ON NECESSARY CONDITIONS FOR STOCHASTIC CONTROL PROBLEMS

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Abstract

We are concerned with necessary conditions for stochastic control problems whose dynamics are described by nonlinear Ito's equation. It is shown that general methods used in deterministic optimization problems are applicable in stochastic case also, even if the diffusion coefficients are depending on the control variable. The adjoint system defines a non-anticipative process with a prescribed final value.

Generally, maximum principle and adjoint system in stochastic control problems are equivalent with Euler's inequation (see (22)).

§1. Introduction

We consider a class of stochastic differential equations

$$1)\quad dx=f(\omega,t,x,u(t))dt+\sum_{i=1}^{k} g_i(\omega,t,x,u(t))dB_i(t),\quad t\in[t_o,t_1],\quad x\in R^n,$$

with given initial condition $x(0)=x_o\in R^n$, where $B(t)=(B_1(t),\ldots,B_k(t))$ is a k-dimensional Brownian motion and the control u is a stochastic process over the probability space $\{\Omega,\mathcal{F},P\}$. For each $(t,x,u)\in(t_o,t_1\ xR^n xR^m$, f and g_i are random vectors and it is marked by explicit dependence on $\omega\in\Omega$. As admissible controls we allow any non-anticipative bounded process $u(t)=(u_1(t),\ldots,u_m(t))\in U$, where U is a convex subset in R^m.

Assume that f, g_i satisfy some growth and Lipschitz condition in x uniformly with respect to $(\omega,t,u)\in\Omega \times [t_o,t_1] \times U_1$ and as random processes they are non-anticipative for each $(x,u)\in R^n \times U$ with respect to σ-algebras $\mathcal{F}_t \subseteq \mathcal{F}$, generated by $\{B(s), t_o \leq s \leq t\}$.

For an admissible control u there is an unique non-anticipative process x^u verifying (1) in integral form a.e. on Ω whose trajectories are continuous functions.

As the functional to be minimized we consider

$$2) \quad J(x,u)=E\left[G(\omega,x(t_1))+\int_{t_o}^{t_1} L(\omega,t,x(t),u(t))dt\right]$$

where G and L verify polinomyal growth conditions.

In [2] it has been given a general characterization of the optimal element in terms of the dynamic programming equation. Unfortunately the method used in [2] is not applicable in our case since the control variable is entering in diffusion coefficients.

When considering stochastic control equations with diffusion coefficients depending on the control we are facing with two alternatives: to use either smooth feedback controls or open loop controls given by non-anticipative processes. In the feedback smooth control case there is a Pontriagin's maximum principle given in [3] but the adjoint system doesn't define a non-anticipative solution.

Deterministic optimization methods in stochastic control problems have been used in [1] where by convex analysis the Pontriagin maximum principle is obtained for the problem where G=0, the functional L is convex and the system (1) is linear in (x,u).

In our opinion, the Pontriagin type variations (or Mac

Shane variations) are not suitable in nonlinear stochastic control problems since the drift and difussion terms get different orders of variation corresponding to the same variation of the control.

Moreover, even if diffusion coefficients are not depending on the control variable, the adjoint system is not defining a nonanticipative process (see [3]).

The most suitable control variations are those small in L_∞-norm.

In this paper we get first order necessary conditions converting the optimality property into the Euler inequation on a Banach space (the Lagragean form) and using the Wiener integral reprezentation of a square integrable martingale we obtain the adjoint system and the Hamiltonian eypression of the optimality (maximum principle).

Since we use local variations in original problems the maximum principle has a local form.

In a forthcoming paper introducing relaxed controls in stochastic control problems we shall get the global maximum principle.

§2. Some definitions and notations

In order to list the conditions under which (1) has an unique solution we need to state more precisely the problem we are concerned with.

On the probability space $\{\Omega,\mathcal{F},P\}$ a k-dimensional Brownian motion $B(t)=(B_1(t),\dots,B_k(t))$ $t\in[t_o,t_1]$ with $B(t_o)=0$, is considered and let $\mathcal{F}_t\subseteq\mathcal{F}$ be the increasing family of σ-algebras generated by $\{B(s),\ t_o\le s\le t\}$. Denote $\mathcal{B}_k$ the σ-algebra of Borelian sets in R^k, $S=\Omega \times [t_o,t_1]$ and consider on S the σ-algebra product $\mathcal{F}\otimes\mathcal{B}_1$ generated by the sets $C=A \times B$, $A\in\mathcal{F}$, B a Borelian set in $[t_o,t_1]$.

Let $\mathcal{S}$ be the σ-algebra consisting of all measurable sets $E\in\mathcal{F}\otimes\mathcal{B}_1$ such that

i) $E_\omega=\{t\in[t_o,t_1]:(\omega,t)\in E\}\in\mathcal{B}_1$ for all $\omega\in\Omega$

ii) $E_t=\{\omega\in\Omega:(\omega,t)\in E\}\in\mathcal{F}_t$ for all $t\in[t_o,t_1]$.

On the space $S \times R^{n+m}$ it is considered the σ-algebra $\mathcal{S}\otimes\mathcal{B}_{n+m}$ generated by the algebra $\mathcal{S}\times\mathcal{B}_{n+m}$.

Assume that $f, g_i : S \times R^{n+m} \longrightarrow R^n$ are $\mathcal{S}\times\mathcal{B}_{n+m}$-measurable.

By definition, for each $(t,x,u)\in[t_o,t_1]\times R^n\times U$ fixed the functions f, g_i are $\mathcal{F}_t$-measurable and therefore $f(\cdot,\cdot,x,u)$, $g_i(\cdot,\cdot,x,u)$ are non-anticipative.

Assume that $G:\Omega\times R^n \longrightarrow R$ and $L:S\times R^{n+m} \longrightarrow R$ are $\mathcal{F}\otimes\mathcal{B}_n$ and $\mathcal{S}\otimes\mathcal{B}_{n+m}$ measurable respectively.

For each $s\in S$, f, g_i, G and L are continuous in (x,u) and they have continuous first derivatives in $(x,u)\in R^n\times R^m$ such that

H_1) the matrix valued functions $\frac{\partial h}{\partial x}(s,x,u)$, $\frac{\partial h}{\partial u}(s,x,u)$ are $\mathcal{S}\otimes\mathcal{B}_{n+m}$-measurable and $\left\|\frac{\partial h}{\partial x}(s,x,u)\right\| + \left\|\frac{\partial h}{\partial u}(s,x,u)\right\| \leq K$,

$(\forall)$ $(s,x,u)\in S \times R^n \times U$ for some constant $K>0$, where $h=f, g_i$;

H_2) $\|h(s,0,0)\| \leq L_1$, $(\forall)$ $s\in S$, for some constant $L_1>0$, where $h=f, g_i$;

H_3) $\left\|\frac{\partial h}{\partial x}(s,x,u)\right\| + \left\|\frac{\partial h}{\partial u}(s,x,u)\right\|$, $\|h(s,x,u)\| \leq L_2(1+\|x\|^p+\|u\|^p)$,

$(\forall)$ $s\in S$, for some constants $L_2>0$, $p>1$, where $h=G, L$.

The simbol $\|\cdot\|$ means the norm of a (nx1)-matrix considered as a vector in R^{nl}.

The admissible class of controls consists of all bounded $\mathcal{S}$-measurable functions $u:S \longrightarrow U$, where $U \subseteq R^m$ is a convex set and denote it by $\mathcal{U}$. By definition any $u \in \mathcal{U}$ is a non-anticipative process with respect to the family $\{\mathcal{F}_t\}$ of σ-algebras.

Under the hypotheses (H_1) and (H_2), for each $u \in \mathcal{U}$, there exists a non-anticipative process $x^u(t)$ with continuous trajectories, verifying (1) in integral form a.e. (P) with respect to $\omega \in \Omega$, and $E \sup_{t_o \le t \le t_1} \|x^u(t)\|^2 < \infty$.

The uniqueness of the solution x^u must be understood in the following sense: any other process x verifying the same conditions as x^u satisfies $P\{\sup_{t \le t_1} \|x^u(t)-x(t)\| > 0\} = 0$. Since (H_1) and (H_2) imply a linear growth condition

$$3) \qquad \|h(s,x,u)\| \le \bar{K}(1+\|x\|) \quad (\forall)\ s \in S, \text{ for } h=f,g_i ,$$

where $\bar{K} \geq \max(L_1 + L\|u\|, K)$, the existence and uniqueness of the solution x^u in (1) is shown in a standard way (see for example [4], p.51).

From now on we shall omit to write explicitely the dependence of ω.

We construct a sequence

$$x_o(t) = x_o,\ x_{j+1}(t) = x_o + \int_{t_o}^{t} f(\tau, x_j(\tau), u(\tau))d\tau + \sum_{i=1}^{k} \int_{t_o}^{t} g_i(\tau, x_j(\tau), u(\tau))dB_i(\tau),$$

and it follows $E\int_{t_o}^{t_1} \|x_j(t)\|^2 dt < \infty$, $E\|x_{j+1}(t) - x_j(t)\|^2 \le \frac{[M(t_1-t_o)]^{j+1}}{(j+1)!}$ for any $j \geq 1$, where $M \geq \max(2K^2(1+t_1-t_o)(1+\|x_o\|),$ $2\tilde{K}^2(1+t_1-t_o))$, $\tilde{K} \geq \max(L_1 + K \max_{s \in S} \|u(s)\|, K)$.

Since

$$\sup_{t\leq t_1} \|x_{j+1}(t)-x_j(t)\|^2 \leq 2(t_1-t_o)K^2 \int_{t_o}^{t_1} \|x_j(t)-x_{j-1}(t)\|^2 dt+$$

$$+2\sup_{t\leq t_1} \left\| \sum_{i=1}^{k} \int_{t_o}^{t} [g_i(\tau,x_j(\tau),\ u(\tau))-g_i(\tau,x_{j-1}(\tau),\ u(\tau))]\,dB_i(\ )\right\|^2$$

we find

$$E\sup_{t\leq t_1} \|x_{j+1}(t)-x_j(t)\|^2 \leq 2K^2(t_1-t_o)\int_{t_o}^{t_1} E\|x_j(t)-x_{j-1}(t)\|^2 dt+$$

$$+8K^2 \int_{t_o}^{t_1} E\|x_j(t)-x_{j-1}(t)\|^2 dt \leq C\frac{[M(t_1-t_o)]^j}{j!}$$

where $C=2K^2(t_1-t_o)(t_1-t_o+4)$

It follows

$$P\left\{ \sup_{t\leq t_1} \|x_{j+1}(t)-x_j(t)\| > \frac{1}{2^j}\right\} \leq 2^{2j} C\frac{[M(t_1-t_o)]^j}{j!}$$

and using the Borel-Cantelli Lemma we obtain that the sequence

$$x_j(t)=x_o+\sum_{p=0}^{j-1}(x_{p+1}(t)-x_p(t))$$

converges a.e. (P) and uniformly with respect to $t\in[t_o,t_1]$. Denote x^u the limit process and it will be a non-anticipative one with continuous trajectories a.e. in $\omega\in\Omega$.

By definition $\int_{t_o}^{t_1} \|x^u(t)\|^2 dt<\infty$ a.e. (P) and the integrals

$$\int_{t_o}^{t} f(\tau,x^u(\tau),\ u(\tau))\,d\tau,\qquad \int_{t_o}^{t} g_i(\tau,x^u(\tau),u(\tau))\,dB_i(\tau)$$

are well defined.

Moreover, for almost all $\omega \in \Omega$ we have

$$\lim_{j\to\infty} f(t,x^j(t),\ u(t))=f(t,x^u(t),u(t)),$$

$$\lim_{j\to\infty} g_i(t,x^j(t),u(t))=g_i(t,x^u(t),u(t))$$

uniformly with respect to $t\in[t_o,t_1]$ and hence

$$\lim_{j\to\infty} \int_{t_o}^{t_1} \| g_i(t,x_j(t),u(t))-g_i(t,x^u(t),u(t))\|^2 dt=0$$

in probability.

Therefore x^u is a solution in (1)

Since $E\,\|x_{j+1}(t)\|^2 \leq C(1+\|x_o\|^2)+C\int_{t_o}^{t} E\,\|x_j(\tau)\|^2 d\tau$, where the constant C is depending on $\overline{K}$, (t_1-t_o), and the norm of the bounded control u, by induction argument we get

$$E\,\|x_{j+1}(t)\|^2 \leq C(1+\|x_o\|^2)\exp C(t_1-t_o) .$$

Using Fatou's lemma we conclude

$$E\,\|x^u(t)\|^2 \leq C(1+\|x_o\|^2)\exp C(t_1-t_o)$$

and therefore $x^u(t)$ belongs to $L_2(\Omega)$ for any $t\in[t_o,t_1]$.

Actually, we have

4) $$E \sup_{t\leq t_1} \|x^u(t)\|^l < \infty, \text{ for any integer } l,\ l \geq 1 .$$

In order to prove (4) it is enough to consider l even.

Denote y_j the j-component of x^u. We have

$$y_j(t)=x_{oj}+\int_{t_o}^{t} f_j(\tau,x^u(\tau),u(\tau))d\tau+\sum_{i=1}^{k}\int_{t_o}^{t} g_{ij}(\tau,x^u(\tau),u(\tau))dB_i(\tau)$$

and using Hölder's inequality for p=1, $q=\frac{l}{l-1}$ it follows

$$(y_j(t))^l\leq \zeta^{l-1}\Big[x_{oj}^l+(t_1-t_o)^{l-1}\int_{t_o}^{t} f_j^l(\tau,x^u(\tau),u(\tau))d\tau +$$

$$+k^{l-1}\sum_{i=1}^{k}\Big(\int_{t_o}^{t} g_{ij}(\tau,x^u(\tau),u(\tau))dB_i(\tau)\Big)^l\Big]$$

Using (3) we get

5) $$g_{ij}^l(t,x^u(t),u(t)),\ f_j^l(t,x^u(t),u(t))\leq C(1+m^l(t)),$$

for some constant $C>0$, where $m^l(t)=\sup_{\substack{j=1,..,n\\ \tau\leq t}} y_j^l(\tau)$

We conclude that

6) $$\int_{t_o}^{t} f_j^l(\tau,x^u(\tau),u(\tau))d\tau\leq C\int_{t_o}^{t}(1+m^l(\tau))d\tau$$

Now, we shall estimate $z^l(t)$ in $L_2(\Omega)$, where

7) $$z(t)=\int_{t_o}^{t} g_{ij}(\tau,x^u(\tau),\ u(\tau))dB_i(\tau)$$

Denote $\sigma_A=\inf\{t\in[t_o,t_1],\ \|z(t)\|\geq A\}$, $g_A(\tau)=\chi_{[t_o,\sigma_A]}(\tau)g_{ij}(\tau,x^u(\tau),u(\tau))$

and $$Z_A(t)=\int_{t_o}^{t\wedge\sigma_A} g_{ij}(\tau,x^u(\tau),u(\tau))dB_i(\tau)$$

$$Em^{l}(t) \leq C_1(1+\|x_o\|^{l})+C_2\int_{t_o}^{t} Em^{l}(\tau)d\tau$$

and by Gronwall's Lemma

$$Em^{l}(t) \leqslant C(1+\|x_o\|^{l})$$

which completes the proof of (4).

§ 3. Some auxiliary results

Let $(\tilde{x}(t),\tilde{u}(t))$ be optimal in the problem defined by the dynamic (1) and functional (2). For $u\in\mathcal{U}$ define

$$u_\varepsilon(t)=\tilde{u}(t)+\varepsilon(u(t)-\tilde{u}(t)), \quad \text{for } \varepsilon\in[0,1]$$

Since $\mathcal{U}$ is convex we have $u_\varepsilon\in\mathcal{U}$ and u_ε, $\varepsilon\in[0,1]$, are uniformly bounded.

It is desirable to know the dependence on ε of the solution x_ε in (1) corresponding to u_ε.

Lemma 1

Assume that (H_1) and (H_2) hold. Let x_ε and $\tilde{x}$ be the solutions in (1) corresponding to the controls u_ε and $\tilde{u}$.

Then $\lim\limits_{\varepsilon\to 0} E \sup\limits_{t\leq t_1} \|x_\varepsilon(t)-\tilde{x}(t)\|^{2}=0$

Proof

We have

By definition Z_A is bounded and

$$Z_A(t)=\int_{t_o}^{t} g_A(\tau)\,dB_i(\tau)$$

Using Ito's formula we get that

$$E\,Z_A^l(t)=\frac{l(l-1)}{2}\int_{t_o}^{t} E\,Z_A^{l-2}(\tau)\,g_A^2(\tau)\,d\tau$$

and $E\,Z_A^l(t)$ is increasing in t.

By Hölder's inequality for $p=\frac{l}{l-2}$, $q=\frac{l}{2}$

$$E\,Z_A^l(t)\leq\frac{l(l-1)}{2}\int_{t_o}^{t}(EZ_A^l(\tau))^{\frac{1}{p}}(Eg_A^l(\tau))^{\frac{1}{q}}d\tau\leq$$

$$\frac{l(l-1)}{2}(E\,Z_A^l(t))^{1/p}\int_{t_o}^{t}(Eg_A^l(\tau))^{1/q}d\tau$$

and hence

$$(E\,Z_A^l(t))^{1/q}\leq\frac{l(l-1)}{2}\int_{t_o}^{t}(Eg_A^l(\tau))^{1/q}d\tau$$

Finally

$$E\,Z_A^l(t)\leqslant C(l)\int_{t_o}^{t}Eg_A^l(\tau)\,d\tau\leqslant C(l)\int_{t_o}^{t}(1+Em^l(\tau))\,d\tau$$

and by Fatou's Lemma

$$E\,Z^l(t)\leq\varliminf_{A\to\infty}EZ_A^l(t)\leq C(l)\int_{t_o}^{t}(1+Em^l(\tau))\,d\tau$$

In conclusion

$$x_\varepsilon(t)-\tilde{x}(t)=\int_{t_o}^{t}\left[f(\tau,x_\varepsilon(\tau),u_\varepsilon(\tau))-f(\tau,\tilde{x}(\tau),\tilde{u}(\tau))\right]d\tau+$$

$$+\sum_{i=1}^{k}\int_{t_o}^{t}\left[g_i(\tau,x_\varepsilon(\tau),u_\varepsilon(\tau))-g_i(\tau,\tilde{x}(\tau),\tilde{u}(\tau))\right]dB_i(\tau)$$

Since each stochastic integral is a continuous martingale we get

$$E\sup_{t\leq T}\|x_\varepsilon(t)-\tilde{x}(t)\|^2\leq 2(t_1-t_o)E\int_{t_o}^{T}\|f(\tau,x_\varepsilon(\tau),u_\varepsilon(\tau))-f(\tau,\tilde{x}(\tau),\tilde{u}(\tau))\|^2d\tau+$$

$$+8k\sum_{i=1}^{k}E\int_{t_o}^{T}\|g_i(\tau,x_\varepsilon(\tau),u_\varepsilon(\tau))-g_i(\tau,\tilde{x}(\tau),\tilde{u}(\tau))\|^2d\tau$$

Using (H_1) it follows

$$h_\varepsilon(T)\overset{\Delta}{=}E\sup_{t\leq T}\|x_\varepsilon(t)-\tilde{x}(t)\|^2\leq N\int_{t_o}^{t}h_\varepsilon(\tau)d\tau+\varepsilon^2N\int_{t_o}^{t_1}E\|u(t)-\tilde{u}(t)\|^2dt\ ,$$

where $N=8(k^2+1)K^2(t_1-t_o+1)$

and by Gronwall's lemma

$$h_\varepsilon(T)\leq\varepsilon^2N\int_{t_o}^{t_1}\|u(t)-\tilde{u}(t)\|^2dt\ \exp N(t_1-t_o),\quad\text{for all }T\leq t_1$$

The proof is complete.

Further we shall prove that $x_\varepsilon(t)$ fulfils

8) $\qquad x_\varepsilon(t)=\tilde{x}(t)+\varepsilon\bar{x}(t)+\theta(\varepsilon,t)$

where $\limsup_{\varepsilon\to 0}\ \sup_{t\leq t_1}E\left\|\dfrac{0(\varepsilon,t)}{\varepsilon}\right\|^2=0$, and $\bar{x}(t)$ is the solution of the following stochastic equation with random coefficients

$$9) \quad dx = A(t)x(t)+B(t)(u(t)-\tilde{u}(t))\,dt+\sum_{i=1}^{k}[C^i(t)x(t)+D^i(t)(u(t)-\tilde{u}(t))]dB_i(t)$$

$$x(t_o)=0$$

where $A(t)=\frac{\partial f}{\partial x}(t,\tilde{x}(t),\ \tilde{u}(t))$, $B(t)=\frac{\partial f}{\partial u}(t,\tilde{x}(t),\ \tilde{u}(t))$,

$$C^i(t)=\frac{\partial g_i}{\partial x}(t,\tilde{x}(t),\ \tilde{u}(t)),\quad D^i(t)=\frac{\partial g_i}{\partial u}(t,\tilde{x}(t),\tilde{u}(t)).$$

To prove (8) we need the following lemma.

Let $(S,\mathcal{S},\mu)$ be a measure space with $0<\mu(S)<\infty$.

Lemma 2

Let $f(s,y):S\times R^k\to R$ be $\mathcal{S}\otimes\mathcal{B}_k$-measurable and continuous in y for each s. Let $\tilde{y},y_n:S\to R^k$ be $\mathcal{S}$-measurable and such that

i) $\tilde{y},y_n\in L_1(S,\mu)$, $\lim\limits_{n\to\infty} y_n=\tilde{y}$ in $L_1(S,\mu)$

ii) $f(s,y_n(s))\leq h(s)$, $s\in S$, where $h\in L_p(\mathcal{S},\mu)$ $(p\geq 1)$

Then $f(s,y(s))$ is $\mathcal{S}$-measurable, for $y=y_n$, $\tilde{y}$, and

$$\lim_{n\to\infty} f(s,y_n(s))=f(s,\tilde{y}(s)) \quad \text{in } L_p(S,\mu)$$

Proof

The proof is almost obvious. If $y:S\to R^k$ is $\mathcal{S}$-measurable then $g(s)\stackrel{\Delta}{=}(s,y(s)):S\to S\times R^k$ is $\mathcal{S}$-measurable, where on $S\times R^k$ one considers the σ-algebra $\mathcal{S}\otimes\mathcal{B}_k$ generated by the algebra $\mathcal{S}\times\mathcal{B}_k$. Since the familly of sets $C\in\mathcal{S}\otimes\mathcal{B}_k$ verifyin $g^{-1}(C)\in\mathcal{S}$ is a

σ-algebra containing the sets $C=A \times B$, $A \in \mathcal{S}$, $B \in \mathcal{B}_k$ it follows $\mathcal{S} \supseteq g^{-1}(\mathcal{S} \otimes \mathcal{B}_k)$.

By hypothesis $f(s,y)$ is $\mathcal{S} \otimes \mathcal{B}_k$-measurable and therefore $f(g(s))$ is $\mathcal{S}$-measurable for $g=g_n(s) \stackrel{\Delta}{=} (s,y_n(s))$ and $g=\tilde{g}(s) \stackrel{\Delta}{=} (s,\tilde{y}(s))$. In order to prove convergence in $L_p(S,\mu)$ of $f(g_n(s))$ we notice that any subsequence of $y_n(s)$ contains a sequence that converges a.e. (μ) to $\tilde{y}(s)$ and since $f(s,y)$ is continuous in y we get the same property for $f_n(s)=f(g_n(s))$ and $\tilde{f}(s)=f(\tilde{g}(s))$. Using (ii) and dominated convergence theorem we obtain the conclusion.

In order to get (8) we have to estimate ~~in~~

$$h_\varepsilon(t)=\frac{x_\varepsilon(t)-\tilde{x}(t)-\varepsilon\bar{x}(t)}{\varepsilon} \quad \text{in } L_2(\Omega,P).$$

Lemma 3

Assume (H_1) and (H_2) hold. Let x_ε and $\tilde{x}$ be solutions in (1) corresponding to u_ε and $\tilde{u}$. Then $\sup\limits_{t\leq t_1} E\|h_\varepsilon(t)\|^2 \longrightarrow 0$ for $\varepsilon \longrightarrow 0$.

Proof

Denote $p_{\varepsilon,\mu}(t)=(\tilde{x}(t)+\mu(x_\varepsilon(t)-x(t)),\ \tilde{u}(t)+\mu\varepsilon(u(t)-\tilde{u}(t)))$,

$$A_\varepsilon(t,\mu)=\frac{\partial f}{\partial x}(t,p_{\varepsilon,\mu}(t)),\quad B_\varepsilon(t,\mu)=\frac{\partial f}{\partial u}(t,p_{\varepsilon,\mu}(t)),$$

$$C^i_\varepsilon(t,\mu)=\frac{\partial g_i}{\partial x}(t,p_{\varepsilon,\mu}(t)),\quad D^i_\varepsilon(t,\mu)=\frac{\partial g_i}{\partial u}(t,p_{\varepsilon,\mu}(t)).$$

By hypothesis f and g_i are of class C^1 in (x,u) and it follows

$$10)\quad f(t,x_\varepsilon(t),u_\varepsilon(t))-f(t,\tilde{x}(t),\tilde{u}(t))=\int_0^1\left[A_\varepsilon(t,\mu)(x_\varepsilon(t)-\tilde{x}(t))+\varepsilon B_\varepsilon(t,\mu)(u(t)-\tilde{u}(t))\right]d\mu$$

11) $g_i(t,x_\varepsilon(t),u_\varepsilon(t))-g_i(t,\tilde{x}(t),\tilde{u}(t))=\int_0^1\left[C^i_\varepsilon(t,\mu)(x_\varepsilon(t)-\tilde{x}(t))+\varepsilon D^i_\varepsilon(t,\mu)(u(t)-\tilde{u}(t))\right]d\mu$

Using (19) and (11) we get

12) $$h_\varepsilon(t)=\int_{t_0}^t\left\{\int_0^1\left[A_\varepsilon(\tau,\mu)\frac{x_\varepsilon(\tau)-\tilde{x}(\tau)}{\varepsilon}+B_\varepsilon(\tau,\mu)(u(\tau)-\tilde{u}(\tau))\right]d\mu\right\}d\tau+$$

$$+\sum_{i=1}^k\int_{t_0}^t\left\{\int_0^1\left[C^i_\varepsilon(\tau,\mu)\frac{x_\varepsilon(\tau)-\tilde{x}(\tau)}{\varepsilon}+D^i_\varepsilon(\tau,\mu)(u(\tau)-\tilde{u}(\tau))\right]d\mu\right\}dB_i(\tau)-\bar{x}(t)$$

Adding and substracting $\int_{t_0}^t\left[\int_0^1 A_\varepsilon(\tau,\mu)\bar{x}(\tau)d\mu\right]d\tau+\sum_{i=1}^k\int_{t_0}^t\left[\int_0^1 C^i_\varepsilon(\tau,\mu)\bar{x}(\tau)d\mu\right]dB_i(\tau)$

in (12) we get

13) $$h_\varepsilon(t)=\int_{t_0}^t\left[\int_0^1 A_\varepsilon(\tau,\mu)h_\varepsilon(\tau)d\mu\right]d\tau+\sum_{i=1}^k\int_{t_0}^t\left[\int_0^1 C^i_\varepsilon(\tau,\mu)h_\varepsilon(\tau)d\mu\right]dB_i(\tau)+$$

$$+\int_{t_0}^t\left\{\int_0^1\left[A_\varepsilon(\tau,\mu)-A(\tau)\right]\bar{x}(\tau)d\mu\right\}d\tau+\sum_{i=1}^k\int_{t_0}^t\left\{\int_0^1\left[C^i_\varepsilon(\tau,\mu)-C^i(\tau)\right]\bar{x}(\tau)d\mu\right\}dB_i(\tau)$$

$$+\int_{t_0}^t\left\{\int_0^1\left[B_\varepsilon(\tau,\mu)-B(\tau)\right](u(\tau)-\tilde{u}(\tau))d\mu\right\}d\tau+\sum_{i=1}^k\int_{t_0}^t\left\{\int_0^1\left[D^i_\varepsilon(\tau,\mu)-D^i(\tau)\right](u(\tau)-\tilde{u}(\tau))d\mu\right\}dB_i(\tau)\,.$$

All integrals with respect to $\mu\in[0,1]$ in (13) are Riemann integrals and they are defined a.e. in $(\omega,\tau)\in\Omega\times[t_0,t_1]$ with respect to the product measure $dP\otimes dt$.

Since the integrals in (13) are $\mathcal{S}$-measurable for each $\mu\in[0,1]$ (see Lemma 2) it follows that Riemann integrals define $\mathcal{S}$-mesurable functions. On the other hand all matrices in (13) are defined by partial derivatives with respect to x_i or u_j and they are bounded by the constant 2K (see (H_1)) and go to zero when $\varepsilon \to 0$.

Therefore all Riemann integrals in (13) define $\mathcal{S}$-measurable functions bounded by $2K \|h_\varepsilon(\tau)\|$ for the first (k+1) terms, by $2K \|\bar{x}(\tau)\|$ for the next (k+1) terms and by $2K \|u(\tau)-\tilde{u}(\tau)\|$ for the last (k+1) terms. It follows that all Lebesgue or Wiener integrals in (13) exist. Denote by I the first (k+1) terms and by II the last 2(k+1) terms in (13).

We have

14) $$E \|I\|^2 \leq N \int_{t_o}^{t} E \|h_\varepsilon(\tau)\|^2 d\tau ,$$

where $N=(k+1)K^2(t_1-t_o+1)$

Let $R_\varepsilon^i(\tau)$, i=1,..., 2(k+1) be the Riemann integrals in II. We obtain

15) $$E \|II\|^2 \leq N_1 \sum_{i=1}^{2(k+1)} E \int_{t_o}^{t_1} \|R_\varepsilon^i(\ )\|^2 d\tau ,$$

where $N_1=2(k+1)(t_1-t_o+1)$.

Any $\|R_\varepsilon^i(\tau)\|^2$ fulfils the conditions in Lemma 2 with $\lim_{\varepsilon\to 0} \|R_\varepsilon^i(\tau)\|^2=0$ a.e. with respect to the measure $\mu=dP \otimes dt$ and we get

16) $$\lim_{\varepsilon\to 0} E \|II\|^2=0$$

Finally, using (14) and (15), from (13) we get

17) $$E \|h_\varepsilon(t)\|^2 \leq 2N \int_{t_o}^{t} E \|h_\varepsilon(\tau)\|^2 d\tau+2E \|II\|^2$$

and using Gronwall's Lemma it follows

18) $\sup_{t \leq t_1} E \|h_\varepsilon(t)\|^2 \leq 2E \|II\|^2 \exp 2N(t_1 - t_o)$

The conclusions (16) and (18) complete the proof.

Lemma 4

Let $(\tilde{x}, \tilde{u})$ be optimal and assume that (H_1), (H_2) and (H_3) hold. Then

$$(\#) \quad dJ(\bar{x}^u, u-\tilde{u}) \hat{=} E\left\{\left\langle \frac{\partial G}{\partial x}(x(t_1)), \bar{x}^u(t_1)\right\rangle + \int_{t_o}^{t_1} \left[\left\langle \frac{\partial L}{\partial x}(t, \tilde{x}(t), \tilde{u}(t)), \bar{x}^u(t)\right\rangle + \right.\right.$$

$$\left.\left.\left\langle \frac{\partial L}{\partial u}(t, \tilde{x}(t), \tilde{u}(t)), u(t) - \tilde{u}(t)\right\rangle\right] dt\right\} \geq 0$$

for any $u \in \mathcal{U}$, where $\bar{x}^u$ is the solution in (9) corresponding to u.

Proof

By hypothesis the conditions in Lemma 3 are satisfied and hence the x_ε in (1) corresponding to $u_\varepsilon(t) = \tilde{u}(t) + \varepsilon(u(t) - \tilde{u}(t))$ fulfils $x_\varepsilon(t) = \tilde{x}(t) + \varepsilon\bar{x}(t) + \theta(\varepsilon, t)$, where $\lim_0 E \left\|\frac{\theta(\varepsilon,t)}{\varepsilon}\right\|^2 = 0$ uniformly with respect to $t \in [t_o, t_1]$.

Since $u_\varepsilon \in \mathcal{U}$, we have

9) $$\lim_{\varepsilon \to 0} \frac{J(x_\varepsilon, u_\varepsilon) - J(\tilde{x}, \tilde{u})}{\varepsilon} \geq 0$$

if this limit exist (actually is enough to exist a sequence $\varepsilon_n \to 0$ such that the $\lim_{\varepsilon_n \to 0} \frac{J(x_{\varepsilon_n}, u_{\varepsilon_n}) - J(\tilde{x}, \tilde{u})}{\varepsilon_n}$ exists.

We shall show that the limit in (9) is equal to the expresssion in the statement. Denote $p_\varepsilon(t, \mu) = (\tilde{x}(t) + \mu(x_\varepsilon(t) - \tilde{x}(t)),\ \tilde{u}(t) + \mu\varepsilon(u(t) - \tilde{u}(t)))$

and since G and L are continuously differentiable in (x,u) we get

$$20)\quad G(x_\varepsilon(t_1))-G(\tilde{x}(t_1))=\int_0^1\left\langle\frac{\partial G}{\partial x}(\tilde{x}(t_1)+\mu(x_\varepsilon(t_1)-\tilde{x}(t_1)),\ x_\varepsilon(t_1)-\tilde{x}(t_1)\right\rangle d\mu$$

$$21)\quad L(t,x_\varepsilon(t),u_\varepsilon(t)-L(t,\tilde{x}(t),\tilde{u}(t))=\int_0^1\Big[\left\langle\frac{\partial L}{\partial x}(t,p_\varepsilon(t,\mu)),\ x_\varepsilon(t)-\tilde{x}(t)\right\rangle +$$

$$+\varepsilon\left\langle\frac{\partial L}{\partial u}(t,p_\varepsilon(t,\mu)),\ u(t)-\tilde{u}(t)\right\rangle\Big]\ d\mu$$

Using Lemma 3 we have $\lim\limits_{\varepsilon\to 0}\dfrac{x_\varepsilon(t)-\tilde{x}(t)}{\varepsilon}=\bar{x}(t)$ in $L_2(\Omega)$ uniformly with respect to $t\in[t_o,t_1]$.

By hypothesis (see (H_3)) $\frac{\partial G}{\partial x}$, $\frac{\partial L}{\partial x}$ and $\frac{\partial L}{\partial u}$ fulfil a polynomial growth condition and since $u_\varepsilon(\mu,t)$ is uniformly bounded for $s\in S$, $\mu\in[0,1]$, $\varepsilon\in[0,1]$ it follows that they are bounded by $C(1+\|x_\varepsilon(\mu,t)\|^p)$ where $C>0$, $p\geq 1$ are constants.

Using (4) we get that the partial derivatives in (20) and (21) are bounded in $L_2(\Omega)$ uniformly with respect to $t\in[t_o,t_1]$, $\varepsilon,\mu\in[0,1]$. On the other hand for any sequence $\varepsilon_n\to 0$ such that $x_{\varepsilon_n}(t)-\tilde{x}(t)\to 0$ a.e. in $\omega\in\Omega$ and uniformly with respect to $t\in[t_o,t_1]$ it follows that $\frac{\partial G}{\partial x}(\tilde{x}(t_1)+\varepsilon(x_\varepsilon(t_1)-\tilde{x}(t_1))$, $\frac{\partial L}{\partial x}(t,p_\varepsilon(t,\mu))$ and $\frac{\partial L}{\partial u}(t,p_\varepsilon(t,\mu))$ converge to $\frac{\partial G}{\partial x}(\tilde{x}(t_1))$, $\frac{\partial L}{\partial x}(t,\tilde{x}(t),\tilde{u}(t))$, $\frac{\partial L}{\partial u}(t,\tilde{x}(t),\tilde{u}(t))$ for all $t\in[t_o,t_1]$ and $\omega\notin\Omega_o$ $(P(\Omega_o)=0)$ uniformly with respect to $\mu\in[0,1]$. Using dominated convergence theorem we get

$$\lim_{\varepsilon\to 0}\int_0^1\left\langle\frac{\partial G}{\partial x}(\tilde{x}(t_1)+\mu(x_\varepsilon(t_1)-\tilde{x}(t_1)),\bar{x}(t_1)\right\rangle d\mu=\frac{\partial G}{\partial x}(\tilde{x}(t_1))\quad\text{in } L_1(\Omega),$$

$$\lim_{\varepsilon\to 0}\int_0^1\left\langle\frac{\partial L}{\partial x}(t,p_\varepsilon(t,\mu)),\ \bar{x}(t)\right\rangle d\mu=\left\langle\frac{\partial L}{\partial x}(t,\tilde{x}(t),\tilde{u}(t)),\ \bar{x}(t)\right\rangle\ \text{in } L_1(S)$$

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$$\lim_{\varepsilon\to 0}\int_0^1 \langle \frac{\partial L}{\partial u}(t, p_\varepsilon(t,\mu)), u(t)-\tilde{u}(t)\rangle d\mu = \langle \frac{\partial L}{\partial u}(t,\tilde{x}(t),\tilde{u}(t)), u(t)-\tilde{u}(t)\rangle \text{ in } L_1(S)$$

Since $\lim_{\varepsilon\to 0}\dfrac{x_\varepsilon(t)-\tilde{x}(t)-\varepsilon\bar{x}(t)}{\varepsilon} = 0$ in $L_2(\Omega)$, uniformly in $t\in[t_0,t_1]$, dividing in (20), (21) by ε and letting $\varepsilon\to 0$ we get the convergence of these expressions in $L_1(\Omega)$ to

$$\langle \frac{\partial G}{\partial x}(\tilde{x}(t_1)), \bar{x}(t_1)\rangle \quad \text{and}$$

$$\int_{t_0}^{t_1}[\langle \frac{\partial L}{\partial x}(t,\tilde{x}(t),\tilde{u}(t)), \bar{x}(t)\rangle + \langle \frac{\partial L}{\partial u}(t,\tilde{x}(t),\tilde{u}(t)), u(t)-\tilde{u}(t)\rangle] dt$$

respectively. The proof is complete.

§ 4. Necessary conditions

under the conditions in Lemma 4 we have

$$22)\quad E\langle \lambda, \bar{x}^u(t_1)\rangle + E\int_{t_0}^{t_1}[\langle L_x(t), \bar{x}^u(t)\rangle dt + \langle L_u(t), u(t)-\tilde{u}(t)\rangle] dt \geq 0$$

for all $u\in\mathcal{U}$, where $\bar{x}^u$ is the solution in (9) corresponding to the control u and $\lambda = \frac{\partial G}{\partial x}(\tilde{x}(t_1))$, $L_x(t) = \frac{\partial L}{\partial x}(t,\tilde{x}(t),\tilde{u}(t))$, $L_u(t) = \frac{\partial L}{\partial u}(t,\tilde{x}(t),\tilde{u}(t))$.

To obtain the corresponding maximum principle from (22) and (9) is not possible since we don't know yet what the adjoint system is in our stochastic problem. As we can see later even if we know the adjoint system, we cannot get directly the maximum principle from (9) and (22). First we have to replace (22) and (9) by the corresponding Euler's inequation fox all $(x(t), u(t)-\tilde{u}(t))$ verifying $E\int_{t_0}^{t_1}\|x(t)\|^2 dt < \infty$, $u\in\mathcal{U}$.

In order to get Euler's inequation (a variational inequality) it is suitable to work on the Hilbert space of the square integrable and $\bar{\mathcal{S}}$- measurable functions (classes) $x: S\to R^n$, where $\bar{\mathcal{S}}$ is the σ- algebra $\mathcal{S}$ completed with respect to the measure product $dP\otimes dt$ and endowed with the usual inner product $\langle x,y\rangle =$

$= \int_{t_0}^{t_1} E\langle x(t),y(t)\rangle dt$.

Denote the measure space $(S,\mathcal{S}\, dP\otimes dt)$ by $\bar{S}$ and $L_2(\bar{S},R^n)$ the Hilbert space.

The functional $E\langle \lambda, x(t_1)\rangle$ doesn't have any meaning for $x\in L_2(\bar{S},R^n)$ and we shall convert it into in integral form for $\bar{x}^u$ a solution in (9).

Since λ is $\mathcal{F}_{t_1}$- measurable and $E\|\lambda\|^2<\infty$ it follows that there exists $\mathcal{S}$- measurable functions $h_i\in L_2(\bar{S},R^n)$, $i=1,\ldots,k$ (see for example [5]) such that

$$23)\ \lambda=\lambda_0 + \sum_{i=1}^{k}\int_{t_0}^{t_1} h_i(t)dB_i(t),\quad \lambda_0=E\lambda,\quad E(\lambda/F_t)=\lambda(t),\quad \lambda(t_1)=\lambda .$$

Since $\bar{x}^u(t)$ verifies (9) in integral form it follows

$$24)\ E\langle\lambda,\bar{x}^u(t_1)\rangle = \int_{t_0}^{t_1} E\langle\lambda, A(t)\bar{x}^u(t)+B(t)(u(t)-\tilde{u}(t))\rangle dt + \sum_{i=1}^{k}\int_{t_0}^{t_1} E\langle h_i(t),C^i(t)\bar{x}^u(t)+D^i(t)(u(t)-\tilde{u}(t))\rangle dt$$

Taking into account that $E\langle\lambda, f(t)\rangle = E\, E(\langle\lambda, f(t)\rangle/\mathcal{F}_t) =$ $= E\langle E(\lambda/\mathcal{F}_t),\, f(t)\rangle = E\langle\lambda(t),f(t)\rangle$ if $f(t)$ is $\mathcal{F}_t$- measurable then changing accordingly the integrand in the first term in (24) we get

$$25)\ E\langle\lambda,\bar{x}^u(t_1)\rangle = \int_{t_0}^{t_1} E\langle\lambda(t), A(t)\bar{x}^u(t)+B(t)(u(t)-\tilde{u}(t))\rangle dt + \sum_{i=1}^{k}\int_{t_0}^{t_1} E\langle h_i(t),\, C^i(t)\bar{x}^u(t)+D^i(t)(u(t)-\tilde{u}(t))\rangle dt$$

Hence $E\langle\lambda,\ \bar{x}^u(t_1)\rangle$ has an integral form given in (25) for any solution $\bar{x}^u$ in (9).

In place of the functional (22) we shall take its equivalent expression

26) $$E\int_{t_0}^{t_1}\left[\langle\tilde{L}_x(t),\tilde{x}^u(t)\rangle dt+\langle\tilde{L}_u(t),u(t)-\tilde{u}(t)\rangle\right]dt \geqslant 0$$

for any $u\in\mathcal{U}_0$, where $\tilde{x}^u$ is in (9), and

27) $$\tilde{L}_x(t) = L_x(t) + \lambda^*(t)A(t) + \sum_{i=1}^{k} h_i^*(t)C^i(t)$$

$$\tilde{L}_u(t) = L_u(t)+\lambda^*(t)B(t)+\sum_{i=1}^{k} h_i^*(t)D^i(t)$$

The simbol "$*$" means transposition of a vector. Now the functional (26) has the advantage that it is defined for all $x\in L_2(\bar{S},R^n)$. Indeed, A,B,C^i and D^i are bounded by the constant K (see (H_1)), and L_x, L_u are in $L_2(\bar{S})$ (see (H_1) and (H_3)); it follows that $\tilde{L}_x$ and $\tilde{L}_u$ are in $L_2(\bar{S})$.

Define $A,C : L_2(\bar{S},R^n)\longrightarrow L_2(\bar{S},R^n)$ by

27') $$(Ax)(t)=\int_{t_0}^{t} A(\tau)x(\tau)d\tau,\quad (Cx)(t)=\sum_{i=1}^{k}\int_{t_0}^{t} C^i(\tau)x(\tau)dB_i(\tau).$$

Let $m\in L_2(\bar{S},R^n)$ be arbitrarily fixed.

We are looking for $x\in L_2(\bar{S},R^n)$ such that

28) $$x = Ax + Cx + m \quad \text{in } L_2(\bar{S},R^n).$$

It is easy to see that the solution x in (28) is unique in $L_2(\bar{S},R^n)$ if it exists.

The definition of a solution in (28) is made in a standard way.

Define a sequence $\{x_p\}$ in $L_2(\bar{S},R^n)$

29) $$x_0=m,\ x_1=Ax_0+Cx_0+m,\ldots,\ x_{p+1} = Ax_p + Cx_p + m,\ldots$$

We prove that $\{x_p\}$ has a limit in $L_2(\bar{S},R^n)$ and this limit fulfils (28). By definition

$$x_{p+1}(t)-x_p(t)=(Ax_p(t)+(Cx_p)(t)$$

and hence $x_{p+1}(t)-x_p(t)$ is continuous in $t\in[t_o,t_1]$ for almost all $\omega\in\Omega$.

By induction argument we get

30) $$E\|x_{p+1}(t)-x_p(t)\|^2\leq K_1^{p+1}\|m\|^2\frac{(t-t_o)^p}{p!}$$

where $K_1=2K^2(1+(t-t_o))$.

In addition

31) $$E\sup_{t\leq t_1}\|x_{p+1}(t)-x_p(t)\|^2\leq 2K^2(t_1-t_o)\int_{t_o}^{t_1}E\|x_p(t)-x_{p-1}(t)\|^2dt+$$
$$+2K^2\int_{t_o}^{t_1}E\|x_p(t)-x_{p-1}(t)\|^2dt\leq K_1\int_{t_o}^{t_1}E\|x_p(t)-x_{p-1}(t)\|^2dt\leq$$
$$K_1^{p+1}\|m\|^2\frac{(t_1-t_o)^p}{(p-1)!}=C\frac{M^{p-1}}{(p-1)!}$$

where $C=\frac{1}{p}K_1^2\|m\|^2(t_1-t_o)$, $M=K_1(t_1-t_o)$.

Using (31) we obtain

$$P\left\{\sup_{t\leq t_1}\|x_{p+1}(t)-x_p(t)\|>\frac{1}{2^p}\right\}\leq C_1\frac{M_1^{p-1}}{(p-1)!},\ M_1=M.2^2,\ C_1=4C$$

and from Borel-Cantelli's lemma we get that for any $\omega\in\Omega\setminus\Omega_o$ where $P\Omega_o=0$, there is $N(\omega)$ such that

$$\sup_{t\leq t_1}\|x_{p+1}(t)-x_p(t)\|\leq\frac{1}{2^p}\quad\text{for any } p\geq N(\omega)$$

We conclude that the sequence $\{x_p(t)\}_{p\geq 1}$,

$$x_{p+1}(t) = x_o(t)+(x_1(t)-x_o(t))+\ldots+x_{p+1}(t)-x_p(t)/$$

converges uniformly in $t\in[t_o,t_1]$, for all $\omega\in\Omega\setminus\Omega_o$.

Let $x(t)=\lim\limits_{p\to\infty} x_p(t)$ for each $(\omega,t)\in(\Omega\setminus\Omega_o)\times[t_o,t_1]$.

By definition, x is $\mathcal{S}$ - measurable, and the integrals $\int_{t_o}^{t} A(\tau)x(\tau)d\tau$, $\int_{t_o}^{t} C(\tau)x(\tau)d\tau$ exist for all $t\in[t_o,t_1]$ and $\omega\in\Omega\setminus\Omega_o$.

Since $\{x_p(t)\}$ converges uniformly in $t\in[t_o,t_1]$ we get

$$\lim_{p\to\infty}\int_{t_o}^{t}A(\tau)x_p(\tau)d\tau=\int_{t_o}^{t}A(\tau)x(\tau)d\tau,\ \lim_{p\to\infty}\int_{t_o}^{t}C^i(\tau)x_p(\tau)d\tau=\int_{t_o}^{t}C^i(\tau)x(\tau)d\tau$$

uniformly in $t\in[t_o,t_1]$ for all $\omega\in\Omega\setminus\Omega_o$.

Therefore $\int_{t_o}^{t}C^i(\tau)x_p(\tau)dB_i(\tau)$ converges in probability to $\int_{t_o}^{t}C^i(\tau)x(\tau)dB_i(\tau)$ uniformly in $t\in[t_o,t_1]$.

Letting $p\to\infty$ in (29) we obtain

$$x(t) = (Ax)(t)+(Cx)(t)+m(t) \quad \text{for all } (\omega,t)\in(\Omega\setminus\Omega_o)\times[t_o,t_1].$$

By construction

$$E\int_{t_o}^{t}\|x_{p+1}(\tau)\|^2d\tau\le 3\left[\int_{t_o}^{t}E\|m(\tau)\|^2d\tau+\int_{t_o}^{t}E\|Ax_p(\tau)\|^2d\tau+\int_{t_o}^{t}E\|Cx_p(\tau)\|^2d\tau\right]$$

and

$$E\|Ax_p(t)\|^2\le K^2(t_1-t_o)\int_{t_o}^{t}E\|x_p(\tau)\|^2d\tau,$$

$$E\|Cx_p(t)\|^2\le K^2\int_{t_o}^{t}E\|x_p(\tau)\|^2d\tau.$$

Finally we obtain

$$\varphi_{p+1}(t) \triangleq \quad E\|x_{p+1}(\tau)\|^2 d\tau \leq C\int_{t_o}^{t} E\|m(\tau)\|^2 d\tau + C\int_{t_o}^{t}\varphi_p(\tau)\,d\tau ,$$

where $C \geqslant 3(K^2+1)(1+t_1-t_o)$, and by induction argument if follows

$$\int_{t_o}^{t} E\|x_{p+1}(\tau)\|^2 d\tau \leq (1+C(t-t_o)+\ldots+\frac{C^{p+1}(t-to)^{p+1}}{(p+1)!})C\int_{t_o}^{t} E\|m(\tau)\|^2 d\tau$$

Using Fatou's lemma we conclude

$$\int_{t_o}^{t} E\|x(\tau)\|^2 d\tau \leq \underline{\lim}_{p\to\infty} \int_{t_o}^{t} E\|x_{p+1}(\tau)\|^2 d\tau \leq C\int_{t_o}^{t} E\|m(\tau)\|^2 d\tau \;\exp C(t_1-t_o)$$

and hence $x \in L_2(\bar{S},R^n)$. The proof is complete.

Define $(Bv)(t) = \int_{t_o}^{t} B(\tau)v(\tau)\,d\tau$, $(Dv)(t) = \sum_{i=1}^{k}\int_{t_o}^{t} D^i(\tau)v(\tau)\,dB_i(\tau)$.

Lemma 5

Under the same conditions as in lemma 4, the conclusion (#) holds if and only if there exists $n \in L_2(\bar{S},R^n)$ and nonanticipative such that

a) $E\int_{t_o}^{t_1}\langle \tilde{L}_x(t),x(t)\rangle dt + E\int_{t_o}^{t_1}\langle n(t),x(t)-(Ax)(t)-(Cx)(t)\rangle dt = 0$

for any $x \in L_2(\bar{S},R^n)$ (A and C are defined in (27')) ;

b) $E\int_{t_o}^{t_1}\langle \tilde{L}_u(t),u(t)-\tilde{u}(t)\rangle dt - E\int_{t_o}^{t_1}\langle n(t),(B(u-\tilde{u}))(t)+(D(u-\tilde{u}))(t)\rangle dt \geqslant 0$

for any $u \in \mathcal{U}$,

where $\tilde{L}_x$ and the operators A, C are defined in (27)', (27').

In addition, there is an Ito process $\psi(t) = \psi_o + \int_{t_o}^{t} n(\tau)\,d\tau + \sum_{i=1}^{k}\int_{t_o}^{t} M_i(\tau)\,dB_i(\tau)$, such that $\psi_o \in R^n$, $\psi(t_1) = (\frac{\partial G}{\partial x}(\tilde{x}(t_1)))^*$, $M_i(t)$ is nonanticipative $M_i \in L_2(\bar{S},R^n)$, and

b') $\langle \frac{\partial H}{\partial u}(t,\psi(t),M(t),\tilde{x}(t),\tilde{u}(t)),u-\tilde{u}(t)\rangle \geqslant 0$ for all $u \in U_1$ a.e. $(dP \otimes dt)$

where $H(t,\psi, M,x,u)=\psi f(t,x,u)+\sum_{i=1}^{k} M_i g_i(t,x,u)+L(t,x,u)$ (ψ, M_i are line vectors).

Proof

We proved that the conclusions in lemma 4 are equivalent with (26) under the conditions $\bar{x}^u$ is a solution in (9).

The equations in (9) and (26) can be represented by

32) $T(x^u, u-\tilde{u})=0,\ l(x^u,u-\tilde{u})\geqslant 0$ for any $u\in\mathcal{U}$,

where l is a linear continous functional and T is a linear continuous operator from $L_2(\bar{S},R^n)\times L_\infty(\bar{S},R^m)$ to $L_2(\bar{S},R^n)$. With the above notations (see lemma 4) $T(x,v)=(I-A-C)x-(B+D)v$ and

$$l(x,v) = E\left[\int_{t_0}^{t_1}\langle\tilde{L}_x(t),x(t)\rangle dt+\int_{t_0}^{t_1}\langle\tilde{L}_u(t),v(t)\rangle dt\right]$$

Since $T(x,0): L_2(\bar{S},R^n)\to L_2(\bar{S},R^n)$ is a surjective one (see (28)) we claim that applying a separation theorem for convex sets from (32) we get that there exists $n\in L_2(\bar{S},R^n)$ such that

33) $l(x,u-\tilde{u})+n(T(x,u-\tilde{u}))\geqslant 0$ for all $x\in L_2(\bar{S},R^n)$, $u\in\mathcal{U}$

Indeed, define the convex sets C_1,C_2 in $R\times L_2(\bar{S},R^n)$

$$C_1=\{(r,0),\ r<0\},\quad C_2=\{(l(x,u-\tilde{u})+\varepsilon,\ T(x,u-\tilde{u})):x\in L_2(\bar{S},R^n),u\in\mathcal{U},\ \varepsilon>0\}.$$

We have $C_1\cap C_2=\emptyset$ otherwise one contradicts (32).

Moreover int $C_2\neq\emptyset$ in $R\times L_2(\bar{S},R^n)$. Since $T(\cdot,0)$ is a surjective application we get that

$\{T(x,0):\|x\|\le 1\}\supseteq\{y:\|y\|<\delta\}$, for $\delta>0$ sufficiently small and in additio $|l(x,0)|<r$ for all $\|x\|<1$, if $r>0$ is sufficiently large. Therefore $(x,\infty)\ x\{y:\|y\|<\delta\}\subseteq C_2$ and we can apply a separation theorem for C_1 and C_2 in $R\times L_2(\bar{S},R^n)$. We get that there exist $\alpha\geq 0$ and $\mu\in L_2(\bar{S},R^n)$ such that

34) $\alpha+\|\mu\|>0,\ \alpha(l(x,u-\tilde{u})+\varepsilon)+\mu(T(x,u-\tilde{u}))\geq 0$ for all $u\in\mathcal{U}$, $x\in L_2(\bar{S},R^n)$

and $\varepsilon>0$.

We have $\alpha>0$ otherwise $\mu\equiv 0$ contradicting (34).

Hence we can divide by α in (34) and letting $\varepsilon\to 0$ we get (33), where $\tilde{n}=\frac{\mu}{\alpha}$. Since $\tilde{n}\in L_2(\bar{S},R^n)$, there is a $\mathcal{S}$- measurable function n such that $n(s)=\tilde{n}(s)$ a.e. $(dP\otimes dt)$.

Taking $u=\tilde{u}$ in (33) we obtain (a) and for $x=0$ we get (b). The sufficieny follows by adding (a) and (b).

The last part in statement we obtain in the followinig way . Define $k(t)=k_o+\int_{t_o}^{t} n(\tau)d\tau$ and we are looking for $k_o\in R^n$ and $\mathcal{F}_t$-nonanticipative n-dimensional processes $H_i(t)$, $i=1,\dots,k$ such that the Ito process

35) $p(t)=k(t)+\sum_{i=1}^{k}\int_{t_o}^{t}H_i(\tau)dB_i(\tau)$

fulfis $p\in L_2(\bar{S},R^n)$, $p(t_1)=0$.

Since $Ek(t_1)$ must be zero we choose $k_o=-E\int_{t_o}^{t_1}n(t)dt$

On the other hand $k(t_1)$ is $\mathcal{F}_{t_1}$-measurable and we get the Wiener integral reprezentation (see for example [5])

36) $-k(t_1)=\sum_{i=1}^{k}\int_{t_o}^{t_1}H_i(t)dB_i(t)$

where H_i are n-dimensional non-anticipative processes.

Define $p(t)=k(t)-E(k(t_1)\mathcal{F}_t)$ and we get (35)

Denote $y^{u}(t) = \int_{t_0}^{t} B(\tau)(u(\tau)-\tilde{u}(\tau))d\tau+\sum_{i=1}^{k}\int_{t_0}^{t} D^i(\tau)(u(\tau)-\tilde{u}(\tau)dB_i(\tau))$

Since $p(t_1)=0$, applying Ito's stochastic rule for computation of $\langle p(t_1),y^{u}(t_1)\rangle$ we get

$$- E\int_{t_0}^{t_1}\langle n(t),y^{u}(t)\rangle dt = E\int_{t_0}^{t_1}\Big[\langle p(t),B(t)(u(t)-\tilde{u}(t)\rangle+\sum_{i=1}^{k}\langle H_i(t), D^i(t)(u(t)-\tilde{u}(t))\rangle\Big]dt.$$

The conclusion (b) is equivalent with

$$E\int_{t_0}^{t_1}\langle\tilde{L}_u(t)+p^*(t)B(t)+\sum_{i=1}^{k}H_i^*(t)D^i(t),u(t)-\tilde{u}(t)\rangle dt\geq 0 \text{ for any } u\in\mathcal{U}$$

and recalling $\tilde{L}_u$ in (27) we obtain (b') where

$\psi(t) = p^*(t)+\lambda^*(t)$ and $M_i(t) = H_i^*(t) + h_i^*(t)$

The proof is complete.

The main result is contain in the following

<u>Theorem</u>

Let $(\tilde{x},\tilde{u})$ be optimal. Assume that (H_1), (H_2) and (H_3) hold. Then there exist n-dimensional nonanticipative processes

$\psi(t)$, $M_i(t)$, $i=1,\dots,k$, such that

a) $d\psi = -\frac{\partial H}{\partial x}(t,\psi,M(t),\tilde{x}(t),\tilde{u}(t))+\sum_{i=1}^{k}M_i(t)dB_i(t)$

$\psi(t_1) = \left(\frac{\partial G}{\partial x}(\tilde{x}(t_1))\right)^*$

and

b) $\left\langle \frac{\partial H}{\partial u}(t,\psi(t),M(t),\tilde{x}(t),\tilde{u}(t)),\ u-\tilde{u}(t)\right\rangle \geq 0$ for any $u \in U$,

a.e. in $(\omega,t)\in \Omega\times[t_o,t_1]$ with respect to the measure $dP(x)\,dt$.

Proof

By hypothesis the conditions in lemma 5 are satisfied. Since (b′) in lemma 5 is (b) in theorem we have to prove that (a) in Lemma 5 is equivalent with adjoint system.

We shall transform the terms in (a) using the reprezentation of p(t) as an Ito's process (see (35)).

Integrating by parts we have

$$37)\quad E\int_{t_o}^{t_1}\left\langle n(t),\ \int_{t_o}^{t} A(\tau)x(\tau)d\tau\right\rangle dt = E\left\langle k(t_1),\ \int_{t_o}^{t_1} A(t)x(t)dt\right\rangle - E\int_{t_o}^{t_1}\left\langle k(t),A(t)x(t)\right\rangle dt$$

for any $x\in L_2(\bar{S},R^n)$.

Since $\int_{t_o}^{t} f(\tau)dB_i(\tau) = \int_{t_o}^{t_1} f(\tau)dB_i(\tau) - \int_{t_o}^{t_1} f(\tau)dB_i(\tau)$

using conditioned expectation with respect to F_t we get

$$\int_{t_o}^{t_1} E\left\langle n(t),\ \sum_{i=1}^{k}\int_{t}^{t_1} C^i(\tau)x(\tau)dB_i(\tau)\right\rangle dt = \int_{t_o}^{t_1} E\left\langle n(t),\ \sum_{i=1}^{k} E\int_{t}^{t_1} C^i(\tau)x(\tau)dB_i(\tau)\right\rangle dt = 0$$

and

$$38)\quad E\int_{t_o}^{t_1}\left\langle n(t),\ \sum_{i=1}^{k}\int_{t_o}^{t} C^i(\tau)x(\tau)dB_i(\tau)\right\rangle dt = E\left\langle k(t_1),\ \sum_{i=1}^{k}\int_{t_o}^{t_1} C^i(t)x(t)dB_i(t)\right\rangle$$

for any $x\in L_2(\bar{S},R^n)$.

Using (37) and (38), the conclusion (a) in lemma 5 becomes

$$39)\quad E\int_{t_0}^{t_1}\langle \tilde{L}_x(t)+n(t),x(t)\rangle dt+E\int_{t_0}^{t_1}\langle k(t),A(t)x(t)\rangle dt -$$
$$- E\langle k(t_1),\int_{t_0}^{t_1}A(t)x(t)dt\rangle - E\langle k(t_1),\sum_{i=1}^{k}\int_{t_0}^{T_1}C^i(t)x(t)dB_i(t)\rangle = 0$$

for any $x\in L_2(\bar{S},R^n)$.

Replacing $k(t)$ in (39) by $p(t)-\sum_{i=1}^{k}\int_{t_0}^{t}H_i(\tau)dB_i(\tau)$ we get

$$40)\quad E\int_{t_0}^{t_1}\langle k(t),A(t)x(t)\rangle dt=E\int_{t_0}^{t_1}\langle p(t),A(t)x(t)\rangle dt-E\langle\sum_{i=1}^{k}\int_{t_0}^{t_1}H_i(t)dB_i(t),\int_{t_0}^{t_1}A(t)x(t)dt\rangle$$

and using $p(t_1)=0$ we obtain

$$41)\quad E\langle k(t_1),\sum_{i=1}^{k}\int_{t_0}^{t_1}C^i(t)x(t)dB_i(t)\rangle = -E\sum_{i=1}^{k}\int_{t_0}^{t_1}\langle H_i(t),C^i(t)x(t)\rangle dt$$

The equation (39) has a simpler form using (40) and (41),

$$42)\quad E\int_{t_0}^{t_1}\langle \tilde{L}_x(t)+n(t)+p^*(t)A(t)+\sum_{i=1}^{k}H_i^*(t)C^i(t),x(t)\rangle dt = 0$$

for all $x\in L_2(\bar{S},R^n)$.

Therefore we have

$$43)\quad \tilde{L}_x(t)+n(t)+p^*(t)A(t)+\sum_{i=1}^{k}H_i^*(t)C^i(t)=0$$

a.e. with respect to $dP\otimes dt$, and

$$44)\quad p_o+\int_{t_0}^{t}n(\tau)d\tau+\int_{t_0}^{t}p^*(\tau)A(\tau)d\tau+\int_{t_0}^{t}\tilde{L}_x(\tau)d\tau+\sum_{i=1}^{k}\int_{t_0}^{t}H_i^*(\tau)C^i(\tau)d\tau=\text{const.}=p_o$$

where $p_o=k_o\in R^n$.

Recall that $p(t)=p_o+\int_{t_0}^{t}n(\tau)d\tau+\sum_{i=1}^{k}\int_{t_0}^{t}H_i(\tau)dB_i(\tau)$ and $\tilde{L}_x$ is defined in (27).

Let $\lambda(t)$ and $h_i(t)$ be those that define $\tilde{L}_x$. Denote

$$45)\ \psi(t)=p^*(t)+\lambda^*(t),\quad M_i(t)=H_i^*(t)+h_i^*(t),$$

and from (44) by computation we obtain

$$46)\ \psi(t)+\int_{t_0}^{t}\psi(\tau)A(\tau)d\tau+\sum_{i=1}^{k}\int_{t_0}^{t}M_i(\tau)C^i(\tau)d\tau+\int_{t_0}^{t}L_x(\tau)d\tau - \\ -\sum_{i=1}^{k}\int_{t_0}^{t}M_i(\tau)dB_i(\tau)=\psi_0=\text{const.}$$

By definition $\psi(t_1)=\lambda^*(t_1)=(\frac{\partial G}{\partial x}(\tilde{x}(t_1)))^*$ (see (23)) and (46) stands for conclusion (a) in the statement.

The proof is complete.

Remark

The conclusions (a) and (b) in theorem are equivalent with (a) and (b) in lemma 5 and with (#) in lemma 4.

R E F E R E N C E S

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