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ALGEBRAIC ANALYSIS OF THE TOPOLOGICAL LOGIC L(I)

by

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The topological logic L(■) was introduced by J.A.Makowsky and M.Ziegler in [8] and by J.Sgro in [15] . In this paper we shall define the polyadic L(■) algebras as adequate algebraic structures for the predicate logic L(■).

The main result of this paper is a representation theorem for the polyadic L(■)-algebras (see[8] and[15]). Another result is an omitting types theorem formulated in the context of the polyadic L(■)-algebras.

§ 1. Polyadic L(■)-algebras

In this paper we shall consider only locally finite polyadic algebras of infinite degree(see[3],[4]and[6]). We shall suppose known the concepts, the results and the notations of[3] and [6].

Let (A,I,S,∃,E) be a locally finite polyadic algebra of infinite degree having the equality E. For any $p \in A$ we shall denote by J_p the minimal support of p, i.e. the intersection of the supports of p.

Let us consider a familiy of unary operations of A:

$$\{ \mathbb{I}(i): A \to A : i \in I \}$$

such that for any $i \in I$ and $p,q \in A$ we have the following properties:

(A1) $\forall(i)(p \leftrightarrow q) \leq \forall(i)(\mathbb{I}(i)p \leftrightarrow \mathbb{I}(i)q)$,

(A2) $\mathbb{I}(i)p \leq p$,

(A3) $\mathbb{I}(i)p \wedge \mathbb{I}(i)q = \mathbb{I}(i)(p \wedge q)$,

(A4) $\mathbb{I}(i)p \leq \mathbb{I}(i)\,\mathbb{I}(i)p$,

(A5) $\mathbb{I}(i)1 = 1$

(A6) $S(j/i)\,\mathbb{I}(i)p = \mathbb{I}(j)S(j/i)p$ for any $j \notin J_p$,

(A7) If J is a support of p, then $J \cup \{i\}$ is a support of $\mathbb{I}(i)p$,

(A8) For any $\sigma \in I^I$ such that $\sigma|_{\sigma^{-1}(\{i\})}$ is injective, we have

$$\mathbb{I}(i)S(\sigma) = S(\sigma)\,\mathbb{I}(j) \text{ where } \sigma(j)=i.$$

<u>Definition</u>. A polyadic algebra $(A,I,S,\exists,E)$ with a family $\{\mathbb{I}(i): i \in I\}$ of unary operations will be called a (polyadic) $L(\mathbb{I})$-algebra if the axioms (A1)-(A8) are verified.

We shall use the notation $\langle A, \mathbb{I}(i);\ i \in I \rangle$.

If $\langle A, \mathbb{I}(i): i \in I \rangle$, $\langle A', \mathbb{I}(i): i \in I \rangle$ are two $L(\mathbb{I})$-algebras then a morphism of $L(\mathbb{I})$-algebras

$$f: \langle A, \mathbb{I}(i): i \in I \rangle \to \langle A', \mathbb{I}(i): i \in I \rangle$$

is a morphism of polyadic algebras with equality $f: A \to A'$ such that $\mathbb{I}(i)f(p) = f(\mathbb{I}(i)p)$.

The Lindenbaum-Tarsky algebra of the topological logic $L(\mathbb{I})$ (see[8],[16]) has a canonical structure of $L(\mathbb{I})$-algebra . The following example of $L(\mathbb{I})$-algebra corresponds to the concept of model for the logic $L(\mathbb{I})$.

Let X be a non-empty set and $\mathbf{O}$ the Boolean algebra $\{0,1\}$. The set $\mathrm{Hom}_{Ens}(X^I,\mathbf{O})$ of the functions $X^I \to \mathbf{O}$ is a polyadic algebra in the following way (see [6]):

$$S(\tau)p(x) = p(x\tau)$$

$$\exists(J)p(x) = \bigvee\{p(y) : y \in X^I,\ y|_{I-J} = x|_{I-J}\}$$

for any $p : X^I \to \mathbf{O}$, $\tau \in I^I$, $J \subseteq I$ and $x \in X^I$. The canonical equality E_o is defined by

$$E_o(i,j)(x) = \begin{cases} 0, & \text{if } x_i \neq x_j \\ 1, & \text{if } x_i = x_j \end{cases}$$

for any $i,j \in I$ and $x \in X^I$. It is known that J is a support of an element p of this polyadic algebra iff for any $x,y \in X^I$ such that $x|_J = y|_J$ we have $p(x) = p(y)$.

We shall denote by $F(X^I,\mathbf{O})$ the polyadic algebra of the elements $p : X^I \to \mathbf{O}$ of finite support.

For any $u \in X, i \in I$ and $x \in X^I$ let $(u/i)_* x$ be the element of $F(X^I,\mathbf{O})$ defined by

$$((u/i)_* x)(j) = \begin{cases} x(j), & \text{if } j \neq i \\ u, & \text{if } j = i. \end{cases}$$

If $(X,\mathcal{O})$ is a topological space, then for any $i \in I$, we shall denote by $\mathbf{I}_{\mathcal{O}} : F(X^I,\mathbf{O}) \to F(X^I,\mathbf{O})$ the function defined by

$$\mathbf{I}_{\mathcal{O}}(i)p(x) = 1 \iff x_i \in \mathrm{Int}\{u \in X : p((u/i)_* x) = 1\}$$

for any $p \in F(X^I,\mathbf{O})$ and $x \in X$.

<u>Lemma 1.</u> $\langle F(X^I,\mathcal{O}),\ \mathbb{I}_{\mathcal{O}}(i) : i\in I\rangle$ is a $L(\mathbf{I})$-algebra.

<u>Proof.</u> We shall prove only (A6), (A7) and (A8).

(A6) We have the equivalences

$$S(j/i)\ \mathbb{I}_{\mathcal{O}}(i)p(x)=1 \Longleftrightarrow \mathbb{I}_{\mathcal{O}}(i)p(x\circ(j/i))=1$$
$$\Longleftrightarrow x_j\in \mathrm{Int}\{u : p((u/i)_{*}(x\circ(j/i))=1\}$$

$$\mathbb{I}_{\mathcal{O}}(j)S(j/i)p(x)=1 \Longleftrightarrow x_j\in \mathrm{Int}\{u:\ S(j/i)p((u/j)_{*}x)=1\}$$
$$\Longleftrightarrow x_j\in \mathrm{Int}\{u : p((u/j)_{*}x)\circ(j/i))\}.$$

But $j\notin J_p$ and

$$(u/i)_{*}(x\circ(j/i))|_{I-\{j\}} = ((u/j)_{*}x)\circ(j/i)|_{I-\{j\}}$$

for any $u\in X$, therefore we have

$$p((u/i)_{*}(x\circ(j/i))=p(((u/j)_{*}x)\circ(j/i)),\ \text{for any } u\in X.$$

It results that $S(j/i)\ \mathbb{I}_{\mathcal{O}}(i)p(x)=\mathbb{I}_{\mathcal{O}}(j)S(j/i)p(x)$.

(A7) If J is a support of $p\in F(X^I,\mathcal{O})$ then for $x,y\in X^I$ we have

$$x|_J = y|_J \Rightarrow p(x)=p(y).$$

From the implications:

$$x|_{J\cup\{i\}} = y|_{J\cup\{i\}} \Rightarrow ((u/i)_{*}x)|_J=((u/i)_{*}y)|_J,\ \text{for any } u\in X$$
$$\Rightarrow p((u/i)_{*}x)=p((u/i)_{*}y),\ \text{for any } u\in X$$

it results that

$$\underset{\mathcal{O}}{\mathbf{I}}(i)p(x)=\mathrm{Int}\{u : p((u/i)_* x)=1\}$$
$$=\mathrm{Int}\{u : p((u/i)_* y) = 1\}= \underset{\mathcal{O}}{\mathbf{I}}(i)p(y).$$

Then $J\cup\{i\}$ is a support of $\underset{\mathcal{O}}{\mathbf{I}}(i)p$.

(A8) Suppose that $\sigma|_{\sigma^{-1}(\{i\})}$ is injective, then there exists an unique $j\in I$ such that $\sigma(j)=i$. For any $p\in F(X^I,\mathcal{O})$ and $x\in X^I$ we have the equivalences:

$$\underset{\mathcal{O}}{\mathbf{I}}(i)S(\sigma)p(x)=1 \Longleftrightarrow x_i\in \mathrm{Int}\{u: S(\sigma)p((u/i)_* x)=1\}$$
$$\Longleftrightarrow x_i\in \mathrm{Int}\{u : p((u/i)_* x)\circ\sigma)=1$$

$$S(\sigma)\ \underset{\mathcal{O}}{\mathbf{I}}(j)p(x)=1 \Longleftrightarrow \underset{\mathcal{O}}{\mathbf{I}}(j)p(x\sigma)=1$$
$$\Longleftrightarrow (x\sigma)_j\in \mathrm{Int}\{u: p((u/j)_* (x\sigma))=1\}.$$

But $((u/i)_* x)\circ\sigma = (u/j)_* (x\sigma)$ then it results that

$$\underset{\mathcal{O}}{\mathbf{I}}(i)S(\sigma)p(x)=S(\sigma)\ \underset{\mathcal{O}}{\mathbf{I}}(j)p(x).$$

<u>Lemma 2.</u> Let <u>a</u> be a polyadic ideal of the $L(\mathbf{I})$-algebra $\langle A, \mathbf{I}(i): i\in I\rangle$. Then the quotient polyadic algebra $A/\underline{a}$ is an $L(\mathbf{I})$-algebra.

<u>Proof</u>. By the axiom (A1) we have:

$$\mathbf{I}(i)p+\mathbf{I}(i)q=\neg(\mathbf{I}(i)p\leftrightarrow \mathbf{I}(i)q)\leqslant \neg\forall(i)(p\leftrightarrow q) =$$
$$=\exists(i)\neg(p\leftrightarrow q) =\exists(i)(p+q)$$

Then we have

$$p\equiv q(\mathrm{mod}\ \underline{a}) \Rightarrow p+q\in\underline{a} \Rightarrow \exists(i)(p+q)\in\underline{a} \Rightarrow$$
$$\Rightarrow \mathbf{I}(i)p+\mathbf{I}(i)q\in\underline{a} \Rightarrow \mathbf{I}(i)p\equiv \mathbf{I}(i)q\ (\mathrm{mod}\ \underline{a}).$$

Let $(A^+, I^+, S\ , \exists^+, E^+)$ be a polyadic algebra with equality and $I \subset I^+$. We shall consider the I-compression of A^+

$$A = \{ p \in A^+ : I \text{ is a support of } p \}.$$

It is known [6] that A has a structure of polyadic algebra $(A, I, S, \exists, E)$. Suppose that $\langle A^+, \mathbf{I}^+(i) : i \in I \rangle$ is an L(▮)-algebra. For any $p \in A$ and $i \in I$ we denote $\mathbf{I}(i)p = \mathbf{I}^+(i)p$. Using the axiom (A+) we can see that this definition is correct.

<u>Lemma 3</u>. $\langle A,\ \mathbf{I}(i) : i \in I \rangle$ is a L(▮) - algebra.

We shall say that $\langle A, \mathbf{I}(i) : i \in I \rangle$ is the I-<u>compression</u> of $\langle A^+,\ \mathbf{I}^+(i) : i \in I^+ \rangle$

Let $\langle A, \mathbf{I}(i) ; i \in I \rangle$ be a L(▮)-algebra, $I \subset I^+$ and $\langle A^+,\ \mathbf{I}^+(i) : i \in I^+ \rangle$ a L(▮)-algebra such that A is a polyadic I - dilation of A. If $\mathbf{I}(i)p = \mathbf{I}^+(i)p$ for any $p \in A$ and $i \in I$ then $\langle A^+, \mathbf{I}^+(i) : i \in I^+ \rangle$ will be called a $\mathbf{I}^+$ - <u>dilation</u> of $\langle A, \mathbf{I}(i) : i \in I \rangle$.

<u>Lemma 4.</u> Let $\langle A, \mathbf{I}(i) : i \in I \rangle$ be an L(▮)-algebra and $I \subset I^+$. Then there exists an I^+-dilation $\langle A^+,\ \mathbf{I}^+(i) : i \in I^+ \rangle$ of $\langle A,\ \mathbf{I}(i) : i \in I \rangle$.

<u>Proof</u>. Exactly as in the proof of the theorem 10.2 of [3] we shall consider two steps.

a) <u>Card (I^+)=Card (I).</u> There exists a bijection $\gamma : I^+ \to I$ and $A^+ = A$ has a structure of polyadic algebra $(A^+, I^+, S^+, \exists^+, E^+)$ (see [3]). For any $p \in A^+$ and $i \in I^+$ we put $\mathbf{I}^+(i)p = \mathbf{I}(\gamma(i))p$.

We can prove that $\langle A^+,\ \mathbf{I}^+(i) ; i \in I^+ \rangle$ is a L(▮)-algebra.

b) Card $(I^+) >$ Card (I). This case follows exactly as in [3].

Let $\langle A,\ \mathbf{I}(i) : i \in I \rangle$ be a L(▮)-algebra and

$\langle A^+, \blacksquare^+(i):i\in I\rangle$ an I^+- dilation of $\langle A, \blacksquare(i):i\in I\rangle$. If $K = I^+ - I$ and $(A^+, I, S, \exists, E)$ is the polyadic algebra obtained by <u>fixing</u> the variables of K, then A^+ is a $L(\blacksquare)$-algebra by puting $\blacksquare'(i)p = \blacksquare^+(i)p$ for any $p\in A^+$ and $i\in I$.

The $L(\blacksquare)$-algebra $\langle A^+, \blacksquare'(i):i\in I\rangle$ will be denoted by $\langle A(K), \blacksquare(i):i\in I\rangle$. A(K) will be called a <u>free extension</u> of A.

<u>Lemma 5.</u> Let $\langle A, \blacksquare(i):i\in I\rangle$ be a $L(\blacksquare)$-algebra and c a constant of the polyadic algebra $(A,I,S,\exists,E)$. Then we have the equalities

a) $\blacksquare(i)c(j) = c(j)\blacksquare(i)$ for $i\neq j$.

b) $c(j)\ \blacksquare(j)p = c(i)\ \blacksquare(i)S(i/j)p$, if $i\notin J_p$.

<u>Proof.</u> We shall use the proof of the theorem 10.17 of [3]. Consider a free extension A(k) of A and let $\underline{n}$ be the ideal which corresponds to the filter generated by E(k,c). In [3] it is shown that $A \to A(k) \to A(k)/\underline{n}$ is an injective polyadic morphism. In accordance to Lemma 2 it results that $A(k)/\underline{n}$ is an $L(\blacksquare)$-algebra and $A\to A(k)/\underline{n}$ is a morphism of $L(\blacksquare)$-algebras. If $\bar{k}$ is the constant of $A(k)/\underline{n}$ induced by k then $c = \bar{k}|_A$ (see [3]).
Since $j\neq i$ it results from (A8) that $S(k/i)\ \blacksquare(j) = \blacksquare(j)S(k/i)$, i.e. $k(i)\ \blacksquare(j) = \blacksquare(j)k(i)$. From this it results a):

For every $p\in A$ we have in A(k):

$$S(k/j)\ \blacksquare(j)p = \blacksquare(k)S(k/j)p \qquad (k\notin J_p)$$

$$S(k/i)\ \blacksquare(i)S(i/j)p = \blacksquare(k)\ S(k/i)\blacksquare(i/j)p = \blacksquare(k)S(k/j)p.$$

therefore we have $k(j)\ \blacksquare(j)p = k(i)\blacksquare(i)S(i/j)p$. Then we obtain b).

<u>Lemma 6.</u> Let $\langle A, \blacksquare(i):i\in I\rangle$ be a $L(\blacksquare)$-algebra. Then

there exists a $L(\blacksquare)$-algebra $\langle A', \mathbf{I}(i) : i\in I\rangle$ such that

(i) A' is a rich polyadic algebra.

(ii) $\langle A, \mathbf{I}(i) : i\in I\rangle$ is a $L(\blacksquare)$-subalgebra of $\langle A', \mathbf{I}(i) : i\in I\rangle$.

Proof. Exactly as in [6], pp.158-160.

§2. Representation theorem

For any polyadic algebra $(A, I, S, \exists, E)$ we shall denote by $E(A)$ the following Boolean algebra

$$E(A) = \{p\in A : J_p = \emptyset\} .$$

Representation theorem. Let $\langle A, \mathbf{I}(i) : i\in I\rangle$ be a $L(\blacksquare)$-algebra and Γ a proper Boolean filter of $E(A)$. Then there exists a topological space $(X, \mathcal{O})$ and a morphism of $L(\blacksquare)$ - algebras

$$\Phi : \langle A, \mathbf{I}(i) : i\in I\rangle \rightarrow \langle F(X^I, \mathcal{O}), \mathbf{I}_{\mathcal{O}}(i) : i\in I\rangle$$

such that $\Phi(p) = 1$ for any $p\in\Gamma$.

Proof: In accordance to Lemma 6, we consider a rich $L(\blacksquare)$ - extension $\langle A_1, \mathbf{I}(i) : i\in I\rangle$ of $\langle A, \mathbf{I}(i) : i\in I\rangle$. Let Δ be an ultrafilter of A_1 such that $\Gamma \subseteq \Delta$. On the set Y of the constants of A_1 we consider the following equivalence relation:

$$c \sim d \iff E(c,d) \in \Delta .$$

Denote $X = Y/\sim$ and let $\hat{c}$ be the equivalence class of

$c \in Y$. If $c \sim d$, $c,d \in Y$ we shall prove that

$$c(i)p \in \Delta \Longleftrightarrow d(i)p \in \Delta$$

for any $i \in I$ and $p \in A_1$. We recall that $E(c,c') = c(i) \wedge c'(j)\ E(i,j)$, where $i \neq j$. For every $c \in Y$ we have (see[3], p.100):

$$c(i)p = \exists(i)(p \wedge E(i,c)),\ i \in I,\ p \in A_1.$$

It results that

$$\begin{aligned} c(i)p \wedge E(c,d) &= c(i)(p \wedge E(i,d)) \\ &= \exists(i)(p \wedge E(i,c) \wedge E(i,d)) \\ &= d(i)p \wedge E(c,d). \end{aligned}$$

Since Δ is ultrafilter and $E(c,d) \in \Delta$ we have

$$\begin{aligned} c(i)p \in \Delta &\Longleftrightarrow c(i)p \wedge E(e,d) \in \Delta \\ &\Longleftrightarrow d(i)p \wedge E(c,d) \in \Delta \\ &\Longleftrightarrow d(i)p \in \Delta. \end{aligned}$$

For any $x \in Y^I$ let $\hat{x}: I \to X$ be the function $x \longmapsto \widehat{x(i)}$. Define $\Psi: A_1 \to F(X^I, \mathbf{O})$ by puting

$$(1)\ \Psi(p)(x) = 1 \Longleftrightarrow x_{i_1}(i_1) \ldots x_{i_n}(i_n)p \in \Delta$$

where $\{i_1, \ldots, i_n\}$ is a support of $p \in A_1$. Exactly as in [3], p.103 it results that Ψ is a polyadic morphism.

We shall denote

$\underline{q}=\{\{\hat{u}\in X: u(i)\ \mathbf{I}(i)p\in\Delta\};\ p\in A_1, i\in I,\ J_p\subseteq\{i\}\}$.

For any $i\in I$ we define $\mathbf{I}'(i):F(X^I,\mathbf{O})\to F(X^I,\mathbf{O})$ by puting

$\mathbf{I}'(i)p(x)=1 \Longleftrightarrow$ there exists $U\in\underline{q}$ such that

$$\hat{x}_i\in U\subseteq\{\hat{u}\in X:\ p((\hat{u}/i)_*x)=1\}.$$

We shall prove that

$$(2)\quad \Psi(\mathbf{I}(i)p)(\hat{x})=\mathbf{I}'(i)\Psi(p)(\hat{x})$$

for any $i\in I$, $p\in A_1$ and $\hat{x}\in X^I$. Suppose that $\{i_1,\dots,i_n\}$ is a support of $p\in A_1$, that $\{i,i_1,\dots,i_n\}$ is a support of $\mathbf{I}(i)p$.

We have the equivalences

$$\Psi(\mathbf{I}(i)p)(\hat{x})=1 \Longleftrightarrow x_i(i)x_{i_1}(i_1)\dots x_{i_n}(i_n)\ \mathbf{I}(i)p\in\Delta$$
$$\Longleftrightarrow x_i(i)\ \mathbf{I}(i)x_{i_1}(i_1)\dots x_{i_n}(i_n)p\in\Delta.$$

$\mathbf{I}'(i)\Psi(p)(\hat{x})=1 \Longleftrightarrow$ there exists $U\in\underline{q}$ such that

$$\hat{x}_i\in U\subseteq\{\hat{u}:\Psi(p)((\hat{u}/i)_*x)=1\}.$$

Supposing $\Psi(\mathbf{I}(i)p)(\hat{x})=1$ we have

$$\hat{x}_i\in\{\hat{u}:\ u(i)\ \mathbf{I}(i)x_{i_1}(i_1)\dots x_{i_n}(i_n)p\in\Delta\}\in\underline{q}.$$

By axiom (A_2):

$$\mathbb{I}(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\leq x_{i_1}(i_1)\ldots x_{i_n}(i_n)p$$

then we obtain

$$U=\{\hat{u}:u(i)\ \mathbb{I}(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta\}\subseteq\{\hat{u}:u(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta\}.$$

It results that

$$\hat{x}_i\in U\subseteq\{\hat{u}:\ \Psi(p)((\hat{u}/i)_*\hat{x})=1\}$$

i.e. $\mathbb{I}'(i)\Psi(p)(x)=1.$

If $\mathbb{I}'(i)\Psi(p)(\hat{x})=1$, let $U\in\underline{q}$ be such that

$$\hat{x}_i\in U\subseteq\{\hat{u}\ u(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta\}.$$

In accordance to the definition of $\underline{q}$, there exists $r\in A_1$, $J_r\subseteq\{j\}$ such that $U=\{\hat{u}:\ u(j)\ \mathbb{I}(j)r\in\Delta\}$. For every $u\in Y$ we have:

$$u(j)\ \mathbb{I}(j)r\in\Delta\Longleftrightarrow u(i)\ x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta\ ,$$

then

$$u(j)\ \mathbb{I}(j)r\wedge u(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta\Longleftrightarrow u(j)\ \mathbb{I}(j)r\in\Delta.$$

From Lemma 5 we have $u(j)\ \mathbb{I}(j)r=u(i)\ \mathbb{I}(i)S(i/j)r$, then

$$u(i)(\ \mathbb{I}(i)S(i/j)r\wedge x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\Leftrightarrow \mathbb{I}(i)S(i/j)r)\in\Delta\ .$$

Since A_1 is rich we have

$\forall(i)(\mathbf{I}(i)S(i/j)r\wedge x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\leftrightarrow \mathbf{I}(i)S(i/j)r)\in\Delta$.

Applying (A_1), (A_3), (A_4) we obtain

$\forall(i)(\mathbf{I}(i)S(i/j)r\wedge \mathbf{I}(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\leftrightarrow \mathbf{I}(i)S(i/j)r)\in\Delta$.

But in a polyadic algebra we have $\forall(i)q\leq c(i)q$ for every constant c and for every element q, then

$x_i(i)\,\mathbf{I}(i)S(i/j)r\wedge x_i(i)\,\mathbf{I}(i)x_{i_1}\ldots x_{i_n}(i_n)p\leftrightarrow x_i(i)\,\mathbf{I}(i)S(i/j)r\in\Delta$.

A new application of Lemma 5 give

$x_i(j)\,\mathbf{I}(j)r\wedge x_i(i)\,\mathbf{I}(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\leftrightarrow x_i(j)\,\mathbf{I}(j)r\in\Delta$.

But $\hat{x}_i\in U\Rightarrow x_i(j)\,\mathbf{I}(j)r\in\Delta$, then we have

$$x_i(i)\,\mathbf{I}(i)x_{i_1}(i_1)\ldots x_{i_n}(i_n)p\in\Delta,$$

i.e. $\Psi(\mathbf{I}(i)p(\hat{x})=1$.

In accordance to (A3), (A5), $\underline{g}$ is a basis of a topology $\mathcal{O}$ on X. Consider the $L(\mathbf{I})$-algebra $\langle F(X^I,\mathbf{O}),\ \mathbf{I}_{\mathcal{O}}(i):i\in I\rangle$ defined by the topology $\mathcal{O}$.

We shall prove that

(3) $\mathbf{I}'(i)p(\hat{x})=\mathbf{I}_{\mathcal{O}}(i)p(\hat{x})$ for any $p\in F(X^I,\mathbf{O})$, $i\in I$ and $\hat{x}\in X^I$.

Since $\underline{g}\subseteq\mathcal{O}$, the following implication is obvious

$$\mathcal{I}'(i)p(\hat{x})=1 \Rightarrow \mathcal{I}_{\mathcal{O}}(i)p(\hat{x})=1.$$

If $\mathcal{I}_{\mathcal{O}}(i)p(\hat{x})=1$ then there exists $U\in\mathcal{O}$ such that

$$\hat{x}_i \in U \subseteq \{\hat{u}: p((\hat{u}/i)_{*}\hat{x})=1\}.$$

But q is the basis of $\mathcal{O}$, then there exists $V\in q$ such that $\hat{x}_i \in V \subseteq U$. It results that $\mathcal{I}'(i)p(\hat{x})=1$.

From (2) and (3) we deduce that

$$\Phi=\Psi|_A : \langle A, \mathcal{I}(i):i\in I\rangle \longrightarrow \langle F(X^I,\mathcal{O}), \mathcal{I}_{\mathcal{O}}(i):i\in I\rangle$$

is a morphism of $L(\mathcal{I})$-algebras. The rest results from the definition of Φ.

Q.E.D.

The $L(\mathcal{I})$-morphisms of the form

$$\Phi : \langle A, \mathcal{I}(i):i\in I\rangle \longrightarrow \langle F(X^I,\mathcal{O}), \mathcal{I}_{\mathcal{O}}(i):i\in I\rangle$$

will be called $L(\mathcal{I})$-representations of $\langle A, \mathcal{I}(i):i\in I\rangle$.

Remark. In the case when $\langle A, \mathcal{I}(i):i\in I\rangle$ is the Lindenbaum-Tarsky algebra of $L(\mathcal{I})$, the representation theorem is exactly the completeness theorem of $L(\mathcal{I})$ (see [8] and [15]).

§3. An omitting types theorem

In [8] and [15] it was proved the omitting types theorem for the topological logic $L(\mathcal{I})$. A cylindrical version of the omitting types theorem for the predicate calculus was given by J.D.Mork in [12].

The aim of this paragraph is to formulate and to prove an omitting types theorem in the context of the L(I)-algebras.

A <u>type</u> U(i) of the L(I)-algebra $\langle A, \exists(i): i\in I\rangle$ is a subset of A such that every element of U(i) has the minimal support $\{i\}$.

We say that a L(I)-representation $\Phi: A \to F(X^I, O)$ <u>omits</u> the type U(i) if for any $u\in X$ and $x\in X^I$ with $x_i=u$, there exists $q\in U(i)$ such that $\Phi(q)(x)=0$.

A subset T of E(A) is <u>consistent</u> if the Boolean filter of A generated by T is proper. We shall say that the proper subset T of A <u>locally omits</u> the type U(i) if for every $p\in A$ with $J_p\subseteq\{i\}$ we have

(*) if $T\cup\{\exists(i)p\}$ is consistent, then there exists $q\in U(i)$ such that

$T\cup\{\exists(i)(p\wedge\neg q)\}$ is consistent.

<u>Theorem</u>. Let $\langle A, \exists(i): i\in I\rangle$ be a countable L(I)-algebra of countable degree. Suppose that $T\subset E(A)$ is consistent and U(i) is a type of A. If T locally omits the type U(i) then there exists a L(I)-representation of $\langle A, \exists(i): i\in I\rangle$ such that

(i) Φ omits the type U(i).
(ii) $\Phi(p)=1$ for any $p\in T$.

<u>Proof</u>. Let us consider a free extension $\langle A(K), \exists(i): i\in I\rangle$ of $\langle A, \exists(i): i\in I\rangle$ where K is countable.

We shall prove that A(K) has the following property:

(**) For any $r\in A(K)$ such that $J_r=\emptyset$ and $p\in A(K)$, $J_p\subseteq\{i\}$, if $T\cup\{r\wedge\exists(i)p\}$

is consistent in $A(K)$ then there exists $q \in U(i)$ such that $T \cup \{r \wedge \exists(i)(p \wedge \neg q)\}$ is consistent.

If $s = r \wedge p$ then we have $r \wedge \exists(i)p = \exists(i)s$. It is known (see [3]) that $s = S(\tau)t$ where $t \in A$, τ is the bijection $(k_1, i_1) \circ \ldots \circ (k_n, i_n)$ and s is independent from $\{i_1, \ldots, i_n\} \subseteq I$. We have also $s = k_1(i_1), \ldots, k_n(i_n)t$.

From $\exists(i)s = \exists(i)k_1(i_1), \ldots, k_n(i_n)t \leq \exists(i)\exists(i_1, \ldots, i_n)t$ it results that $T \cup \{\exists(i)\exists(i_1, \ldots, i_n)t\}$ is consistent. Since T locally omits $U(i)$ in A then there exists $q \in U(i)$ such that

$$T \cup \{\exists(i)(\exists(i_1, \ldots, i_n)t \wedge \neg q)\}$$

is consistent. We shall prove that $T \cup \{\exists(i)(s \wedge \neg q)\}$ is consistent. If not, then there exist $u_1, \ldots, u_n \in T$ such that

$$u_1 \wedge \ldots \wedge u_n \wedge \exists(i)(s \wedge \neg q) = 0.$$

Denoting $u = u_1 \wedge \ldots \wedge u_n$ it results that $\exists(i)(u \wedge s \wedge \neg q) = u \wedge \exists(i)(s \wedge \neg q) = 0$, then we obtain $u \wedge s \wedge \neg q = 0$. But $J_u = \emptyset$, $J_{\neg q} = \{i\}$ and $t = S(\tau)s$ then it follows

$$u \wedge t \wedge \neg q = u \wedge S(\tau)s \wedge \neg q = S(\tau)(u \wedge s \wedge \neg q) = 0.$$

It results that

$$u \wedge \exists(i)(\exists(i_1, \ldots, i_n)t \wedge \neg q) = \exists(i)\exists(i_1, \ldots, i_n)(u \wedge t \wedge \neg q) = 0$$

then $T \cup \{\exists(i)(\exists(i_1, \ldots, i_n)t \wedge \neg q)\}$ is inconsistent. The contradiction is obvious. Since $\exists(i)(s \wedge \neg q) = r \wedge \exists(i)(p \wedge \neg q)$ the property (**) is proved.

We can deduce that there exists a countable $L(\mathbb{I})$-algebra $\langle A^*, \mathbb{I}(i) : i \in I \rangle$ of countable degree such that

a) $\langle A^*, \mathbb{I}(i):i\in I\rangle$ is a rich extension of $\langle A, \mathbb{I}(i):i\in I\rangle$.

b) The property (**) holds in A^*.

This results by observing that the Halmos construction of a rich extension of A preserves the property (**).

Consider a countable set $K=\{k_1,k_2,\dots\}$ of constants of A^* such that every element of A^* has a witness $\{k_{j_1},\dots,k_{j_m}\}\subset K$.

We shall construct by induction a sequence $T_o=T\subseteq T_1\subseteq T_2\subseteq\dots$ of subsets of $E(A^*)$.

Since $\exists(i)E(i,k_1)=1$ (see [3], Lemma 10.15). $T\cup\{\exists(i)E(i,k_1)\}$ is consistent, then there exists $q_1\in U(i)$ such that

$$T\cup\{\exists(i)(\neg q_1\wedge E(i,k_1))\}$$

is consistent. But $k_1(i)\neg q_1=\exists(i)(\neg q_1\wedge E(i,k_1))$ then we can take $T_1=T_o\cup\{k_1(i)\neg q_1\}$. Suppose that there exist $q_1,\dots,q_n\in U(i)$ such that

$$T=T_o\cup\{k_1(i_1)\neg q_1,\dots,k_n(i)\neg q_n\}$$

is consistent. From the property (**) it results that there exists $q_{n+1}\in U(i)$ such that

$$T\cup\{\bigwedge_{t=1}^{n}k_t(i)\neg q_t\wedge\exists(i)(E(i,k_{n+1})\wedge\neg q_{n+1})\}$$

is consistent. Then $T_{n+1}=T_n\cup\{k_{n+1}(i)\neg q_{n+1}\}$ is consistent. It results that the following set

$$\bigcup_{n=0}^{\infty}T_n=T\cup\{k_1(i)\neg q_1,\dots,k_n(i)\neg q_n,\dots\}$$

is consistent. Let Δ be an ultrafilter of A^* such that $\bigcup_{n=0}^{\infty}T_n\subseteq\Delta$.

Consider the following equivalence relation $\sim$ on K

$$k \sim k' \iff E(k, k') \in \Delta \ .$$

Exactly as in §2 we can construct a structure of $L(\blacksquare)$-algebra on $F((K/\sim)^I, \mathbf{O})$ and a $L(\blacksquare)$-representation

$$\Psi : A^* \longrightarrow F(K/\sim)^I, \mathbf{O}) .$$

The $L(\blacksquare)$-morphism $A \longrightarrow A^* \xrightarrow{\Psi} F((K/\sim)^I, \mathbf{O})$ verifies the conditions of the theorem.

Q.E.D.

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