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1. Introduction. The main result of this paper is a local existence theorem for integral solutions in the sense of P. Benilan and H. Brezis [3] to the initial-value problem :

$$(1)\quad \begin{cases} \dfrac{du(t)}{dt} \in Au(t) + f(t,u(t)) , & o \leq t \leq T , \\ u(o) = u_o , \; u(t) \in D & \text{for } o \leq t \leq T , \end{cases}$$

where A is a m-dissipative (possibly multivalued) operator acting on a real Banach space X, operator that generates a strongly continuous semigroup of nonlinear contractions $S(t) : \overline{D(A)} \longrightarrow \overline{D(A)}$, with S(t) compact for all $t > o$, f is a X-valued continuous function defined on $[o,T] \times D$, D being a given nonempty subset of X which generally is not open, and $u_o \in \overline{D(A)} \cap D$.

Problems of this kind have been studied previously by A. Pazy [13] under the additional assumptions that A is linear and D is open, by I. I. Vrabie [14] in the case in which A is nonlinear and D is open, by N. H. Pavel [9] in the case in which A is linear and D is locally closed, and by N. H. Pavel and I. I. Vrabie [10] , [11] , [12] in the case in which A is linear, D is semi locally closed (see Definition 1) and f is a demiclosed and locally bounded multivalued mapping .

We note also the pioneering work of M. Nagumo [8] on flow-invariance problems in finite dimensional spaces, and the papers of H. Brezis [5] and R. H. Martin Jr. [7] on flow-invariance problems in infinite dimensional spaces .

We assume familiarity with the basic concepts of the nonlinear semigroup theory in general Banach space, and we recall for easy references some definitions and results we shall use in the sequel. For further details, see V. Barbu's book [2].

Let X be a real Banach space whose norm is denoted by $\|\cdot\|$, X^* its dual with the corresponding norm $\|\cdot\|_*$, $G : X \longrightarrow 2^{X^*}$ the duality mapping, i.e. :

(2) $G(x) = \{ x^* \in X^* \; ; \|x\|^2 = \|x^*\|_*^2 = x^*(x) \}$,

for $x \in X$, and $\langle \cdot , \cdot \rangle : X \times X \longrightarrow R$ given by :

(3) $\langle y,x \rangle = \sup \{ x^*(y) \; ; \; x^* \in G(x) \}$,

for all $(y,x) \in X \times X$. It is well known (see [2] Ch.I, §1.1, Proposition 1.2) that $\langle \cdot , \cdot \rangle$ is upper semicontinuous on $X \times X$, and the following inequality :

(4) $\langle y,x \rangle \leq \|y\| \, \|x\|$,

holds, for all $(y,x) \in X \times X$.

If $r > o$, $x \in X$, $y \in X$, $D \subset X$ and $E \subset X$, then :

$S(x,r)$ represents the open ball with center x and radius r ;

$B(x,r)$ represents the closed ball with center x and radius r ;

$d(x,D)$ represents the usual distance between x and D ;

$\overline{D}$ represents the closure of D, and

$\rho(D,E)$ represents the Hausdorff-Pompeiu distance between D and E, i.e. :

(5) $\rho(D,E) = \inf \{ h > o \; ; \; D \subset \bigcup_{x \in E} S(x,h), \; E \subset \bigcup_{y \in D} S(y,h) \}$.

<u>DEFINITION 1</u>. The set $D \subset X$ is called <u>semi locally closed</u> if D satisfies :

(i) $D = \bigcup_{\varepsilon \in]o,1[} D_\varepsilon$;

(ii) for each $x \in D$ there exists $r > o$, such that $B(x,r) \cap D_\varepsilon$ is closed in X for all $\varepsilon \in]o,1[$;

(iii) for each $\varepsilon_o \in]o,1[$ and $x \in D_{\varepsilon_o}$ there exist $r > o$ and $\delta > o$ such that the mapping $\varepsilon \longmapsto B(x,\overline{r}) \cap D_\varepsilon$ is continuous in the

Hausdorff-Pompeiu metric on $[\varepsilon_o, \varepsilon_o + \delta]$ for each fixed $o < \bar{r} \leq r$.

We recall that o <u>locally closed</u> set is a set $D \subset X$, such that for each $x \in D$ there exists $r > o$ with $B(x,r) \cap D$ - closed in X, and let us remark that each locally closed set is semi locally closed, but the converse is not true, as we can easily deduce from Lemma 1 below.

Consider the following initial-value problem :

$$(6) \quad \begin{cases} \dfrac{du(t)}{dt} \in Au(t) + f(t) \ , \ a \leq t \leq b \ , \\ u(a) = u_o \ , \end{cases}$$

where $A : D(A) \subset X \longrightarrow X$ is a m-dissipative (possibly multivalued) operator, $u_o \in \overline{D(A)}$ and $f \in L^1(a,b;X)$.

<u>DEFINITION 2</u>. A continuous function $u : [a,b] \longrightarrow \overline{D(A)}$ is called <u>integral solution</u> for the problem (6), if $u(a) = u_o$ and :

$$(7) \quad \|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2\int_s^t \langle f(\theta) + y, u(\theta) - x \rangle d\theta \ ,$$

for all $a \leq s \leq t \leq b$ and $(x,y) \in D(A) \times X$ with $y \in Ax$.

It is well known that if A is m-dissipative, $f \in L^1(a,b;X)$ and $u_o \in \overline{D(A)}$, then the problem (6) has a unique integral solution u on $[a,b]$. Moreover, if u is the integral solution of the problem (6) and v is the integral solution of the problem :

$$(7) \quad \begin{cases} \dfrac{dv(t)}{dt} \in Av(t) + g(t) \ , \ a \leq t \leq b \ , \\ v(a) = v_o \ , \end{cases}$$

where $v_o \in \overline{D(A)}$ and $g \in L^1(a,b;X)$, then, the following inequality :

$$(8) \quad \|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(\theta) - g(\theta)\| d\theta \ ,$$

holds, for all $a \leq s \leq t \leq b$.

For the proof of this fundamental result due to P. Benilan, see [2] Ch.III, §2.1, Theorem 2.1.

DEFINITION 3. A continuous function $u : [o,T] \longrightarrow \overline{D(A)} \cap D$ is called integral solution for the problem (1), if $u(o) = u_o$ and :

$$(9) \quad \|u(t) - x\|^2 \leq \|u(s) - x\|^2 + 2\int_s^t \langle f(\theta,u(\theta)) + y, u(\theta) - x \rangle d\theta ,$$

for all $o \leq s \leq t \leq T$ and $(x,y) \in D(A) \times X$ with $y \in Ax$.

We recall that a strong solution for the problem (1) is a continuous function $u : [o,T] \longrightarrow \overline{D(A)} \cap D$ which is absolutely continuous and almost everywhere differentiable on each compact in $]o,T[$, $u(t) \in D(A)$ a.e. on $]o,T[$, and u verifies (1) .

If $t \in [o,T]$, $h > o$ and $x \in \overline{D(A)} \cap D$, then $u(t,t+h,x)$ represents the value at $t+h$ of the integral solution u for the problem :

$$(10) \quad \begin{cases} \dfrac{du(s)}{ds} \in Au(s) + f(t,x) , & t \leq s \leq t+h , \\ u(t) = x . \end{cases}$$

2. The main result. We begin with the hypotheses we shall use in the sequel .

(H_1) X is a real Banach space .

(H_2) $A : D(A) \subset X \longrightarrow X$ is a m-dissipative (possibly multivalued) operator that generates a C_o - semigroup of contractions $S(t) : \overline{D(A)} \longrightarrow \overline{D(A)}$, with $S(t)$ compact for all $t > o$.

(H_3) $f : [o,T_o] \times D \longrightarrow X$ is a continuous and bounded function, $M \geq \sup\{\|f(t,u)\|; (t,u) \in [o,T_o] \times D\}$, $D = \bigcup_{\varepsilon \in]o,1[} D_\varepsilon$ D_ε being a semi locally closed subset in X .

(H_4) For each $o < \varepsilon < 1$, there exists $\delta > o$, such that for each $\varepsilon' \in [\varepsilon, \varepsilon+\delta]$, $x \in D_{\varepsilon'}$ and $o \leq t \leq T_o$, one has :

$$(11) \quad \lim_{h \to o_+} \frac{1}{h} \cdot d(u(t,t+h,x), D_{\varepsilon'+hM}) = o ,$$

uniformly with respect to $\varepsilon' \in [\varepsilon, \varepsilon+\delta]$, $x \in D_{\varepsilon'}$ and $o \leq t \leq T_o$

We shall see latter that the boundedness assumption on the function f is not so restrictive as it seems to be, since in many specific problems this condition is fulfilled by choosing an appropriate semi locally closed subset D .

Our main result is the following :

<u>THEOREM 1. Assume that</u> (H_1), (H_2), (H_3) <u>and</u> (H_4) <u>are satisfied</u> . <u>Then, for each</u> $u_o \in \overline{D(A)} \cap D$, <u>there exists</u> $T \in]o,T_o]$, <u>such that the problem</u> (1) <u>has at least one integral solution</u> u <u>on</u> $[o,T]$.

<u>Proof</u> : Let $u_o \in \overline{D(A)} \cap D$. By (H_3) it follows that there exists $\varepsilon_o \in]o,1[$ with $u_o \in D_{\varepsilon_o}$. Choose $T \in]o,T_o]$ and $r > o$, such that :

$B(u_o,r) \cap D_\varepsilon$ is closed in X for all $o < \varepsilon < 1$,

(12) $\varepsilon_o + M\cdot T < 1$,

(13) $\|S(t)u_o - u_o\| + M\cdot T \leq r/2$,

for all $o \leq t \leq T$, and in addition, the mapping $\varepsilon \longmapsto B(u_o,r) \cap D_\varepsilon$ is continuous in the Hausdorff-Pompeiu metric on $[\varepsilon_o, \varepsilon_o + M\cdot T]$.

We suppose also that T is small enough, such that :

(14) $\lim_{h \to o_+} \frac{1}{h}\cdot d(u(t,t+h,x), D_{\varepsilon+M\cdot h}) = o$,

uniformly with respect to $\varepsilon_o \leq \varepsilon \leq \varepsilon_o + M\cdot T$, $x \in D_\varepsilon$ and $o \leq t \leq T$.

Fix any natural number n satisfying :

(15) $T/n \leq r/2$, $\varepsilon_o + (T + 1/n)\cdot M < 1$,

and choose the largest number $d^n \in]o,1/n]$ verifying :

(16) $d(u(t,t+d^n,x), D_{\varepsilon+M\cdot d^n}) \leq d^n/2n$,

for all $\varepsilon_o \leq \varepsilon \leq \varepsilon_o + M\cdot T$, $x \in D_\varepsilon$ and $o \leq t \leq T$.

Set $t_i^n = i\cdot d^n$, for $i = o,1,\ldots,I(n)$, where $I(n) \in N$ satisfies $\varepsilon_o + M\cdot I(n)\cdot d^n < 1$, $(I(n)-1)\cdot d^n < T$ and $I(n)\cdot d^n \geq T$. Let us define $u_i^n \in D_{\varepsilon_o+M\cdot i\cdot d^n}$, $i = o,1,\ldots,I(n)$, as follows :

Set $u_o^n = u_o$ and suppose that we have constructed u_j^n belonging to $D_{\varepsilon_o+M\cdot j\cdot d^n}$ with $j< I(n)$. Then, using (16) we easily get :

(17) $d(u(t_j^n,t_j^n+d^n,u_j^n),D_{\varepsilon_o+M\cdot(j+1)\cdot d^n}) \leq d^n/2n$.

Now, define u_{j+1}^n as an arbitrary, but fixed, element in $D_{\varepsilon_o+M\cdot(j+1)\cdot d^n}$ which satisfies :

(18) $\| u(t_j^n,t_j^n+d^n,u_j^n) - u_{j+1}^n \| \leq d^n/n$,

element whose existence is assured by (17) .

Consider the step functions $a_n : [o,T] \longrightarrow [o,T]$ and $u_n : [o,T] \longrightarrow \overline{D(A)}\cap D$ given by :

$a_n(t) = t_i^n$ for $t_i^n \leq t < t_{i+1}^n$, $i = o,1,\ldots,I(n)-1$,

$u_n(t) = u_i^n$ for $t_i^n \leq t < t_{i+1}^n$, $i = o,1,\ldots,I(n)-1$,

and let us observe that in view of P. Benilan's existence and uniqueness Theorem, the initial-value problem :

$$(19)\quad \begin{cases} \dfrac{dy_n(t)}{dt} \in Ay_n(t) + f(a_n(t),u_n(t)) , & o \leq t \leq T , \\ y_n(o) = u_o , \end{cases}$$

has a unique integral solution $y_n : [o,T] \longrightarrow \overline{D(A)}$.

Using (8) and (19) we get :

(2o) $\| y_n(t) - u(t_i^n,t,u_i^n) \| \leq \| y_n(t_i^n) - u_i^n \| \leq \| y_n(t_i^n) - u(t_{i-1}^n,t_{i-1}^n+d^n,u_{i-1}^n) \| + \| u(t_{i-1}^n,t_{i-1}^n+d^n,u_i^n) - u_i^n \| \leq \| y_n(t_{i-1}^n) - u_{i-1}^n \| + d^n/n$,

for each $t_i^n \leq t < t_{i+1}^n$, $i = o,1,\ldots,I(n)-1$.

From (2o) we easily deduce :

(21) $\| y_n(t) - u(t_i^n,t,u_i^n) \| \leq T/n$,

for all $t_i^n \leq t < t_{i+1}^n$, $i = o,1,\ldots,I(n)-1$, relation which in view of (13) and (15), implies :

(22) $\| u_i^n - u_o \| \leq \| u_i^n - y_n(t_i^n) \| + \| y_n(t_i^n) - S(t_i^n)u_o \| + \| S(t_i^n)u_o - u_o \| \leq T/n + \| S(t_i^n)u_o - u_o \| + M\cdot T \leq r/2 + r/2 = r$, $i = o,1,\ldots,I(n)$. Thus, $u_i^n \in D_{\varepsilon_o+M\cdot i\cdot d^n} \cap B(u_o,r)$, for $i = o,1,\ldots,I(n)$.

Now, taking into account the compactness assumption (H_2), and reasoning as in the proof of Theorem 2.1 in [14] (see also the main result of P. Baras [1]), we conclude that the set $\{y_n \; ; n\in N\}$ is relatively compact in $C(o,T;X)$. Let $\{y_{n_k}\}_{k\in N}$ be a convergent subsequence of $\{y_n\}_{n\in N}$ to an element $u\in C(o,T;X)$. From (2.1) it follows that the sequence of step functions $\{u_{n_k}\}_{k\in N}$ converges uniformly to u . As $u_{n_k}(t)\in D_{\varepsilon_o+M\cdot a_{n_k}(t)}\cap B(u_o,r)$ for all $o\leq t\leq T$, and $\varepsilon\longmapsto D_\varepsilon\cap B(u_o,r)$ is continuous in the Hausdorff-Pompeiu metric on $[\varepsilon_o, \varepsilon_o + M\cdot T]$, one has :

(23) $$u(t)\in D_{\varepsilon_o+M\cdot t}\cap B(u_o,r) ,$$

for all $o\leq t\leq T$. Therefore :

(24) $$\lim_{k\to\infty} f(a_{n_k}(t),u_{n_k}(t)) = f(t,u(t)) ,$$

uniformly on $[o,T]$, and consequently, u is an integral solution for the problem (1) on $[o,T]$, as claimed .

3. An example. Let $\Omega\subset R^n$ be any nonempty, bounded and open set whose boundary Γ is a C^∞ - manifold, and consider the following nonlinear parabolic equation :

(25) $$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + f(t,x,u(t,x)) & \text{a.e. on }]o,T[\times\Omega \\ -\dfrac{\partial u}{\partial n}\in\beta(u) & \text{a.e. on }]o,T[\times\Gamma \\ u(o,x) = u_o(x) & \text{a.e. on } \Omega \end{cases} ;$$

where $f : [o,+\infty[\times\bar{\Omega}\times R \longrightarrow R$ is a given continuous function, $\beta\subset R\times R$ is a maximal monotone graph with $o\in\beta(o)$, Δ is the Laplace operator, $\dfrac{\partial u}{\partial n}$ is the outward normal derivative and $u_o\in L^\infty(\Omega)$.

Now, using Theorem 1 one may prove :

THEOREM 2. Assume that $f : [o,+\infty[\times\bar{\Omega}\times R \longrightarrow R$ is continuous, $\beta\subset R\times R$ is maximal monotone with $o\in\beta(o)$. Then, for each $u_o\in L^\infty(\Omega)$, there exists $T>o$, such that the problem (25) has at least one strong solution $u : [o,T] \longrightarrow L^{\infty}(\Omega)$, verifying :

(i) $u \in W^{1,2}(\delta,T;L^2(\Omega))$ for all $o < \delta < T$,

(ii) $t^{1/2}\frac{\partial u}{\partial t} \in L^2(o,T;L^2(\Omega))$, $u(t) \in H^2(\Omega)$ a.e. on $]o,T[$,

$-\frac{\partial u}{\partial n} \in \beta(u)$ a.e. on $]o,T[\times \Gamma$,

(iii) $\frac{1}{2}\int_{\Omega}|\text{grad}u|^2dx + \int_{\Gamma} j(u)ds \in L^1(o,T;R)$, where $j : R \longrightarrow [o,+\infty]$ is a lower semicontinuous, convex function with $\partial j = \beta$.

If in addition $u_o \in H^1(\Omega)$ and $j(u) \in L^1(\Gamma)$, then u satisfies :

(iv) $\frac{\partial u}{\partial t} \in L^2(o,T;L^2(\Omega))$, $\frac{1}{2}\int_{\Omega}|\text{grad}u|^2dx + \int_{\Gamma} j(u)ds \in L^{\infty}(o,T;R)$.

In order to prove Theorem 2, we need the following lemma, which is interesting by itself.

LEMMA 1. Let $S_{\infty}(o,k+1)$ be the open ball with center o and radius k+1 in $L^{\infty}(\Omega)$, where k is a fixed positive number and Ω is a measurable subset of R^n, whose Lebesgue measure is finite . Then, $S_{\infty}(o,k+1)$ is semi locally closed in $L^2(\Omega)$, but is not locally closed in $L^2(\Omega)$.

Proof of Lemma 1 : Set $S_{\infty}(o,k+1) = \bigcup_{\varepsilon \in]o,1[} B(o,k+\varepsilon)$, where for each $\varepsilon \in]o,1[$ $B_{\infty}(o,k+\varepsilon)$ is the closed ball with center o and radius $k+\varepsilon$ in $L^{\infty}(\Omega)$. As $B_{\infty}(o,k+\varepsilon)$ is closed in $L^2(\Omega)$, it follows that the conditions (i) and (ii) in Definition 1 are satisfied .

Let $x \in S_{\infty}(o,k+1)$, $r > o$ and denote by $B_2(x,r)$ the closed ball with center x and radius r in $L^2(\Omega)$. To prove (iii) in Definition 1, it suffices to show that for each $x \in S_{\infty}(o,k+1)$, each $\varepsilon_o \in]o,1[$ with $x \in B(o,k+\varepsilon_o)$ and each $r > o$, there exists $\delta > o$, such that the mapping $\varepsilon \longmapsto B(o,k+\varepsilon) \cap B_2(x,r)$ is continuous in the Hausdorff-Pompeiu metric ρ_2 on $[\varepsilon_o,\varepsilon_o+\delta]$. Here ρ_2 represents the Hausdorff-Pompeiu metric defined by using the norm of $L^2(\Omega)$. As $L^{\infty}(\Omega)$ is continuously imbedded in $L^2(\Omega)$, for proving the last assertion, it suffices to check out the continuity of the mapping

$\varepsilon \longmapsto B(o,k+\varepsilon)\cap B_2(x,r)$ on $[\varepsilon_o,\varepsilon_o+\delta]$ in the Hausdorff - Pompeiu metric ρ_∞ defined by using the norm of $L^\infty(\Omega)$.

Let $x\in B(o,k+\varepsilon_o)$, $r>o$, $\delta>o$ such that $k+\varepsilon_o+\delta<1$ and let $\bar{h}>o$ be such that $[\bar{h}/(k+\varepsilon+\bar{h})]\cdot M\leq r/2$ for all $\varepsilon\in[\varepsilon_o,\varepsilon_o+\delta]$, where $M=\sup\{\|y\|_{L^2(\Omega)}\ ;\ y\in S_\infty(o,k+1)\}$, and $k+\varepsilon_o+\delta+\bar{h}<1$.

Denote by :

$A_\varepsilon=B_\infty(o,k+\varepsilon)\cap B_2(x,r)$ and

$A_{\varepsilon+h}=B_\infty(o,k+\varepsilon+h)\cap B_2(x,r)$, where $o<h\leq\bar{h}$.

From (5), taking into account that $A_\varepsilon\subset A_{\varepsilon+h}$, one easily deduce :

(26) $\rho_\infty(A_\varepsilon,A_{\varepsilon+h})=\sup\{p>o\ ;\ A_{\varepsilon+h}\subset\bigcup_{y\in A_\varepsilon}S_\infty(y,p)\}$.

We shall prove that there exists $g:]o,\bar{h}]\longrightarrow R_+$ with $\lim_{h\to o}g(h)=o$ and :

(27) $\rho_\infty(A_\varepsilon,A_{\varepsilon+h})\leq g(h)$, for all $o<h\leq\bar{h}$.

Let $y\in A_{\varepsilon+h}$ and define :

(28) $$y_{\varepsilon h}=\frac{k+\varepsilon}{k+\varepsilon+h}\cdot y.$$

Define also :

(29) $$\lambda(h,y)=\begin{cases}\dfrac{\|x-y\|_{L^2(\Omega)}}{\|x-y_{\varepsilon h}\|_{L^2(\Omega)}} & \text{if } r/2<\|x-y\|_{L^2(\Omega)}\leq\|x-y_{\varepsilon h}\|_{L^2(\Omega)}\\ 1 \text{ if } \|x-y\|_{L^2(\Omega)}\leq r/2, \text{or } \|x-y_{\varepsilon h}\|_{L^2(\Omega)}\leq \\ \quad\leq\|x-y\|_{L^2(\Omega)}, & \end{cases}$$

and let us observe that $\lambda(h,y)\cdot y_{\varepsilon h}+(1-\lambda(h,y))\cdot x\in A_\varepsilon$.

Moreover:

(3o) $$\|y-\lambda(h,y)\cdot y_{\varepsilon h}-(1-\lambda(h,y))\cdot x\|_{L^\infty(\Omega)}\leq\lambda(h,y)\cdot\|y-y_{\varepsilon h}\|_{L^\infty(\Omega)}+(1-\lambda(h,y))\cdot\|x-y\|_{L^\infty(\Omega)},$$

relation which implies :

(31) $\|y - \lambda(h,y)y_{\varepsilon h} - (1-\lambda(h,y))x\|_{L^\infty(\Omega)} \leq h + 2\cdot(1-\lambda(h,y))\cdot(k+1)$.

From (31), takind into account that :

(32) $\lambda(h,y) \geq \frac{m}{m+K\cdot h}$, where $K = \frac{M}{k+\varepsilon}$ and $m = r/2$,

we deduce :

(33) $\|y - \lambda(h,y)y_{\varepsilon h} - (1-\lambda(h,y))x\|_{L^\infty(\Omega)} \leq g(h)$,

where $g :]0,\bar{h}] \longrightarrow R_+$ is given by :

$$g(h) = h + 2\cdot(1-\frac{m}{m+K\cdot h})\cdot(k+1) \quad .$$

As (33) implies (27), it follows that $S_\infty(o,k+1)$ is semi locally closed in $L^2(\Omega)$.

Now, let us remark that for proving that $S_\infty(o,k+1)$ is not locally closed in $L^2(\Omega)$, it suffices to show that for each positive number r with :

$$o < \frac{r^2}{(k+1)^2} < \text{mes}\,\Omega \quad ,$$

there exists a sequence $\{u_n\}_{n\in N}$, $u_n \in S_\infty(o,k+1)\cap B_2(o,r)$ for all $n \in N$, and :

(34) $\lim_{n\to\infty} u_n = u$ in $L^2(\Omega)$,

(35) $u \notin S_\infty(o,k+1)$.

Let Ω_o be any measurable subset of Ω with :

(36) $o < \text{mes}\,\Omega_o \leq \frac{r^2}{(k+1)^2}$.

Define :

(37) $$u_n(x) = \begin{cases} \frac{n}{n+1}\cdot(k+1) & , \text{ if } x \in \Omega_o \\ o & , \text{ if } x \in \Omega\setminus\Omega_o , \end{cases}$$

and let us observe that $\{u_n\}_{n\in N}$ given by (37) satisfies (34) and (35), where

$$u(x) = \begin{cases} k+1 & , \text{ if } x \in \Omega_o \\ o & , \text{ if } x \in \Omega\setminus\Omega_o , \end{cases}$$

thereby completing the proof of Lemma 1 .

<u>Proof of Theorem 2</u> : Take $X = L^2(\Omega)$, $A = \Delta$ with $D(A) = \{u ;$

$u \in H^2(\Omega)$, $-\frac{\partial u}{\partial n} \in \beta(u)$ a.e. on $\Gamma\}$, $D = S_\infty(o,k+1) = \bigcup_{\varepsilon \in]o,1[} B_\infty(o,k+\varepsilon)$, where $k = \|u_o\|_{L^\infty(\Omega)}$, and $F : [o,T_o] \times D \longrightarrow L^2(\Omega)$, defined by :

$$F(t,u)(x) = f(t,x,u(x)) .$$

Let $j : R \longrightarrow [o,+\infty]$ be any convex, lower semicontinuous and proper function, with $\partial j = \beta$. It is well known that

$Au = \partial\varphi(u)$ for all $u \in D(A)$, where $\varphi : L^2(\Omega) \longrightarrow R$ is given by :

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2 dx + \int_\Gamma j(u)ds , & \text{if } u \in H^1(\Omega) \text{ and } j(u) \in L^1(\Gamma) \\ +\infty & \text{in rest .} \end{cases}$$

Using Y. Konishi's main result in [6] , we deduce that A verifies (H_2) . Let us observe that F is continuous from $[o,T_o] \times D$ into $L^2(\Omega)$ (in the topology of $L^2(\Omega)$), and also that F is bounded.

Let $M = \max\{ \sup\{ \|F(t,u)\|_{L^2(\Omega)} ; (t,u) \in [o,T_o] \times D \} , \sup\{ \|F(t,u)\|_{L^\infty(\Omega)} ; (t,u) \in [o,T_o] \times D \}\}$ and let us remark that the problem (25) may be rewritten as :

$$(38) \quad \begin{cases} \frac{du(t)}{dt} = Au(t) + F(t,u(t)) , & o \le t \le T , \\ u(o) = u_o & . \end{cases}$$

From Lemma 1 it follows that $D = S_\infty(o,k+1)$ is semi locally closed in $L^2(\Omega)$, and thus (H_1), (H_2) and (H_3) in Theorem 1 are verified . For proving (H_4), it suffices to show that for each $o < \varepsilon < 1$, there exists $\delta > o$ such that for each $\varepsilon' \in [\varepsilon, \varepsilon+\delta]$, $x \in B_\infty(o,k+\varepsilon')$ and $o \le t \le T_o$, one has :

$$(39) \quad \|u(t,t+h,x)\|_{L^\infty(\Omega)} \le k + \varepsilon' + h\cdot M ,$$

uniformly with respect to $\varepsilon' \in [\varepsilon,\varepsilon+\delta]$, $x \in B\ (o,k+\varepsilon')$ and $o \le t \le T_o$

Let $o < \varepsilon < 1$ and choose $\delta > o$ such that $\varepsilon + \delta < 1$. Consider $\varepsilon' \in [\varepsilon,\varepsilon+\delta]$ and $x \in B_\infty(o,k+\varepsilon')$. As the restriction of the operator A to $L^{\infty}(\Omega)$ is m-dissipative (see H. Brezis [4] Remarque I34 p. 6o), one may assert that u(t,t+h,x) is in $L^{\infty}(\Omega)$ and in addition that :

(4o) $\|u(t,t+h,x)\|_{L^{\infty}(\Omega)} \leq \|u(t,t+h,x) - S(h)x\|_{L^{\infty}(\Omega)} + \|S(h)x\|_{L^{\infty}(\Omega)}$ $\leq k + \varepsilon' + h \cdot M$.

Now, using Theorem 1, we deduce that the problem (25) has at least one (local) integral solution $u : [o,T] \longrightarrow S_{\infty}(o,k+1)$, which in view of [2] , Cap.IV, §2.1, Theorem 2.1, verifies (i), (ii), (iii) and (iv), as claimed .

For another proof of Theorem 2 , see I. I. Vrabie [15] .

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