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ISSN 0250-3638

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PREPRINT SERIES IN MATHEMATICS

No.19/1980

BUCURESTI

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April 1980

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EXISTENCE FOR A CLASS OF STOCHASTIC PARABOLIC VARIATIONAL INEQUALITIES

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Abstract. In this paper we deal with the existence of a nonanticipative stochastic process u, solution of the stochastic parabolic variational inequality:

$$E\int_0^T \langle v'+A(t)u-f,v+m+M-u\rangle dt + \frac{1}{2}E|v(0)-u_o|^2 + \frac{1}{2}E|m(T)|^2$$
$$+ E\int_0^T \varphi(t,v+m+M)dt \geq E\int_0^T \varphi(t,u)dt$$

for all v and m in some fixed spaces of stochastic processes (m and M are martingales). If φ is a convex indicator function , we obtain a maximal solution.

§0. Introduction

Let V and H be two real separable Hilbert spaces, and V densely and continuously embedded in H. It is known that, for the stochastic differential equations:

$$(0.1)\qquad \begin{cases} du + A(t,u)dt + \partial\varphi(t,u)dt \ni f(t)dt + dM(t) \\ u(0) = u_o \end{cases}$$

where $A(t,.)+\partial\varphi(t,.): V \to V^*$ is an unbounded operator and M is a martingale with values in H (for example a H-valued Wiener process), we cannot generally find strong stochastic solutions (Itô solutions). On the other hand if we denote u-M=z, Eq.(0.1) becomes

$$(0.2)\qquad \begin{cases} \frac{dz}{dt} + A(t,z+M(\omega,t)) + \partial\varphi(t,z+M(\omega,t)) \ni f(\omega,t) \\ z(\omega,0) = u_o(\omega) \end{cases}$$

which is a time dependent random differential equation. By a strong solution of Eq.(0.2) we mean a function $z(\omega,t)$ which is absolutely continuous in t almost sure (a.s.) in $\omega\in\Omega$, and which satisfies Eq.(0.2) almost everywhere (a.e.) on $]0,T[$ ($u=z+M$ is Itô solution for (0.1)). In general the time dependent random differential equation (0.2) does not admit a strong solution.

For these reasons we introduced a concept of weak solution for Eq.(0.1) (which will be given in Def.2.4). For M=0, any deterministic process which is a weak solution in this stochastic sense, is also a weak solution in the deterministic sense from [3],[6],[14].

We shall prove the existence of weak solutions in Theorem 2.7 and Theorem 3.3 . Example 5.1 shows that the weak solution is not unique. That is why in Section 4 we try to individualize a weak solution.

In order to find a maximal element in the set of weak solutions some assumptions of noncorrelation would have been necessary, which would have been difficult to verify on examples. That is why we abandoned this idea, and instead, we have sought for a majorant of the set of weak solutions which is a solution of Eq.(0.1) in a certain sense, quite close to the weak one. Def. 2.11 introduces the concept of "almost weak" solution, and in Theorems 2.13 and 3.7 are given existence results for an "almost weak" solution of Eq.(0.1). Under some ordering hypotheses, for $\varphi(t,u)=I_{K(t)}(u)$ (the convex indicator function for K(t)), an "almost weak" solution of Eq.(0.1), which maximizes the set of weak solutions will be found (Th.4.2). Also we give a method to approximate the maximal solution.

In this paper we use certain results of Pardoux from [15],[16] and also some results of nonlinear analysis in infinite dimensional spaces from [6],[7],[8],[17]. Our technique is based on Itô's formula (energy equality).

The plan of the paper is the following. In Section 1 we give the basic notions and notations. In Section 2 we prove existence theorems (Th.2.7 and 2.13) for weak and "almost weak" solutions. Proposition 2.6 shows that Itô solutions are weak solutions. In Section 3 we prove the existence of weak solutions (Th.3.3) and "almost weak" solutions (Th.3.7) under weaker coercivity conditions on A, than those used Theorems 2.7 and 2.13, but in some additional ordering hypotheses The "almost weak" solution, $\bar{u}$, in Theorems 2.13 and 3.7 "maximizes" the set of weak and "almost weak" solutions (Th.4.2 in Section 4). Finally in Section 5 we give some examples of stochastic variational problems.

§1. Preliminaries

A. Troughout this paper we shall consider a given probabilized stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\geq 0})$ which means that $(\Omega,\mathcal{F},P)$ is a complete probability space and $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sub-σ-algebras of $\mathcal{F}$, such that $A\in\mathcal{F}_0$ if $P(A)=0$

B. (see [18]) If X is a reflexive separable Banach space and $\xi\in L^p(\Omega,\mathcal{F},P;X)$, $1\leq p\leq+\infty$, then there exists a unique element $E(\xi|\mathcal{F}_t)$ $\in L^p(\Omega,\mathcal{F}_t,P;X)$, called conditional expectation of with respect to $\mathcal{F}_t$, and defined by

$$(1.1)\qquad E(1_F\xi) = E(1_F E(\xi|\mathcal{F}_t)),$$

for every $F\in\mathcal{F}_t$, where

$$(1.2)\qquad E\xi = \int_\Omega \xi(\omega)dP(\omega) \qquad (= \text{expectation of } \xi)$$

and

$$(1.3)\qquad 1_F(\omega) = 1 \text{ if } \omega\in F, \quad = 0 \text{ if } \omega\notin F.$$

C. Let $L^p_t(\Omega\times]0,T[;X)$, $1\leq p\leq+\infty$, be the space of stochastic processes $f\in L^p(\Omega\times]0,T[,\mathcal{F}\times\mathcal{L},P\times dt;X) = L^p(\Omega\times]0,T[;X)$ with the property: there is a representative $\bar{f}$ of the equivalence class f such that $\bar{f}(t)$ is $\mathcal{F}_t$-measurable a.e. on $]0,T[$ ($\bar{f}$ is called nonanticipative process). We denoted by $\mathcal{L}$ the σ- algebras of Lebesgue measurable sets on $]0,T[$ and with dt the Lebesgue's measure. The space $L^p_t(\Omega\times]0,T[;X)$ is a linear closed subspace of $L^p(\Omega\times]0,T[;X)$.

We denote by $L^p_t(\Omega;C(0,T;X))$, $1\leq p\leq+\infty$, the space of all stochastic processes $f\in L^p(\Omega,\mathcal{F},P;C([0,T];X)) = L(\Omega;C(0,T;X))$ with the property: there is a function $\bar{f}$ belonging to the class f such that $\bar{f}(t)$ is $\mathcal{F}_t$-measurable for every $t\in[0,T]$ ($\bar{f}$ is called adapted process). The space $L^p_t(\Omega;C(0,T;X))$ is a linear closed subspace of $L^p(\Omega;C(0,T;X))$.

It is quite easy to show that in for each of the two cases any representative of the class f has the same property as the corresponding $\bar{f}$.

D. (see [9]-[13],[15],[16]) If $(H,|\cdot|)$ is a real separable Hilbert space, we denote by $\mathcal{M}^p(0,T;H)$ the space of stochastic processes $M\in L^p_t(\Omega;C(0,T;H))$ with the properties:

$$(1.4)\qquad \begin{aligned}&\text{(i)}\ \ M(0) = 0 \ \text{ a.s.}\\ &\text{(ii)}\ E(M(t)|\mathcal{F}_s) = M(s) \ \text{ a.s. , if } 0\leq s\leq t\leq T .\end{aligned}$$

The elements of this space are called continuous p-martingales.

Let M be a martingale belonging to $\mathcal{M}^2(0,T;H)$. By the Doob -Meyer's decomposition, there exists a unique stochastic process $\langle M\rangle \in L^1_t(\Omega;C(0,T;R))$ such that:

$$\text{(1.5)}\quad \begin{cases} \text{(i)} & t \longrightarrow \langle M\rangle(\omega,t) \text{ is increasing } \omega\text{-a.s.}, \\ \text{(ii)} & |M|^2 - \langle M\rangle \in \mathcal{M}^1(0,T;R). \end{cases}$$

Moreover the following inequality is satisfied

$$\text{(1.6)}\quad E \sup_{s\in[0,t]} |M(s)|^2 \leq 3E\sqrt{\langle M\rangle(t)} ,$$

for all $t \in [0,T]$.

For any $f \in L^2_t(\Omega;C(0,T;H))$ and $M \in \mathcal{M}^2(0,T;H)$ the stochastic integral

$$\int_0^t (f(s),dM(s))$$

is defined. In this case (see [15]- I-Partie, Th.3.3)

$$\int_0^{\cdot} (f(s),dM(s)) \in \mathcal{M}^1(0,T;R)$$

and

$$\text{(1.7)}\quad E \sup_{s\in[0,t]} \left|\int_0^s (f(\tau),dM(\tau))\right| \leq 3E \sup_{s\in[0,t]} |f(s)|\sqrt{\langle M\rangle_t} .$$

E. Let $(V,\|\cdot\|)$ and $(H,|\cdot|)$ be two real separable Hilbert spaces such that V is densely and continuously embedded in H. Let us denote by V^* the dual of V and identify H with its own dual. Then the following relation holds

$$V \subset H \subset V^*$$

algebraically and topologically. Denote by $\|\cdot\|_*$ the norm (dual) of V^* and by $\langle w,v\rangle$ the value of $w \in V^*$ in $v \in V$; if $w,v \in H$ then $\langle w,v\rangle = (w,v)$, where $(\cdot,\cdot)$ is the inner product in H.

<u>Proposition 1.1</u> (energy equality). If

$$\text{(1.8)}\quad \begin{cases} \text{(i)} & u \in L^2(\Omega\times]0,T[;V), \\ \text{(ii)} & u(t) = u_0 + \int_0^t v(s)ds + M(t), \\ \text{(iii)} & u_0 \in L^2(\Omega,\mathcal{F}_0,P;H), \\ \text{(iV)} & v \in L^2_t(\Omega\times]0,T[;V^*), \end{cases}$$

(V) $M \in \mathcal{M}^2(0,T;H)$,

then

(c_1) $u \in L^2_t(\Omega; C(0,T;H))$,

(1.9)

(c_2) $|u(t)|^2 = |u_0|^2 + 2\int_0^t \langle v(s),u(s)\rangle ds + 2\int_0^t (u(s), dM(s)) + \langle M\rangle_t$, for all $t \in [0,T]$; ω-a.s.,

(c_3) $E|u(t)|^2 = E|u_0|^2 + 2E\int_0^t \langle v(s),u(s)\rangle ds + E|M(t)|^2$, for all $t \in [0,T]$.

The proof may be found in [15] (Chap.II, §3).

Proposition 1.2 (see [15], Chap.III, §1) Consider the equation

(1.10)
$$du(t) + A(t,u(t))dt = f(t)dt + dM(t)$$
$$u(0) = u_0$$

If $A(t,.)$ is a family of nonlinear (single-valued) operators from V to V^* defined for almost every $t \in]0,T[$ (t-a.e.) such that :

(1.11)

(A_1)(coercivity) There exist $\sigma>0$, $\alpha \in R$, $\gamma \in R$ such that
$$\langle A(t,u),u\rangle + \alpha|u|^2 + \gamma \geq \sigma\|u\|^2, \text{ for all } u \in V; \text{ t-a.e.}$$

(A_2)(monotonicity) $\langle A(t,u) - A(t,v), u - v\rangle + \alpha|u - v|^2 \geq 0$, for all $u,v \in V$; t-a.e.

(A_3)(boundedness) There exists $C>0$ such that
$$\|A(t,u)\|_* \leq C\|u\| , \text{ for all } u \in V; \text{ t-a.e.}$$

(A_4)(hemicontinuity) $\theta \longrightarrow \langle A(t,u+\theta v),w\rangle$ is continuous from R to R, for all $u,v,w \in V$; t-a.e.

(A_5)(measurablity) For all $u \in V$, $t \rightarrow A(t,u)$ is strong measurable from $]0,T[$ to V^*,

and

(1.12)

(i) $u_0 \in L^2(\Omega,\mathcal{F}_0,P;H)$

(ii) $f \in L^2_t(\Omega\times]0,T[;V^*)$

(iii) $M \in \mathcal{M}^2(0,T;H)$,

then there exists a unique process

$$(1.13)\qquad u \in L^2_t(\Omega\times]0,T[;V) \cap L^2_t(\Omega;C(0,T;H))$$

which verifies the equation (1.10) in the following sense

$$(1.14)\qquad u(t) + \int_0^t A(s,u(s))ds = u_o + \int_0^t f(s)ds + M(t),$$

for all $t\in[0,T]$; ω-a.s.

Remark 1.3 By using Lemma III-14 from [7] and the Pettis theorem, or Lemma 2.2 from [15] (Chap.III), we obtain that, for each $u\in L^2_t(\Omega\times]0,T[;V)$, $A(.,u(.,.))\in L^2_t(\Omega\times]0,T[;V^*)$ in the hypotheses (A_2)-(A_5).

Finally in addition for what follows we introduce the spaces

$$(1.15)\qquad W(0,T;V) = \{u: u\in L^2(0,T;V),\ u'=\frac{du}{dt}\in L^2(0,T;V^*)\},$$

and

$$(1.16)\qquad W_t(\Omega\times]0,T[;V) = \{u : u\in L^2(\Omega\times]0,T[;V) \text{ for which there exist } u_o\in L^2(\Omega,\mathcal{F}_o,P;H) \text{ and } v\in L^2_t(\Omega\times]0,T[;V^*) \text{ such that } u(t) = u_o + \int_0^t v(s)ds\};$$

in (1.16) u_o and $v=u'$ are uniquely determined by the stochastic process u.

It may be shown (see [4] Chap.1, §3.4) that $W(0,T;V)$ can be identified with a linear subspace of the space $C(0,T;H)$, and $W_t(\Omega\times]0,T[;V)$ can be identified (see Prop.1.1) with a linear subspace of the space $L^2_t(\Omega;C(0,T;H))$. Also $W(0,T;V)$ is embedded in $W_t(\Omega\times]0,T[;V)$.

The space $W_t(\Omega\times]0,T[;V)$ is dense in the space $L^2_t(\Omega\times]0,T[;V)$ because for $v\in L^2_t(\Omega\times]0,T[;V)$ the sequence

$$v_n(t) = n\int_0^t e^{n(s-t)}v(s)ds$$

belongs to $W_t(\Omega\times]0,T[;V)$ and $v_n\to v$ (for $n\to+\infty$) in $L^2_t(\Omega\times]0,T[;V)$.

§2. Existence of the weak and "almost weak" solutions in the case A coercive

2.1 Hypotheses The Hilbert spaces V and H will be the same as those given in Section 1(E). Hence $V\subset H\subset V^*$, where the inclusion mappings are continuous, and V and H are dense in H′ and V^* respectively.

Consider that for almost every $t\in]0,T[$, there are defined

(H_1)

(i) $A(t):V\to V^*$ linear, continuous; t-a-e.;

(ii) There exist $\sigma>0$, $\alpha\in R$ such that
$\langle A(t)v,v\rangle + \alpha|v|^2 \geq \sigma\|v\|^2$, for all $v\in V$; t-a.e.;

(iii) There exists $C>0$ such that $\|A(t)v\|_* \leq C\|v\|$, for all $v\in V$; t-a.e.;

(iV) $t\to A(t)v$ is strong measurable from $]0,T[$ to V^*, for all $v\in V$.

Let $\varphi(t,.):V\to]-\infty,+\infty]$, $t\in[0,T]$, a family of functions satisfying the conditions

(H_2)

(i) φ is a normal convex integrand (see [1],[16]), i.e. $\varphi(.,.)$ is $\mathcal{L}\times\mathcal{B}(V)$-measurable, and for every $t\in[0,T]$, $\varphi(t,.)$ is a proper convex lower-semicontinuous function;

(ii) There exists $v_0\in W(0,T;V)$ such that $\varphi(.,v_0(.))\in L^1(0,T)$;

(iii) There exist $a\in]0,\frac{\sigma}{2}[$, $b\in L^1(0,T)$ such that for all $v\in V$ $\varphi(t,v)+a\|v\|^2+b(t)\geq 0$ a.e. on $[0,T]$.

<u>Remark 2.1</u> a) We denoted by $\mathcal{L}$ the σ-algebras of Lebesgue measurable subsets of $[0,T]$, and with $\mathcal{B}(V)$ the Borel σ-algebras on V.
b) The hypothesis $(H_2$-iii) is satisfied e.g. if there exist $b_1\in L^2(0,T;V^*)$ and $b_2\in L^1(0,T)$ such that

(2.1) $\varphi(t,v)\geq\langle b_1(t),v\rangle + b_2(t)$, for all $v\in V$; t-a.e.

Under the hypotheses (H_2) (see [1],[2],[17]) the function from $L^2(\Omega\times]0,T[;V)$ to $]-\infty,+\infty]$ given by

(2.2) $$\phi(u)=\begin{cases} E\int_0^T\varphi(t,u(t))dt, & \text{if } \varphi(.,u(.,.))\in L^1(\Omega\times]0,T[) \\ +\infty, & \text{otherwise}\end{cases}$$

is proper convex and lower-semicontinuous (hence it is also weakly lower-semicontinuous) and $v_0 \in D(\phi) \cap L^2_t(\Omega \times]0,T[;V)$ where

$$D(\phi) = \{u : u \in L^2(\Omega \times]0,T[;V),\ \phi(u) < +\infty\}$$

$$(2.3) \qquad = \{u : u \in L^2(\Omega \times]0,T[;V),\ u(\omega,t) \in D\varphi(t,.)\ (\omega,t)\text{-a.e.},$$

$$\text{and } \varphi(u) \in L^1(\Omega \times]0,T[)\}.$$

Also for every $u \in L^2(\Omega \times]0,T[;V)$ and $\lambda > 0$, the mapping

$$(2.4) \qquad (\omega,t) \longrightarrow J_\lambda(t,u(\omega,t)) = (F + \lambda\partial\varphi(t,.))^{-1}Fu(\omega,t)$$

is in $L^2(\Omega \times]0,T[;V)$, and

$$(2.5) \qquad \|J_\lambda(t,v)\| \leqslant \|v\| + (1+\lambda)\|J_1(t,0)\|, \text{ for all } v \in V \text{ and } t \in [0,T].$$

We denoted by F the duality mapping of V defined by $Fu = \{u^* : u^* \in V^*,\ \langle u^*,u\rangle = \|u\|^2 = \|u^*\|_*^2\}$, and by $\partial\varphi(t,u)$ the subdifferential of $\varphi(t,.)$ at u:

$$(2.6) \qquad \partial\varphi(t,u) = \{u^* \in V^* : \langle u^*, v-u\rangle + \varphi(t,u) \leqslant \varphi(t,v), \text{ for all } v \in V\}.$$

For a function φ satisfying (H_2) there always exists a family of operators $\beta_\lambda : [0,T] \times V \longrightarrow V^*$, $\lambda > 0$, with the properties:

(B)

(β_1) For every $t \in [0,T]$, $\beta_\lambda(t,.) : V \longrightarrow V^*$ are monotone and Lipschitz-continuous with the Lipschitz constant independent of t and λ;

(β_2) $\beta_\lambda(.,v)$ $L^2(0,T;V^*)$ for every $v \in V$, and $\lambda \longrightarrow \beta_\lambda(.,0)$ is bounded;

(β_3) $\frac{1}{\lambda}\langle \beta_\lambda(t,u), v-u\rangle + \varphi(t,J_\lambda(t,u)) \leqslant \varphi(t,v)$ for all $t \in [0,T]$ and $u, v \in V$;

(β_4) If there exists a sequence $\lambda_n \in]0,+\infty]$, $\lambda_n \downarrow 0$, such that for $u_{\lambda_n}, u \in L^2_t(\Omega \times]0,T[;V)$ we have

1°. $u_{\lambda_n} \longrightarrow u$ (weakly) in $L^2(\Omega \times]0,T[;V)$,

2°. $E\int_0^T \langle \beta_{\lambda_n}(t,u_{\lambda_n}), v-u_{\lambda_n}\rangle dt \longrightarrow 0$ for every $v \in L^2_t(\Omega \times]0,T[;V)$,

then $\liminf\limits_{\lambda_n \longrightarrow 0} \phi(J_{\lambda_n}(u_{\lambda_n})) \geqslant \phi(u)$.

Remark 2.2 (I) Indeed, by taking $\beta_\lambda(t,u) = \lambda\partial\varphi_\lambda(t,u) =$ $= F(u-J_\lambda(t,u))$ as an example, where

$$\varphi_\lambda(t,u) = \inf\left\{\frac{\|u-v\|^2}{2\lambda} + \varphi(t,v);\ v\in V\right\} = \frac{1}{2\lambda}\|u-J_\lambda(t,u)\|^2 + \varphi(t,J_\lambda(t,u)) \tag{2.7}$$

the properties (B) are immediately verified since the following inequalities hold :

$$\langle\partial\varphi_\lambda(t,u),v-u\rangle + \varphi_\lambda(t,u) \leq \varphi_\lambda(t,v)$$

$$\varphi_\lambda(t,J_\lambda(t,v)) \leq \varphi_\lambda(t,v) \leq \varphi(t,v),$$

which give

$$\frac{1}{\lambda}\langle\beta_\lambda(t,u),v-u\rangle + \frac{1}{2\lambda}\|u-J_\lambda(t,u)\|^2 + \varphi(t,J_\lambda(t,u)) \leq \varphi(t,v)$$

where by using (H_2-iii) and (2.5) we obtain

$$\langle\beta_\lambda(t,u),v-u\rangle + \frac{1}{2}\|u-J_\lambda(t,u)\|^2 \leq 2\lambda|a|(\|u\|^2 + (\lambda+1)^2\|J_1(t,0)\|^2) + \lambda|b(t)| + \lambda\varphi(t,v)$$

Remark 2.2 (II) Let $\{K(t),\ t\in[0,T]\}$ be a family of nonempty convex closed subsets of V such that :

(2.8)

(i) There exists $v_0\in W(0,T;V)$, $v_0(t)\in K(t)$ t-a.e. ;

(ii) There exists $\beta(.,.):[0,T]\times V\longrightarrow V^*$ with the following properties :

a) $\beta(.,v)\in L^2(0,T;V^*)$, for all $v\in V$;

b) $\beta(t,.):V\longrightarrow V^*$ is monotone and uniformly Lipschitz continuous with respect to $t\in[0,T]$;

c) $K(t)=\{v:v\in V,\ \beta(t,v) = 0\}$.

These properties imply that $\beta(t,.)$ is a penalty operator for $K(t)$. (By a penalty operator for a nonempty convex closed subset $K\subset V$ we understand a monotone hemicontinuous bounded operator $\beta:V\longrightarrow V^*$ such that $K = \{v\in V:\ \beta(v)=0$)

The convex indicator fonction

$$\varphi(t,u) = I_{K(t)}(u) = \begin{cases} 0, & \text{if } u\in K(t) \\ +\infty, & \text{if } u\in V\setminus K(t) \end{cases}$$

verifies the assumption (H_2) because it is relativly easy to show that (2.8-ii) implies that $t \longrightarrow K(t)$ is a measurable multifunction (see [1] or [7] for definitions and properties) and then φ is a normal convex integrand.

In this case $\beta_\lambda(t,u) = \beta(t,u)$, for all $\lambda>0$, verifies the properties (B); e.g., the property (β_3) is satisfied because $J_\lambda(t,u) = P_{K(t)}(u)$, where $P_{K(t)}$ is the projection operator of V on K(t), $\varphi(t,J_\lambda(t,u)) = 0$ and $\langle\beta(t,u),v-u\rangle = \langle\beta(t,u)-\beta(t,v),v-u\rangle \leq 0$ for every $v \in K(t)$.

Concerning the property (β_4) since β is monotone, for every $v \in L_t^2(\ \times\ 0,T\ ;V)$ we have

$$E\int_0^T \langle\beta(t,u_{\lambda_n})-\beta(t,v),u_{\lambda_n}-v\rangle dt \geqslant 0$$

from which, for $\lambda_n \longrightarrow 0$, we obtain

$$E\int_0^T \langle\beta(v),v-u\rangle dt \geqslant 0$$

In this relation we replace v by $u+\varepsilon v$, $\varepsilon>0$, and by letting $\varepsilon \longrightarrow 0$ we find

$$E\int_0^T \langle\beta(u),v\rangle dt \geqslant 0$$

for all $v \in L_t^2(\Omega\times]0,T[;V)$. Hence $\beta(u) = 0$ (ω,t)-a.e. , i.e. , $u(\omega,t) \in K(t)$ (ω,t)-a.e. and then $\Phi(u) = \Phi(J_{\lambda_n}(u_{\lambda_n})) = 0$.

<u>Remark 2.3</u> The properties (B) can be satisfied only for $\lambda \in]0,\varepsilon_o]$ where $\varepsilon_o > 0$ is a fixed real number, or just only for $\lambda \in \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots\}$, where $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n > \ldots > 0$ and $\varepsilon_n \longrightarrow 0$.

Consider on [0,T] the equation

$$(2.9)\qquad \begin{cases} du(t) + A(t)u(t)dt + \partial\varphi(t,u(t))dt \ni f(t)dt + dM(t) \\ u(0) = u_o\ , \end{cases}$$

where

(H_3)
- (i) $u_o \in L^2(\Omega,\mathcal{F}_o,P;H)$
- (ii) $f \in L_t^2(\Omega\times]0,T[;V^*)$
- (iii) $M \in \mathcal{M}^2(0,T;H)$

2.2 Weak Solutions. Definition 2.4 A stochastic process u is a weak solution of Eq.(2.9) if

$$\text{(i)} \quad u \in L_t^2(\Omega \times]0,T[;V) \;;$$

$$\text{(ii)} \quad u(\omega,t) \in D\varphi(t,.) \quad (\omega,t)\text{-a.e. in } \Omega \times]0,T[\;;$$

$$(2.10) \quad \text{(iii)} \quad E\int_0^T \langle v'+A(t)u-f, v+m+M-u\rangle dt + \frac{1}{2}E|m(T)|^2 + \frac{1}{2}E|v(0)-u_0|^2 + \Phi(v+m+M) \geqslant \Phi(u), \quad \text{for all}$$

$$v \in W_t(\Omega \times]0,T[;V),\; m \in \mathcal{M}^2(0,T;H) \text{ such that } m+M \in L^2(\Omega \times]0,T[;V).$$

Remark 2.5 a) For (2.10-iii) we have

$$E\int_0^T \langle v', v+m+M-u\rangle dt = E\int_0^T \langle v', v-u\rangle dt + E\langle v(T), m(T)+M(T)\rangle \;;$$

(see [15]: Lemma 1.3 in §1 Chap.II)

b) From (2.10-iii) it follows that $u \in D(\Phi)$.

Proposition 2.6 If $u \in L_t^2(\Omega \times]0,T[;V)$ is a strong solution (Itô) for the equation (2.9) i.e.

$$\text{(i)} \quad u(\omega,t) \in D\varphi(t,.) \quad (\omega,t)\text{-a.e.} \;;$$

$$\text{(ii)} \quad \text{There exists } \eta \in L_t^2(\Omega \times]0,T[;V^*) \text{ such that}$$

$$(2.11) \quad \eta(\omega,t) \in \partial\varphi(t,u(\omega,t)) \quad (\omega,t)\text{-a.e.} \;;$$

$$\text{(iii)} \quad u(t) + \int_0^t A(s)u(s)ds + \int_0^t \eta(s)ds = u_0 + \int_0^t f(s)ds + M(t)$$

$$\text{for all } t \in [0,T] \;;\; \omega\text{-a.s.} \;;$$

then u is a weak solution for Eq.(2.9).

Proof. By Prop.1.1 and Remark 1.3 it follows that $u \in L_t^2(\Omega;C(0,T;H))$ and $Au \in L_t^2(\Omega \times]0,T[;V^*)$. Let $v \in W_t(\Omega \times]0,T[;V)$ and $m \in \mathcal{M}^2(0,T;H)$ be arbitrary, such that $m+M \in L^2(\Omega \times]0,T[;V)$. By using (2.11-iii) we have

$$v(t)+m(t)+M(t)-u(t) = v(0)-u_0+\int_0^t [v'(s)+A(s)u(s)+\eta(s)-f(s)]ds + m(t),$$

and by Prop.1.1 we obtain

$$E|v(T)+m(T)+M(T)-u(T)|^2 = E|v(0)-u_0|^2 + 2E\int_0^T \langle v'+Au+\eta-f, v+m+M-u\rangle dt \,.$$

Hence, taking into account the definition of the subdifferential $\partial\varphi$, we can write

$$E\int_0^T \langle v'+Au-f, v+m+M-u\rangle dt + \frac{1}{2}E|m(T)|^2 + \frac{1}{2}E|v(0)-u_0|^2 + \Phi(v+m+M) \geq \Phi(u),$$

which concludes the proof.

Theorem 2.7 Under the hypotheses (H_1, with $\alpha=0$) (H_2) and (H_3), the equation (2.9) has at least one weak solution in $L^2(\Omega;L^\infty(0,T;H))$ (the variational problem (2.10) has at least a solution).

Remark 2.8 The case in which $\alpha<0$ may be reduced to the case in which $\alpha=0$. We shall also prove Th.2.7 for $\alpha>0$, but making additional assumptions.

Proof of Theorem 2.7 Let for the moment be, $\alpha\in R$ arbitrary. Later in the proof we are going to specify the moment when we shall assume $\alpha=0$.

Consider for the function φ a family of operators β_λ which satisfy the conditions (B), and also consider the approximating equations

$$(2.12)\quad \begin{cases} du_\lambda(t) + A(t)u_\lambda(t)dt + \frac{1}{\lambda}\beta_\lambda(t,u_\lambda(t))dt = f(t)dt + dM(t) \\ u\ (0) = u_0 \end{cases}$$

where we take only those $\lambda\in]0,1]$ (see also Remark 2.3).

The equations (2.12) have unique solutions

$$(2.13)\quad u_\lambda \in L_t^2(\Omega\times]0,T[;V) \cap L_t^2(\Omega;C(0,T;H))$$

because the hypotheses of Prop.1.2 are satisfied with $A(t,v)$ substituted by $A(t)v+\frac{1}{\lambda}\beta_\lambda(t,v)-\frac{1}{\lambda}\beta_\lambda(t,0)$ and $f(t)$ by $f(t)-\frac{1}{\lambda}\beta_\lambda(t,0)$.

By Remark 1.3, $Au_\lambda + \frac{1}{\lambda}\beta_\lambda(u_\lambda) \in L_t^2(\Omega\times]0,T[;V^*)$.

Let $v\in W_t(\Omega\times]0,T[;V)$ and $m\in\mathcal{M}^2(0,T;H)$ be arbitrary processes such that $m+M\in L^2(\Omega\times]0,T[;V)$.

From (2.12) it follows that :

$$v(t) + m(t) + M(t) - u_\lambda(t) = v(0) - u_0 + \int_0^t (v'+Au_\lambda+\frac{1}{\lambda}\beta_\lambda(u_\lambda)-f)ds + m(t),$$

for all $t\in[0,T]$; ω-a.s. .

By using here the energy equality in the form (1.9-c_2), we are led to :

$$(2.14)\quad |v(t)+m(t)+M(t)-u_\lambda(t)|^2 = |v(0)-u_o|^2 + 2\int_0^t \langle v' + Au_\lambda + \frac{1}{\lambda}\beta_\lambda(u_\lambda) - f, v+m+M-u_\lambda\rangle ds + 2\int_0^t (v+m+M-u_\lambda, dm) + \langle m\rangle_t ,$$

for all $t\in[0,T]$; ω-a.s.

Next

$$(2.15)\quad \int_0^t \langle v' + Au_\lambda - f, v+m+M-u_\lambda\rangle ds + \int_0^t (v+m+M-u_\lambda, dm) + \frac{1}{2}\langle m\rangle_t + \frac{1}{2}|v(0)-u_o|^2 + \int_0^t \varphi(s,v+m+M)ds \geqslant \int_0^t \varphi(s,J_\lambda(s,u_\lambda(s)))ds + \frac{1}{2}|v(t)+m(t)+M(t)-u_\lambda(t)|^2, \quad \text{for all } t\in[0,T];\ \omega\text{-a.s.}$$

By virtue of $(H_2\text{-iii})$, (2.5) and because $\lambda\in]0,1]$ the following inequalities hold:

$$(2.16)\quad \varphi(s,J_\lambda(s,u_\lambda)) \geqslant -a\|J_\lambda(s,u_\lambda)\|^2 - b(s) \geqslant -2a\|u_\lambda\|^2 - b_1(s),$$

(ω,s) - a.e.,

where $b_1(s) = 4\|J_1(s,0)\|^2 + b(s)$, $b_1\in L^1(0,T)$.
On the other hand one has:

$$(2.17)\quad |x-y|^2 \geqslant \frac{1}{2}|y|^2 - |x|^2, \qquad \text{for every } x,y\in H.$$

By combining now the hypotheses $(H_1\text{-ii})$, $(H_2\text{-ii})$ and the inequalities (2.16) and (2.17) we obtain from (2.15) for $m=-M$ and $v=v_o$ the following inequality:

$$(2.18)\quad |u_\lambda(t)|^2 + 4(\sigma-2a)\int_0^t \|u_\lambda(s)\|^2 ds \leqslant g(t) + 4|\alpha|\int_0^t |u_\lambda(s)|^2 ds + 4\int_0^t |\langle -v_o' + A^* v_o + f, u_\lambda\rangle| ds + 4\left|\int_0^t (u_\lambda, dM)\right|,$$

for all $t\in[0,T]$; ω-a.s.

where

$$g(t) = 4\int_0^t (|\langle v_o', v_o\rangle| + |\langle f, v_o\rangle| + |\varphi(s,v_o)| + |b_1(s)|)ds + 4\left|\int_0^t (v_o, dM)\right| + 2|\langle -M\rangle_t| + 2|v_o(t)| + 2|v(0)-u_o|^2$$

and $g\in L^1(\Omega;C(0,T;R))$
In the second member of (2.18) we make the follwing estimation:

$$(2.19)\quad \int_0^t |\langle -v_o' + A^* v_o + f, u_\lambda\rangle| ds \leqslant \frac{1}{2\varepsilon^2}\int_0^T \|-v_o' + A^* v_o + f\|_*^2 ds + \frac{\varepsilon^2}{2}\int_0^t \|u_\lambda(s)\|^2 ds,$$

where $\varepsilon^2 = 6(\sigma-2a)>0$, and then from (2.18) we get

$$\text{(2.20)}\quad \begin{aligned} &E\sup_{t\in[0,\tau]}|u_\lambda(t)|^2 + (\sigma-2a)E\int_0^\tau \|u_\lambda(s)\|^2 ds \leqslant C_1 + 4|\alpha| E\int_0^\tau |u_\lambda(s)|^2 ds \\ &+ 4E\sup_{t\in[0,\tau]}\left|\int_0^t (u_\lambda(s),dM(s)\right|, \end{aligned}$$

where $C_1 = E\sup_{t\in[0,T]} g(t) + \frac{1}{12(\sigma-2a)}\int_0^T \|-v_o'+A^* v_o+f\|_*^2 dt.$

By using now the inequality (1.7) we have

$$\begin{aligned} &4E\sup_{t\in[0,\tau]}\left|\int_0^t (u_\lambda(s),dM(s))\right| \leqslant 12E[\sup_{t\in[0,\tau]}|u_\lambda(t)|\sqrt{\langle M\rangle_\tau}] \leqslant \\ &\leqslant \frac{12}{2\delta^2}E\sup_{t\in[0,\tau]}|u_\lambda(t)|^2 + \frac{12\delta^2}{2}E\langle M\rangle_T, \end{aligned}$$

and for $\delta=\sqrt{12}$ we obtain from (2.20):

$$\text{(2.21)}\quad E\sup_{t\in[0,\tau]}|u_\lambda(t)|^2 \leqslant C_2 + C_3\int_0^\tau E\sup_{\theta\in[0,s]}|u_\lambda(\theta)|^2 ds$$

and

$$\text{(2.22)}\quad E\int_0^T \|u_\lambda(s)\|^2 ds \leqslant C_4 + C_5\int_0^T E\sup_{\theta\in[0,s]}|u_\lambda(\theta)|^2 ds.$$

By applying Gronwall's inequality in (2.21), the inequelities (2.21),(2.22) imply:

$$\text{(2.23)}\quad \begin{aligned} &\text{a)}\quad E\sup_{t\in[0,T]}|u_\lambda(t)|^2 \leqslant C_6 \\ &\text{b)}\quad E\int_0^T \|u_\lambda(t)\|^2 dt \leqslant C_7 \end{aligned}$$

(the constants C_i are independent of λ).

Hence there exist $\lambda_n \downarrow 0$ and $u\in L^2(\Omega\times]0,T[;V)\cap L^2(\Omega;L^\infty(0,T;H))$ such that:

$$\text{(2.24)}\quad \begin{aligned} &\text{a)}\quad u_{\lambda_n} \xrightarrow{*} u \text{ (weakly star) in } L^2(\Omega;L^\infty(0,T;H)), \\ &\text{b)}\quad u_{\lambda_n} \longrightarrow u \text{ (weakly) in } L^2(\Omega\times]0,T[;V). \end{aligned}$$

But $L^2_t(\Omega\times]0,T[;V)$ is a closed linear subspace of $L^2(\Omega\times]0,T[;V)$ and then, since $u_\lambda\in L^2_t(\Omega\times]0,T[;V)$ for all $\lambda>0$, $u\in L^2_t(\Omega\times]0,T[;V)$. From (2.14) and (2.23) we obtain:

$$\lim_{\lambda\to 0} E\int_0^T \langle\beta_\lambda(u_\lambda),v-u_\lambda\rangle dt = 0$$

for every $v\in W_t(\Omega\times]0,T[;V)$.

This limit just holds for every $v\in L^2_t(\Omega\times]0,T[;V)$ because $W_t(\Omega\times]0,T[;V)$ is dense in $L^2_t(\Omega\times]0,T[;V)$, and $(B-\beta_1,\beta_2)$ are hold.

Hence by $(B-\beta_4)$ we have :

$$\text{(2.25)} \quad \liminf_{\lambda_n \downarrow 0} \Phi(J_{\lambda_n}(u_{\lambda_n})) \geqslant \Phi(u) .$$

By applying the expectation in (2.15) for t=T, one obtains :

$$\text{(2.26)} \quad \begin{aligned} & E\int_0^T \langle v'-f, v+m+M-u_\lambda\rangle dt + E\int_0^T \langle A(s)u_\lambda, v+m+M-u_\lambda\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ & + \tfrac{1}{2}E|v(0)-u_0|^2 + \Phi(v+m+M) \geqslant \Phi(J_\lambda(.,u_\lambda(.,.))) , \end{aligned}$$

and therefore

$$\text{(2.27)} \quad \begin{aligned} & E\int_0^T \langle v'-f, v+m+M-u_\lambda\rangle dt + E\int_0^T \langle u_\lambda, A^*(v+m+M)\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ & + \tfrac{1}{2}E|v(0)-u_0|^2 + \Phi(v+m+M) + \alpha E\int_0^T |u_\lambda(t)|^2 dt \geqslant \\ & \Phi(J_\lambda(u_\lambda)) + E\int_0^T (\langle A(t)u_\lambda, u_\lambda\rangle + \alpha|u_\lambda|^2) dt . \end{aligned}$$

The mapping

$$v \longrightarrow E\int_0^T (\langle Av,v\rangle + \alpha|v|^2) ds$$

from $L^2(\Omega\times]0,T[;V)$ into $]-\infty,+\infty]$ is a proper convex lower-semicontinuous function, and hence it is also lower-semicontinuous with respect to the weak topology on $L^2(\Omega\times]0,T[;V)$.

<u>We assume now $\alpha=0$</u>

Letting $\lambda=\lambda_n \longrightarrow 0$ in (2.27) we obtain (2.10-iii) for u from (2.24). The proof is complete.

<u>Remark 2.9</u> a) In (2.27) it is not necessary to suppose $\alpha=0$ if u_λ (or u_{λ_n}) is strongly convergent to u in $L^2(\Omega\times]0,T[;H)$. We shall see that, under certain additional ordering hypotheses, one obtains this strong convergence of u_λ.

b) In general the weak solution is not unique (see Example 5.1).

<u>Remark 2.10</u> a) Theorem 2.7 can be extended with no essential changes in the proof, to the case in which V is a smooth strictly convex separable reflexive Banach space, A is a nonlinear operator satisfying the assumption (1.11), and the mapping

$$\text{(2.28)} \quad v \longrightarrow E\int_0^T \langle A(s,v(s)), z(s)-v(s)\rangle ds$$

is an upper-semicontinuous function $L^2(\Omega\times]0,T[;V)$ into $]-\infty,+\infty]$, for every $z \in L^2(\Omega\times]0,T[;V)$. (Props.1.1 and 1.2 are also true for such a space V - see [15], II Partie Chap.II).

b) If V has the properties specified in Remark 2.10-a, and A is a nonlinear operator verifying (1.11) with $\alpha=0$, then the following problem has at least a solution:

(P_1) $u \in L^2_t(\Omega\times]0,T[;V) \cap L^2(\Omega;L^\infty(0,T;H))$

(P_2) $u(\omega,t)\in D\varphi(t,.)$ (ω,t)-a.e.

$$(2.29)\quad (P_3)\quad E\int_0^T \langle v'-f + A(t,v+m+M), v+m+M-u\rangle dt + \frac{1}{2}E|m(T)|^2 + \frac{1}{2}E|v(0)-u_0|^2 + \Phi(v+m+M) \geq \Phi(u)$$

for all $v\in W_t(\Omega\times]0,T[;V)$, $m\in\mathcal{M}^2(0,T;H)$ such that

$m+M\in L^2(\Omega\times]0,T[;V)$.

The proofs for Remarks (a) and (b) repeat word for word that of Th.2.7 with $\beta_\lambda(t,v) = \partial\varphi_\lambda(t,v)$ up to the relation (2.25). Then we pass to limit $\lambda=\lambda_n\to 0$ in (2.26) for (a), while for (c) we first use the inequality $\langle A(t,u_\lambda), v+m+M-u_\lambda\rangle \leq \langle A(t,v+m+M), v+m+M-u_\lambda\rangle$ and then we pass to limit.

2.3 "Almost weak" solutions. In order to find a maximal element of the set of weak solutions we should need some hypotheses of noncorrelation which would be dificult to verify on examples. That is why we abandoned this idea, and, instead, we have sought a majorant of the set of weak solutions which is a solution for Eq.(2.9) in a certain sense, quite close to the weak one. Thus we justify the introduction of the concept of "almost weak" solutions.

Definition 2.11 A stochastic process u is said to be an "almost weak" solution of Eq.(2.9) if :

(i) $u\in L^2_t(\Omega\times]0,T[;V)$

(ii) $Eu(t)\in D\varphi(t,.)$ a.e. on $[0,T]$

$$(2.30)\quad (iii)\ E\int_0^T \langle v'+Au-f, v+m+M-u\rangle dt + \frac{1}{2}E|m(T)|^2 + \frac{1}{2}E|v(0)-u_0|^2 + \int_0^T \varphi(t,Ev(t))dt \geq \int_0^T \varphi(t,Eu(t))dt,\ \text{for all } m\in\mathcal{M}^2(0,T;H),$$

$v\in W_t(\Omega\times]0,T[;V)$ such that $m+M\in L^2(\Omega\times]0,T[;V)$.

Remark 2.12 If the process u is an "almost weak" solution such that

$$(2.31)\quad \varphi(t,Eu(t))\geq E\varphi(t,u(t))\quad t\text{-a.e.},$$

then it is a weak solution. For example if $\varphi(t,v) = I_{K(t)}(v)$ (see Remark 2.2(II)) and if u is an "almost weak" solution such that $u(\omega,t)\in K(t)$ (ω,t)-a.e., then u is a weak solution.

Theorem 2.13 Under the hypotheses (H_1, with = 0)(H_2) and (H_3), Eq.(2.9) has at least an "almost weak " solution (the variational problem (2.30) has at least a solution).

<u>Proof.</u> As in the proof of Th.2.7, at the begining we assume R arbitrary. We are going to specify later in the proof the moment when we shall take $\alpha = 0$. According to the proof of Th.2.7 the approximating equation

$$(2,32)\qquad \begin{aligned} & y_\lambda' + A(t)y_\lambda + \frac{1}{\lambda}\beta_\lambda(t,y_\lambda) = Ef \\ & y_\lambda(0) = Eu_0, \quad (0<\lambda\leq 1), \end{aligned}$$

in which β_λ satisfy the properties (B), has a unique solution $y_\lambda \in W(0,T;V)$.

Moreover

a) $\{y_\lambda, 0<\lambda\leq 1\}$ is bounded in $L^2(0,T;V)$ $L^\infty(0,T;H)$;

b) there exist $\lambda_n \downarrow 0, \bar{y}\in L^2(0,T;V)\cap L^\infty(0,T;H)$ such that

$$(2.33)\qquad \begin{aligned} & y_{\lambda_n} \longrightarrow \bar{y} \text{ in } L^2(0,T;V) \text{ (weakly)} \\ & y_{\lambda_n} \overset{*}{\longrightarrow} \bar{y} \text{ in } L^\infty(0,T;H) \text{ (weak star);} \end{aligned}$$

c) $\bar{y}(t)\in D\varphi(t,.)$ t-a.e. in $[0,T]$.

These assertions may be obtained directly too. Now consider the approximating equation:

$$(2.34)\qquad \begin{aligned} & dv_\lambda + A(t)v_\lambda dt + \frac{1}{\lambda}\beta_\lambda(t,y_\lambda)dt = f(t)dt + dM(t) \\ & v_\lambda(0) = u_0, \end{aligned}$$

where y_λ is the solution of Eq.(2.32).

By Prop.1.2, Eq.(2.34) has a unique solution $v_\lambda\in L_t^2(\Omega\times]0,T[;V)\cap$ $\cap L_t^2(\Omega;C(0,T;H))$ in the following sense:

$$(2.35)\qquad \begin{aligned} & v_\lambda(t) + \int_0^t A(s)v_\lambda(s)ds + \frac{1}{\lambda}\int_0^t \beta_\lambda(s,y_\lambda(s))ds = \\ & = u_0 + \int_0^t f(s)ds + M(t), \end{aligned}$$

for all $t\in[0,T]$; ω-a.s. in Ω.

After we take the expectation in (2.35), the relations (2.32) and (2.35) yield:

$$(2.36)\qquad Ev_\lambda(t) = y_\lambda(t).$$

Let $v \in W_t(\Omega \times]0,T[;V)$ and $m \in \mathcal{M}^2(0,T;H)$ arbitrary processes such that $m+M \in L^2(\Omega \times]0,T[;V)$. From (2.35) it follows that

$$v(t)+m(t)+M(t)-v_\lambda(t) = v(0)-u_o+\int_0^t (v'+Av_\lambda + \frac{1}{\lambda}\beta_\lambda(y_\lambda)-f)ds+m(t),$$

and by using the energy equality (Prop.1.1) we get

$$\frac{1}{2}E|v(t)+m(t)+M(t)-v_\lambda(t)|^2 = \frac{1}{2}E|v(0)-u_o|^2 + \tag{2.37}$$
$$+ E\int_0^t \langle v'+Av_\lambda + \frac{1}{\lambda}\beta_\lambda(y_\lambda)-f, v+m+M-v_\lambda\rangle ds + \frac{1}{2}E|m(t)|^2.$$

Next by (2.36) and (B) we have :

$$E\int_0^t \langle \frac{1}{\lambda}\beta_\lambda(y_\lambda), v+m+M-v_\lambda\rangle ds + E\int_0^t \varphi(J_\lambda(y_\lambda))ds \leq \int_0^t \varphi(Ev)ds.$$

The boundedness of y_λ and the relation (2.37), where we put $v = v_o$ and $m = -M$, imply in the same manner as in the proof of Th.2.7 that:

$$E|v_\lambda(t)|^2 + E\int_0^t \|v_\lambda(s)\|^2 ds \leq C_1 + C_1\int_0^t E|V_\lambda(s)|^2\, ds, \tag{2.39}$$

where the constant C_1 is independent of λ.

Consequently, v_λ is bounded in $L^2_t(\Omega \times]0,T[;V)$ and there exist $\lambda_n \downarrow 0$, $\bar{u} \in L^2_t(\Omega \times]0,T[;V)$ such that:

$$v_{\lambda_n} \longrightarrow \bar{u} \text{ in } L^2(\Omega \times]0,T[;V) \text{ (weakly)},$$
$$v_{\lambda_n} \longrightarrow \bar{u} \text{ in } L^2(\Omega \times]0,T[;H) \text{ (weakly)}, \tag{2.40}$$
$$Ev_{\lambda_n} = y_{\lambda_n} \longrightarrow E\bar{u} = \bar{y} \text{ in } L^2(0,T;V) \text{ (weakly)}.$$

By using the relations (2.37) (2.36), the boundedness of v_λ, and the density of $W_t(\Omega \times]0,T[;V)$ in $L^2_t(\Omega \times]0,T[;V)$ we obtain:

$$\lim_{\lambda \to 0}\int_0^T \langle \beta_\lambda(y_\lambda), Ev-y_\lambda\rangle dt = 0 \tag{2.41}$$

for every $v \in L^2_t(\Omega \times]0,T[;V)$.

Hence by (B-β_4) it follows that:

$$\lim_{\lambda_n \to 0} \inf E\int_0^T \varphi(t, J_{\lambda_n}(t, y_{\lambda_n}))dt \geq \int_0^T \varphi(t,\bar{y}) = \int_0^T \varphi(t, E\bar{u})dt \tag{2.42}$$

On the other hand from the relations (2.37) (2.38) we have

$$E\int_0^T \langle v'-f, v+m+M-v_\lambda\rangle dt + E\int_0^T \langle v_\lambda, A^*(v+m+M)\rangle dt + \frac{1}{2}E|m(T)|^2$$
$$+ \frac{1}{2}E|v(0)-u_o|^2 + \int_0^T \varphi(Ev)dt + \alpha E\int_0^T |v_\lambda(t)|^2 dt \geq \int_0^T \varphi(J_\lambda(y_\lambda))dt \tag{2.43}$$
$$+ E\int_0^T (\langle Av_\lambda, v_\lambda\rangle + \alpha|v_\lambda|^2)dt$$

Now we assume $\alpha = 0$.
Letting $\lambda = \lambda_n \to 0$ in (2.43) we find that $\bar{u}$ is an "almost weak" solution of Eq.(2.9), that is to say $\bar{u}$ is a solution of the variational problem (2.30). This finishes the proof.

Remark 2.14 Remarks 2.9 and 2.10 are also valid for the "almost weak" solutions.

§3. Existence of the weak and "almost weak" solutions when A is α- coercive.

3.1 Hypotheses. Throughout this section we shall suppose that the Hilbert space H is ordered by a nonempty closed convex cone C. In addition, we assume that the cone C satisfies the following properties

$$(3.1)\qquad \begin{aligned} &(c_1) \quad C = \{u \in H : (u,v) \geq 0 \text{ for all } v \in C\} \\ &(c_2) \quad (u^+ - v^+ ; (u-v)^+) \geq 0 \text{ for every } u, v \in H, \end{aligned}$$

where u^+ is the projection of $u \in H$ onto C.
In general the assumption (3.1-c_1) does not imply the assumption (3.1-c_2). Indeed, for $H = R^3$, $C = \{(x,y,z) : z \geq 0, x^2 + y^2 - z^2 \leq 0\}$, $u = (1,0,-1)$, $v = (0,1,0)$, (c_1) is verified but (c_2) does not hold.

If we denote $u^- = (-u)^+$, then $(u^+, u^-) = 0$ and $u = u^+ - u^-$ for every $u \in H$. The space V satisfies the hypotheses from Section 2 and

$$(3.2)\qquad u^+ \in V \text{ for every } u \in V$$

Example 3.1. Hypotheses (3.1) and (3.2) are easily verified in the following situations:

a) If $\{e_k : k \in I\}$, $I \subset N$, is an orthonormal basis of H and $\{\lambda, \lambda_k ; k \in I\}$ is a sequence of real numbers such that $0 < \lambda_k < \lambda$, for every $k \in I$, then denoting $Tv = \sum_{k \in I} \lambda_k (v, e_k) e_k$ we can take

$$C = \{u \in H : (u, e_k) \geq 0, \text{ for all } k \in I\}$$

$$V = T(H), \quad \|Tv\| = |v|$$

b) If D is a bounded open subset of R^m with a sufficiently smooth boundary, we can take $H = L^2(D)$, $C = \{u \in L^2(D) : u(x) \geq 0 \text{ a.e. in } D\}$ and $V = H^1_o(D)$, or $V = H^1(D)$, or $V = \{u \in H^1(D) : u_{|\Gamma_o} = 0, \Gamma_o \subset \Gamma \text{ sufficiently smooth}\}$ etc.

Now we assume that $A(t)$, $t \in [0,T]$, in Section 2, has the additional property:

(3.3) $\langle A(t)v^+, v^-\rangle \leq 0$ for every $v \in V$; a.e. on $[0,T]$,

and that for the function φ there exists a family of operators, ,
which verify the conditions (B) and moreover

(3.4) $\langle \frac{1}{\lambda}\beta_\lambda(t,u) - \frac{1}{\mu}\beta_\mu(t,v), (u-v)^+\rangle \geq 0$

for all $u,v \in V$, $t \in [0,T]$, $0 < \lambda \leq \mu$

Example 3.2 a) For φ and β_λ from Remark 2,2(II) ($\varphi(t,u) = I_{K(t)}(u)$, $\beta_\lambda(t,u) = \beta(t,u)$) if

(3.5)
a) $\langle \beta(t,v), u^+\rangle \geq 0$

b) $\langle \beta(t,u) - \beta(t,v), (u-v)^+\rangle \geq 0$

for all $u,v \in V$, $t \in [0,T]$, then the condition (3.4) it is satisfied because

$$\langle \frac{1}{\lambda}\beta(u) - \frac{1}{\mu}\beta(v), (u-v)^+\rangle = \frac{1}{\lambda}\langle \beta(u) - \beta(v), (u-v)^+\rangle$$

$$+ \frac{\mu-\lambda}{\lambda\mu}\langle \beta(v), (u-v)^+\rangle \geq 0$$

b) If $\psi : [0,T] \longrightarrow H$ is a measurable function such that there exists $v_o \in W(0,T;V)$ with the property

$$v_o(t) \leq \psi(t) \quad \text{a.e. on } [0,T],$$

then for the mapping $\varphi(t,v) = I_{K(t)}(v)$, where $K(t) = \{v \in V \; v \leq \psi(t)\}$, (H_2), (B) and (3.4) are verified by taking $\beta_\lambda(t,v) = \beta(t,v) = (v- \;(t))^+$ (see Ex.3.2(a)).

c) If $V=H$ and $\varphi(u) = \frac{1}{2}|u^+|^2$, we have $\partial\varphi(u) = u^+$, $J_\lambda(u) = \frac{1}{1+\lambda}u^+ - u^-$, $\partial\varphi_\lambda(u) = \frac{1}{\lambda}(u - J_\lambda u) = \frac{u^+}{1+\lambda}$ and by taking $\beta_\lambda(u) = \lambda\partial\varphi_\lambda(u) = \frac{\lambda}{1+\lambda}u^+$, (H_2), (B) and (3.4) are satisfied.

d) If $V=H=R$ and $\varphi(u) = \frac{1}{2}|u^+|^2 + I_{]-\infty,a]}(u)$, $a \in R$, then by taking

$$\beta_\lambda(u) = \lambda\partial\varphi_\lambda(u) = \frac{\lambda u^+ + (u-(1+\lambda)a - \lambda u^-)^+}{1+\lambda},$$

(H_2) (B) and (3.4) are also verified.

We shall prove subsequently the existence of weak solution and "almost weak" solutions under weaker coercivity conditions on A (A is α-coercive), than those of Ths.2.7 and 2.13.

Hence, let

(2.9)
$$du(t) + A(t)u(t)dt + \partial\varphi(t,u(t))dt \ni f(t)dt + dM(t)$$
$$u(0) = u_o,$$

where

(H_1') A satisfies the hypotheses (H_1) and (3.3);

(H_2') φ satisfies the hypotheses (H_2), (B), (3.4);

(H_3) $u_0 \in L^2(\Omega, \mathcal{F}_0, P; H)$, f $L_t^2(\Omega \times]0,T[; V^*)$, $M \in \mathcal{M}^2(0,T;H)$.

3.2 Weak solutions. Theorem 3.3 If the hypotheses (H_1'), (H_2') and (H_3) hold, then Eq.(2.9) has at least one weak solution u $L^2(\Omega; L^\infty(0,T;H))$.

For the proof we first give:

Lemma 3.4 In the assumption of Th.3.3 the solutions of the equations

$$(3.7)\quad \begin{aligned} &du_\lambda(t) + A(t)u_\lambda(t)dt + \frac{1}{\lambda}\beta_\lambda(t,u_\lambda(t))dt = f(t)dt + dM(t) \\ &u_\lambda(0) = u_0 \end{aligned}$$

have the property

(3.8) if $0<\lambda\leq\mu$ then $u_\lambda(\omega,t)\leq u_\mu(\omega,t)$ for all $t\in[0,T]$; ω-a.s. .

Proof (see also [14]). Since $u_\lambda \in L_t^2(\ \times\ 0,T\ ;V) \cap L_t^2(\Omega; C(0,T;H))$ we have

$$(3.9)\quad \begin{aligned} &\frac{dw_\lambda}{dt} + A(t)(w_\lambda + M) + \frac{1}{\lambda}\beta_\lambda(t, w_\lambda + M) = f \\ &w_\lambda(0) = u_0\ , \end{aligned}$$

and

$$(3.10)\quad w_\lambda \in L_t^2(\Omega; C(0,T;H)),\ \frac{dw_\lambda}{dt} \in L_t^2(\Omega \times]0,T[; V^*),$$

where $w_\lambda = u + M$.

By denoting

$$(3.11)\quad z_\lambda(t) = e^{-\alpha t}w_\lambda(t)$$

we obtain

$$\begin{aligned} &\frac{dz_\lambda}{dt} + \alpha z_\lambda + A(t)(z_\lambda + e^{-\alpha t}M) + \frac{e^{-\alpha t}}{\lambda}\beta_\lambda(t, e^{\alpha t}z_\lambda + M) = fe^{-\alpha t} \\ &z_\lambda(0) = u_0\ . \end{aligned}$$

Hence

$$\frac{d(z_\lambda - z_\mu)}{dt} + \alpha(z_\lambda - z_\mu) + A(t)(z_\lambda - z_\mu) + \frac{e^{-\alpha t}}{\lambda}\beta_\lambda(t, e^{\alpha t}z_\lambda + M) -$$

$$(3.12) \qquad - \frac{e^{-\alpha t}}{\mu} \beta_\mu(t, e^{\alpha t} z_\mu + M) = 0$$

$$(z_\lambda - z_\mu)(0) = 0,$$

where

$$(3.13) \qquad z_\lambda - z_\mu = e^{-\alpha t}(w_\lambda - w_\mu) = e^{-\alpha t}(u_\lambda - u_\mu) \in W_t(\Omega \times]0,T[;V)$$

Let $\lambda \leq \mu$. We multiply the equation (3.12) by $(z_\lambda - z_\mu)^+$ and integrate from 0 to T. By using the hypotheses $(H_1\text{-ii})$,(3.3) and (3.4) we get

$$\sigma \int_0^T \|(z_\lambda(s) - z_\mu(s))^+\|^2 ds \leq 0 \quad \omega\text{-a.s.} ,$$

and then

$$(3.14) \qquad (z_\lambda(\omega,s) \leq z_\mu(\omega,s), \ s\text{-a.e.}) \ \omega\text{-a.s.} \ .$$

Now by (3.13),(3.14) and $u_\lambda \in L^2(\Omega; C(0,T;H))$ we have (3.8). Q.E.D.

Proof of Th.3.3 The proof is that of Th.2.7 until $\alpha=0$ is supposed. By Lemma 3.4 and (2.24) we find for "the full sequence" u_λ that

if $\lambda \to 0$ then :

(3.15) a) $u_\lambda \to u$ (strongly) in $L^2(\Omega \times]0,T[;H)$

b) $u \rightharpoonup u$ (weakly) in $L^2(\Omega \times]0,T[;V)$

by using the following abstract result (see [8]-§6.12,Prop.3).

If $(X, \|\cdot\|)$ is an ordered normed linear space such that $0 \leq u \leq v$ implies $\|u\| \leq \|v\|$, and if $\{u, u_\lambda; \lambda > 0\} \subset X$ has the properties:

(i) there exists $\lambda_n \downarrow 0$ such that $u_{\lambda_n} \rightharpoonup u$ in X (weakly);

(3.16) (ii) $\lambda \leq \mu$ implies $u_\lambda \leq u_\mu$;

(iii) $u \leq u_\lambda$ for all $\lambda > 0$

then $\lim_{\lambda \to 0} u_\lambda = u$ in X (strongly)

In our case we take $X = L^2(\Omega \times]0,T[;H)$ with the order $v_1 \leq v_2$ if $v_1(\omega,t) \leq v_2(\omega,t)$ (ω,t)-a.e.

Now by (3.15) and Remark 2.9 we obtain that u is a weak solution of Eq.(2.9) and the solutions u_λ of the approximating equations (3.7) converge to u strongly in $L^2(\Omega \times]0,T[;H)$ and weakly in $L^2(\Omega \times]0,T[;V)$ when $\lambda \to 0$. This finishes the proof.

3.3 "Almost weak" solutions. From Lemma 3.4 where M=0, f is substituted by Ef and u_o by Eu_o we have:

Lemma 3.5 Under the hypotheses (H_1', H_2', H_3) the solutions of the equations

$$(3.17)\quad \begin{aligned} & y_\lambda' + A(t)y_\lambda(t) + \frac{1}{\lambda}\beta_\lambda(t, y_\lambda(t)) = Ef \\ & y_\lambda(0) = Eu_o \end{aligned}$$

have the property:

(3.18) if $0<\lambda\leqslant\mu$ then $y_\lambda(t)\leqslant y_\mu(t)$ in H for every $t\in[0,T]$.

Now we can state the following result:

Lemma 3.6 Under the hypotheses (H_1', H_2', H_3) the solutions of the equations:

$$(3.19)\quad \begin{aligned} & dv_\lambda(t) + A(t)v_\lambda(t)dt + \frac{1}{\lambda}\beta_\lambda(t, y_\lambda(t))dt = fdt + dM(t) \\ & v_\lambda(0) = u_o \end{aligned}$$

have the property:

(3.20) If $0<\lambda\leqslant\mu$ then $v_\lambda(\omega,t)\leqslant v_\mu(\omega,t)$ for every $t\in[0,T]$; ω-a.s.

Proof By (2.36) we have $Ev_\lambda(t) = y_\lambda(t)$ and by (3.19)

$$d(v_\lambda - v_\mu) + A(t)(v_\lambda - v_\mu)dt = \frac{1}{\mu}\beta_\mu(y_\mu) - \frac{1}{\lambda}\beta_\lambda(y_\lambda)$$

$$(v_\lambda - v_\mu)(0) = 0$$

But the equation $\quad v' + A(t)v = g \in L^2(0,T;V^*)$

$$v(0) = 0$$

has a unique solution $v\in W(0,T;V)$. Consequently, $v_\lambda - v_\mu$ is a deterministic process and then, for $\lambda\leqslant\mu$

$$v_\lambda(t) - v_\mu(t) = E[v_\lambda(t) - v_\mu(t)] = Ev_\lambda(t) - Ev_\mu(t) =$$

$$= y_\lambda(t) - y_\mu(t) \leqslant 0, \quad \text{for all } t\in[0,T];\ \omega\text{-a.s.}$$

This finishes the proof.

Theorem 3.7 If the hypotheses (H_1', H_2', H_3) hold, then the equation (2.9) has at least one "almost weak" solution.

Proof. The proof is the same as that of Th.2.13 until it is assumed $\alpha=0$. We can pass to the limit in (2.43), when $\alpha\in R$ is arbitrary if $v_\lambda \to \bar{u}$ strongly in $L^2(\Omega\times]0,T[;H)$ for $\lambda \to 0$. But according to the proof of Th.3.3 and Lemma 3.6 we have this strong convergence. That completes the proof.

§4. Maximal solutions.

By $\mathcal{S}$ we denote the set of weak solutions and by $\mathcal{S}_a$ the set of "almost weak" solutions of Eq.(2.9).

We have seen that in the hypotheses (H_1 with =0, H_2, H_3) (see Th.2.7 and 2.13) or in the hypotheses (H_1', H_2', H_3) (see Th.3.3 and 3.7) $\mathcal{S}$ and $\mathcal{S}_a$ are nonempty.

In order to find a solution which in a certain sense maximize the sets $\mathcal{S}$ and $\mathcal{S}_a$, we focus our attention on the process $\bar{u}$ from Th.2.13 (or Th.3.7 for real arbitrary α).

We shall suppose that

$$\varphi(t,v-u^+) \leq \varphi(t,v), \tag{4.1}$$

for every $t\in[0,T]$ and $u,v\in V$.

Let y_λ and v_λ be the solutions of Eqs.(2.32) (or 3.17) and Eqs.(2.34) (or 3.19) respectively.

Lemma 4.1 If the hypotheses (H_1 with $\alpha=0$, H_2, H_3) or (H_1', H_2', H_3) and (4.1) hold and we denote $w_\lambda = \exp(-\alpha t)(u-v_\lambda)$ where $u\in\mathcal{S}\cup\mathcal{S}_a$, then

$$E\int_0^T \langle\theta',w_\lambda\rangle dt + \frac{1}{\lambda}E\int_0^T e^{-\alpha t}\langle\beta_\lambda(t,y_\lambda),\theta\rangle dt \geq E\int_0^T[\langle A(t)w_\lambda,\theta\rangle + \alpha(w_\lambda,\theta)]\,dt, \text{ for all } \theta\in W_t(\Omega\times]0,T[;V),\theta(T)=0, \text{ such that} \tag{4.2}$$

$\theta(\omega,t)\geq 0$ (ω,t)-a.e. if $u\in\mathcal{S}$, or $E\theta(t)\geq 0$ t-a.e. if $u\in\mathcal{S}_a$.

Proof We denote $z_\lambda=v_\lambda-M$. From (2.34) (or 3.19) we have

$$\begin{cases}\frac{dz_\lambda}{dt} + A(t)v_\lambda + \frac{1}{\lambda}\beta_\lambda(t,y_\lambda) = f\\ z_\lambda(0) = u_0\end{cases} \tag{4.3}$$

and

$$z_\lambda\in L_t^2(\Omega;C(0,T;H)),\quad z_\lambda' = \frac{dz_\lambda}{dt}\in L_t^2(\Omega\times]0,T[;V^*) \tag{4.4}$$

Multiplying Eq.(4.3) by $v+m+M-u$, where $v\in W_t(\Omega\times]0,T[;V)$ and $m\in\mathcal{M}^2(0,T;H)$ such that $m+M\in L^2(\Omega\times]0,T[;V)$ and

$v+m+M\in D(\phi)$ if $u\in\mathcal{S}$,

or

$Ev\in D(\phi)$ if $u\in\mathcal{S}_a$,

we obtain :

$$(4.5)\qquad E\int_0^T \langle z_\lambda' + Av_\lambda + \frac{1}{\lambda}\beta_\lambda(y_\lambda),\ v+m+M-u\rangle dt = E\int_0^T \langle f, v+m+M-u\rangle dt$$

But from Def.2.4 for $u\in\mathcal{S}$ we have

$$(4.6)\qquad \begin{aligned} &E\int_0^T \langle f, v+m+M-u\rangle dt \leq E\int_0^T \langle v'+Au, v+m+M-u\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ &+ \tfrac{1}{2}E|v(0)-u_o|^2 + \Phi(v+m+M) - \Phi(u) \end{aligned}$$

and from Def.2.11 for $u\in\mathcal{S}_a$

$$(4.7)\qquad \begin{aligned} &E\int_0^T \langle f, v+m+M-u\rangle dt \leq E\int_0^T \langle v'+Au, v+m+M-u\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ &+ \tfrac{1}{2}E|v(0)-u_o|^2 + \Phi(Ev) - \Phi(Eu). \end{aligned}$$

The relations (4.5) (4.6) and (4.7) get to:

$$(4.8)\qquad \begin{aligned} &E\int_0^T \langle v'-z_\lambda' + e^{\alpha t}Aw_\lambda - \tfrac{1}{\lambda}\beta_\lambda(y_\lambda), v+m+M-u\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ &+ \tfrac{1}{2}E|v(0)-u_o|^2 + \Phi(v+m+M) - \Phi(u) \geq 0 \end{aligned}$$

if $u\in\mathcal{S}$, and

$$(4.9)\qquad \begin{aligned} &E\int_0^T \langle v'-z_\lambda' + e^{\alpha t}Aw_\lambda - \tfrac{1}{\lambda}\beta_\lambda(y_\lambda), v+m+M-u\rangle dt + \tfrac{1}{2}E|m(T)|^2 \\ &+ \tfrac{1}{2}E|v(0)-u_o|^2 + \Phi(Ev) - \Phi(Eu) \geq 0 \end{aligned}$$

if $u\in\mathcal{S}_a$.

Let $z\in W_t(\Omega\times]0,T[;V)$, $z(T)=0$, such that:

$$z(\omega,t)\geq 0 \quad (\omega,t)\text{-a.e. if } u\in\mathcal{S},$$

or

$$Ez(t)\geq 0 \quad t\text{-a.e. if } u\in\mathcal{S}_a .$$

Thanks to the assumption (4.1) one can substitute v in (4.8) and (4.9) by $v-\frac{1}{\varepsilon}z, (\varepsilon>0)$, and so it follows:

$$(4.10)\qquad \begin{aligned} &B_\lambda + E\int_0^T \langle -\tfrac{1}{\varepsilon}z', v-\tfrac{1}{\varepsilon}z+m+M-u\rangle dt + E\int_0^T \langle v'-z_\lambda', -\tfrac{1}{\varepsilon}z\rangle dt \\ &+ \tfrac{1}{2}E|v(0)-\tfrac{1}{\varepsilon}z(0)-u_o|^2 - \tfrac{1}{2}E|v(0)-u_o|^2 + \tfrac{1}{\lambda\varepsilon}E\int_0^T \langle \beta_\lambda(y_\lambda), z\rangle dt \geq \\ &\geq \tfrac{1}{\varepsilon}E\int_0^T e^{\alpha t}\langle Aw_\lambda, z\rangle dt \end{aligned}$$

Where B_λ is the left member in (4.8) if $u\in\mathcal{S}$, or in (4.9) if $u\in\mathcal{S}_a$.

On the other hand we see that

$$(4.11)\quad v(t)-\frac{1}{\varepsilon}z(t)+m(t)-z_\lambda(t) = v(0)-\frac{1}{\varepsilon}z(0)-u_o + \int_0^t (v'-\frac{1}{\varepsilon}z'-z'_\lambda)ds + m(t)$$

and by Prop.1.1 (the energy equality) we obtain:

$$(4.12)\quad \frac{1}{2}E|v(T)-\frac{1}{\varepsilon}z(T)+m(T)-z_\lambda(T)|^2 = \frac{1}{2}E|v(0)-\frac{1}{\varepsilon}z(0)-u_o|^2 + E\int_0^T\langle v'-\frac{1}{\varepsilon}z'-z'_\lambda, v-\frac{1}{\varepsilon}z+m-z_\lambda\rangle dt + \frac{1}{2}E|m(T)|^2.$$

The relation (4.12), for z=0, reduces to:

$$(4.13)\quad \frac{1}{2}E|v(T)+m(T)-z_\lambda(T)|^2 = \frac{1}{2}E|v(0)-u_o|^2 + E\int_0^T\langle v'-z'_\lambda, v+m-z_\lambda\rangle dt + \frac{1}{2}E|m(T)|^2.$$

By subtracting the equalities (4.12) and (4.13) member by member, we infer that:

$$(4.14)\quad E\int_0^T\langle -\frac{1}{\varepsilon}z', v-\frac{1}{\varepsilon}z+m-z_\lambda\rangle dt + E\int_0^T\langle v'-z'_\lambda, -\frac{1}{\varepsilon}z\rangle dt + \frac{1}{2}E|v(0)-\frac{1}{\varepsilon}z(0)-u_o|^2 - \frac{1}{2}E|v(0)-u_o|^2 = 0.$$

Now from (4.10) and (4.14) we get:

$$(4.15)\quad B_\lambda + E\int_0^T\langle -\frac{1}{\varepsilon}z', M-u+z_\lambda\rangle dt + \frac{1}{\lambda\varepsilon}E\int_0^T\langle\beta_\lambda(y_\lambda), z\rangle dt \geq \frac{1}{\varepsilon}E\int_0^T e^{\alpha t}\langle Aw_\lambda, z\rangle dt$$

where $M-u+z_\lambda = M-u+v_\lambda-M = -e^{\alpha t}w_\lambda$.

Let us multiply the inequality (4.15) by ε and afterwards make $\varepsilon\to 0$. We obtain:

$$(4.16)\quad E\int_0^T e^{\alpha t}\langle z', w_\lambda\rangle dt + \frac{1}{\lambda}E\int_0^T\langle\beta_\lambda(y_\lambda), z\rangle dt \geq E\int_0^T e^{\alpha t}\langle Aw_\lambda, z\rangle dt$$

Which yields (4.2) by making the change $z(t)=e^{-\alpha t}\theta(t)$. This completes the proof.

Now we suppose that the space H is ordered like in the Section 3 and that the conditions (3.1) and (3.2) are satisfied.

<u>Theorem 4.2</u> Under the hypotheses (H_1' with $\alpha=0, H_2, H_3$) or (H_1', H_2', H_3), (4.1), and

$$(4.17)\quad \langle\beta_\lambda(t,u), (v-u)^+\rangle \leq 0,$$

for every $v\in D\varphi(t,.)$, $t\in[0,T]$, $u\in V$, $\lambda>0$, it holds

$$(4.18)\quad Eu(t)\leq E\bar{u}(t),\ \ t\text{-a.e.}\qquad \text{for all } u\in\mathcal{J}\cup\mathcal{J}_a,$$

where $\bar{u}$ is the "almost weak" solution of Eq.(2.4) obtained as the limit of the approximating solutions v_λ of Eqs.(2.34),or 3.19, for $\lambda \longrightarrow 0$.

Remark 4.3 The conditions (4.1) and (4.17) are verified for $\varphi(t,u)=I_{K(t)}(u)$ and $\beta_\lambda(t,u)=\beta(t,u)$, from Ex.3.2(a), if, in addition, we have

$$(4.19) \qquad v-u^+ \in K(t) \quad \text{for every } v \in K(t), u \in V.$$

Since other examples we did not find yet, we put the problem of maximal solutions first of all for such functions φ.

Proof.of Th.4.2 Let θ_ε be, the solution of the equation

$$-\varepsilon\theta_\varepsilon'(t) + \theta_\varepsilon(t) = (Ew_\lambda(t))^+$$

$$\theta_\varepsilon(T) = 0, \quad t \in [0,T],$$

where $w_\lambda(t)=e^{-\alpha t}(u(t)-v_\lambda(t))$.
Hence

$$\theta_\varepsilon(t) = \frac{1}{\varepsilon}\int_t^T e^{\frac{t-s}{\varepsilon}}(Ew_\lambda(s))^+ ds \geqslant 0$$

$$\theta_\varepsilon \in W(0,T;V)$$

$$\theta_\varepsilon \longrightarrow (Ew_\lambda)^+ \text{ in } L^2(0,T;V) \text{ (when } \varepsilon \longrightarrow 0).$$

Since

$$E\int_0^T \langle\theta_\varepsilon', w_\lambda\rangle dt = \int_0^T \langle\theta_\varepsilon', (Ew_\lambda)^+\rangle dt - \int_0^T \langle\theta_\varepsilon', (Ew_\lambda)^-\rangle dt$$

$$= -\varepsilon\int_0^T |\theta_\varepsilon'|^2 dt - \frac{1}{2}|\theta_\varepsilon(0)|^2 - \frac{1}{\varepsilon}\int_0^T (\theta_\varepsilon, (Ew_\lambda)^-)dt \leqslant 0$$

We have from (4.2) for $\theta=\theta_\varepsilon$ the following inequality:

$$(4.20) \qquad \frac{1}{\lambda}\int_0^T e^{-\alpha t}\langle\beta_\lambda(t,y_\lambda),\theta_\varepsilon\rangle dt \geqslant \int_0^T [\langle A(t)(Ew_\lambda),\theta_\varepsilon\rangle + \alpha(Ew_\lambda,\theta_\varepsilon)]dt$$

and by passing to the limit ($\varepsilon \longrightarrow 0$) in (4.20) we obtain:

$$0 \geqslant \sigma\int_0^T \|(Ew_\lambda)^+\|^2 dt,$$

by using (4.17) also the hypotheses concerning A.
Consequently $(Ew_\lambda)^+=0$, i.e. $Eu(t) \leqslant Eu_\lambda(t)$ t-a.e., which implies $Eu(t) \leqslant E\bar{u}(t)$ a.e. on $[0,T]$. Q.E.D.

§5. Examples

Example 5.1 We shall reconsider in the context of this paper the example given by Mignot and Puel in [14], in order to show that, in general $\mathcal{S}\cap\mathcal{S}_a$ has more than one element.

Let V=H=R, A(t)u=u, T=2,

$$(5.1)\qquad \Psi(t) = \begin{cases} 0, & t\in[0,1[\\ -1, & t\in[1,2] \end{cases}$$

and $K(t) = \{v : v\in R, v\leq\Psi(t)\}$.

We have the equation

$$(5.2)\qquad \begin{cases} du(t)+u(t)dt+\partial I_{K(t)}(u(t))dt \ni 0 \\ u(0) = 0 \end{cases}$$

$(f=0, M=0, u_0=0)$.

The deterministic processes

$$(5.3)\qquad \begin{aligned} z_\alpha(t) &= 0, && \text{if } t\in[0,1[\\ &= -\alpha e^{1-t}, && \text{if } t\in[1,a] \\ &= -1, && \text{if } t\in]a,2], \end{aligned}$$

where $a=1+\log\alpha$ and $\alpha\in[1,2]$, are weak and "almost weak" solutions for the Eq.(5.2).

Indeed

$$(5.4)\qquad z_\alpha\in L^2_t(\Omega\times]0,2[;R)$$

$$(5.5)\qquad z_\alpha(t) = Ez_\alpha(t)\in K(t),\ \text{for all } t\in[0,2]$$

are obvious.

From (5.5) and Remark 2.12 it follows that if $z\in\mathcal{S}_a$, then $z_\alpha\in\mathcal{S}\cap\mathcal{S}_a$

Therefore, it remains only to show that:

$$(5.6)\qquad (F_\alpha(v,m)=)\ E\int_0^2(v'+z_\alpha)(v+m-z_\alpha)dt + \tfrac{1}{2}E|v(0)|^2 + \tfrac{1}{2}E|m(2)|^2 \geq 0$$

for every $v\in W_t(\Omega\times]0,2[;R)$, $m\in\mathcal{M}^2(0,2;R)$ such that $Ev(t)\leq\Psi(t)$ a.e. on [0,2]. First we remark that, since $Ev(t)$ and $\Psi(t)$ are continuous and continuous from the right on [0,2], respectively, the inequality $Ev(t)\leq\Psi(t)$ holds for all $t\in[0,2]$.

For proving (5.6) we have:

$$v(t)+m(t) = v(0) + \int_0^t v'(s)ds + m(t),$$

and by applying Prop.1.1 we obtain:

$$\tfrac{1}{2}E|v(2)+m(2)|^2 = \tfrac{1}{2}E|v(0)|^2 + E\int_0^2 v'(v+m)dt + \tfrac{1}{2}E|m(2)|^2.$$

Hence

$$F_{\alpha}(v,m) = \tfrac{1}{2}E|v(2)+m(2)|^2 + E\int_0^2 z_{\alpha}(v+m-z_{\alpha})dt - E\int_0^2 v' z_{\alpha}dt \tag{5.7}$$

But Em(t)=0,

$$E\int_0^2 (z_{\alpha}v - v' z_{\alpha})dt = Ev(2) - \alpha Ev(1) - E\int_a^2 v\,dt \geqslant Ev(2) - Ev(1) + 2 - a,$$

and

$$E\int_0^2 z_{\alpha}(m-z_{\alpha})dt = \tfrac{1}{2} - \frac{\alpha^2}{2} - 2 + a .$$

Consequently

$$\begin{aligned} F_{\alpha}(v,m) &\geqslant \tfrac{1}{2}E|v(2)+m(2)|^2 + Ev(2) + \tfrac{1}{2} - Ev(1) - \frac{\alpha^2}{2} \geqslant \\ &\geqslant \tfrac{1}{2}E|1+v(2)+m(2)|^2 + \alpha - \frac{\alpha^2}{2} \geqslant \frac{\alpha(2-\alpha)}{2} \geqslant 0 \end{aligned} \tag{5.8}$$

Thus we have shown that $z_{\alpha} \in \mathcal{S} \cap \mathcal{S}_a$ for every $\alpha \in [1,2]$.

The approximating equation for (5.2) is the following:

$$\begin{aligned} & u'_{\lambda} + u_{\lambda} + \tfrac{1}{\lambda}(u_{\lambda} - \psi(t))^+ = 0 \\ & u_{\lambda}(0) = 0 \end{aligned} \tag{5.9}$$

which has the solution

$$\begin{aligned} u_{\lambda}(t) &= 0 , && \text{if } t \in [0,1[\\ &= \frac{1}{1+\lambda}\left(e^{\frac{\lambda+1}{\lambda}(1-t)} - 1\right), && \text{if } t \in [1,2] \end{aligned} \tag{5.10}$$

The maximal solution is $\bar{u}(t)=\psi(t)$.

<u>Example 5.2</u> Let D be a bounded and open domain of the Euclidean space R^N with a sufficiently smooth boundary , and $Q=]0,T[\times D$

Let $H=L^2(D)$, $V=H^1_0(D)$ and $A(t): V \longrightarrow V^*$ given by:

$$\begin{aligned} \langle A(t)u,v \rangle = & \sum_{i,j=1}^{N} \int_D a_{ij}(t,x)\frac{\partial u}{x_i}\frac{\partial v}{x_j}dx + \sum_{i=1}^{N}\int_D b_i(t,x)\frac{\partial u}{x_i}v\,dx \\ & + \int_D a_0(t,x)uv\,dx \end{aligned} \tag{5.11}$$

where $a_{ij}, b_i, a_0 \in L^{\infty}(Q)$.

We assume that

$$\begin{aligned} & \text{there exists } \sigma > 0 \text{ such that} \\ & \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geqslant \sigma \sum_{i=1}^{N} \xi_i^2 \end{aligned} \tag{5.12}$$

for all $\xi \in R^N$.

Let $\psi:[0,T]\longrightarrow L^2(D)$ be a measurable mapping with the property:

$$(5.13)\qquad \begin{array}{l} \text{there exists } v_o\in W(0,T;H_o^1(D))\subset C(0,T;L^2(D)) \text{ such that:} \\ v_o(t)\le\psi(t), \text{ in } L^2(D), \text{ for all } t\in[0,T]. \end{array}$$

We denote

$$(5.14)\qquad \begin{array}{l} K(t) = \{ v\in H_o^1(D):\ v\le\psi(t) \quad \text{a.e. in } D\} \\ \beta(t,v) = (v-\psi(t))^+ , \end{array}$$

and we consider the equation

$$(5.15)\qquad \begin{array}{l} \frac{\partial u}{\partial t}dt+A(t)udt + \partial I_{K(t)}(u)dt \ni fdt+dM(t) \\ u(0,x) = u_o(x) \end{array}$$

where $f\in L^2(Q)$, $u_o\in L^2(D)$, and $M\in\mathcal{M}^2(0,T;L^2(D))$ is a Wiener process.

Since the hypotheses of Ths. 3.3, 3.7, 4.2 are verified, it follows that Eq.(5.15) has weak solutions ($\mathcal{S}\neq\emptyset$) and "almost weak" solutions ($\mathcal{S}_a\neq\emptyset$) and moreover, that there exists $\bar u\in\mathcal{S}_a$ such that

$$Eu\le E\bar u \quad \text{a.e. on } Q$$

($\bar u$ is a maximal solution).

Hence the variational problem:

$$(5.16)\qquad \begin{array}{l} E\int_Q(\frac{\partial v}{\partial t} + \sum_{i=1}^{N} b_i\frac{\partial u}{\partial x_i} + a_ou)(v+m+M-u)dxdt + \\ \sum_{i,j=1}^{N} E\int_Q a_{ij}\frac{\partial u}{\partial x_i}\,\frac{\partial(v+m+M-u)}{\partial x_j}dxdt + \frac{1}{2}E\int_D|m(T,x)|^2dx \\ + \frac{1}{2}E\int_D|v(0,x)-u_o(x)|^2dx \ge E\int_Q f(v+m+M-u)dxdt, \end{array}$$

for all $v\in W_t(\Omega\times]0,T[;H_o^1(D))$, $m\in\mathcal{M}^2(0,T;L^2(D))$ such that:

$$(5.17)\qquad \begin{array}{l} (V_1)\quad v+m+M\le\psi \quad \text{a.e. on } \Omega\times Q \\ \text{or} \\ (V_2)\quad Ev\le\psi \quad \text{a.e. on } Q, \end{array}$$

has a solution $u\in L_t^2(\Omega\times]0,T[;H_o^1(D))$ which satisfies:

$$(5.18)\quad \begin{array}{l}(S_1)\ u \leq \psi \text{ a.e. on } \Omega \times Q \text{ in the condition } (V_1), \text{ or} \\ (S_2)\ Eu \leq \psi \text{ a.e. on } Q \text{ in the condition } (V_2).\end{array}$$

The weak solutions ($\mathcal{S}$) are those that satisfy (S_1) and the "almost weak" solutions ($\mathcal{S}_a$) are those that satisfy (S_2).
(5.16) is a stochastic problem of the Dirichlet type, with unilateral constraints in the interior.

<u>Example 5.3</u> If in Example 5.2 we replace $H_0^1(D)$ with $H^1(D)$ we shall have a stochastic problem of the Neumann type with unilateral constraints at the interior, for which we also find a "maximal" solution.

In the context of the same spaces for $\psi:[0,T] \longrightarrow L^2(\Gamma)$ such that there exists $v_0 \in W(0,T;H^1(D))$ $v_0|_\Gamma \leq \psi(t)$, $K(t) = \{v \in H^1(D): v|_\Gamma \leq \psi(t)\}$ and $\beta(t,v)=(v|_\Gamma - \psi(t))^+$ the hypotheses of Th.3.3,3.7,4.2 are verified
Thus the stochastic problem (5.16) with boundary constraints

$$(5.17)\quad \begin{array}{l}(V_1)\ (v+m+M)|_\Gamma \leq \psi \text{ a.e. on } \Omega \times]0,T[\times \Gamma \\ \text{or} \\ (V_2)\ (Ev)|_\Gamma \leq \psi \text{ a.e. on }]0,T[\times \Gamma\end{array}$$

has a solution u $L_t^2(\Omega \times]0,T[;H^1(D))$ such that:

$$(5.18)\quad \begin{array}{l}(S_1)\ u|_\Gamma \leq \psi \text{ a.e. on } \Omega \times]0,T[\times \Gamma \text{ in the constraint } (V_1), \text{ or} \\ (S_2)\ (Eu)|_\Gamma \leq \psi \text{ a.e. on }]0,T[\times \Gamma \text{ in the constraint } (V_2).\end{array}$$

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