$$

Put $S = \bigoplus_{m=o}^{\infty} \Gamma(L^{\otimes m})$; then $S/\sigma S \cong \bigoplus_{m=o}^{\infty} \Gamma(O(m,m))$ is a graded k-algebra generated by its part of degree one. On the other hand

$$(L^{\cdot 3})_X = 1/r.(L^{\cdot 2}.Y)_X = 1/r.(O(1,1).O(1,1))_Y = 2/r.$$

Therefore r = 2 or r = 1.

a_1) Case r = 2. If in (8) we take m = 1 we get $\dim\Gamma(L) = 4$. Since $(L^{\cdot 3})_X = 1$ theorem 3 implies $X = P^3$ and we get case i).

a_2) Case r = 1. Then $\deg(\sigma) = 1$ and since $S/\sigma S$ is generated by its homogeneous part of degree one, the same is true for S. In particular L is very ample. Again take m = 1 in (8) and get $\dim\Gamma(L) = 5$. Thus $\varphi_L : X \longrightarrow P(\Gamma(L)) \cong$ $\cong P^4$ and since $\deg\varphi_L(X) = 2$ we get case ii).

b) $Pic(X) \cong \mathbb{Z}\times\mathbb{Z}$. Then the map $Pic(X) \xrightarrow{\alpha} Pic(Y)$ is an isomorphism. Therefore there are two invertible O_X-modules L_1 and L_2 such that $L_1\otimes O_Y \cong O(1,o)$ and $L_2\otimes O_Y = O(o,1)$. If $O_X(Y)\otimes O_Y \cong O(s_1,s_2)$ with $s_1 > o$ and $s_2 > o$ (Y is ample on X), then since the map α is injective, $O_X(Y) \cong L_1^{\otimes s_1}\otimes L_2^{\otimes s_2}$. Let $\sigma\in\Gamma(O_X(Y)) \cong$ $\cong \Gamma(L_1^{\otimes s_1}\otimes L_2^{\otimes s_2})$ be a section such that $div_X(\sigma) = Y$. Then the exact sequence

$$o \longrightarrow O_X((m-1)Y) \xrightarrow{\sigma} O_X(mY) \longrightarrow O(ms_1,ms_2) \longrightarrow o$$

yields the exact sequence (exactly as in the proof of theorem 1)

(9) $$o \longrightarrow \Gamma(O_X((m-1)Y)) \xrightarrow{\sigma} \Gamma(O_X(mY)) \longrightarrow \Gamma(O(ms_1,ms_2)) \longrightarrow o.$$

Put $S = \bigoplus_{m=o}^{\infty}\Gamma(O_X(mY))$; then $\sigma\in S_1$ and $S/\sigma S \cong \bigoplus_{m=o}^{\infty}\Gamma(Y,O(ms_1,ms_2))$ is generated by its homogeneous part of degree one. Therefore S itself is generated by S_1 and hence Y is very ample on X. If in (9) we take m = 1 we get

(1o) $$\dim|Y| = (s_1+1)(s_2+1).$$

If $s_1 = s_2 = 1$ then $|Y| = P^4$ and X would be a smooth hypersurface in P^4. But then Lefschetz's theorem yields $Pic(X) \cong \mathbb{Z}$, a contradiction. Thus at least one s_1 or s_2 is > 1.

Suppose $s_1 > 1$. Then the exact sequence

$$o \longrightarrow L_1^{\otimes(1-s_1)}\otimes L_2^{(-s_2)} \xrightarrow{\sigma} L_1 \longrightarrow O(1,o) \longrightarrow o$$

yields the exact sequence

$$(11)\quad o\longrightarrow\Gamma(L_1^{\otimes(1-s_1)}\otimes L_2^{\otimes(-s_2)})\longrightarrow\Gamma(L_1)\longrightarrow\Gamma(O(1,o))\longrightarrow H^1(L_1^{\otimes(1-s_1)}\otimes L_2^{\otimes(-s_2)}).$$

Since $1-s_1<o$ and $-s_2<o$ we have $H^i(L_1^{\otimes(1-s_1)}\otimes L_2^{\otimes(-s_2)}) = o$ for $i\leqslant 1$. Indeed, $H^i(F\otimes O_X(mY)) = o$ for $i\leqslant 1$ and $m\ll o$ (with $F = L_1$), and from the exact sequence

$$o\longrightarrow F\otimes O_X((m-1)Y)\longrightarrow F\otimes O_X(mY)\longrightarrow O(ms_1+1,ms_2)\longrightarrow o$$

we deduce for every $m<o$ and $i\leqslant 1$:

$$H^i(F\otimes O_X((m-1)Y))\longrightarrow H^i(F\otimes O_X(mY))\longrightarrow H^i(O(ms_1+1,ms_2)).$$

By Künneth's formulae we get $H^i(O(ms_1+1,ms_2)) = o$ for $i\leqslant 1$ and $m<o$, and the affirmation results by induction on m.

Now recalling (11) we get that the map of restriction $\Gamma(L_1)\longrightarrow\Gamma(O(1,o))$ is an isomorphism if $s_1>1$. In particular, for every $\Delta,\Delta'\in|L_1|$ $(\Delta\neq\Delta')$ we have $\Delta\cap\Delta'\cap Y=\phi$. Since Y is ample on X, $\Delta\cap\Delta'$ is at most a finite set of closed points. Since X is smooth we cannot have $\Delta\cap\Delta'\neq\phi$ because otherwise

$$3 = \operatorname{codim}_X(\Delta\cap\Delta')\leqslant\operatorname{codim}_X(\Delta)+\operatorname{codim}_X(\Delta') = 1+1 = 2.$$

Therefore $\Delta\cap\Delta'=\phi$. Thus the linear system $|L_1|$ has no base points and hence the corresponding map $p = \varphi_{L_1}:X\longrightarrow|L_1|= P^1$ (such that $p^*O_{P^1}(1)\cong L_1$) is a morphism. Moreover, for every invertible O_X-module L, $(L_1^{\cdot 2}.L) = o$.

Now look at the equalities

$$1 = (O(1,o).O(o,1))_Y = (L_1.L_2.Y)_X = s_1(L_1^{\cdot 2}.L_2)+s_2(L_1.L_2^{\cdot 2}).$$

One deduces $s_2(L_1.L_2^{\cdot 2}) = 1$, i.e. $s_2 = 1$ and $(L_1.L_2^{\cdot 2}) = 1$. Set $s_1 = s$.

Let $\Delta\in|L_1|$ be arbitrary. Then $(O_X(Y)^{\cdot 2}.\Delta) = s^2(L_1^{\cdot 3})+2s(L_1^{\cdot 2}.L_2)+(L_1.L_2^{\cdot 2}) = (L_1.L_2^{\cdot 2}) = 1$. Therefore, denoting by $M = O_X(Y)\otimes O_\Delta$, we get $(M^{\cdot 2})_\Delta = 1$, M is ample on Δ and Δ is a Cohen-Macaulay scheme of pure dimension 2. Moreover,

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for every $i = o, 1, \ldots, s-1$ one has the exact sequence (since $L_1 \otimes O_\Delta \cong O_\Delta$)

$$o \longrightarrow L_1^{\otimes(s-i-1)} \otimes L_2 \longrightarrow L_1^{\otimes(s-i)} \otimes L_2 \longrightarrow M \longrightarrow o$$

and hence

$$o \longrightarrow \Gamma(L_1^{\otimes(s-i-1)} \otimes L_2) \longrightarrow \Gamma(L_1^{\otimes(s-i)} \otimes L_2) \longrightarrow \Gamma(M).$$

<u>Claim.</u> $\mathrm{Dim}\,\Gamma(M) \geqslant 3$.

Indeed, assuming the contrary we get

$$2 \geqslant \dim \Gamma(L_1^{\otimes(s-i)} \otimes L_2) - \dim \Gamma(L_1^{\otimes(s-i-1)} \otimes L_2), \quad i = o, 1, \ldots, s-1,$$

and therefore taking the sum:

(12) $\qquad 2s \geqslant \dim \Gamma(O_X(Y)) - \dim \Gamma(L_2)$.

But the exact sequence

$$o \longrightarrow L_1^{\otimes(-s)} \xrightarrow{\ \sigma\ } L_2 \longrightarrow O(o,1) \longrightarrow o$$

yields

$$o = \Gamma(L_1^{\otimes(-s)}) \longrightarrow \Gamma(L_2) \longrightarrow \Gamma(O(o,1))$$

and thus $\dim \Gamma(L_2) \leqslant 2$. Therefore (12) becomes $\dim \Gamma(O_X(Y)) \leqslant 2(s+1)$, or else $\dim |Y| \leqslant 2s+1$, which contradicts (1o). The claim is proved.

By theorem 3 we deduce then that $\Delta \cong P^2$ and $O_\Delta(1) \cong L_2 \otimes O_\Delta$. Now Hironaka has shown that in these circumstances p is the projection of the projective bundle $P(E)$ associated to a locally free O_{P^1}-module E of rank 3 (see [9] theorem (1.8)). Moreover $O_X(Y) \otimes O_\Delta \cong L_1^{\otimes s} \otimes L_2 \otimes O_\Delta \cong L_2 \otimes O_\Delta \cong O_\Delta(1)$, and therefore we can take $E = p_* O_X(Y)$. Then $O_{P(E)}(1) = O_X(Y)$ and the exact sequence

$$o \longrightarrow O_X \longrightarrow O_X(Y) \longrightarrow O(s,1) \longrightarrow o$$

yields

$$o \longrightarrow p_* O_X \cong O_{P^1} \longrightarrow p_* O_X(Y) = E \longrightarrow p_{1*} O(s,1) \cong O(s) \oplus O(s) \longrightarrow R^1 p_* O_X = o,$$

where $p_1 : P^1 \times P^1 \longrightarrow P^1$ is the first projection. In other words we get the exact sequence of locally free O_{P^1}-modules

$$o \longrightarrow O_{P^1} \longrightarrow E \longrightarrow O(s) \oplus O(s) \longrightarrow o.$$

In particular $\deg(E) = 2s$. By a theorem of Grothendieck (see [4] for $k = \mathbb{C}$, but the same result holds in arbitrary characteristic) there are three integers a, b, c (uniquely determined up to a permutation) such that $E \cong O(a) \oplus O(b) \oplus O(c)$. Finally, since $O_X(Y)$ is ample on X, E is ample on P^1, and therefore $a > o$, $b > o$ and $c > o$. In other words we get situation iii). Q.E.D.

Remarks. 1) The case iii) of theorem 5 really occurs. Indeed, we shall construct an exact sequence as in case iii) with $c = s$, i.e. with $a+b = s$ ($a > o$, $b > o$ and $c > o$). It will be sufficient to construct a surjection of the form $\varphi' : O(a) \oplus O(b) \longrightarrow O(a+b) = O(s)$, because one can take $\varphi = \varphi' \oplus \mathrm{id}_{O(s)}$ (and then taking the degrees one sees that $\mathrm{Ker}(\varphi) \cong O_{P^1}$). Let x_o and x_1 homogeneous coordinates on P^1 and define $\varphi'(p,q) = x_o^b p + x_1^a q$. We claim that $\Gamma(\varphi') : \Gamma(O(a)) \oplus \Gamma(O(b)) \longrightarrow \Gamma(O(a+b))$ is surjective. For, if $u \in \Gamma(O(a+b)) = k[x_o, x_1]_{a+b}$ is of the form $u = \sum_{i=0}^{a+b} a_i x_o^i x_1^{a+b-i}$, then $u = x_o^b p + x_1^a q$, where $p = \sum_{i=0}^{a-1} a_i x_o^i x_1^{a-i} \in \Gamma(O(a))$ and $q = \sum_{i=a}^{a+b} a_i x_o^{i-a} x^{a+b-i} \in \Gamma(O(b))$. Now since $\Gamma(\varphi')$ is surjective and $O(a+b)$ is generated by its global sections, φ' is also surjective (and thus φ is surjective).

2). Note that the theorem asserting that P^n is the unique smooth projective variety containing P^{n-1} ($n \geq 3$) as an ample divisor was known for $n \geq 4$ and char(k) arbitrary, and for $n = 3$ and char(k) $\neq 3$ (see [12]).

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