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April 1980

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CONTENTS

1. INTRODUCTION

It is the purpose of this paper to treat the problem of flow-invariance of a set with respect to the autonomous second order differential equation (2.4)(or, to the force field f) and then to give several applications in Mechanics. From the point of view of applications, this theory is important at least for two reasons.

The first one consists in the fact that it unifies many classical results of Mechanics. Indeed, by flow-invariance method (via Theorem 2.4) we can prove e.g. Bonnet's theorem, the equivalence of the first two laws of Kepler to the law of force varying inversely as the square of the distance conceived by Newton, as well as:

A mass particle moving under a central field of force describes an orbit wich lies in a plane. (see Theorem 5.6)

Furthermore, via this method one can determine all "g-smooth" field of force (in the sense of Definition 5.2) under which a given orbit can be described (Theorem 5.3). In this manner the results of Dainelli are derived (Remark 5.6). Finally, the method allows to get several geometrical properties of some particular flow-invariant sets (like the curves whose curvature is different from zero).

The second reason is the following:

This theory allows us to prove that some mathematical characterizations of the motion in R^3 or R^2 remain valid in any $R^n(n>3)$ and even more-in any real Hilbert or Banach space X (see e.g. Theorem 5.4, Remark 5.6 and Theorem 5.2 which asserts that the set of all g-smooth force fields under which D_g can be described is a convex cone). Therefore Theorem 5.2 is a generalization of Bonnet's theorem from R^3 to the real Banach space X. The paper is based on a result (Theorem 2.1) on flow-invariance of a set with respect to the first order differential equation (2.1). In a slightly different form (see Remark 2.1), Theorem 2.1 has been proved by Nagumo [14] and independently by Brezis [4].

A significant generalization of Theorem 2.1 is given by Martin [12]. This result of Martin is extended to the time-dependent closed subsets, in [24]. On this subject the bibliography indicates references to which the interested reader may turn.

This bibliography (e.g. [1], [3], [6], [7], [10], [12], [16-25]) shows that actually, the problem of flow-invariance of a set came to the atention of the people interested in it, after Brezis' paper [4]. The first major application of flow-invariance theory is that given by Bourgouignon and Brezis [3] to the study of Euler's equation (which describes the motion of an incompressible perfect fluid.). Such a theory has been applied also by Abraham and Marsden [1] to Hamiltonian systems as well as in [20], to get necessary and sufficient conditions for the existence of positive solutions. The proof of our results is based on the theory of "tangent sets". This efficient technique has also been used in optimization and control theory. For example, using such a technique, Clarke [6] and Ursescu [27] have obtained significant extentions of the Pontryagin maximum principle. The idea of the second section goes back to [16] and [21]. The core of the section is the introduction of the subset M_D (by (2.5)). It is shown that for the existence of a D-valued solution u of (2.4) corresponding to the initial dates $u(o) = x \in D$ and $u'(o) = y \in X$, it is necessary that $(x,y) \in M_D$ (but this is not sufficient). Then the following question arises:

What is the best condition we must add to the hypothesis (x,y)

$\in M_D$,in order that the corresponding solution u be D-valued (i.e. $u(t) \in D$ as long as it exists)?

Such a condition is (2.6)(call it "the invariance condition of D under f").It consists in the requirement that for each $(x,y) \in M_D$, $(y,f(x))$ be "tangent" to M_D at (x,y).In the case of $D = D_g$ given by (2.7),the "invariance condition of D_g under f" becomes (2.13).This equation (2.13),in important particular cases (Remark 2.2,Remark 5.8) is a partial differential equation in f,which allows the determinatio of all g-smooth force field f,under which D_g can be described (see (5.28),(5.82)).Therefore the main results of this section are:the way in which the notion of flow-invariance of D is defined ((2.5) and Definition 2.3),Theorem 2.4 as well as its consequences.

In a slightly different form,M_D and Theorem 2.4 has been presented in [21] ,but not too efficiently.For applications,the consequences of Theorem 2.4 are useful (as it is shown in §5).

In the third section,several preliminaries on tangent sets are discussed.Such results are used in the proof of those of second section.Some of them seem to be new (e.g.Theorem 3.2,Corollary 3.2,Lemma 3.5,see also Remark 3.2).

In § 4,the results of second section are proved.

Applications of the results of second sections are given in § 5. Some of these applications has already been discussed at the begining. In addition we want to point out that (in view of Theorems 5.5 and 5.6) any conic is a flow-invariant set under the Newtonian field of force.Therefore,via flow-invariance method one obtains a qualitative explanation (based on the second law of Dynamics) of the motion of planets as well as of the launching of man-made satellites (Theorem 5.6,Corollary 5.4).Note also that the formula

(1.1) $y^2 = -\frac{\dot{g}(x)(f(x))}{c(x)\|\dot{g}(x)\|}$ (c(x)- the curvature of D_g at $x \in D_g$)

appearing in (2.24), is the square magnitude of the speed y of "projection" from $x \in D_g$. More precisely, if f satisfies the invariance condition (2.25) (or (2.27)) of D_g, then a mass particle projected from any point $x \in D_g$ in the tangent direction to D_g at x, with the speed y of square norm (1.1), describes (under the action of the force field f) an orbit which lies in D_g. The formula (1.1) contains in particular, the well-known cosmic speeds (see 5.47), (5.51), (5.63) and perhaps is classical too.

The case of the inverse square repulsion (important in physics at the bombardament of atoms by α- particles) is included in this theory, too (see (5.65) and Theorem 5.5). Finally, in the last section 6, some open problems and suggestions for further study are given.

2. STATEMENT OF THE MAIN RESULTS

Let X be a real Banach space of norm $\|\cdot\|$. Throughout this paper $A \subset X$ is a nonempty open subset of X and f: $A \to X$ - a locally Lipschitz function (i.e. for each $x \in A$, there is a neighborhood of x, on which f is Lipschitz). Let us consider the first order autonomous differential equation:

(2.1) $u'(t) = f(u(t)), t \geqslant 0.$

Let D be a nonempty subset of A.

The following definition is well-known

Definition 2.1 The set $D \subset A$ is said to be a "flow-invariant set" for the equation (2.1) if every solution u: $[0,T) \to A$ of (2.1) with $u(o) \in D$, verifies $u(t) \in D$ for all $t \in [0,T)$.

In other words, D is said to be a flow-invariant set for (2.1), if every solution of (2.1) starting from D, remains in D as long as it exists.

Definition 2.2. The nonempty set D is said to be closed in A if $D = \overline{D} \cap A$, where $\overline{D}$ denotes the closure of D.

Obviously, if $D \subset A$ and $D = \overline{D}$, then D is closed in A (the converse statement is not true).

For $z \in X$, denote

(2.2) $d[z;D] = \inf\{ \|z - x\| \; ; \; x \in D\}$.

It is easy to check that

(2.2)' $\left| d[z;D] - d[w;D] \right| \leq \|z - w\| \quad , \forall z,w \in X$

The following theorem is a consequence of a result of Nagumo [14] (independently considered by Brezis [4] and generalized by Martin [12]. See also [24]).

Theorem 2.1 Assume that D is closed in A. Then D is a flow-invariant set for (2.1) iff :

(2.3) $\lim\limits_{h \downarrow 0} \frac{1}{h} d[x + hf(x);D] = 0, \quad \forall x \in D.$

Remark 2.1 (2.3) means that for each $x \in D$, $f(x)$ is "tangent" to D at x. Theorem 2.1 holds even f is defined (and locally Lipschitz) on D only. In this case Theorem 2.1 is in fact a result of existence rather than a problem of invariance. If $D \subset A$ is closed (in X) and f is globally Lipschitz on D, then "the tangent condition"(2.3), assures the existence (and uniqueness) of solution u to (2.1), on $[0,+\infty)$ (i.e. $u: [0,+\infty) \to D$, $u(0) = x \in D$. See Martin [12]).

In what follows we are concerned with the notion of flow-invariant set for the autonomous second order differential equation

(2.4) $u''(t) = f(u(t)), t \geqslant 0$

We shall give necessary and suficient conditions in order for the solution to the Cauchy problem of (2.4)(determined by the initial condition $u(o) = x, u'(o) = y, x \in D$) to remain in D as long as it exists. To do that we need the following set

$$(2.5) \qquad M_D = \left\{ (x,y) \in A \times X \ ; \ \lim_{h \downarrow o} \frac{1}{h^2} d\left[x + hy + \frac{h^2}{2} f(x); D\right] = 0 \right\}$$

A necessary condition for the existence of a solution $u: [0,T) \to D$ to the equation (2.4) is that $(u(o), u'(o)) \in M_D$. This condition is not sufficient (as we shall see in Theorem 2.4 or Theorem 5.4). A first result in this direction is givem by

Theorem 2.2 *Let $u: [0,T) \to A$ be a solution of the equation* (2.4)

(i) *If* $u(t) \in D$ *for all* $t \in [0,T)$, *then* $(u(t), u'(t)) \in M_D$ *for all* $t \in [0,T)$.

(ii) *If* D *is closed in* A, *then* $u(t) \in D$ *for all* $t \in [0,T)$ *iff* $(u(t), u'(t)) \in M_D$ *for all* $t \in [0,T)$.

The proof of this theorem (as well as of all results of this section) is given in section 4. The following definition of a flow-invariant set for (2.4) is now quiet natural.

Definition 2.3 *The set* D *is said to be a flow-invariant set for* (2.4) *if* M_D *is nonempty and if for every solution* $u: [0,T) \to A$ *of* (2.4) *with* $(u(o), u'(o)) \in M_D$, *we have* $(u(t), u'(t)) \in M_D$ *for all* $t \in [0,T)$.

The result below justifies, both the introduction of M_D and the Definition 2.3.

Theorem 2.3 *Assume that* D *is closed in* A. *Then* D *is a flow-invariant set for* (2.4) *iff* M_D *is nonempty and every solution* $u: [0,T) \to A$ *of* (2.4) *with* $(u(o), u'(o)) \in M_D$ *is* D-*valued* (i.e. $u(t) \in D$ *for all* $t \in [0,T)$).

A direct consequence of Theorem 2.3 is given by

Corollary 2.1 *Let* $D_i, i \in I$ *be a family of nonempty (closed in A) sets with each* D_i-*flow-invariant set for* (2.4) *where* I *is a nenempty* set of indexes).Assume in addition that

$D = \bigcap_{i\in I} D_i$ is nonempty too.Then

(*) $M_D = \bigcap_{i\in I} M_{D_i}$

(2) Moreover,D is flow-invariant set for (2.3) iff M_D is a nonempty set.

We now state one of the main results of this paper.

Theorem 2.4 Assume that M_D is closed in $A\times X$.Then D is a flow-invariant set for the equation (2.4) iff M_D is nonempty and

(2.6) $$\lim_{h\downarrow 0} \frac{1}{h} d\left[(x,y) + h(y,f(x));M_D\right] = 0, \ \forall (x,y)\in M_D.$$

(i.e iff for each $(x,y)\in M_D$,$(y,f(x))$ is "tangent" to M_D at (x,y)).

A first simple example of applications of the above result is the following

Theorem 2.5. Let S be a closed linear subspace of X and $D=A\cap S$. Then

(i) $M_D = (D\cap f^{-1}(S))\times S$

(ii) Assume in addition that $f(D)\subset S$.Then $M_D = D\times S$ and D is a flow-invariant set for (2.4)

(As usual,$f^{-1}(S) = \{z\in X;f(z)\in S\}$ and $f(D) = \{f(x);x\in D\}$).

Next we are interested to examine (2.5) and (2.6) in significant particular cases.

Let Y be a real normed space (who's norm is denoted by $\|\cdot\|$,too). We shall give some consequences of Theorem 2.4 in the case which

(2.7) $$D = D_g = \{x\in A, g(x) = 0\}$$

where $g:A\to Y$ is a function.

For this purpose we need some elements of Fréchet differential calculus.Such elements can be found in Cartan's book ([5] Ch.I,§5). However,let us recall here some basical aspects in Fréchet differentiability (strictly necessary in what follows)

The function $g:A\to Y$ is said to be Fréchet differentiable at $x\in A$ if there is a linear continuous function (say $\dot{g}(x)$) from X into Y such that:

$$(2.8)\quad \lim_{\substack{y\to 0\\ y\neq 0}} \frac{1}{\|y\|}\|g(x+y)-g(x)-\dot{g}(x)(y)\| = 0$$

(All over the paper the differentiability of a function is considered in Fréchet sense, only).

The function g is said to be differentiable on A if it is differentiable at every point $x\in A$.

By definition, $\dot{g}:A\to L(X,Y)$ (the space of all linear continuous operators from X into Y, endowed with the standard linear normed structure). Similarly, $\dot{g}$ is said to be differentiable at $x\in A$ if there is a linear continuous function (call it $\ddot{g}(x)$) from X into $L(X,Y)$, such that

$$(2.9)\quad \lim_{\substack{y\to 0\\ y\neq 0}} \frac{1}{\|y\|} \|\dot{g}(x+y)-\dot{g}(x)-\ddot{g}(x)(y)\| = 0$$

where $\|z\|$ is the norm in $X(L(X,Y))$ if $z\in X$ (resp. $z\in L(X,Y)$). g is said to be twice differentiable on A if both g and $\dot{g}$ are differentiable at any $x\in A$.

Therefore, $\ddot{g}:A\to L(X,L(X,Y))$. Inductively one defines $\dddot{g}:A\to L(X,L(X,L(X,Y)))$ a.s.o. We shall use also the following consequence of Taylor formula (when g is twice differentiable on $x\in A$).

$$(2.10)\quad \lim_{\substack{y\to 0\\ y\neq 0}} \frac{1}{\|y\|^2} \|g(x+y)-g(x)-\dot{g}(x)(y)-\tfrac{1}{2}\ddot{g}(x)(y)(y)\| = 0$$

Finally, by $C^k(A,Y)$ we mean (as usual) the set of all k-times differentiable functions $g:A\subset X\to Y$, with $g^{(k)}$ (the derivative of k-order) continuous on A.

In the sequel we shall suppose that $Y = R^n$ (the Euclidian n-space) and that the function $w: A \to R^n$ given by

(2.11) $w(x) = \dot{g}(x)(f(x)), x \in A$

is differentiable on A. For application in mechanics the following three results are useful

Theorem 2.6 Assume that the function $u \to \dot{g}(x)(u)$ from X into R^n is surjective (for each $x \in D_g$) and g is twice differentiable on A. Then

(2.12) $M_{D_g} = \{(x,y) \in A \times X; g(x) = 0, \dot{g}(x)(y) = 0, \ddot{g}(x)(y)(y) + \dot{g}(x)(f(x)) = 0\}$

(i.e. M_D given by (2.5) has the form (2.12)).

Assume in addition that g is three times differentiable on A, w (given by (2.11)) is differentiable on A, M_{D_g} is nonempty and the function $u \to (\dot{g}(x)(u), \ddot{g}(x)(y)(u))$ from X into $R^n \times R^n$ is surjective, for each $(x,y) \in M_{D_g}$. Then D_g is a flow invariant set for (2.4) iff

(2.13) $\dddot{g}(x)(y)(y)(y) + 2\ddot{g}(x)(f(x))(y) + \dot{w}(x)(y) = 0, \forall (x,y) \in M_{D_g}$.

Furthermore we consider the case n = 1 (i.e. Y = R - the real axis)

Theoreme 2.7 (1) Let $g: A \subset X \to R$ be twice differentiable and $w: A \to R$ (given by (2.11)) satisfying the condition $w(x) \neq 0$ for all $x \in D_g$. Then (2.12) holds.

(2) Assume in addition that g is three times differentiable on A, w is differentiable on A, and M_{D_g} is nonempty. Then D_g is flow-invariant for (2.4) iff (2.13) holds.

(3) Moreover, for each $x, \bar{y} \in X$ with the properties:

(2.14) $g(x) = 0, \dot{g}(x)(\bar{y}) = 0, \dot{g}(x)(f(x))\ddot{g}(x)(\bar{y})(\bar{y}) < 0$,

the pair $(x,y) \in M_{D_g}$ where

(2.15) $$y = \sqrt{-\frac{\dot{g}(x)(f(x))}{\ddot{g}(x)(\bar{y})(\bar{y})}}\ \bar{y}.$$

We now consider the case $X = R^2$, $Y = R$. Let us precise several notations.

$$(2.15)' \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2, f(x) = \begin{pmatrix} f^1(x_1,x_2) \\ f^2(x_1,x_2) \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; y^2 = y_1^2 + y_2^2 = \|y\|^2$$

$$(2.16) \begin{cases} g_i = g_i(x) = \dfrac{\partial g}{\partial x_i}(x), g_{ij} = g_{ij}(x) = \dfrac{\partial^2 g}{\partial x_i \partial x_j}(x), g_{ijk} = \\ \qquad = g_{ijk}(x) = \dfrac{\partial^3 g}{\partial x_i \partial x_j \partial x_k}(x) \\ w_i = w_i(x) = \dfrac{\partial w}{\partial x_i}(x), i,j,k = 1,2. \end{cases}$$

In this case it is well-known that

$$(2.17) \begin{cases} g(x) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} (\text{i.e.}\, \dot g(x) = \text{grad } g(x)), \ddot g(x) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \\ \dot g(x)(y) = \langle g(x), y \rangle = g_1 y_1 + g_2 y_2, \dot g(x)(\dot g(x))^{\perp} = 0, \dot g(x)^{\perp} = \begin{pmatrix} g_2 \\ -g_1 \end{pmatrix} \end{cases}$$

For the sake of simplicity, denote

$$(2.18) \begin{cases} a_1 = a_1(x) = g_1^2 + g_2^2 (\text{i.e.}\, a_1(x) = \|\dot g(x)\|^2 = \|\dot g(x)^{\perp}\|^2) \\ a_2 = a_2(x) = g_{11}g_2^2 - 2g_{12}g_1g_2 + g_{22}g_1^2 \\ a_3 = a_3(x) = g_{111}g_2^3 - 3g_{112}g_1 g_2^2 + 3g_{122}g_1^2 g_2 - g_{222}g_1^3 \end{cases}$$

where g_i^2 is the square of the number $g_i(x), i = 1,2.$

Obviously

$$(2.19) \quad \ddot g(x)(y)(y) = (y_1, y_2) \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = g_{11}y_1^2 + 2g_{12}y_1y_2 + g_{22}y_2^2 \; (\text{if } g_{12} = g_{21})$$

$$(2.20) \quad \dddot g(x)(y)(y)(y) = \sum_{i,j,k=1}^{2} g_{ijk}y_iy_jy_k$$

Therefore with $y = \dot g(x)^{\perp}$, it follows

$$(2.21) \quad a_2 = \ddot g(x)(\dot g(x)^{\perp})(\dot g(x)^{\perp}), a_3 = \dddot g(x)(\dot g(x)^{\perp})(\dot g(x)^{\perp})(\dot g(x)^{\perp})$$

Denote by $c(x)$ the curvature of D_g at x. It is well-known that

$$(2.22) \quad c(x) = \frac{a_2(x)}{(a_1(x))^{3/2}},$$

From Theorem 2.7 we can derive a result with a unifying effect in the theory of flight space, namely (with the above notations):

Theorem 2.8 *Assume that* $g: A \subset R^2 \to R$ *is twice differentiable and*

$$(2.23)\quad \dot{g}(x)(f(x))\cdot\ddot{g}(x)(y)(y) < 0,$$

for all (x,y) *with* $g(x) = 0$ *and* $\dot{g}(x)(y) = 0, y \neq 0$.
Then, M_{D_g} *given by* (2.12) *has the form*

$$(2.24)\quad M_{D_g} = \left\{ (x,y) \in A \times R^2; g(x) = 0, g_1 y_1 + g_2 y_2 = 0, y^2 = -\frac{\dot{g}(x)(f(x))}{c(x)\|\dot{g}(x)\|} \right\}$$

and for each $x \in D_g$ *there is* $y \in R^2$ *such that* $(x,y) \in M_{D_g}$. *Assume in addition that* g *is three times differentiable on* A *and* $w: A \subset R^2 \to R$ *(given by* (2.11)) *is differentiable on* A. *Then* D_g *is flow-invariant for* (2.4) *iff*

$$(2.25)\quad -\dot{g}(x)(f(x))\frac{a_3(x)}{a_2(x)} + 2(g_{11}g_2 - g_{12}g_1)f^1(x) + 2(g_{21}g_2 - g_{22}g_1)f^2(x) + w_1 g_2 - w_2 g_1 = 0$$

for all x *with* $g(x) = 0$

Remark 2.2 If $f: A \subset R^2 \to R^2$ is differentiable on A, the invariance condition (2.13) becomes obviously

$$(2.26)\quad \dddot{g}(x)(y)(y)(y) + 3\ddot{g}(x)(f(x))(y) + \dot{g}(x)(\dot{f}(x)(y)) = 0$$

since

$$\dot{w}(x)(y) = \ddot{g}(x)(f(x))(y) + \dot{g}(x)(\dot{f}(x)(y)) .$$

In this case (2.25) can be rewritten under the form

$$(2.27)\quad -(g_1 f^1 + g_2 f^2)\frac{a_3(x)}{a_1(x)} + 3(g_{11}g_2 - g_{12}g_1)f^1 + 3(g_{21}g_2 - g_{22}g_1)f^2 + g_1 g_2\left(\frac{\partial f^1}{\partial x_1} - \frac{\partial f^2}{\partial x_2}\right) + g_2^{\,2}\frac{\partial f^2}{\partial x_1} - g_1^2\frac{\partial f^1}{\partial x_2} = 0$$

for all x with $g(x) = 0$

Indeed, this partial differential equation in f^i is a direct consequence of (2.25), in which $w_i (i=1,2)$ is replaced by the expressions

(2.29) below.

First of all, in this case (2.11) is the following function

(2.28) $w(x)=\langle\dot{g}(x),f(x)\rangle=g_1f^1+g_2f^2$

Consequently,

(2.29) $\dot{w}_i=\dot{w}_i(x)=\frac{\partial w}{\partial x_i}=g_{1i}f^1+g_{2i}f^2+g_1\frac{\partial f^1}{\partial x_i}+g_2\frac{\partial f^2}{\partial x_1},i=1,2.$

(2.30) $\dot{w}(x)(\dot{g}(x)^{\perp})=\langle\dot{w}(x),\dot{g}(x)^{\perp}\rangle=w_1g_2-w_2g_1.$ ∎

3. PRELIMINARIES ON TANGENT SETS

For the proof of the results stated in section 2 it is necessary to present some aspects on tangent sets. We shall use the notations of the previous section.

Let us consider the conditions:

(3.1) $$\lim_{h\downarrow 0}\frac{1}{h}d[x+hy;D]=0\ ,$$

(3.2) $$\lim_{h\downarrow 0}\frac{1}{h^2}d\left[x+hy+\frac{h^2}{2}z;D\right]=0$$

where D is a nonempty subset of the real Banach space X and $x,y,z\in X$.

Using (2.2)' it follows

$$\frac{1}{h}\left|d\left[x+hy+\frac{h^2}{2}z;D\right]-d[x+hy;D]\right|\leq\frac{h}{2}\ \|z\|$$

Which shows that (3.2) implies (3.1)

The set of all y satisfying (3.1) is said to be a "tangent set" to D at x.

In some particular cases (e.g. when D is a smooth set) the tangent set is just the tangent space to D at x (in classical sense). On this subject we refere to Ursescu [27-29] .

Let E be a nonempty subset of real normed space $Y, A\subset X$ an open subset of X and $g:A\to Y$ a differentiable function at $x\in A$.

We now introduce the condition

$$(3.3)\qquad \lim_{h\downarrow 0}\frac{1}{h}\, d[g(x)+h\dot g(x)(y);E]=0$$

and, if g is twice differentiable at $x\in A$,

$$(3.4)\qquad \lim_{h\downarrow 0}\frac{1}{h^2}\, d[g(x)+h\dot g(x)(y)+\frac{h^2}{2}(\ddot g(x)(y)(y)+\dot g(x)(z));E]=0$$

The key of the proof of the results from section 2 is given by the next two theorems.

Theorem 3.1 If $D=g^{-1}(E)$ and g is differentiable at $x\in A$ then (3.1) implies (3.3). If in addition to the above hypothesis we assume that g is continuous on A, $u\to\dot g(x)(u)$ from X into Y is surjective and Y is finite dimensional, then (3.3) implies (3.1) (therefore in this case (3.1) and (3.3) are equivalent).

Theorem 3.2 If $D=g^{-1}(E)$ and g is twice differentiable at $x\in A$, then (3.2) implies (3.4). If in addition, $u\to\dot g(x)(u)$ from X into Y is surjective and Y is finite dimensional, then (3.2) is equivalent to (3.4).

For the proof of these theorems, the following lemmas are useful.

Lemma 3.1. The condition (3.1) is equivalent to each of the statements (3.5), (3.6) below

(3.5) For every $\varepsilon>0$, there is $\delta=\delta(\varepsilon)>0$, such that for each $h\in(0,\delta)$ there exists $u=u(\varepsilon,h)\in X$ satisfying $\|u\|<\varepsilon$ and $x+h(y+u)\in D$

(3.6) For each $h>0$ there is $u(h)\in X$ such that $u(h)\to 0$ as $h\downarrow 0$ and $x+h(y+u(h))\in D$.

Lemma 3.2 Each of the statements (3.7) and (3.8) are equivalent to (3.2), where

(3.7) For every $\varepsilon>0$, there is $\delta>0$ such that for each $h\in(0,\delta)$, there is $u=u(\varepsilon,h)\in X$ with the properties

$$\|u\|<\varepsilon \text{ and } x+hy+\frac{h^2}{2}(z+u)\in D$$

(3.8) For each $h>0$, there is $u(h)\in X$ such that $u(h)\to 0$ as $h\downarrow 0$ and $x+hy+\frac{h^2}{2}(z+u(h))\in D$

In the cartesian product $X\times X$, the last part of Lemma 3.1 becomes

Lemma 3.3 Let M be a nonempty subset of $X\times X$. For each $(x,y)\in M$ and $(z,w)\in X\times X$, the following conditions (i) and (ii) are equivalent

(i) $$\lim_{h\downarrow 0}\frac{1}{h}\, d\big[(x,y)+h(z,w);M\big]=0$$

(ii) For each $h>0$, there exist $r_j(h)\in X$ $(j=1,2)$ with $r_j(h)\to 0$ as $h\downarrow 0$, such that

$$(x+h(z+r_1(h)),y+h(w+r_2(h)))\in M$$

The proof of these lemmas is elementary and can be found e.g. in [21], so we omit it. A second set of results establish some consequences of the relations (2.8) and (2.9). First of all let us consider (2.8).

If $g\colon A\subset X\to Y$ is differentiable at $x\in A$ then from (2.8) it follows that

(3.9) $$\lim_{h\downarrow 0}\frac{1}{h}\,(g(x+hy)-g(x)-h\dot g(x)(y))=0,$$

uniformly with respect to y from bounded subsets of X.

Replacing (in 3.9)) y by $y+u$, one obtains

(3.10) $$\lim_{\substack{h\downarrow 0\\ u\to 0}}\frac{1}{h}\,(g(x+h(y+u))-g(x)-h\dot g(x)(y))=0$$

The lemma below states a reciprocal (in a certain sense) relation to (3.10), namely.

Lemma 3.4 Assume that g is continuous on A differentiable at $x\in A$, $u\to\dot g(x)(u)$ from X into Y is surjective and Y is finite dimensional. Then for every $\varepsilon>0$ there is $\delta>0$ such that for each $h\in(0,\delta)$ and $v\in Y$ with $\|v\|<\delta$, there is $u\in X$ with the properties:

(3.11) $\|u\|<\varepsilon$, $x+h(y+u)\in A$, $g(x+h(y+u))=g(x)+h(\dot g(x)(y)+v)$

In the proof of Lemma 3.4, the following simple result is needed

Lemma 3.5 Let $L\colon X\to Y$ be a linear surjective operator. If Y is finite dimensional, then there is a linear continuous operator $l: Y \to X$ such that

(3.12) $L(l(y)) = y$, for all $y \in Y$.

Proof. Let $e_1, \ldots, e_n$ be a basis for Y. Choose $x_i \in X$ such that $L(x_i) = e_i$, $i=1,2,\ldots,n$. If y is arbitrary in Y, there exists $a_i \in R$, $i=1,\ldots,n$ such that $y = \sum_{i=1}^{n} a_i e_i$. With $l: Y \to X$ given by $l(y) = \sum_{i=1}^{n} a_i x_i$, the assertion of lemma is proved.

<u>Proof of Lemma</u> 3.4 Since $\dot{g}(x)$ is surjective, by Lemma 3.5 (with $L = \dot{g}(x)$) there is a linear continuous operator $l: Y \to X$, such that

(3.13) $\dot{g}(x)(l(w)) = w$, for all $w \in Y$.

Let $\varepsilon > 0$ arbitrary. Inasmuch as $l(o) = 0$ and l is continuous at 0, there is $r > 0$ such that

(3.13)' $\|l(w)\| < \varepsilon$, for all $w \in B(r) = \{w \in Y; \|w\| \leq r\}$.

According to (3.9), there is $\delta = \delta(\varepsilon) > 0$, such that

(3.14) $x + h(y + l(w)) \in A$

(3.15) $\frac{1}{h}\|g(x+h(y+l(w))) - g(x) - h\dot{g}(x)(y+l(w))\| < \frac{r}{2}$

for all $h \in (0,\delta)$ and $w \in B(r)$.

We may assume (without loss of generality) that $\delta \leq \frac{r}{2}$.

Let us show that this δ satisfies the condition required by our lemma Take an arbitrary $h \in (0,\delta)$ and $v \in Y$ with $\|v\| < \delta$ and denote by $F: B(r) \to Y$ the function

(3.16) $F(w) = \frac{1}{h}\left[(-g(x+h(y+l(w)))) + g(x) + h\dot{g}(x)(y+l(w)))\right] + v$

Using the linearity of $\dot{g}(x)$ and (3.13), (3.16) yields

(3.17) $g(x+h(y+l(w))) = g(x) + h(\dot{g}(x)(y)+v) + h(w - F(w))$

In view of (3.15) and of $\delta \leq \frac{r}{2}$, $\|v\| \leq \delta \leq \frac{r}{2}$, we have $\|F(w)\| \leq r$, for all $w \in B(r)$, i.e. $F: B(r) \to B(r)$. Since F is continuous on

B(r), by Brower fixed point theorem, there is an element $w \in B(r)$ such that $F(w) = w$. With this w and $u = l(w)$, (3.13)', (3.14) and (3.17) show that the requirements of the lemma are satisfied.

We now assume that g is twice differentiable at $x \in A$. Replacing in (2.10) y by hy (y by, $hy + \frac{h^2}{2} z$) and taking into account $\ddot{g}(x) \in L(X, L(X,Y))$ we easily get (3.18) (resp. (3.19)) below,

$$(3.18) \quad \lim_{h \downarrow o} \frac{2}{h^2}\left[g(x+hy)-g(x)-h\dot{g}(x)(y)-\frac{h^2}{2}\ddot{g}(x)(y)(y)\right] = 0$$

uniformly with respect to y from bounded subsets of X,

$$(3.19) \quad \lim_{h \downarrow o} \frac{2}{h^2}\left[g(x+hy+\frac{h^2}{2}z)-g(x)-h\dot{g}(x)(y)-\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z))\right]=0$$

uniformly with respect to (y,z) from bounded subsets of $X \times X$.

Finally, if in (3.19) we replace z by $z+u$ and we have in mind $\dot{g}(x) \in L(X,Y)$, we obtain

$$(3.20) \quad \lim_{\substack{h \downarrow o \\ u \to o}} \frac{2}{h^2}\left[g(x+hy+\frac{h^2}{2}(z+u))-g(x)-h\dot{g}(x)(y)-\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z))\right]=0$$

The lemma below establishes a reciprocal (in a certain sense) relation to (3.20)

Lemma 3.5 Assume that g is continuous on A, twice differentiable at $x \in A$, $u \to \dot{g}(x)(u)$ from X into Y is surjective and Y is finite dimensional. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that for each $h \in (0,\delta)$ and $v \in Y$ with $\|v\| < \delta$, there is $u \in X$ with the properties

$$(3.21) \begin{cases} \|u\| < \varepsilon, \; x+hy+\frac{h^2}{2}(z+u) \in A, \\ g(x+hy+\frac{h^2}{2}(z+u)) = g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z)+v) \end{cases}$$

Proof. Let $l : Y \to X$ as in the proof of the previous lemma, satisfying (3.13). For an arbitrary $\varepsilon > 0$, let $r > 0$ be such that (3.13)' holds. Since A is open and (3.19) holds (with $z+l(w)$ instead of z) it follows that there is $\delta > 0$ such that

$$(3.22) \quad x+hy+\frac{h^2}{2}(z+u) \in A$$

and

$$(3.23)\quad \frac{2}{h^2}\|g(x+hy+\frac{h^2}{2}(z+l(w)))-g(x)-h\dot{g}(x)(y)-\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z+l(w)))\|\leq\frac{r}{2}$$

for all $h\in(0,\delta)$ and $w\in B(r)$(see(3.13))

Without loss of generality,we may assume that $\delta\leq\frac{r}{2}$.Let us show that this δ is a suitable one.To do that,let $h\in(0,\delta)$ and $v\in Y$ with $\|v\|<\delta$.Define the function $F:B(r)\to Y$ by

$$(3.24)\quad F(w)=\frac{2}{h^2}\left[-g(x+hy+\frac{h^2}{2}(z+l(w)))+g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z+l(w)))\right]+v$$

The linearity of $\dot{g}(x)$ and (3.24) implies

$$(3.25)\quad g(x+hy+\frac{h^2}{2}(z+l(w)))=g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z)+v+w-F(w)).$$

Clearly (3.23) and (3.24) yields $\|F(w)\|\leq r$ for all $w\in B(r)$,that is $F: B(r)\to B(r)$.Since F is continuous on $B(r)$,there is $w\in B(r)$ such that $F(w)=w$.With this w and $u=l(w)$,(3.13),(3.22) and (3.25) conclude the proof.

We now can proceded to the proof of the theorems 3.1 and 3.2.

<u>Proof of Theorem</u> 3.1 Let us assume that g is differentiable at $x\in A$ and that (3.1) holds.In order to get (3.3)we shall use Lemma 3.1 By this Lemma,for each $h>0$,there is $u(h)\in X$ such that $u(h)\to 0$ as $h\downarrow 0$ and $x+h(y+u(h))\in D=g^{-1}(E)$.This means that

$$g(x+h(y+u(h)))\in E,\ \forall\ h>0 \text{ and therefore}$$

$$\frac{1}{h}\,d\left[g(x)+h\dot{g}(x)(y);E\right]\leq\frac{1}{h}\|g(x)+h\dot{g}(x)(y)-g(x+h(y+u(h)))\|$$

which according to (3.10)) implies (3.3).

Assume now in addition that g is continuous on A,$\dot{g}(x):X\to Y$ is surjective,Y is finite dimensional and that (3.3) holds.Then by the part (3.6) of Lemma 3.1,for each $h>0$,there is $r(h)\in Y$ with $r(h)\to 0$ as $h\downarrow 0$, such that

(3.26) $g(x)+h(\dot{g}(x)(y)+r(h))\in E$

For an arbitrary $\varepsilon>0$, let $\delta=\delta(\varepsilon)>0$ be a number with the property given by Lemma 3.4.

Since $r(h)\to 0$, as $h\downarrow 0$, there is $\delta_1=\delta_1(\varepsilon)\in(0,\delta)$ such that

(3.27) $\|r(h)\|<\delta,\ \forall h\in(0,\delta_1)$

According to Lemma 3.4, for each h with $0<h<\delta_1$ ($\delta_1<\delta$) there is $u=u(h)\in X$ with the properties:

(3.28) $\|u\|<\varepsilon, x+h(y+u)\in A, g(x+h(y+u)) = g(x)+h(\dot{g}(x)(y)+r(h))$

First of all (3.28) shows (taking into account (3.26)) that $x+h(y+u)\in g^{-1}(E) = D$, consequently, in view of Lemma 3.1, (3.1) is proved.

<u>Proof of Theorem</u> 3.2 The proof of this theorem is very similar to that of theorem 3.1. However, since it is rather technical, we shall give it.

Let $D = g^{-1}(E)$, $g: A\to Y$ twice differentiable at $x\in A$ and assume that (3.2) holds.

In view of Lemma 3.2, there is $u(h)\in X$ with the property (3.8) which gives

$$g(x+hy+\frac{h^2}{2}(z+u(h)))\in E,$$

for all $h>0$.

Therefore

$$\frac{1}{h^2}d[g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z));E]\leq\frac{1}{h^2}\|g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z))-g(x+hy+\frac{h^2}{2}(z+u(h))\|$$

which (according to (3.20)) implies (3.4).

Assume now in addition that, $\dot{g}(x): X\to Y$ is surjective, Y is finite dimensional and (3.4) holds.

Then by the part (3.8) of Lemma 3.2, there is $r(h)\in Y$ with $r(h)\to 0$ as $h\downarrow 0$ and

(3.29) $g(x)+h\dot{g}(x)(y)+\frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z)+r(h))\in E$

for all $h > 0$.

For $\varepsilon > 0$, let $\delta > 0$ be a number with the property given by Lemma 3-5 and $\delta_1 \in (0, \delta)$ satisfying (3.27). Since $0 < \delta_1 < \delta$, according to Lemma 3.5, for each $h \in (0, \delta_1)$, there is $u \in X$ with the properties

$$\|u\| < \varepsilon, \quad x + hy + \frac{h^2}{2}(z+u) \in A \tag{3.30}$$
$$g(x+hy+\frac{h^2}{2}(z+u)) = g(x) + h\dot{g}(x)(y) + \frac{h^2}{2}(\ddot{g}(x)(y)(y) + \dot{g}(x)(z) + r(h))$$

which implies (using (3.29)),

$$x + hy + \frac{h^2}{2}(z+u) \in g^{-1}(E) = D \tag{3.31}$$

In view of Lemma 3.2 (with δ_1 instead of δ), (3.31) and the first part of (3.30), prove (3.2).

From theorems 3.1 and 3.2 we get easily the results below

<u>Corollary</u> 3.1 <u>Assume that</u> $g: A \subset X \to Y$ <u>is differentiable at</u> $x \in A$ <u>continuous on</u> A, $\dot{g}(x): X \to Y$ <u>is surjective and</u> Y <u>is finite dimensional. Then the following conditions are equivalent</u>

$$\lim_{h \downarrow 0} \frac{1}{h} d[x+hy; D_g] = 0, \quad y \in X \tag{3.32}$$

$$g(x) = 0, \quad \dot{g}(x)(y) = 0 \tag{3.33}$$

Here D_g is given by (2.7).

<u>Corollary</u> 3.2 <u>In addition to the hypotheses of Corollary</u> 3.1 <u>we assume that</u> g is <u>twice differentiable at</u> $x \in A$. <u>Then the following conditions are equivalent</u>

$$\lim_{h \downarrow 0} \frac{1}{h^2} d[x+hy+\frac{h^2}{2}z; D_g] = 0, \quad y, z \in X \tag{3.34}$$

$$g(x) = 0, \dot{g}(x)(y) = 0, \ddot{g}(x)(y)(y) + \dot{g}(x)(z) = 0 \tag{3.35}$$

<u>Proof of Corollary 3.1</u>

Set $E = \{0\}$. Then $D_g = g^{-1}(E)$.

In view of Theorem 3.1, (3.32) is equivalent to

(3.36) $\lim_{h\downarrow o} \frac{1}{h} d\left[g(x)+h\dot{g}(x)(y); \{0\}\right] = 0$

Obviously (3.36) holds iff (3.33) holds.

Proof Of Corollary 3.2 In view of Theorem 3.2, (3.32) is equivalent to

(3.37) $\lim_{h\downarrow o} \frac{1}{h^2} d\left[g(x)+h\dot{g}(x)(y)+ \frac{h^2}{2}(\ddot{g}(x)(y)(y)+\dot{g}(x)(z)); \{0\}\right] = 0$

and the results follows.

Remark 3.1 In the case in which Y is a general Banach space and g is if class $C^1(A,Y)$, the equivalence of (3.32) and (3.33) is proved in [11] p.483. ∎

Denote by $\langle x,y\rangle_+$, the usual one-sided directional derivative of the norm $\|\cdot\|$ of X at x, i.e.

(3.38) $\langle x,y\rangle_+ = \lim_{h\downarrow o} \frac{\|x+hy\|-\|x\|}{h}$, $x,y \in X$

We shall give a characterization of (3.32) in terms of (3.38), when

(3.39) $g(x) = \frac{1}{2}(\|x\|^2-r^2), D_g = S(r) = \{x \in X, \|x\|= r\}, \ r>0$

Proposition 3.1 The following conditions are equivalent

(i) $\liminf_{h\downarrow o} \frac{1}{h} d\left[x+hy; S(r)\right] = 0,$

(ii) $\langle x,y\rangle_+ = 0, \ \|x\| = r$

(iii) $\lim_{h\downarrow o} \frac{1}{h} d\left[x+hy; S(r)\right] = 0$

Proof Let us assume that (i) holds. Then there exists two sequences $h_n \downarrow 0$ and $r_n \in X$ with $r_n \to 0$ as $n \to \infty$, such that

$$\|x+h_n(y+r_n)\| = r$$

This implies $\|x\| = r$ and

$$\big|\|x+h_ny\|-\|x\|\big| = \big|\|x+h_ny\| - \|x+h_n(y+r_n)\|\big| \leq h_n\|r_n\|$$

which yields (ii).

We now prove that (ii) implies (iii). For each $h>0$, choose $r(h)\in X$, such that

$$(3.40)\quad x+hy+hr(h) = \frac{r(x+hy)}{\|x+hy\|}$$

Clearly

$$(3.41)\quad \|x+hy+hr(h)\| = r,\ \|r(h)\| = \frac{|\|x+hy\|-\|x\||}{h}$$

Since $\langle x,y\rangle_+ = 0$, (3.41) shows that $r(h)\to 0$ as $h\downarrow 0$. In view of Lemma 3.1, (3.--), (iii) follows from (3.41).

Remark 3.2 (1) When the norm of X is not differentiable, proposition 3.1 cannot be obtained from Corollary 3.1.

(2) In the case of Gâteaux differentiability of the norm of X, with a proof very similar to that of Proposition 3.1, one can prove the equivalence of the condition below

$$(3.42)\quad \lim_{\substack{t\to 0\\ t\neq 0}} \frac{1}{t}\, d[x+ty;S(r)] = 0,$$

$$(3.43)\quad \lim_{\substack{t\to 0\\ t\neq 0}} \frac{\|x+ty\|-\|x\|}{t} = 0,\ \|x\| = r,$$

(3) The condition (ii) of Proposition 3.1 is related to Example 4.1 from [9]. ∎

In the case of a real Hilbert space H of inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, the condition (ii) of Proposition 3.1 means $\langle x,y\rangle = 0$. Moreover, we have

Corollary.3.3 Let $x,y,z\in H$.

Then

$$(3.44)\quad \lim_{h\downarrow 0} \frac{1}{h^2}\, d\left[x+hy+\frac{h^2}{2}z;S(r)\right] = 0$$

iff

$$(3.45)\quad \|x\| = r,\ \langle x,y\rangle = 0,\ \|y\|^2 + \langle x,z\rangle = 0$$

Proof. One applies Corollary 3.2 with g given by (3.39), observing that

(3.46) $\dot{g}(x)(y) = \langle x,y\rangle$, $\ddot{g}(x)(y)(v) = \langle y,v\rangle$, $\forall v \in H$.

4. PROOF OF THE MAIN RESULTS

Proof of the Theorem 2.2 Assume that $u: [0,T) \to A$ is a solution of (2.4). It is known that (see e.g. [5], Ch. I) if $u''(t)$ exists then

(4.1) $$\lim_{h \downarrow 0} \| u(t+h)-u(t)-hu'(t)- \frac{h^2}{2} u''(t)\|/h^2 = 0$$

where the derivatives are taken in strong sense.

Therefore

(4.2) $$\lim_{h \downarrow 0} \|u(t+h)-u(t)-hu'(t)- \frac{h^2}{2} f(u(t))\|/h^2 = 0$$

(i) If $u(t) \in D$ for all $t \in [0,T)$, then for each $t \in [0,T)$, $u(t+h) \in D$ for all $h \in [0,T-t)$

Consequently

(4.3) $$\frac{1}{h^2} d\left[u(t)+hu'(t)+ \frac{h^2}{2} f(u(t)); D\right] \leq \frac{1}{h^2} \| u(t)+hu'(t)+ \frac{h^2}{2} f(u(t)) - u(t+h) \|$$

for all $h \in (0,T-t)$ and $t \in [0,T)$.

Combining (4.2) and (4.3) it follows $(u(t),u'(t)) \in M_D$. We now assume (in addition) that D is closed in A (in the sense of Definition 2.2) and $(u(t),u'(t)) \in M_D$ (given by (2.5)). This means (by Lemma 3.2) that there is $r(h) \in X$ with $r(h) \to 0$ as $h \downarrow 0$, such that

(4.4) $$u(t)+hu'(t)+ \frac{h^2}{2}(f(u(t))+r(h)) \in D = \bar{D} \cap A$$

for all $h > 0$.

Since $u(t) \in A$, (4.4) implies $u(t) \in \bar{D} \cap A = D$. ∎

Remark 4.1 With the same proof it follows that the projection $pr_1(M_D)$ of $M_D \subset A \times X$ on the first factor space of $X \times X$ satisfies

$pr_1(M_D) \subset \bar{D} \cap A$.

Therefore, the case $D = \bar{D} \cap A$ yields $pr_1(M_D) \subset D$.

m The proof of theorem 2.3 is a simple combination of Theorem 2.2 with Definition 2.3, so it is left to the reader.

<u>Proof of Corollary</u> 2.1 Assume that we are in the hypothesis (1). Since $D \subset \bigcap_{i \in I} D_i$, it follows directly from (2.5) that $M_D \subset \bigcap_{i \in I} M_{D_i}$. It remains to prove the converse inclusion.

Let $(x,y) \in \bigcap_{i \in I} M_{D_i}$. Since $f\colon A \subset X \to X$ is locally Lipschitz the then solution $u\colon [0,T) \to A$ of (2.4) with $u(o) = x, u'(o) = y$ is uniquely determined by $(x,y) \in M_{D_i}$. According to Theorem 2.3 $(u(o),u'(o)) \in M_{D_i}$ implies $u(t) \in D_i$ for all $t \in [0,T)$, whenever $i \in I$. Consequently, $u(t) \in D$ for all $t \in [0,T)$, which implies (by Theorem 2.2) $(u(t),u'(t)) \in M_D, \forall t \in [0,T)$. For $t = 0$, this gives $(x,y) \in M_D$ and (*) is proved.

(2) Assume that M_D is nonempty. If $u\colon [0,T) \to A$ is a solution of (2.2) with $(u(o),u'(o)) \in M_D$, then $(u(o),u'(o)) \in M_{D_i}$. Hence (by Definition 2.2) $(u(t),u'(t)) \in M_{D_i}$, $i \in I$, $t \in [0,T)$ which implies (using (*)) $(u(t),u'(t)) \in M_D$. This means just the fact that D is flow-invariant set for (2.4).

<u>Proof of Theorem</u> 2.4 By a standard device, consider the first order autonomous differential system

$$(4.5) \quad \begin{cases} u'(t) = v(t) \\ v'(t) = f(u(t)) \end{cases} \quad t \geq 0$$

Obviously this system is equivalnt to (2.4). In view of definitions 2.1 and 2.3 it follows that D is a flow-invariant set for (2.4) if and only if M_D is a nonempty flow-invariant set for (4.5).

Since the function $f\colon A \subset X \to X$ is locally Lipschitz, the function $(x,y) \to (y,f(x))$ from $A \times X$ into $X \times X$ is locally Lipschitz too. By Theorem 2.1, the nonempty (closed in $A \times X$) set M_D is flow-invariant for (4.5) iff (2,6) holds. ∎

Remark 4.1 Essentially, the result given by Theorem 2.4 (as well as it proof) has been presented in [21]. Here, both the proof of Theorem 2.4 and the notion of "flow-invariant set" for a second order differential equation, are much more precisely given. The limit appearing in (2.5) has been considered (before [21]) in [16] but not efficiently used. However, the idea to investigate the flow-invariance of a set for (2.4), goes back to [16]. ∎

Proof of Theorem 2.5 Let $(x,y)\in M_D$ and u the function defined by (3.8). Then

(4.6) $x+hy+\frac{h^2}{2}(f(x)+u(h))\in D = A\cap S\subset S,\ \forall\ h>0$

and $u(h)\to 0$ as $h\downarrow 0$,

Since S is closed (4.6) implies $x\in S$ so $x\in A\cap S = D$.

Further, because S is a linear space, (4.6) implies now

(4.7) $y+\frac{h}{2}(f(x)+u(h))\in S\ ,\ \forall\ h>0$

Arguing as above, (4.7) gives $y\in S$ and then $f(x)\in S$. Therefore $x\in f^{-1}(s)$ and $(x,y)\in (D\cap f^{-1}(S))\times S$.

Finally let $(x,y)\in (D\cap f^{-1}(S))\times S$. Then for every $h>0$ we have (inasmuch as $D = A\cap S$)

$v_h= x+hy+\frac{h^2}{2}f(x)\in S$

Since A is open, there is $\delta>0$ such that $v_h\in A$ (hence $v_h\in A\cap S$) $\forall\ h\in(0,\delta)$. According to the definition of M_D (see (2.5)) it now trivially follows that $(x,y)\in M_D$ and the part (i) off the theorem is proved. If we assume that $f(D)\subset S$ (i.e. $D\subset f^{-1}(S)$) then obviously, (i) becomes $M_D= D\times S$. Therefore M_D is a nonempty closed in $A\times X$ set. To get the last assertion of (ii), we apply Theorem 2.4.

Let $(x,y)\in M_D$ (which means $x\in A\cap S$ and $y\in S$). Then for every $h>0$, we have

$$(x,y)+h(y,f(x))\in S\times S$$

and there is $\delta>0$ such that $x+hy\in A$, $\forall h\in(0,\delta)$. Therefore

$$(x,y)+h(y,f(x)) \in (A \cap S)\times S = D \times S = M_D, \forall h \in (0,\delta)$$

which trivially implies (2.6). The theorem is proved.

Proof of Theorem 2.6 The form (2.12) of M_D given by (2.5) follows from Corollary 3.2. To prove the second part of the theorem, observe that M_{D_g} can be written in a form similar to (2.7) namely:

$$(4.8) \quad M_{D_g} = \{(x,y) \in A \times X;\ k(x,y) = 0\}$$

where $k\colon A \times X \to R^{3n}$ is given by

$$k(x,y) = (g(x), \dot{g}(x)(y), \ddot{g}(x)(y)(y)+w(x))$$

with w given by (2.11).

It is easy to check that k is differentiable on $A\times X$ and that for each $(x,y) \in A \times X$,

$$\dot{k}(x,y)(u,v) = (\dot{g}(x)(u), \ddot{g}(x)(y)(u)+\dot{g}(x)(v), \dddot{g}(x)(u)(y)(y)+2\ddot{g}(x)(y)(v)+ +\dot{w}(x)(u))$$

We now prove that the function $\dot{k}(x,y)\colon X \times X \to R^{3n}$ is surjective (for each $(x,y) \in M_{D_g}$).

To do that let $y_i \in R^n$, $i=1,2,3$. We have to prove that there is $(u,v) \in X \times X$ such that $\dot{k}(x,y)(u,v) = (y_1, y_2, y_3)$, i.e.

$$(4.9) \quad \begin{cases} \dot{g}(x)(u) = y_1 \\ \ddot{g}(x)(y)(u)+\dot{g}(x)(v) = y_2 \\ \dddot{g}(x)(u)(y)+2\ddot{g}(x)(y)(v)+\dot{w}(x)(u) = y_3 \end{cases}$$

The first hypothesis of the theorem is the surjectivity of $\dot{g}(x)\colon X \to R^n$. Therefore, there is $u \in X$ verifying the first equation of (4.9). The existence of $v \in X$ verifying the other two equations of (4.9) is a direct consequence of surjectivity of

$$u \to (\dot{g}(x)(u), \ddot{g}(x)(y)(u)).$$

According to Corollary 3.1, with $k(M_{D_g})$ instead of g (resp. D) we conclude that (2.6) holds iff $\dot{k}(x,y)(y, \dot{f}(x)) = 0$ holds (which led us to (2.13)).

Proof of Theorem 2.7 The fact that (2.12) holds too, is a direct consequence of Theorem 2.6. Indeed, since $\dot{g}(x): X \to R$ and $\dot{g}(x)(f(x)) \neq 0$ on D_g (according to one of the hypothesis), $\dot{g}(x)$ is not the null fuctional and consequently is surjective.

To prove the second part of the Theorem, it suffices to check that in this case, the linear function $u \to (\dot{g}(x)(u), \ddot{g}(x)(y)(u))$ from X into R^2 is surjective (for each $(x,y) \in M_{D_g}$), or equivalently that the linear functions

(4.10) $u \to \dot{g}(x)(u), \quad u \to \ddot{g}(x)(y)(u)$

from X to R are linear independent.

To this aim, let us consider the linear combination,

(4.11) $r\,\dot{g}(x)(u) + s\ddot{g}(x)(y)(u) = 0, \quad \forall u \in X$

where r and s are real numbers.

The hypothesys $w(x) \neq 0$ for all $x \in D_g$ implies (in view of (2.12)) $\ddot{g}(x)(y)(y) \neq 0$ for all $(x,y) \in M_{D_g}$. Therefore, with u=y, (4.11) gives $s = 0$ (since $\dot{g}(x)(y) = 0$). Furthermore, for $u = f(x)$, (4.11) gives $r = 0$ and hence, by Theorem 2.6, the flow-invariance of D_g is equivalent to (2.13). Finally, the third part of the theorem is obvious. ∎

Proof of Theorem 2.8 Let $(x,y) \in M_{D_g}$ given by (2.12). Clearly, with $\bar{y} = \dot{g}(x)^{\perp}$ given by (2.17), (2.23) is satisfied. In this case $\dot{g}(x)(y) = 0$ and $\dot{g}(x)(\bar{y}) = 0$, shows that the vectors y and $\bar{y}$ are parallel, therefore there is $a = a(x) \in R$ such that $y = a\bar{y}$. Since $\ddot{g}(x)(y)(y) + \dot{g}(x)(f((x)) = 0$ it follows

(4.12) $a^2\ddot{g}(x)(\bar{y})(\bar{y}) + \dot{g}(x)(f(x)) = 0$

Having in mind the notations (2.15)-(2.22) and

$$\|y\|^2 = a^2\|\bar{y}\|^2 = a^2\|\dot{g}(x)^{\perp}\|^2 = a^2 a_1, \ddot{g}(x)(\bar{y})(\bar{y}) = a_2$$

it is clear that (4.12) and (2.22) implies $(x,y) \in M_{D_g}$ given by (2.24). Note that the hypothesis (2.23) implies $w(x) \neq 0$ and (for $y = \dot{g}(x)^{\perp}$) $a_2 \neq 0$

Actually we have proved that for $x \in D_g$, there is $y \in X$ namely

$$(4.13) \quad y = a\bar{y} = \left(-\frac{\dot{g}(x)(f(x))}{a_2}\right)^{1/2} \dot{g}(x)^{\perp}$$

such that $(x,y) \in M_{D_g}$ and conversely (i.e. if $(x,y) \in M_{D_g}$ then y is given by (4.13).

The last assertion of the theorem follows by replacing y given by (4.13) in (2.13) and using (2.30).

5. APPLICATIONS IN MECHANICS

It is the goal of this section to show that Theorem 2.4 (by its consequences)unifies some fundamental results of Dynamics.

Moreover, some of our results (like Theorem 5.2, Corollary 5.1, Theorem 5.4, Remark 5.6) allow us to see that some mathematical characterization of the motion on a set D_g (given by (2,7)) in R^2 or R^3, remain valid in any R^n with $n > 3$, or even more, in any real Hilbert (Banach) space.

5.1 <u>Geometrical properties of a flow-invariant set</u>.

First of all we recall that a function $f: A \subset X \to X$ is regarded as a field of force on A, in the sense that with each vector position $x \in A$ is associated the vector force $f(x) \in X$.

<u>Remark</u> 5.1 If $g: A \subset X \to R$ is continuous, then $D_g = \{x \in A, g(x) = 0\}$ is closed in A.

According to Definition 2.3 and Theorem 2.3, the notion

(5.1)"<u>D_g is a flow-invariant set for the equation</u> (2.4)"

can be restated in terms of Dynamics as follows

(5.2) A "mass particle", "projected" <u>from a point</u> $x \in D_g$ <u>with a "speed"</u> $y \in X$ <u>such that</u> $(x,y) \in M_{D_g}$ <u>(given by</u> (2.12)), <u>describes (under the action of the force field</u> f) <u>an orbit which lies in</u> D_g.

Denote by N_x the null space of the linear continuous operator $\dot{g}(x): X \to R$ (where $x \in A$ and g is differentiable at x), that is

(5.3) $N(\dot{g}(x)) = N_x = \{y \in X,\ \dot{g}(x)(y) = 0\}$

In terms of geometry, $N(\dot{g}(x))$ can be identified with the tangent space at x to the manifold D_g.

Definitions 5.1 (1) A function $g: A \subset X \to R$ is said to be "smooth" if it satisfies the properties

(a) g is three times (Fréchet) differentiable on A.

(b) For each $x \in D_g$, $\dot{g}(x) : X \to R$ is not the null functional (i.e. there is $u \in X$, such that $\dot{g}(x)(u) \neq 0$).

(c) For each $x \in D_g$, $\ddot{g}(x)(y)(y) \neq 0, y \in N_x, y \neq 0$.

(2) g is said to be "completely smooth", if it is smooth and if for each $x \in D_g$, the functional F(x) defined by

(5.4) $$F(x)(y) = \begin{cases} \dfrac{\dddot{g}(x)(y)(y)(y)}{\ddot{g}(x)(y)(y)}, & y \in N_x,\ y \neq 0 \\ 0, & y = 0 \in N_x \end{cases}$$

is a linear continuous functional from N_x into R.

Definition 5.2 A field of force f on A is said to be "g-smooth" if it locally Lipschitz, the function $w: A \to R$ (given by (2.11)) is differentiable on A and for each $x \in D_g$,

(5.5) $\dot{g}(x)(f(x)) \cdot \ddot{g}(x)(y)(y) < 0, \quad \forall y \in N_x, \ y \neq 0$

Theorem 5.1 Let A be an open subset of the real Banach space X and let $g: A \to R$ be a smooth function.

(1) If there is a g-smooth force field of on A such that a "mass particle" "projected" from any point $X \in D_g$ with a "speed" y such that $(x,y) \in M_{D_g}$, "describes" (under the action of f) an "orbit" which lies in D_g, then g is completely smooth.

(2) <u>If</u> g <u>is completely smooth and</u> f <u>is g-smooth, then</u> D_g <u>is a flow-invariant set for</u> (2,4) <u>iff for each</u> $x \in D_g$, <u>there is</u> a $(x) \in R$ <u>with the property</u>

$$(5.6) \quad -w(x)F(x)+2\ddot{g}(x)(f(x))+\dot{w}(x) = a(x)\dot{g}(x) \ , \ x \in D_g.$$

Proof.(1) Let g be a smooth function from A into R. If there is a g-smooth force field f on A with the property that D_g is a flow-invariant set for (2.4), then in view of Theorem 2.7, (2.13) holds. Take $x \in D_g$. If $y \neq 0$ is an arbitrary element of N_x (given by (5.3) and

$$(5.7) \quad \bar{y} = \left(-\frac{w(x)}{\ddot{g}(x)(y)(y)}\right)^{1/2} y = ay \ , \ a = \left(-\frac{w(x)}{\ddot{g}(x)(y)(y)}\right)^{1/2}$$

then obviously $(x,\bar{y}) \in M_{D_g}$ (given by (2.12))

Replacing (x,y) in (2.13) and dividing by a it follows that

$$(5.8) \quad - w(x)F(x)(y)+2\ddot{g}(x)(f(x))(y)+\dot{w}(x)(y) = 0$$

for all $y \in N_x$

By definition of differentiability, the functions $y \to \dot{w}(x)(y)$, $y \to \ddot{g}(x)(f(x))(y)$ are linear continuous functionals from X into R. Therefore, (5.8) implies that F(x) given by (5.4) is a linear continuous from N_x into R, hence g is completely smooth.

(2) This part of the Theorem is a consequence of (5.8).

To show that, let $L_i: X \to R, i = 1,2$ be two linear functionals (different from the null functional). Denote

$$N(L_i) = \{y \in X, L_i(y) = 0\} \ , \ i = 1,2.$$

It is well-known that if $N(L_2) \subset N(L_1)$ then there is $a \in R, a \neq 0$, such that $L_1 = aL_2$ (which implies $N(L_1) = N(L_2)$).

Since $N_x = N(g(x))$, (5.6) follows from (5.8) with

$$L_1 = - w(x)F(x)+2\ddot{g}(x)(f(x))+\dot{w}(x) \ , \ L_2 = \dot{g}(x) \ ,$$

<u>Remark</u> 5.2 If g is smooth and f is g-smooth then M_{D_g} is nonempty. Moreover, for each $x \in D_g$, there is $y \in X$ (e.g. y given by (5.7)) such that

$(x,y) \in M_{D_g}$.

We now give some examples of completely smooth functions and g-smooth force field.

Let us consider the function

$$(5.9) \quad g(x) = \frac{1}{2}\left(\|x - \bar{a}\|^2 - (\langle \bar{b}, x - \bar{a}\rangle + d)^2\right),\ x \in R^n, n \geqslant 2$$

where $\bar{a},\bar{b} \in R^n$ and $d \in R, d \neq 0$, $\langle , \rangle$ - the inner product of R^n. In the case $n = 2$, $C = D_g = \left\{ x \in R^2; g(x) = 0 \right\}$ is a conic with $\bar{a}$ as one of the foci (i.e. circle, ellipse, hyperbola and parabola).

<u>Proposition</u> 5.1 <u>The function</u> $g\colon R^n \to R$, $n = 2,3$ <u>given by</u> (5.9) <u>is a completely smooth function. Moreover</u>

$$(5.10) \quad \ddot{g}(x)(y)(y) = y^2(1-\cos^2[x - \bar{a}, y]) > 0$$

<u>for each</u> x <u>and</u> $y \in R^n (n = 2,3)$ <u>satisfying</u> $g(x) = 0, \dot{g}(x)(y) = 0, y \neq 0$ <u>and Newtonian field on</u> $A \supset D_g$ <u>is a g-smooth.</u>

Here $\langle x-\bar{a}, y\rangle$ denotes the angle determined by the vectors y and $x - \bar{a}$ (which joins $x \in D_g$ with the focus $\bar{a}$).

Proof. The property (a) appearing in Definition 5.1 is obviously satisfied in this case. (b) is satisfied too, since the gradient $\dot{g}(x)$ of a conic is different from the null vector. Of course (5.10) is also a well-known property. However, we prove it here (since, some of the formulae involved in its proof are needed in 5.2 - 5.4).

Clearly,

$$(5.11) \quad \dot{g}(x)(y) = \langle x-\bar{a}, y\rangle - (\langle \bar{b}, x-\bar{a}\rangle + d)\langle \bar{b}, y\rangle$$

The condition $g(x) = 0$, means

$$(5.12) \quad \|x-\bar{a}\|^2 = (\langle \bar{b}, x - \bar{a}\rangle + d)^2$$

which implies $x \neq \bar{a}$ (since $d \neq 0$.

Now, $\dot{g}(x)(y) = 0$ and $g(x) = 0$, yield

$$(5.13) \quad \langle \bar{b}, y\rangle^2 = \frac{\langle x-\bar{a}, y\rangle^2}{\|x-\bar{a}\|^2} = \|y\|^2 \cos^2[x-\bar{a}, y]$$

On the other hand

(5.14) $\ddot{g}(x)(y)(z) = \langle z,y\rangle - \langle \bar{b},z\rangle\langle \bar{b},y\rangle, \forall y,z \in R^n.$

Therefore $\dddot{g}(x) = 0$ and

(5.14)' $\ddot{g}(x)(y)(y) = \| y\|^2 - \langle \bar{b},y\rangle^2$

Combining (5.13) and (5.14)' one obtains (5.10)(since $[x-\bar{a},y]$ is neither 0 nor 180^o).In this case F(x) defined by (5.4) is the trivial functional.In 5.5 we shall see that the Newtonian field (5.73) is g-smooth.

<u>Proposition</u> 5.2 (1) <u>Any smooth function</u> $g: R^2 \to R$ <u>is completely smooth</u>.

(2) <u>Any smooth polinomyal</u> $g: R^n \to R$ of (<u>at most</u>) <u>second degree is completely smooth</u> (e.g.(5.9)).

Proof (1) In this case $N_x = N(\dot{g}(x))$ given by (5.3) is an one dimensional subspace of R^2.Therefore there is $r = r(x) \neq 0 (r \in R)$ such that for any $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in N(\dot{g}(x))$ we have $y_1 = ry_2$.Taking into account (2.19) and (2.20),it follows that there is a constant $b = b(x) \in R$ such that

$$F(x)(y) = \begin{cases} b(x)\ y_2, \text{if } y \in N_x, y \neq 0 \\ 0, \text{if } y = 0 \end{cases}$$

hence $y \to F(x)(y)$ is a linear functional on R^2.The proof pf the second part is obvious since in this case $\dddot{g}(x)(y)(y)(y) = 0, y \in R^n$.

<u>Remark</u> 5.3.It would be interesting to give an example of "completely smooth" function in $R^n (n \geqslant 3)$,other than the smooth polinomyals of (at most) second degree.

<u>Proposition</u> 5.3. <u>Let</u> $g: A \subset X \to R$ <u>be three times differentiable on</u> A.

(1) <u>If there is a</u> g-<u>smooth force field on</u> A,<u>then</u> g <u>is necessarly</u> a g-<u>smooth function</u>.

(2) <u>If</u> g <u>is smooth,then for each</u> $x \in D_g$,<u>the linear continuous function</u> $\begin{pmatrix} u \\ s \end{pmatrix} \to L_x\begin{pmatrix} u \\ s \end{pmatrix}$ <u>from</u> $X \times R$ <u>into</u> $L(X,R) \times R$ <u>defined by</u>

$$(*)\qquad L_x\begin{pmatrix}u\\ s\end{pmatrix}=\begin{pmatrix}\ddot g(x)(u)+s\dot g(x)\\ \dot g(x)(u)\end{pmatrix}=\begin{pmatrix}\ddot g(x) & \dot g(x)\\ \dot g(x) & 0\end{pmatrix}\begin{pmatrix}u\\ s\end{pmatrix},u\in X,s\in R$$

is one to one.

(3) Let $X = R^n$ and $g\colon A\subset R^n \to R$ a smooth function.
Then for each $x\in D_g$, L_x is a bijection (i.e. one to one and surjective)

(4) For $n = 2$, the curvature $c(x)$ of D_g is different from zero.

Proof. (1) This part is a direct consequence of the definitions 5.1 and 5.2.

(2) Let $u\in X$ and $s\in R$ be such that $L_x\begin{pmatrix}u\\ s\end{pmatrix}=0$, i.e.

$$\ddot g(x)(u)+s\dot g(x)=0, 0\in L(X,R), \dot g(x)(u)=0 \text{ (i.e, } u\in N_x)$$

Then $u = 0$ and consequently $s= 0$, too. Indeed if $u\neq 0$, then $0 = \ddot g(x)(u)(u)+s\dot g(x)(u) = \ddot g(x)(u)(u)$ and $u\in N_x$ contradict the hypothesis (c) of Definition 5.1.

(3) In this case L_x can be identified with the $(n+1)\times(n+1)$ matrix

$$(5.14)''\qquad L_x^{n+1}=\begin{pmatrix}g_{11}\cdots g_{1n} & g_1\\ g_{n1}\cdots g_{nn} & g_n\\ g_1\ \cdots g_n & 0\end{pmatrix}$$

since $\dot g(x)$, $\ddot g(x)$, $L(R^n,R)$ are identified with gradient, Hessian matrix and R^n respectively.

The fact that L_x^{n+1} is one to one is equivalent to $\det(L_k^{n+1})\neq 0$ (hence, tp the surjectivity of L_x^{n+1})

(4) If $n = 2$ it is easy to check that $-a_2(x) = \det(L_x^3)\neq 0$ which implies (in view of (2.22) and of $a_1(x)\neq 0$), $c(x)\neq 0$. Actually, the fact that $a_1(x)\neq 0$ ($a_2(x)\neq 0$) follows directely from hypotheses (b) and (c) (with $y = \dot g(x)^{\perp}$) of Definition 5.1, respectively.
(see the notations (2.15) -(2.21).

5.2. A generalization of Bonnet's theorem.

As a first applications of our results we shall give an extension of Bonnet's theorem (see e.g. [32] p.95) Given $g: A\subset X\to R$, set

(5.15) $$K_g=\Big\{f: A\subset X\to X;\ f - \text{"g-smooth"} \Big\}$$
$$\text{and (5.1)(or equivalently, (5.2)) holds}\Big\}$$

For the statements of the results below we shall use definitions 5.1 and 5.2 and the notations (2.7) and (2.11).

Remark 5.4 If g is a completely smooth, then in view of Theorem 5.1, it follows

(5.15)' $$K_g=\Big\{f: A\to X \text{ is g-smooth and (5.6) holds}\Big\}.$$

The null function 0 doesn't satisfy (5.5), hence $0\,\bar{\in}\,K_g$.

Theorem 5.2 Let X be a real Banach space, $A\subset X$ an open subset and $g: A\to R$ a completely smooth function. Then K_g is a convex cone (which doesn't contain the null function).
More precisely, if $f_i\in K_g$ and $b_i\geqslant 0$, $i = 1,\dots,m$ with $\sum_{i=1}^{m} b_i^2>0$
Then f given by

(5.16) $$f = \sum_{i=1}^{m} b_i f_i$$

belongs to K_g, too.
Moreover, if $x\in D_g$ and $v = v(x)$ is such that $(x,v)\in M_{D_g}$ (given by (2.12) with f defined by (5.16), then for each $i=1,\dots,m$ there is v_i having the properties

(5.18) $$(x,y_i)\in M^i_{D_g}\ ,\ v^2=\sum_{i=1}^{m} b_i v_i^2.$$

Here we denoted by $M^i_{D_g}$ the subset M_{D_g} corresponding to f_i, i.e.

(5.19) $$M^i_{D_g}=\Big\{(x,y)\in A\times X;\ g(x) = 0, \dot{g}(x)(y) = 0, \ddot{g}(x)(y)(y)+ + \dot{g}(x)(f_i(x)) = 0\Big\}.$$

Proof of Theorem 5.2. Since f_i are g-smooth, then obviously f given by (5.16) is g-smooth, too. The fact that $f_i\in K_g$ given by (5.15), means also that for each $x\in D_g$, there is $a_i = a_i(x)\in R$ such that

$$(5.20)\qquad -w_i(x)\, F(x)+2\ddot{g}(x)(f_i(x))+\dot{w}_i(x) = a_i\dot{g}(x)\ ,\ x\in D_g$$

where (similarly to (2.11)).

$$(5.21)\qquad w_i(x) = \dot{g}(x)(f_i(x)),\ x\in A(\text{hence } \sum_{i=1}^{m} b_i w_i(x) = w(x))$$

Multiplying (5.20) by b_i and then summing up over 1 to m we get (5.6) with f given by (5.16) and $a = \sum_{i=1}^{m} b_i a_i$.

To prove the last assertion of the theorem let $x\in D_g$ and $v\in X$ be such that $(x,v)\in M_{D_g}$. The existence of such that v is shown in Remark 5.2. Consequently

$$(5.22)\qquad g(x) = 0,\dot{g}(x)(v) = 0,\ddot{g}(x)(v)(v)+\dot{g}(x)(f(x)) = 0$$

with f given by (2.16) (therefore $w(x) = \dot{g}(x)(f(x))$)

It is easy to check that v_i given by

$$(5.23)\qquad v_i = \mp\left(\frac{w_i(x)}{w(x)}\right)^{1/2} v$$

has the property that $(x,v_i)\in M^i_{D_g}$.

Note that $f\in K_g$ implies $w(x)\neq 0$ (according to (5.5)) and then $v\neq 0$ (by(5.22)).

In view of (5.23) we have $b_i v_i^2 = \frac{b_i w_i(x)}{w(x)} v^2$ which implies (5.18) (using (5.21). ∎

We now give (in terms of Dynamics) a consequence of Theorem 5.2.

<u>Corollary</u> 5.1 <u>Let</u> $X = R^2$ <u>and</u> $g\colon A\subset R^2\to R$ <u>a completely smooth function. If the orbit</u> D_g <u>can be described in each g-smooth field of force</u> $f_i, i=1,\dots,m,$ <u>the velocity of any point</u> P <u>of the orbit being</u> v_i<u>, then the same orbit can be described in the field of force</u> $f=\sum_{i=1}^{m} b_i f_i$ $(b_i\geq 0,\ \sum_{i=1}^{m} b_i^2\neq 0)$, <u>the velocity</u> v <u>of</u> P (<u>in the field of force</u> f)<u>being</u>

$$(5.24)\qquad v^2 = \sum_{i=1}^{m} b_i v_i^2$$

Proof. In view of Theorems 5.2, the only fact we have to prove is (5.24). Let v_i be as in the statement of the corollary. In our framework this means (see e.g. Theorem 2.2 and the explanation (5.2)) $(x,v_i) \in M^i_{D_g}$ and $(x,v) \in M_{D_g}$, where x is the vector position of P. Therefore v_i and $v \in N(\dot{g}(x))$ given by (5.3). In this case (i.e. $X=R^2$) $N(\dot{g}(x))$ is a subspace of R^2 of one dimension, hence v_i and v are parallel vectors. Consequently, there is $d_i \in R$ such that $v_i = d_i v$. Combining (5.22) and (5.25) below

$$(5.25) \qquad \ddot{g}(x)(v_i)(v_i)+\dot{g}(x)(f_i(x)) = 0$$

one obtains at once $d_i{}^2 = \frac{w_i(x)}{w(x)}$ (i.e. (5.23)) which implies (arguing as for (5.18))(5.24).

Remark 5.5 In the case $b_i = 1, i=1,\dots,m$. Corollary 5.1 is the Bonnet's theorem (under the form presented in [32] p.95)

5.3. Determination of the field of force under which a given orbit can be desccibed.

In this subsection we are concerned with the solution of the following problem (call it (P)).

(P) Given a curve $D_g = \{x \in A \subset R^2, g(x) = 0\}$ (with g completely smooth, find all (g-smooth) force field $f: A \subset R^2 \to R^2$ with the property that for each $x \in D_g$, there is $v = v(x,f) \in R^2$, such that a mass particle projected from x with the speed v, describes (under the action of f) an orbit which lies in D_g.

In our framework, the solution of the problem (P) consists in the determination of all elements f of K_g given by (5.15).

Remark 5.6. Given a function $f: A \to R^2$, denote by f/D_g the restriction of f to D_g. Let $f \in K_g$. If $f_1: A \subset X \to X$ is g-smooth and $f_1/D_g = f/D_g$, then $f_1 \in K_g$, too. This fact follows from Theorem 2.7 (or Theorem 2.8). Therefore, we are interested to determine merely the restrictions (of elements of K_g) to D_g. When there is now danger of confusion we denote f/D_g by f, too (for the simplicity of writing).

With the notatations of § 2 and

$$(5.26)\qquad B = \begin{pmatrix} -g_{22} & g_{21} \\ g_{11} & -g_{11} \end{pmatrix}.$$

we give the following solution to (P).

Theorem 5.3 Let $g\colon A \subset R^2 \to R$ be completely-smooth (1) If $f \in K_g$, then its restriction (denoted by f, too) to K_g satisfies the system

$$(5.27)\qquad \begin{cases} (g_{11}g_2 - g_{21}g_1)f^1(x) + (g_{21}g_2 - g_{22}g_1)f^2(x) = \frac{1}{2}(w_2 g_1 - w_1 g_2) + \frac{1}{2}\frac{a_3 w(x)}{a_2} \\ g_1 f^1 + g_2 f^2 = w(x) \end{cases}$$

for each $x \in D_g$.

(2) The solution of (5.26) can be written under the following three equivalent forms:

$$(5.28)\qquad f(x) = -\frac{w(x)}{a_2} B\,\dot{g}(x) + \frac{1}{2}\frac{a_3 w(x)}{a_2^2}\dot{g}(x)^{\perp} - \frac{1}{2a_2}\langle \dot{w}(x), \dot{g}(x)^{\perp}\rangle \dot{g}(x)^{\perp}$$

$$(5.28)'\qquad f(x) = -\frac{w(x)}{a_2} B\,\dot{g}(x) + \frac{1}{2}\frac{a_3 w(x)}{a_2^2}\dot{g}(x)^{\perp} - \frac{a_1}{2a_2}\dot{w}(x) + \frac{\langle \dot{w}(x), \dot{g}(x)\rangle}{2a_2}\dot{g}(x)$$

$$(5.28)''\qquad f(x) = z(x) B\,\dot{g}(x) + \frac{1}{2}\|\dot{g}(x)\|^2 \dot{z}(x) - \frac{1}{2}\langle \dot{z}(x), \dot{g}(x)\rangle \dot{g}(x)$$

for each $x \in D_g$

Where $z\colon A \to R$ is a differentiable function and $z(x) > 0$ for all $x \in D_g$.

(3) Conversely, if $w\colon A \to R$ and $z\colon A \to R$ are differentiable function such that

$$(5.29)\qquad a_2 \cdot w(x) < 0,\ z(x) > 0,\ \forall x \in D_g$$

and f given (5.28), (5.28)' or (5.28)" is locally Lipschitz, then this $f \in K_g$.

(4) In particular, if in addition to the above hypotheses, we assume that $w, z \in C^2(A,R)$ and $g \in C^4(A,R)$, then f given by (5.28)-(5.28)" belongs to K_g, therefore K_g is nonempty.

Proof.(1) The first part of the theorem is a consequence of Theorem 2.8.

(2) If in the system (5.27), w is regarded as a parameter function, then solving this elementary system we get easily (5.28).

(First all, one observes that the determinant of the system is just a_2 and that (2.30) holds).

Since $y = \dot{g}(x)^{\perp} \in N_x$ (given by (5.3)) and $a_2 = \ddot{g}(x)(\dot{g}(x)^{\perp})(\dot{g}(x)^{\perp})$, then (5.5) implies the first inequality of (5.29).

To prove that the form (5.28) of f is equivalent to that given by (5.28)' let us observe that for each fixed $x \in D_g$, there exist $a(x), b(x) \in R$ such that

$$\dot{w}(x) = a(x)\, \dot{g}(x) + b(x)\, \dot{g}(x)^{\perp}$$

Namely

$$a(x) = \frac{1}{a_1}\langle \dot{w}(x), \dot{g}(x)\rangle \ , \ b(x) = \frac{1}{a_1}\langle \dot{w}(x), \dot{g}(x)^{\perp}\rangle$$

where (2.18) has been used, as well as $\langle \dot{g}(x), \dot{g}(x)^{\perp}\rangle = 0$

Consequently

$$(5.30) \ \langle \dot{w}(x), \dot{g}(x)^{\perp}\rangle \dot{g}(x)^{\perp} = a_1 \dot{w}(x) - \langle \dot{w}(x), \dot{g}(x)\rangle \dot{g}(x)$$

which shows the equivalence of (5.28) and (5.28)'. We now prove the equivalence of (5.28)" and (5.28). Set

$$(5.31) \quad z(x) = -\frac{w(x)}{a_2} \ , \ x \in D_g$$

Since we have already proved the first inequality of (5.29), it follows that $z(x) > 0$, $x \in D_g$.

Let us prove that the derivative of $a_2 = a_2(x)$ in the direction $\dot{g}(x)^{\perp}$ is just a_3, i.e.

$$(5.32) \ \langle \dot{a}_2(x), \dot{g}(x)^{\perp}\rangle = \dddot{g}(x)(\dot{g}(x)^{\perp})(\dot{g}(x)^{\perp})(\dot{g}(x)^{\perp}) = a_3$$

Indeed, since $a_2(x) = \ddot{g}(x)(\dot{g}(x)^{\perp})(\dot{g}(x)^{\perp})$ and

$$(5.33) \ \langle \ddot{g}(x)(\dot{g}(x)^{\perp}), (\dot{g}(x)^{\perp})(\dot{g}(x)^{\perp})\rangle = 0$$

(which we shall prove below)

we have

$$\langle \dot a_2(x), \dot g(x)^{\perp}\rangle = a_3 + 2\langle \ddot g(x)(\dot g(x)^{\perp}), (\dot g(x)^{\perp})^{\cdot}(\dot g(x)^{\perp})\rangle = a_3$$

We have to prove (5.33). Indeed, since

$$\dot g(x) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \dot g(x)^{\perp} = \begin{pmatrix} g_2 \\ -g_1 \end{pmatrix}$$

it follows

$$\ddot g(x)(\dot g(x)^{\perp}) = \begin{pmatrix} \dot g_1(x)(\dot g(x)^{\perp}) \\ \dot g_2(x)(\dot g(x)^{\perp}) \end{pmatrix}, \quad (\dot g(x)^{\perp})^{\cdot}(\dot g(x)^{\perp}) = \begin{pmatrix} \dot g_2(x)(\dot g(x)^{\perp}) \\ -\dot g_1(x)(\dot g(x)^{\perp}) \end{pmatrix}$$

which yield us to (5.32).

Furthermore, by (5.31) we see that

$$(5.34)\ \langle \dot w(x), \dot g(x)^{\perp}\rangle = -z(x)\langle \dot a_2(x), \dot g(x)^{\perp}\rangle - a_2(x)\langle \dot z(x), \dot g(x)^{\perp}\rangle$$

Using (5.30)(with $\dot z(x)$ instead of $\dot w(x)$) and (5.31), (5.34) gives

$$(5.35)\ \langle \dot w(x), \dot g(x)^{\perp}\rangle \dot g(x)^{\perp} = -a_3 z(x)\dot g(x)^{\perp} - a_2(x)a_1(x)\dot z(x) + a_2(x)\langle \dot z(x), \dot g(x)\rangle \dot g(x)$$

In view of (5.35) and (5.31), it is easy to check that (5.28) is equivalent to (5.28)".

(3) Let w and z be differentiable on Λ real-valued functions, satisfying (5.29) and f given by (5.28). With an elementary calculus we verify that

$$(5.36)\ \langle B\, \dot g(x), \dot g(x)\rangle = -a_2(x) \quad .$$

Therefore, if f is given by (5.28)(or(5.28)") it follows

$$(5.37)\ \langle \dot g(x), f(x)\rangle = w(x), \langle \dot g(x), f(x)\rangle = -a_2(x)z(x)$$

respectively.

Inasmuch as $X = R^2$, any $y \in N_x$ is parallel to $\dot g(x)^{\perp}$, hence (5.37) and (5.29) imply (5.5). Thus, if f given by (5.28) is locally Lipschitz too then it is a g-smooth force field.

The fact that the (unique) solution of (5.27) is given by (5.28), means that for every w as above, f given by (5.28) satisfies (5.27) and therefore (according to Remark 5.6), we may conclude that this $f\in K_g$.

(4) In these hypotheses, f given by (5.28) is of class $C^1(A,R)$ and consequently it is locally Lipschitz. According to (3) it follows that $f\in K_g$.

Remark 5.6 (1) The solution (5.28)" of the problem (P) is essentially due to Dainelli [8] (see also [32, p.96])

(2) In the case of a general (real) Banach space X, given $g: A\subset X \to R$ completely smooth, then in view of Theorem 5.1, the elements f of K_g are given by the g-smooth solutions of the system

$$(5.38)\begin{cases} \ddot{g}(x)(f(x)) - \frac{a(x)}{2}\dot{g}(x) = \frac{w(x)}{2}F(x) - \frac{\dot{w}(x)}{2} \\ \dot{g}(x)(f(x)) = w(x) \qquad , x\in D_g \end{cases}$$

If w is regarded as a parameter function, then the unknowns of (5.38) are the vector f and the real valued function a = a(x). Using Proposition 5.3, we conclude that (at least) in the case $X = R^n$ (5.38) admits an unique solution. Indeed, in this case the matrix of (5.38) is just the nonsingular matrix L_x^{n+1} (see (5.14)'). If $g\in C^4(A,R)$ is completely smooth and $w\in C^2(A,R)$ satisfies

$$w(x)\ddot{g}(x)(y)(y) < 0, \forall y\in N(\dot{g}(x))$$ then the solution $(f^1,\dots,f^n,-a/2)$ of (5.38) is locally Lipschitz and $f=(f^1,\dots,f^n)\in K_g$. Therefore, in the above conditions on g the set K_g is always nonempty. The formula (5.28) can be derived via the system (5.38), too.

5.4. Uniform motions

Throughout this subsection, X is a real Hilbert space H of inner product $\langle\ ,\ \rangle$ and norm $\|\cdot\|$.

Corollary 5.2 In the case of the sphere

$$(5.39)\quad S(r) = \{x\in H, \|x\| = r\},\ r>0$$

the subset given by (2.12), becomes

$$(5.40)\quad M_{S(r)} = \{(x,y)\in A\times H, \|x\| = r, \langle x,y\rangle = 0, \|y\|^2 + \langle x, f(x)\rangle = 0\}.$$

Proof. In the case $S(r) = D_g$, with

(5.41) $g(x) = \frac{1}{2}(\|x\|^2 - r^2)$

For $z = f(x)$ Corollary 3.3 and (2.12) give (5.40). Another proof of (5.40) can be found in [21].

Corollary 5.3 Let $A \subset H$ be an open subset with $S(r) \subset A$. Assume that $f: A \to H$ is locally Lipschitz and in addition satisfies the properties

(i) $\langle x, f(x)\rangle < 0, \ \forall x \in S(r)$

(ii) $w(x) = \langle x, f(x)\rangle$ is differentiable on A.

Then $S(r)$ is a flow-invariant set for (2.4) iff

(5.42) $2\langle f(x), y\rangle + \dot{w}(x)(y) = 0, \ \forall (x,y) \in M_{S(r)}$

Proof. This result is a direct consequence of Theorem 2.7. Indeed, taking into account (3.46), (2.13) becomes (5.42).

Definition 5.1 We say that the motions on $S(r)$ are uniform, if there is a constant $k = k(f) > 0$, such that each $S(r)$-valued solution u of (2.4) satisfies:

(5.43) $\|u'(t)\| = k, k > 0, \ \forall t \geq 0$

We now give a characterization of the locally Lipschitz force field on $S(r)$, under the action of which the motions ("without friction") on $S(r)$ are uniform.

Theorem 5.4 Let $f: S(r) \to H$ be a Lipschitz function such that

$\langle x, f(x)\rangle < 0, \ \forall x \in S(r)$

The necessary and sufficient conditions in order for $S(r)$ to be a flow-invariant set for (2.4) and the motions on $S(r)$ be uniform are the following

(i) $\langle x, f(x)\rangle = -k^2, \ \forall x \in S(r)$

where $k = k(r)$ is a (positive) constant independent of $x \in S(r)$

(ii) $\langle f(x), y\rangle = 0$

for all (x,y) with $\|x\| = r$, and $\langle x, y\rangle = 0$

Remark 5.7 Under differentiability assumption on $x\to\langle x,f(x)\rangle$ on $A\supset S(r)$, this theorem can be derived from Corollary 5.3. Otherwise it requires a special proof. The proof which we shall give here makes use (directly) of Theorem 2.4. Note that Theorem 2.4 remains valid and in the case in which f is defined merely on D (see also Remark 1.1). In this case, in (2.5) we have to consider only $x\in D$, and (2.6) is a necessary and suficient condition of the existence of the solution of (2.4) for each initial condition $(u(o),u'(o))\in M_D$. Thus, in this case the problem of flow-invariance of D is a problem of existence. In such a way Theorem 5.4 must be understood (with $D = S(r)$). Moreover, in this case $D = S(r)$, any solution of (2.4) with $f: D\to H$ Lipschitz, is defined on the whole $[0,+\infty)$.

Proof of Theorem 5.4 The necessity. Let $x\in S(r)$ and $\bar{y}\in H$ be such that $\langle x,\bar{y}\rangle = 0$ and $\|\bar{y}\| = 1$. Then with $y = (-\langle x,f(x)\rangle)^{1/2}\bar{y}$, it follows $(x,y)\in M_{S(r)}$. In view of the above remark, the flow-invariance of S(r) means the existence of a S(r)-valued solution for each initial conditions $(u(o),u'(o))\in M_{S(r)}$. Denote by $u = u(t,x,y)$ the solution of (2.4) with $u(o) = x, u'(o) = y$. According to (5.43) and to $\langle x,f(x)\rangle = -\|y\|^2$ we get (i) (since $\|y\| = \|u'(o)\| = k$).

Therefore, $M_{S(r)}$ can be written under the form

$$(5.44)\quad M_{S(r)} = \{(x,y)\in S(r)\times H, \|x\| = r, \|y\| = k, \langle x,y\rangle = 0\}$$

and (2.6) holds for each $(x,y)\in M_{S(r)}$.

According to Lemma 3.3, there exist $r_j(h)\in H$, with $r_j(h)\to 0$ as $h\downarrow 0$ ($j = 1,2$) such that

$$(5.45)\quad (x+h(y+r_1(h)), y+h(f(x)+r_2(h)))\in M_{S(r)}$$

for all $h>0$.

Consequently, $\|y+h(f(x)+r_2(h))\|^2 = k^2$, ($\|y\| = k$) for all $0<h$, which yields (ii).

The sufficiency. We now assume that (i) and (ii) holds. First of all

(i) implies the form (5.44)(of $M_{S(r)}$ given by (5.40)).For any solution u of (2.4) we have $(u(t),u'(t))\in M_{S(r)}$ (Theorem 2.2),therefore (5.43) holds.Therefore,it remains to prove the existence of the solution,i.e. to verify that (i) and (ii) implies (2.6).Using once again Lemma 3.3,we have to prove the existence of $r_j(h)$ sytisfying (5.45) (for each $(x,y)\in M_{S(r)}$).

Claim that for $(x,y)\in M_{S(r)}$,r_j defined by

(5.46) $\tilde{x} = x+hy+hr_1(h) = \dfrac{r(x+hy)}{\|x+hy\|}$,$\tilde{y}=y+hf(x)+hr_2(h)=\dfrac{k(y+hf(x))}{\|y+hf(x)\|}$

satisfies (5.45).

Indeed,$(x,y)\in M_{S(r)}$ and (ii) implies $\langle x+hy,y+hf(x)\rangle = 0$,and consequently $\langle x,y\rangle = 0$.Since $\|\tilde{x}\|= r$, $\|\tilde{y}\|= k$ it remains to prove that $r_j(h) \to 0$ as $h\downarrow 0$.

Inasmuch as $(x,y)\in M_{S(r)}$ and $\langle f(x),y\rangle = 0$,with an elemetary calculus, from (5.46) we get

$$\|r_1(h)\| = \frac{1}{h}|r-\|x+hy\|| = \frac{|r^2-\|x+hy\|^2|}{h(r+\|x+hy\|)} = \frac{h\|y\|^2}{r+\|x+hy\|}$$

$$\|r_2(h)\| = \frac{h\|f(x)\|^2}{k+\|y+hf(x)\|}$$

which shows clearly that $r_j(h)\to 0$ as $h\downarrow 0, j = 1,2$.The proof is complete.

Now let us show briefly how Theorem 5.4 explains the launching of an Earth's satellite in a circular orbit (when the oblateness of the Earth,air resistence,the attraction of other celestial bodies,a.s.o. are neglected).

In this case f is the Newtonian gravitational field i.e.

(5.47) $f(x) = - GM \dfrac{x}{\|x\|^3}$

where $GM > 0$,is the power of the force center,

$S(r)\subset R^3$ is the sphere of radions r about Earth's center O (as the force center)) and $r = R+r_o$ ($r_o > 0$,R - the radius of the Earth).

The equation of motion is supposed to be (2.4) with f given by (5.47) As a corollary of Theorem 5.4 we obtain at once the following classical result of flight space.

Corollary 5.4 If a body is projected from the point x_0 at a distance $r \hat{=} R+r_0$ from the Earth's center, with the speed y_0 (parallel to the Earth's surface) of magnitude

(5.47)' $\|y_0\| = \sqrt{\frac{GM}{R+r_0}}$,

then the body describes (uniformly) the circle

$C(r) = P(x_0,y_0) \cap S(R+r_0)$,

where $P(x_0,y_0)$ is the plane spanned by the vectors x_0 and y_0.

Proof. In this case we have

$$\langle x,f(x)\rangle = -\frac{GM}{r}, \quad \forall x \text{ with } \|x\| = r = R+r_0,$$

therefore the condition (i) of Theorem 5.4 holds with $k^2 = \frac{GM}{r}$. Obviously, $(x_0,y_0) \in M_{C(r)}$ given by (5.44). Since (ii) is clearly satisfied, it follows that $C(r)$ is a flow-invariant set, which concludes the proof.

Other applications of this type will be given in the sequel.

5.5. The invariance of the conic in Newtonian field.

We shall apply Theorem 2.8 in the case of the conic

(5.48) $C = \{x \in R^2; \|x-\bar{a}\|^2 - (\langle\bar{b},x-\bar{a}\rangle + d)^2 = 0\}$,

where $\bar{a},\bar{b} \in R^2$, $d \in R$ and $d \neq 0$, $\bar{a}$ as one of the foci.

In this case $C = D_g$ with g given by (5.9). A general result on C is given by Proposition 5.1. First of all we shall discuss (2.24) for each particular conic and f given by (5.47) and then we shall verify (2.13). We have already considered the circle C(r) in Corollary 5.4.

5.5.1 Elliptic orbit

Let us consider the ellipse

(5.49) $E = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2, \ \frac{(x_1+c)^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0 \right\}$

where the rectangular axes are taken through one of the foci, 2 a(resp. 2 b) is major (minor) axis and $a^2 - b^2 = c^2$.

Denote by 0 the intersection of axes and

(5.50) $g(x) = \frac{1}{2}\left(\frac{(x_1+c)^2}{a^2} + \frac{x_2^2}{b^2} - 1 \right)$, (hence $D_g = E$).

Lemma 5.1 (1) Let $A = R^2 - \{0\}$. With f given by (5.47)(2.24) and (2.11) become respectively

(5.51) $M_E = \left\{ (x,y) \in E \times R^2, \ \frac{(x_1+c)y_1}{a^2} + \frac{x_2 y_2}{b^2} = 0, y_1^2 + y_2^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \right\}$

where y_1, y_2 are the coordinates of y and $r = (x_1^2 + x_2^2)^{1/2} = \|x\| < 2a$

(5.52) $w(x) = \dot{g}(x)(f(x)) = -\frac{GM}{ar^2}, \ x \in E.$

Proof. It is easy to check that

(5.53) $r = \frac{1}{a}(b^2 - cx_1), \ a(r-a) = -c(x_1+c), x \in E$

In this case (with the notations (2.15)-(2.22),

(5.54) $g_1(x) = \frac{x_1+c}{a^2}, \ g_2(x) = \frac{x_2}{b^2}, g_{11} = \frac{1}{a^2}, \ g_{22} = \frac{1}{b^2}, \ g_{12} = 0$

and therefore

(5.55) $a_1(x) = \frac{1}{a^2 b^2}(2ar - r^2), \ a_2(x) = \frac{1}{a^2 b^2}, \ x \in E$

Finally

(5.56) $\dot{g}(x)(f(x)) = -\frac{GM}{r^3}\left(\frac{x_1(x_1+c)}{a^2} + \frac{x_2^2}{b^2} \right) = -\frac{GM}{ar^2}, \ x \in E$

Combining (5.54),(5.55)(5.56)(2.22) and (2.24) one obtains (5.51).

5.5.2 Hiperbolic orbit

If we take

(5.57) $g(x) = \frac{1}{2}\left(-\frac{x_1+c}{a^2} + \frac{x_2^2}{b^2} + 1 \right), \ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$

then D_g is the hyperbola

(5.58) $$H = \left\{ x \in R^2 - \{0\}, \frac{(x_1+c)^2}{a^2} - \frac{x_2^2}{b^2} - 1 = 0 \right\}.$$

where the rectangular axes are taken through one of the foci, O(o,o) and

(5.59) $$r = \|x\| = \frac{1}{a} | b^2 + cx_1 | , x \in H, c^2 = a^2 + b^2$$

It is necessary to consider the following two cases (for A)

(5.60) $$A_1 = \left\{ x \in R^2 - \{0\}, x_1 > -\frac{b^2}{c} \right\}$$

(5.61) $$A_2 = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2 \quad , x_1 < -\frac{b^2}{c} \right\}$$

Denote by $H_i = H \cap A_i$ the branch of H contained in A_i, i=1,2.
Obviously O is regarded as the center of the force field (5.47) and H_1 is the branch of H around this center.
We shall prove (in 5.5.3) that H_1 is a flow-invariant set under f given by (5.47), while H_2 has this property under "repulsive" field (of center O),

(5.62) $$\bar{f}(x) = GM \frac{x}{\|x\|^3} , x \in A_2$$

this type of field (5.62)(of the inverse square repulsion $K/\|x\|^2, K > 0$ per unit mass) is useful in physics in connection with the bombardament of atoms by α- particles.

<u>Lemma</u> 5.2 (1) <u>With f given by</u> (5.47) <u>and</u> $D_g = H_1$, (2.24) <u>and</u> (2.11), <u>become respectively</u>

(5.63) $$M_{H_1} = \left\{ (x,y) \in H_1 \times R^2, \frac{y_1(x_1+c)}{a^2} - \frac{y_2 x_2}{b^2} = 0, y_1^2 + y_2^2 = GM\left(\frac{2}{r} + \frac{1}{a}\right) \right\}$$

(5.64) $$\dot{g}(x)(f(x)) = -\frac{GM}{a^2 r^3}(b^2 + cx_1) = -\frac{GM}{ar^2}, x \in H_1 = H \cap A_1$$

(2) <u>In the repulsive case</u> (5.62) <u>and</u> $D_g = H_2$,

(5.65) $$M_{H_2} = \left\{ (x,y) \in H_2 \times R^2, \frac{y_1(x_1+c)}{a^2} - \frac{y_2 x_2}{b^2} = 0, y_1^2 + y_2^2 = GM\left(\frac{1}{a} - \frac{2}{r}\right) \right\}$$

and

(5.66) $\dot{g}(x)(\bar{f}(x)) = \frac{GM}{a^2r^3}(b^2+cx_1) = -\frac{GM}{ar^2}$, $x\in H_2 = H\cap A_2$.

Proof.(1)First of all, in this case

(5.67) $r = \frac{1}{a}(b^2+cx_1)$, $a(a+r) = c(c+x_1)$, $x\in H_1$

$$g_1(x) = -\frac{x_1+c}{a^2}, \quad g_2(x) = \frac{x_2}{b^2},$$

and (5.64) follows at once. Further, by (5.67) we have (for $x\in H_1$)

(5.68) $a_1(x) = g_1^2+g_2^2 = \frac{(x_1+c)^2}{a^4} + \frac{x_2^2}{b^4} = \frac{1}{a^4b^2}(c^2(x_1+c)^2-a^4) = \frac{1}{a^2b^2}(r^2+2ar)$

(5.68)' $\quad a_2(x) = \frac{1}{a^2b^2}$

which yield (5.63).

(2) In this case $b^2+cx_1<0$, therefore (5.59) gives

(5.69) $r = -\frac{1}{a}(b^2+cx_1)$ (hence $a(a-r) = -c(c+x_1)$), $x\in H_2$.

Therefore

$$a_1(x) = \frac{1}{a^4b^2}\left[c^2(x_1+c)^2-a^4\right] = \frac{1}{a^2b^2}(r^2-2ar),\ x\in H_2$$

while a_2 is given by (5.68)', too.

According to (5.69) we get easily (5.66). Replacing a_1, a_2 and $\dot{g}(x)(\bar{f}(x))$ in (2.24), (5.65) follows.

5.5.3 Parabolic orbit

Denote by P the parabola

(5.70) $$P = \left\{x\in R^2,\ x_2^2 = p^2+2p\,x_1\right\}$$

where p is the distance from the focus O (as the origin of the rectangular axes) to the directrix.

Clearly $P = D_g$ with

$$g(x) = \frac{1}{2}(x_2^2-2px_1-p^2)$$

Using the elementary fact that in this case

$$r = \|x\| = p+x_1, \text{ if } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in P$$

we get immediately (for f given by (5.47))

(5.71) $\dot{g}(x)(f(x)) = -GM\, p/r^2 \;, \; a_1 = 2pr, a_2 = p^2$

In view of (5.71),(2.24) becomes

(5.72) $M_P = \left\{ (x,y) \in P \times R^2;\; py_1 = x_2y_2 . y_1^2 + y_2^2 = \frac{2GM}{r} \right\}$.

The speed y of a mass particle (at the point $x \in P$) such that $(x,y) \in M_P$ (therefore having the magnitude $y^2 = 2GM/r$, $r = \|x\|$) is so called "the velocity of escape from the force center" (cf.McCuskey [13] p.27). The speed appearing in ((5.51),(5.63) and (5.72) is discussed also in [15] p.28,29 and in [32],p.88.

5.5.4 Flow-invariance of the conic.

The general equation of the conic C is given by (5.48).We now are able to prove (via flow-invariance method) the following result.

Theorem 5.5 (1) Any conic C is a flow-invariant set for the equation (2.4) with

(5.73) $f(x) = -GM \frac{x-\bar{a}}{\|x-\bar{a}\|^3} \;, \; x \neq \bar{a}$

where in the case of hyperbola, $C = H_1$ (i.e. the branch around the focus $\bar{a}$ as the center of the attractive field (5.73)).

(2) The other branch H_2 of the hyperbola is flow-invariant for (2.4) with the repulsive field

(5.74) $\bar{f}(x) = GM \frac{x-a}{\|x-a\|^3} . \; x \neq \bar{a}$

where GM is the power of the force center $\bar{a}$.

Proof. We apply Theorem 2.8.We have already seen that in this case M_{D_g} (with g given by (5.9)) is non↕empty (actually,that (2.23) holds). This fact is proved by (5.40)(or 5.44),(5.51),(5.63),(5.65) and (5.72).

emains to verify (2.25) or equivalently (2.13). In this case $\dddot{g}(x)=$ and f is differentiable on $R^2 - \{\bar{a}\}$, therefore, (2.13) becomes

$$(5.75)\quad 3\ddot{g}(x)(f(x))(y)+\dot{g}(x)(\dot{f}(x)(y)) = 0, \text{ for all } (x,y) \text{ with } g(x) = 0, \dot{g}(x)(y) = 0$$

Indeed, in this case the derivative of w given (by (2.11)) in the direction y is the following one

$$\dot{w}(x)(y) = \ddot{g}(x)(f(x))(y)+\dot{g}(x)(\dot{f}(x)(y)).$$

and $N(\dot{g}(x))$ defined by (5.3) is a subspace of one dimension). Let us observe that the derivative of the function

$$h(x) = \|x-\bar{a}\|, x \in R^2 , x \neq \bar{a} \text{ is just}$$

$$\dot{h}(x)(y) = \frac{\langle x-\bar{a},y\rangle}{\|x-\bar{a}\|}$$

Using this remark as well as (5.11), (5.12) and (5.13), we have successively

$$\dot{f}(x)(y) = -\frac{GM\ y}{\|x-\bar{a}\|^3}+\frac{3GM\langle x-\bar{a},y\rangle}{\|x-\bar{a}\|^5}(x-\bar{a}) ,$$

$$\ddot{g}(x)(f(x))(y) = 3\langle f(x),y\rangle-3\langle\bar{b},f(x)\rangle\langle\bar{b},y\rangle$$

$$I = 3\ddot{g}(x)(f(x))(y)+\dot{g}(x)(\dot{f}(x)(y)) = -\frac{3GM}{\|x-\bar{a}\|^3}(\langle x-\bar{a},y\rangle- \langle\bar{b},x-\bar{a}\rangle\langle\bar{b},y\rangle)+\frac{3GM\ d\langle x-\bar{a},y\rangle}{\|x-\bar{a}\|^5}(d+\langle\bar{b},x-\bar{a}\rangle),$$

$$- GM\ \dot{g}(x)(y)\frac{1}{\|x-\bar{a}\|^3}+\frac{3GM\langle x-\bar{a},y\rangle}{\|x-a\|^5}\dot{g}(x)(x-\bar{a})$$

where we have also used

$$\dot{g}(x)(x-\bar{a}) = d(d+\langle\bar{b},x-\bar{a}\rangle), \forall x \in C$$

Since $\dot{g}(x)(y) = 0$ gives,

$$\langle x-\bar{a},y\rangle-\langle\bar{b},x-\bar{a}\rangle\langle\bar{b},y\rangle= d\langle\bar{b},y\rangle , \quad \forall x \in C,$$

I becomes,

$$I = -\frac{3GM\ d}{\|x-\bar{a}\|^5}\left[\|x-\bar{a}\|^2\ \langle\bar{b},y\rangle -\langle x-\bar{a},y\rangle\ (d+\langle\bar{b},x-\bar{a}\rangle)\right] = \frac{3GM\ d}{\|x-\bar{a}\|^5}(d + \langle\bar{b},x-\bar{a}\rangle)\dot{g}(x)(y) = 0$$

for $g(x) = 0$ and $\dot{g}(x)(y) = 0$, hence (5.75) holds for f given by (5.73). Since $\bar{f} = -f$, the proof is complete.

5.5.5 Central force field

Recall that $f: A \subset R^3 \to R^3$ (see 5.1) is said to be a "central force field" of center O if for each vector position $x \in A$, the force $f(x)$ (associated with x) acts along the vector which joind O with x. The central force field is said to be "attractive" ("repulsive") if for each $x \in A$, the vector force $f(x)$ is directed toward or away from O, respectively.

In terms of dynamics, Theorem 5.5. asserts that a body P projected from a point x C with a velocity y such that $(x,y) \in M_C$ (given by (5.51) a.s.o) describes (under the action of the field (5.73)) the orbit C. Or, in Dynamics it is known much more, namely the following famous result holds:

Theorem 5.6 (1) A mass particle Q moving under a central field of force describes on orbit which lies in a plane.

(2) If Q describes the conic C with constant areal velocity, then the force acting on it varies inversely as the square of the distance from Q to the focus $\bar{a}$ of C.

(3) Conversely, if Q is projected from any point $x \in C$ with the speed y such that $(x,y) \in M_C$ given by (5.44) (with $k = \sqrt{\frac{GM}{r}}$, $r = \|x\|$), (5.51) a.s.o., describes (under the action of f (5.73) or (5.74)), the conic C with constant areal velocity (relative to the focus $\bar{a}$.).

In our framework (i.e. via flow-invariance method) this theorem can easily be proved as follows.

(1) Let Q move under the central force field f of center O. Denote by x_o (resp y_o) the initial position (velocity) of Q and by S-the closed linear subspace of R^3 spanned by the vectors x_o and y_o. If $y_o \neq 0$ and y_o is not parallel to x_o then S is a plane (otherwise S is a straight line containing $x_o \neq 0$). Let $A = R^3 - \{0\}$ and $x \in D = A \cap S$. Since f(x) acts along the vector which joins x with O and $x, O \in S$, it follows that f(x)

S(i.e. $f(D)\subset S$) and therefore (by Theorem 2.5) D is a flow-invariant set for (2.3). Obviously $D = \bar{D}\cap A$ (i.e. D is closed in A) since $D = S - \{0\}$ and hence $\bar{D} = S$. Inasmuch as $(x_o, y_o)\in D\times S = M_D$, in view of Theorem 2.3 the orbit $u = u(t), t\geqslant 0$ of Q lies in D, hence in S.

(2) For the sake of simplicity we shall make a choise, supposing e.g. that C is the Ellipse E (5.49).

Therefore let Q describe E with constant areal velocity (cf. [13] p.5), relative to the focus O as the origin of the rectangular axes. First of all, this implies that the force field f acting on it is central, i.e. for each $x\in E$, there is $h(x)\in R$ such that

(5.76) $\quad f(x) = h(x)x$

If $u = (u_1(t), u_2(t))$ is the law of motion of Q, then the constance of areal velocity (relative to the focus O) means (cf. [13], p.6)

$$(5.77)\qquad u_1(t)u'_2(t) - u_2(t)u'_1(t) = K = \text{const}, \quad t\geqslant 0$$

Since u is E-valued then $(u(t), u'(t))\in M_E$ (Theorem 2.2) therefore

$$(5.78)\quad \begin{cases} y_1(x_1+c)/a^2 + x_2y_2/b^2 = 0 \\ x_1y_2 - x_2y_1 = K \end{cases}$$

for all $(x,y)\in M_E$ given by (5.51), where we have denoted $x_i = u_i(t)$ and $y_i = u'_i(t), i = 1,2$.

Using some of (5.47)-(5.53) one observes that the solution (y_1, y_2) of (5.78) is the following one

$$y_1 = -\frac{aKx_2}{rb^2}, \quad y_2 = \frac{K(x_1+c)}{ar}, \quad (r = \|x\|,\ x\in E)$$

and hence

$$(5.79)\qquad y^2 = y_1^2 + y_2^2 = \frac{K^2}{b^2r^2}(2ar - r^2), \quad x\in E.$$

On the other hand, by (2.24), (5.74) and (5.76) we have

(5.79)' $y^2 = -\frac{a_1}{a_2}\langle \dot{g}(x), f(x)\rangle = -h(x)\frac{a_1}{a_2}(g_1x_1 + g_2x_2) = -h(x)r(2ar - r^2)/_a$

for each $x \in E$.

Comparing (5.79) and (5.79)' we conclude that

(5.80) $h(x) = -\frac{aK^2}{b}\frac{1}{r^3}$ $(r = \|x\|)$,

(3) We now assume that f is given by (5.47).If Q is projected from any point $x \in C$ as mentioned in the theorem,then Q describes under f, $S(r), E, H_1$ and P respectively (and H_2 under repulsion (5.62)).This aspect has been already proved (Theorem 5.5 as well as Corollary 5.4 in the case $C = S(r)$).It remains to prove that Q obeys the law of areas. (i.e. that the areal velocity relative to the focus O is constant). Indeed,the areal velocity (denote it 2K(t)) is given by

$$K(t) = u_1(t)u_2'(t) - u_2(t)u_1'(t)$$

as we have seen above (see (5.77)).

Working in the case of E too,(5.80) holds with $h(x) = -GM/_{r^3}$(by (5.47)),therefore

$$K^2(t) = -r^3bh(x)/_a = GM\ b/_a = \text{const}$$

5.5.6 <u>The force field under which a conic can be described</u>

Applying Theorem 5.3 we can derive all g-smooth force field under which the conic C can be described (with g given by (5.9)

For the simplicity we shall treat this problem in the case of C(r) only,where

(5.81) $C(r) = \{x \in R^2, \|x\| = r\}$, $r > 0$

The rest of discussion is left to the reader.Obviously,C(r) corresponds to D_g with

$$g(x) = \frac{1}{2}(\|x\|^2 - r^2) = \frac{1}{2}(x_1^2 + x_2^2 - r^2)$$

In this case $\dot{g}(x) = x$ and (5.26) becomes

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

In view of theorem 5.3 (formula (5.28)") we get

Corollary 5.5 (1) All force field under which C(r) can be described are given by

$$(5.82) \quad f(x) = -z(x)x + \frac{r^2}{2}\dot{z}(x) - \frac{1}{2}\langle x, \dot{z}(x)\rangle x,$$

where $z\colon R^2 - \{0\} \to R$ is a continuously differentiable function and

$$z(x) > 0 \text{ for } \|x\| = r$$

(2) If in addition we assume that the function z is positively homogeneous of degree 0(i.e. $z(tx) = z(x), \forall t > 0$), then for each natural number n, the sphere

$$S(r) = \left\{x \in R^n, \|x\| = r\right\}$$

can be described under the force field

$$(5.83) \quad f(x) = -z(x)\, x + \frac{r^2}{2}\dot{z}(x)\ , \ x \in R^n\ , \ x \neq 0$$

Proof (1) This assertion follows directely from (5.28)", as we have already mentioned.

(1) In the case n = 2, this assertion follows from (5.82) and the Euler's theorem on continuously differentiable, positively homogeneous functions (of degree zero), namely

$$(5.84) \quad \langle x, \dot{z}(x)\rangle = 0, \ x \in R^n (\dot{z}(x) = \text{grad } z(x))$$

For $n \geq 3$, we have to prove it. First of all, for f given by (5.83), the subset $M_{S(r)}$ given by (5.40) is the following one

$$(5.85) \quad M_{S(r)} = \left\{ (x,y) \in S(r) \times R^n, \langle x,y\rangle = 0, \|y\|^2 = r^2 z(x) \right\}$$

where (5.84) has been used.

We have to prove that any solution u of the equation

$$u'' = -z(u)u + \frac{r^2}{2}\dot{z}(u)$$

with $(u(o),u'(o)) \in M_{S(r)}$) is S(r)-valued (i.e. $\|u(t)\| = r$ as long as it exists).

To do that, set

$$v(t) = \frac{1}{2}(\|u(t)\|^2 - r^2).$$

Using (5.85) it is easy to check that v satisfies the linear (scalar) differential equation

(5.86) $\quad v'''(t) = -4z(u(t))v'(t) - 2\langle u'(t), \dot{z}(u(t))\rangle v(t)$ and the initial conditions

$$v(o) = 0, v'(o) = \langle x,y\rangle = 0,\ v''(o) = \|y\|^2 - r^2 z(x) = 0$$

where $u(o) = x, u'(o) = y$ (i.e. $(x,y) \in M_{S(r)}$).

Therefore v is the trivial solution of (5.86), which implies $\|u(t)\| = r, t \geq 0$.

Remark 5.8 (1) The notion "S(r) can be described under f" is obviously interpreted as "S(r) is flow-invariant set for (2.4)" (see (5.1) and (5.2)).

(2) By Theorem 5.4 it follows that the motions on S(r) under force field (5.83) are uniform iff

$$\langle x, f(x)\rangle = -r^2 z(x) = -k^2 = \text{const}, \forall x \in S(r)$$

(i.e. iff f is the attractive field $f(x) = -kx$)

(3) According to (2.27) it follows that f given by (5.82) is the solution of the partial differential equation

(5.87) $\quad 3x_2 f^1 - 3x_1 f^2 + x_1 x_2 \left(\frac{\partial f^1}{\partial x_1} - \frac{\partial f^2}{\partial x_2}\right) - x_1^2 \frac{\partial f^1}{\partial x_2} + x_2^2 \frac{\partial f^2}{\partial x_1} = 0$

where $x_1^2 + x_2^2 = r^2$.

which characterizes the continuously differentiable force field $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ under which C(r) can be described.

6. FINAL REMARKS

1. In this paper the force field $f: A \subset X \to X$, (X a real Banach space $Y = R^n$) is supposed to depend only on the position x of the "particle" on which it acts (i.e. $f = f(x)$). The methos we have developed allows however to treat (at least up to a certain point) the flow-invariance of $D \subset A$ in a more general case in which $f = f(x,y)$ i.e. f depending both on the position x and of the velocity y (of the particle on which it acts). Indeed, it is easy to check that in this case we have to put $f(x,y)$ instead of $f(x)$, in (2.5), in Theorem 2.4 and (after corresponding modifications) in its consequences.

In this case the speed y of "projection" from $x \in D_g$ is a solution of the system

$$(6.1) \quad \begin{cases} \ddot{g}(x)(y)(y) + \dot{g}(x)(f(x,y)) = 0 \\ \dot{g}(x)(y) = 0 \quad , \quad (g: A \to R^n) \end{cases}$$

Let us assume that such a solution y exists and (2.6) holds (with $f(x,y)$ instead of $f(x)$ and $D = D_g$). Then a body projected from $x \in D_g$ with the speed y, describes an orbit $u(t) = u(t,x,y)$ which lies in D_g (i.e. $g(u(t)) = 0$, as long as u exists) Practically, an exact solution y of (6.1) is difficult to obtain. Then the problem of stability of D_g arises, namely:

Given $\varepsilon > 0$, is there $\delta > 0$ such that if

$$\| y_0 - y \| < \delta \text{, then } \| u(t,x,y_0) - u(t,x.y) \| < \varepsilon \text{ , } \forall t \geq 0?$$

In words, the stability of the orbit D_g means that given a neighborhood V of D_g (see (2.7)), if y_0 is an approximate solution to (6.1) sufficiently close by the exact solution y, then a body projected from $x \in D_g$ with the speed y_0, describes (under f) an orbit which lies in V. Therefore it would be interesting to study the problem of flow-invariance of the set D_g with to the equation

(6.2) $u'' = f(u,u')$

and then the stability of this set. The above remarks hold in the non-autonomous case $f = f(t,u,u')$ too.

2. The (sufficient) surjectivity conditions required in the Theorem 2.6 (which is a consequence of Theorem 2.4) are not the best. This, because in some important cases, these are imposible (Namely, when $X = R^m, Y = R^n, m < 2n$).

Indeed, e.g. in the case $X = R^3, Y = R^2, g: A \subset X \to R^2$ is a vector function, therefore

$$g = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}, \dot{g}(x) = \begin{pmatrix} \dot{g}^1(x) \\ \dot{g}^2(x) \end{pmatrix}, g^i: A \to R, \ \dot{g}^i(x) \in L(R^3, R)$$

Consequently, the surjectivity of $u \to (\dot{g}(x)(u), \ddot{g}(x)(y)(u))$, is equivalent to the linear independency of the following four linear functionals $L_i: R^3 \to R$

$$L_i = \dot{g}^i(x), \ L_{i+2} = \ddot{g}^i(x)(y) \quad , \ i = 1,2.$$

Or, four (2n) linear functionals on R^3 (resp $R^m, m < 2n$) are always linear dependent. Therefore, another open problem is to derive a consequence of Theorem 2.4, better than Theorem 2.6, to cover the cases $X = R^m, X = R^n$ with $m < 2n$ (not excludet by the first theorem)

3. We are also interested in the problem mentioned in Remark. 5.3.

4. The conclusion of Theorem 2.4 remains valid if f is defined only on D (see Remark 5.7). In this case, to obtain its consequences (as in § 2) we have to use the directional derivatives of $x \to \dot{g}(x)(f(x))$, e.g. in the sense of [27-29] (since the Fréchet derivative is defined on open subsets, only). We didn't investigate this (rather akward) fact, although it seems to be possible.

5. Does the conclusion of Theorem 2.6 hold if R^n is replaced by any Banach space Y ?

6. It would be important to study the problem of flow-invariance of a time-dependent set D = D(t). A suggestion to treat such a problem is to make use of the generalization of Theorem 2.1(of Nagumo-Brezis-Martin) to the case D = D(t) given in [24] . First of all it is necessary the adaption of the results of § 3 to time dependent case D(t).

7. For constrained control problem would be important to have results on flow-invariance of D_g for (2.1) with f = f(t,u)(i.e.f -time dependent) in Carathéodory type conditions. A rezult in this direction has been given by C. Ursescu, but only in a general case $D \subset X$ (see [17], p.190).

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