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On distribution semi-groups of subnormal operators.

Ioana Cioranescu.

Abstract.

We shall prove that every distribution semi-group of subnormal operators in a Hilbert space may be extended to a distribution semi-group of normal operators in a larger Hilbert space; as a corollary we get that each exponential distribution semi-group of subnormal operators is an usual continuous semi-group of subnormal operators.

§ 1. Introduction.

Let X be a Banach space and A a closed, densely defined operator in X with domain D(A); an $\mathcal{L}(X)$-valued distribution $\mathcal{E}$ with the support contained in $[0,+\infty)$ is said to be a regular distribution semi-group (R.D.S.G., in short) of generator A if $\mathcal{E}$ and A satisfy the equations :

$$(1.1) \qquad (A- {}^{d}/_{dt}) * \mathcal{E} = \delta \otimes I_X \quad \text{and} \quad \mathcal{E} * (A- {}^{d}/_{dt}) = \delta \otimes I_{D(A)} .$$

An R.D.S.G. $\mathcal{E}$ is said to be an exponential distribution semi-group of type $\leq \omega$ (E.D.S.G., in short) if $\mathcal{E}$ satisfies the following condition: there exists a real ω such that $e^{-\xi t}\mathcal{E}$ is an $\mathcal{L}(X)$-valued tempered distribution, for any $\xi > \omega$.

Distribution semi-groups were defined and studied by J.L.Lions in [7].

Let us denote $Y = \bigcap_{n=1}^{\infty} D(A^n)$ and endowe Y with the Fréchet topology determined by the norms $\|x\|_n = \sum_{j=0}^{n} \|A^j x\|$.

Then the following conditions on the operator A are equivalent:

(i) A is the generator of a R.D.S.G.;

(ii) the resolvent $R(\lambda;A)$ exists in a logarithmic region of the form

$$\Lambda = \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq \alpha \log|\mathrm{Im}\,\lambda| + \beta,\ \mathrm{Re}\,\lambda \geq \gamma\}$$

where $\alpha, \beta \geq 0, \gamma \in R$ are some given constants, and satisfies

$$\| R(\lambda;A) \| \leq p(|\lambda|) \quad , \quad \lambda \in \Lambda$$

$p(\lambda)$ beeing a polynom with positive coefficients.

(iii) the resolvent set $\rho(A)$ is not empty and the restriction of A to Y, A_Y, is the generator of a locally-equicontinuous semi-group $\{U_t\}_{t\geq 0}$ of class (C_0) in Y.

The equivalence (i)$\Longleftrightarrow$(ii) was proved by J.Chazarain in [2] and the equivalence (i)$\Longleftrightarrow$(iii) was obtained by T.Ushijima in [11].

The R.D.S.G. $\mathcal{E}$ and the semi-group $\{U_t\}_{t\geq 0}$ generated by A_Y may be expressed in terms of the resolvent $R(\lambda;A)$ as follows:

(1.2) $$\mathcal{E}(\varphi) = \int_{\partial\Lambda} \tilde{\varphi}(\lambda) R(\lambda;A)\,d\lambda \qquad \text{for each } \varphi \in \mathcal{D}$$

where $\mathcal{D}$ is the space of all indefinitely differentiable functions on the real line with compact support , $\tilde{\varphi}(\lambda) = \int_{R} e^{\lambda t}\varphi(t)\,dt$, and $\partial\Lambda$ is the boundary of Λ ;

(1.3) $$U_t x = \lim_{h\to 0_+} (I-hA)^{-[t/h]} x \qquad \text{for each } x \in Y ,$$

where the convergence is uniform with respect to t in every finite interval in $[0,+\infty)$ (see [8] and [9]) .

Moreover, we recall that holds :

(1.4) $$\mathcal{E}(\varphi) = \int_{R} \varphi(t) U_t x\,dt \qquad \text{for each } x \in Y ,\ \varphi \in \mathcal{D}$$

(see [11]) and that the semi-group property is given by $\mathcal{E}(\varphi * \psi) = \mathcal{E}(\varphi)\mathcal{E}(\psi)$ for $\varphi, \psi \in \mathcal{D}_0 = \{\varphi \in \mathcal{D},\ \mathrm{supp}\,\varphi \subset [0,+\infty)\}$

Further, the following conditions on the operator A are equivalent:

(i') A is the generator of an E.D.S.G. of type $\leq \omega$;

(ii') the resolvent $R(\lambda;A)$ exists for $\mathrm{Re}\,\lambda > \omega$ and satisfies

$$\| R(\lambda;A) \| \leq p(|\lambda|) \quad , \quad \mathrm{Re}\,\lambda > \omega$$

for a polynom $p(\lambda)$;

(iii') the resolvent set $\rho(A)$ is not void and A_Y is the generator of an equicontinuous semi-group of class (C_0) in Y .

J.L.Lions did prove [7] that (i') $\Longleftrightarrow$ (ii') and D.Fujiwara [4] get that (i') $\Longleftrightarrow$ (iii').

Finally we recall that in [3] C.Foiaş studied distribution semi-groups of normal operators and obtained the following result :

if $\mathcal{E}$ is an E.D.S.G.of normal operators in a Hilbert space,then $\mathcal{E}$ is an usual continuous semi-group of normal operators .

After these preliminaries on distribution semi-groups ,we shall now give some elementary facts on subnormal operators.

Let $\mathfrak{X}$ be a Hilbert space;a linear operator T with domain D(T) is called subnormal if there exists a larger Hilbert space H containing $\mathfrak{X}$ and a normal operator N in H which extends T (one says olso that H reduces T).This definition is due to P.R.Halmos [5]. J.Bram [1] did show that a bounded operator on $\mathfrak{X}$ is subnormal if and only if for every finite sequence of vectors $x_0, x_1, \ldots, x_n$ in $\mathfrak{X}$ holds :

$$\sum_{i,j=0}^{n} \langle T^i x_j, T^j x_i \rangle \geq 0 \quad .$$

He also proved that if N is a minimal normal extension of the bounded subnormal operator T,then $\|T\| = \|N\|$ and $\sigma(N) \subset \sigma(T)$ ($\sigma(N)$ denotes the spectrum) .

Let $\{T_t\}_{t \geq 0}$ be a continuous semi-group of bounded subnormal operators in $\mathfrak{X}$;then there exists a Hilbert space $H \supset \mathfrak{X}$ and a continuous semi-group of normal operators in H extending $\{T_t\}_{t \geq 0}$.

This theorem was first proved by T.Ito [6];recently a short,differnt proof was obtained by E.Nussbaum [10].

Adapting coveniently the method from [10],we shall extend in this note the above theorem to distribution semi-groups of subnormal operators and as a corollary we shall get a generalization of Foiaş' result to E.D.S.G. of subnormal operators in a Hilbert space.

§ 2. Main result.

In all this paragraph X will be a Hilbert space.

Proposition. Let $\{T_t\}_{t\geq 0}$ be a continuous semi-group of bounded subnormal operators in X; then for each $\varphi \in \mathcal{D}$, the operator

$$\mathcal{E}(\varphi) = \int_0^{+\infty} \varphi(t) T_t dt$$

is a subnormal operator.

Proof. By a result from [10], Proposition 2, for each $a > 0$ and each function $f: [0,a] \longrightarrow X$, we have:

$$\int_0^a \int_0^a \langle T_t f(s), T_s f(t) \rangle dt ds \geq 0 .$$

On the other hand, using the semi-group property, it is easy to prove that $\mathcal{E}(\varphi \circledast \psi) = \mathcal{E}(\varphi)\mathcal{E}(\psi)$, for each $\varphi, \psi \in \mathcal{D}$, where $\varphi \circledast \psi(t) = \int_0^t \varphi(s)\psi(t-s)ds$

Then, $x_0, x_1, \ldots, x_n$ beeing arbitrary n+1 vectors in X, holds:

$$\sum_{i,j=0}^{n} \langle \mathcal{E}^i(\varphi)x_j, \mathcal{E}^j(\varphi)x_i \rangle =$$

$$= \sum_{i,j=0}^{n} \langle \mathcal{E}(\varphi_i)x_j, \mathcal{E}(\varphi_j)x_i \rangle =$$

$$= \int_0^{na} \int_0^{na} \sum_{i,j=0}^{n} \langle \varphi_i(t) T_t x_j, \varphi_j(s) T_s x_i \rangle dt ds =$$

$$= \int_0^{na} \int_0^{na} \langle T_t f(s), T_s f(t) \rangle dt ds \geq 0$$

where $\varphi_i = \underbrace{\varphi \circledast \varphi \circledast \cdots \circledast \varphi}_{i}$, $f(t) = \sum_{i=0}^{n} \varphi_i(t) x_i$ and a is such that $\operatorname{supp} \varphi \subset [-a,a]$.

q.e.d.

We shall say that the R.D.S.G. $\mathcal{E}$ is a distribution semi-group of subnormal operators on X if for every $\varphi \in \mathcal{D}$, $\mathcal{E}(\varphi)$ is a subnormal operator on X.

By the above proposition, it is quaite natural generalization of the notion of continuous semi-group of bounded subnormal operators.

Then we have the following

Lemma. The generator A of a R.D.S.G. of subnormal operators is a subnormal operator.

Proof. We start by using some arguments from [2] to get a convenient form of the resolvent $R(\lambda;A)$.

Let $0 < a < a'$ and $\theta \in \mathcal{D}$ such that $\theta(t) \equiv 1$ for $t \in [0,a]$ and $\theta(t) \equiv 0$ for $t \notin [-1,a']$. Denote $\theta_\lambda(t) = e^{-\lambda t}\theta(t)$, $\lambda \in \mathbb{C}$. Then, using the first equation from (1.1), we get :

$$(A-\lambda)\,\mathcal{E}(\theta_\lambda) = I - \mathcal{E}(e^{-\lambda t}\theta'(t)) .$$

Put $\Psi_\lambda(t) = e^{-\lambda t}\theta'(t)$; then in [2] it is proved that for λ belonging to some logarithmic region Λ , $\|\mathcal{E}(\Psi_\lambda)\| \leq 1/2$, that is

$$[I - \mathcal{E}(\Psi_\lambda)]^{-1} = \sum_{n=0}^{\infty} \mathcal{E}^n(\Psi_\lambda) = \sum_{n=0}^{\infty} \mathcal{E}^n(\Psi_\lambda^+) =$$

$$= \sum_{n=0} \mathcal{E}(\Psi_\lambda^+ * \cdots * \Psi_\lambda^+) =$$

$$= \lim_{k \to \infty} \mathcal{E}(\varphi_{\lambda,k})$$

where $\varphi_{\lambda,k} = \sum_{n=0}^{k} \Psi_\lambda^+ * \cdots * \Psi_\lambda^+ \in \mathcal{D}_0$, $\Psi_\lambda^+ = \Psi_\lambda|_{[0,+\infty)}$

Finally we get using in the same way the second equation from (1.1) that $R(\lambda;A)$ exists in a logarithmic region Λ and is given by

$$R(\lambda;A) = \mathcal{E}(\theta_\lambda)\,[I - \mathcal{E}(\Psi_\lambda)]^{-1} =$$

$$= \mathcal{E}(\theta_\lambda) \lim_{k \to \infty} \mathcal{E}(\varphi_{\lambda,k}) = \lim_{k \to \infty} \mathcal{E}(\theta_\lambda^+).\mathcal{E}(\varphi_{\lambda,k})$$

$$= \lim_{k \to \infty} \mathcal{E}(\Phi_{\lambda,k})$$

where $\Phi_{\lambda,k} = \theta_\lambda^+ * \varphi_{\lambda,k} \in \mathcal{D}_0$.

(we used the fact proved in [7], that putting for $\Psi \in \mathcal{D}$, $\mathcal{E}(\Psi^+)\mathcal{E}(\varphi) = \mathcal{E}(\Psi^+ * \varphi)$, we get a closed densely defined operator such that $\mathcal{E}(\Psi^+) = \mathcal{E}(\Psi)$)

As for each $\lambda \in \Lambda$ and $k \in \mathbb{N}$, the operator $\mathcal{E}(\varphi_{\lambda,k})$ is subnormal, it is clear that $R(\lambda;A)$ is a subnormal operator on X.

Let $\lambda_0 \in \Lambda$ be fixed and let N_{λ_0} be a minimal normal extension of $R(\lambda_0;A)$ acting on a Hilbert space H. Then $N_{\lambda_0}^{-1}$ exists, by an argument given in [10], Proposition 3 and we give it for completness. If $\mathcal{N}_{\lambda_0}^{\perp} = \mathcal{N}(N_{\lambda_0})$ is the null space of N_{λ_0}, then $\mathcal{N}_{\lambda_0} = \mathcal{N}(N_{\lambda_0}^{*})$ and $\mathcal{N}_{\lambda_0} = \overline{\mathcal{R}(N_{\lambda_0})} \supset \mathcal{R}(N_{\lambda_0}) = X$ ($\mathcal{R}$ denotes the range). Since $\mathcal{N}_{\lambda_0}^{\perp}$ reduces N_{λ_0} and N_{λ_0} is a minimal normal extension of $R(\lambda_0;A)$, $\mathcal{N}_{\lambda_0} = H$ and therefore $\mathcal{N}_{\lambda_0}^{\perp} = \{0\}$. Hence $N_{\lambda_0}^{-1}$ exists, is closed and densely defined and is a minimal normal extension of $\lambda_0 - A$. Hence $N = \lambda_0 - N_{\lambda_0}^{-1}$ is a minimal normal extension of A .

q.e.d.

We can now give the

Theorem. Let $\mathcal{E}$ be a R.D.S.G. of bounded subnormal operators in a Hilbert space X; then there exists a Hilbert space H containing X and a R.D.S.G. $\tilde{\mathcal{E}}$ of normal operators in H such that $\tilde{\mathcal{E}}(\varphi)_X = \mathcal{E}(\varphi)$, for each $\varphi \in \mathcal{D}$.

Proof. Let N be a minimal normal extension of A, associated as in the Lemma to a fixed $\lambda_0 \in \Lambda$, acting in a Hilbert space $H \supset X$. Then $\sigma((\lambda_0 - N)^{-1}) \subset \sigma((\lambda_0 - A)^{-1})$ and by the spectral mapping theorem, it results that $\sigma_e(N) \subset \sigma_e(A)$ (σ_e is the extended spectrum). Hence $\sigma(N) \subset \sigma(A)$, whence $\rho(A) \subset \rho(N)$. So $\rho(N)$ contains the logarithmic region Λ and for $\lambda \in \Lambda$ holds:

$$\|R(\lambda;N)\| = \|R(\lambda;A)\| \leq p(|\lambda|) \quad .$$

Therefore, by the equivalence (i)$\Longleftrightarrow$(ii), N is the generator of a R.D.S.G. of normal operators in H, given by (1.2):

$$\tilde{\mathcal{E}}(\varphi) = \int_{\partial\Lambda} \tilde{\varphi}(\lambda) R(\lambda;N)\, d\lambda \quad , \varphi \in \mathcal{D} ,$$

where by the Lemma, $R(\lambda;N)$ is normal. It is clear that each $\tilde{\mathcal{E}}(\varphi)$ extends $\mathcal{E}(\varphi)$, $\varphi \in \mathcal{D}$.

q.e.d.

Corollary. Let $\mathcal{E}$ be an E.D.S.G. of subnormal operators in a Hilbert space X; then $\mathcal{E}$ is given by an usual continuous semi-group of bounded subnormal operators in X.

Proof. Let N be a minimal normal extension of the generator A of acting on the Hilbert space H X. Then by a similar argument as in the above theorem, it results that $R(\lambda;N)$ exists for Re $\lambda > \omega$, where ω is the type of $\mathcal{E}$ and is majorized by a polynom.
Hence, by the equivalence (i') $\Longleftrightarrow$ (ii'), N is the generator of an E.D.S.G. of normal operators in H, $\tilde{\mathcal{E}}$. By the result of C.Foiaş, $\tilde{\mathcal{E}}$ is given by an usual continuous semi-group $\{T_t\}_{t\geq 0}$ of normal operators in H.

Let $x \in Y = \bigcap\limits_{n=0}^{\infty} D(A^n)$ and $\{U_t\}_{t\geq 0}$ the equicontinuous semi-group of class (C_0) generated by A_Y in Y (see (iii')); then by (1.3) we have :

$$U_t x = \lim_{h \to 0_+} (I-hA)^{-[t/h]} x =$$

$$= \lim_{h \to 0_+} (I-hN)^{-[t/h]} x = T_t x \ .$$

As Y is dense in X, it is clear that $\{U_t\}_{t\geq 0}$ is a continuous semi-group of subnormal operators on X which by (1.4), coincide, in the distributional sense with $\mathcal{E}$.

q.e.d.

References.

1. Bram, J., Subnormal operators, Duke Math. J., 22 (1955), 75-94.
2. Chazarain, J., Problèmes de Cauchy abstraits et applications à quelques problèmes mixtes, J. Funct. Anal., 7 (1972), 386-446.
3. Foias, C., Remarques sur les semi-groupes distributions d'opérateurs normaux, Portugal. Math., 19, (1960), 227-242.
4. Fujiwara, D., A caracterisation of exponential distribution semi-groups, J. Math. Soc. Japan, 18, (1966), 267-274.
5. Halmos, P.R., Normal dilations and extensions of operators, Summa Brasil. Math., 2, (1950), 125-134.
6. Ito, T., On the commutative family of subnormal operators, J. Fac. Sci. Hokkaido Univ. Ser. I, 14, (1958), 1-15.
7. Lions, J.L., Les semi-groupes distributions, Portugal. Math., 19, (1960), 141-164.
8. Oharu, S., Semi-groups of linear operators in a Banach space, Publ. R.I.M.S., Kyoto Univ., 7 (1971 72), 205-260.
9. Oharu, S., Eine Bemerkung zur Charakterisierung der Distributionenhalbgruppen, Math. Ann., 204, (1973), 189-198.
10. Nussbaum, E., Semi-groups of subnormal operators, J. London Math. Soc. (2), 14, (1976), 340-344.
11. Ushijima, T., On the generation and smoothness of semi-groups of linear operators, J. Fac. Sci. Univ. Tokyo, 19, (1972) 65-127.