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by

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Abstract. Necessary conditions of optimality for boundary control problems governed by parabolic equations with nonlinear boundary value conditions are derived.

1. INTRODUCTION

We are concerned here with first order necessary conditions of optimality for convex control problems governed by nonlinear boundary-value problems of the form

$$
\begin{aligned}
& y_t + Ay = 0 && \text{in } Q = \Omega \times]0,T[\\
(1.1)\quad & \frac{\partial y}{\partial \nu} + \beta(y) \ni u + f && \text{in } \Sigma = \Gamma \times]0,T[\\
& y(x,0) = y_0(x) && \text{in } \Omega .
\end{aligned}
$$

Here Ω is a bounded and open subset of the Euclidean space R^N, A is a second order elliptic and symetric operator on Ω and β is a maximal monotone graph (in general multivalued) in $R \times R$. The control u which is exercised through the boundary Γ of Ω, is taken from the space $L^2(\Sigma)$ and the functions y_0 and f are fixed in $L^2(\Omega)$ and $L^2(\Sigma)$, respectively.

Problems of this type occur in heat conduction, mechanics of fluids in porous media and temperature control trough the boundary (see, Duvaut and Lions [7].) The generalized optimality conditions are obtained by an approach similar to that used in [1],[2]. Particular cases of the problem described above have been studied in [12], [13] among others. In particular Theorem 1 includes and refines some results of [13]. The following notation will be used. Let k,r,s, be real numbers. We shall denote by $H^k(\Omega)$ $H^k(\Gamma)$, $H^{r,s}(Q)$ and $H^{r,s}(\Sigma)$ the usual Sobolev spaces on Ω, Γ, Q and Σ, respectively (see e.g. [9]). By $H^k_o(\Omega)$ we shall denote the space of all elements of $H^k(\Omega)$ of trace zero on Γ. Given a Banach space E we shall denote by $L^p(0,T;E)$, $1 \leq p \leq \infty$ the space of all p-integrable functions from $[0,T]$ to E. By $C([0,T];X)$ we shall denote the space of all continuous functions from $[0,T]$ to E. By $L^p(\Omega)$, $L^p(\Gamma)$ and $L^p(\Sigma)$ we shall denote the usual space of p-integrable real valued functions on Ω, Γ and Σ. Given a lower semicontinuous convex function φ from a Banach space X to $\bar{R} = R \cup \{+\infty\}$ we shall denote by $D(\varphi)$ its effective domain i.e., the set $\{x \in X; \varphi(x) < +\infty\}$ and by $\partial\varphi(x)$ the subdifferential of φ at x. We refer to the books [3], [5], [8] and to the survey [11] of Rockafellar for concepts and basic results on convex analysis relevant to this paper.

2. DESCRIPTION OF THE BOUNDARY CONTROL SYSTEM

Let Ω be a bounded and open subset of R^N with a sufficently smooth boundary Γ. Let A be a second order differential operator on Ω of the form

$$Ay = -\sum_{i,j=1}^{N} (a_{ij} y_{x_i})_{x_j} + ay \tag{2.1}$$

where $a_{ij} \in C^1(\bar{\Omega})$, $a \in L^\infty(\Omega)$, $a_{ij} = a_{ji}$ for all i,j and for some $\omega > 0$,

$$\sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j \geq \omega |\xi|^2 \quad \text{a.e. on } \Omega, \ \xi \in R^n .$$

We shall denote by $\partial/\partial\nu$ the outward normal derivative corresponding to A and we shall denote by a: $H^1(\Omega) \times H^1(\Omega) \longrightarrow R$ the bilinear functional

$$(2.2) \qquad a(y,z) = \sum_{i,j=1}^{N} \int_{\Omega} (a_{ij} y_{x_i} z_{x_i} + ayz)dx \ ; \ y,z \in H^1(\Omega).$$

Let us now give a meaning to state system (1.1) where $u,f \in L^2(\Sigma)$ and $y_o \in L^2(\Omega)$.

By definition, a function $y \in L^2(0,T;H^1(\Omega))$ is a solution to (1.1) if there exists a function $v \in L^2(\Sigma)$ such that

$$(2.2) \qquad v(\sigma,t) \in \beta(y(\sigma,t)) \quad \text{a.e.} (\sigma,t) \in \Sigma$$

and

$$(2.3) \quad -\int_Q y\varphi_t dxdt + \int_0^T a(y,\varphi)dt + \int_{\Sigma} (v-u-f)\varphi \, d\sigma \, dt = \int_{\Omega} y_o(x)\varphi(x,0)dx$$

for all $\varphi \in L^2(0.T);H^1(\Omega))$ such that $\varphi_t \in L^2(Q)$ and $\varphi(x,T) = 0$.

We see that $\frac{d}{dt} y \in L^2(0,T;(H^1(\Omega))')$. Hence in particular $y \in C([0,T]; L^2(\Omega))$ and (2.3) can be equivalently defined as

$$(2.4) \quad \begin{aligned} &\frac{d}{dt}(y,\psi) + a(y,\psi) + \int_{\Gamma} (v-u-f)\psi \, d\sigma = 0, \ \text{a.e.} \ t \in]0,T[\\ &y(0) = y_o \qquad\qquad\qquad \text{on } \Omega \end{aligned}$$

for all $\psi \in H^1(\Omega)$, where $(.,.)$ denotes the usual inner product in $L^2(\Omega)$.

Let ρ be a C_o^∞ - function on R satisfying: $\int_{-\infty}^{\infty} \rho(x)dx = 1$, $\rho \geq 0$, support $\rho \subset]-1,1[$ and $\rho(x) = \rho(-x)$ for $x \in R$.

We define for every $\varepsilon > 0$,

$$(2.5) \qquad \beta^{\varepsilon}(y) = \int_{-\infty}^{\infty} \beta_{\varepsilon}(y-\varepsilon\theta) \, \rho(\theta)d\theta, \quad y \in R$$

where

$$\beta_\varepsilon(y) = \varepsilon^{-1}(y - (1+\varepsilon\beta)^{-1}y), \quad y \quad R$$

and β is the maximal monotone graph arising in Eq.(1.1). Obviously, for each $\varepsilon > 0$, β^ε is a monotone increasing C^∞-function on R. Moreover β^ε is Lipschitzian with Lipschitz constant $1/\varepsilon$. Then by a standard fixed point argument it follows that the boundary value problem

$$(2.6)\qquad \begin{aligned} &y_t + Ay = 0 && \text{in } Q \\ &\frac{\partial y}{\partial \nu} + \beta^\varepsilon(y) = u+f && \text{in } \Sigma \\ &y(0) = y_0 && \text{in } \Omega \end{aligned}$$

has a unique solution $y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ (If $y_0 \in H^{1/2}(\Omega)$ then by Theorem 15.2 in [9] it follows that $y_\varepsilon \in H^{3/2,3/4}(Q)$.) Since β is maximal monotone, there exists a lower-semicontinuous convex functions $j:R \longrightarrow \bar{R} =]-\infty,+\infty]$ uniquely determined up to an additive constant such that $\partial j = \beta$ (∂j denotes the subdifferential of j).

PROPOSITION 1. Let $u,f \in L^2(\Sigma)$ and $y_0 \in L^2(\Omega)$ be given such that $j(y_0) \in L^1(\Omega)$. Then problem (1.1) has a unique solution $y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Moreover, we have

$$(2.7)\qquad \|y_\varepsilon - y\|_{C([0,T];L^2(\Omega))} + \|y_\varepsilon - y\|_{L^2(0,T;H^1(\Omega))} \le C\,\varepsilon^{1/2}.$$

Proof. Without any loss of generality we may assume that $y_\varepsilon \in H^{2,1}(Q)$ (this can be achivied assuming for a while that y_0, f and u are sufficiently regular). Then by (2.6) it follows that

$$(2.8)\qquad \|y_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_0^t \|y_\varepsilon(s)\|^2_{H^1(\Omega)}\,ds \le C(\|u\|^2_{L^2(\Sigma)} + 1)$$

and

$$(2.9)\quad \| y_\varepsilon(t) - y_\lambda(t)\|^2_{L^2(\Omega)} + \| y_\varepsilon - y_\lambda\|^2_{L^2(0,T;H^1(\Omega))} + $$

$$+ C \int_\Sigma (\beta^\varepsilon(y_\varepsilon) - \beta^\lambda(y_\lambda))\,(y_\varepsilon - y_\lambda)\, d\sigma\, dt \quad 0$$

for all $\varepsilon, \lambda > 0$, where C is a positive constant independent of ε, λ and u. Next we multiply Eq.(2.6) by $\beta^\varepsilon(y_\varepsilon)$ and integrate over Q. By Green's formula we get

$$(2.10)\quad \int_\Omega j^\varepsilon(y_\varepsilon)dx + \int_\Sigma \beta^\varepsilon(y_\varepsilon)(\beta^\varepsilon(y_\varepsilon) - f - u)\, d\sigma\, dt \leq \int_\Omega j^\varepsilon(y_0)dx$$

where

$$j^\varepsilon(y) = \int_0^y \beta^\varepsilon(r)dr.$$

By (2.8), (2.9) and (2.10) one finds by a standard calculation involving (2.5),

$$(2.11)\quad \| y_\varepsilon(t)\|_{L^2(\Omega)} + \| \beta^\varepsilon(y_\varepsilon)\|_{L^2(\Sigma)} + \| y_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq$$

$$C(\| u\|_{L^2(\Sigma)} + 1)$$

and

$$(2.12)\quad \| y_\varepsilon(t) - y_\lambda(t)\|^2_{L^2(\Omega)} + \| y_\varepsilon - y_\lambda\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\varepsilon+\lambda),\ t\in[0,T]$$

where C is a positive constant independent of ε, λ and u. Hence $y = \lim_{\varepsilon \to 0} y_\varepsilon$ exists in the strong topology of $C([0,T];L^2(\Omega)) \cap$ $\cap\ L^2(0,T;H^1(\Omega))$. By (2.10) we may also suppose that

$$(2.13)\qquad \beta^\varepsilon(y_\varepsilon) \longrightarrow w \quad \text{weakly in } L^2(\Sigma)$$

for $\varepsilon \longrightarrow 0$. Since the realization $\tilde\beta$ of β in $L^2(\Sigma) \times L^2(\Sigma)$ is a maximal monotone graph we may infer that $[y,w]\in\tilde{\beta}$, i.e.,

$$w(\sigma,t)\in\beta(y(\sigma,t)) \qquad \text{a.e. } (\sigma,t)\in\Sigma. \tag{2.14}$$

Then letting ε tend to 0 in Eq.(2.6) (under its weak form) we see that y is a solution to (1.1). As regards estimate (2.7) it is implied by (2.12).

Remarks 1°. Letting ε tend to zero in inequality (2.11) we see that the solution y to Eq.(1.1) satisfies

$$\|y(t)\|_{L^2(\Omega)}+\|y\|_{L^2(0,T;H^1(\Omega))}+\left\|\frac{\partial y}{\partial t}\right\|_{L^2(0,T;(H^1(\Omega)')}+ \tag{2.15}$$
$$+\|\beta(y)\|_{L^2(\Sigma)}\le C(\|u\|_{L^2(\Sigma)}+1) \text{ for all } u\in L^2(\Sigma).$$

In particular this implies that the map $u\longrightarrow y$ is weakly continuous from $L^2(\Sigma)$ to $L^2(0,T;H^1(\Omega))$ and compact from $L^2(\Sigma)$ to $L^2(Q)$.

2°. It must be emphasized that more general boundary control systems of the form

$$\begin{aligned} & y_t+Ay=F && \text{in } Q\\ & \frac{\partial y}{\partial\nu}+\beta(y)\ni u+f_o && \text{in } \Sigma\\ & y(o)=y_o && \text{in } \Omega \end{aligned} \tag{2.16}$$

where $F\in L^2(Q)$ and $f_o\in L^2(\Sigma)$ can be put into the form (1.1) where $f=f_o-\frac{\partial z}{\partial\nu}+u$ and $z\in H^{2,1}(Q)$ is the solution to

$$\begin{aligned} & z_t+Az=f && \text{in } Q\\ & z=0 && \text{in } \Sigma\\ & z(0)=0 && \text{in } \Omega \end{aligned} \tag{2.17}$$

In particular, it follows by Proposition 1 that for every $u \in L^2(\Sigma)$, Eq.(2.14) has a unique solution $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

3. THE MAIN RESULTS

We shall study the following control problem: minimize

$$\frac{1}{2}\int_Q \ell(x,t)\,(y(x,t)-y_d(x,t))^2\,dxdt+\int_\Sigma g(\sigma,u(\sigma,t))\,d\sigma dt+\varphi(y(\cdot,T)) \tag{3.1}$$

over all $u \in L^2(\Sigma)$ and $y \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ subject to state equation (1.1) where y_o and f are fixed as in Proposition 1. In problem (3.1) y_d is fixed element of $L^2(Q)$, φ is a continuous convex function from $L^2(\Omega)$ to R and $\ell \in L^\infty(Q)$ is a given function. As regards the function $g: \Gamma \times R \longrightarrow \bar{R}$ it is a convex normal integrand on $\Gamma \times R$ in the sense of Rockafellar, i.e.,

(i) For each $\sigma \in \Gamma$, $g(\sigma,\cdot)$ is convex, lower semicontinuous and $\not\equiv +\infty$.

(ii) g is measurable with respect to the σ-field of subsets of $\Gamma \times R$ generated by products of Lebesgue sets in Γ and Borel sets in R.

(iii) There exist $\alpha_1 \in L^2(\Gamma)$ and $\alpha_2 \in L^1(\Gamma)$ such that

$$g(\sigma,u) \geq \alpha_1(\sigma)u + \alpha_2(\sigma) \quad \text{a.e. } \sigma \in \Gamma,\ u \in R. \tag{3.2}$$

The first optimality result is concerned with the situation in which β is locally Lipschitzian on R and satisfies the condition

$$\beta'(y)|y| \leq C(|\beta(y)| + y^2+1) \quad \text{a.e. } y \in R. \tag{3.3}$$

Here β' denotes the derivative of β and C is a positive constant.

THEOREM 1. Let (y^*,u^*) be an optimal pair in problem (3.1) where the functions g,φ satisfy Assumptions (i) up to (iii) and β is a monotonically non decreasing locally Lipschitzian function satisfying condition (3.4). Then there exists a function $p\in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ with $p_t\in L^2(0,T;(H^1(\Omega))')$ and $\frac{\partial p}{\partial \nu}\in L^1(\Sigma)$ which satisfies along with y^* and u^* the system

$$(3.4)\qquad y^*_t + Ay^* = 0 \qquad \text{in } Q$$

$$(3.5)\qquad \frac{\partial y^*}{\partial \nu} + \beta(y^*) = u^* + f \qquad \text{in } \Sigma$$

$$(3.6)\qquad y^*(\cdot,0) = y_o \qquad \text{in } \Omega$$

$$(3.7)\qquad p_t - Ap = (y^*-y_d)\ell \qquad \text{in } Q$$

$$(3.8)\qquad \frac{\partial p}{\partial \nu} + \partial\beta(y^*)p \ni 0 \qquad \text{in } \Sigma$$

$$(3.9)\qquad p(\cdot,T) + \partial\varphi(y^*(\cdot,T)) \ni 0 \qquad \text{in } \Omega$$

$$(3.10)\qquad p(\sigma,t)\in\partial g(\sigma,u^*(\sigma,t)) \qquad \text{a.e. } (\sigma,t)\in\Sigma .$$

Here $\partial\beta$ denotes the generalized gradient of β in the sense of Clarke (see [7], [13]) and ∂g is the subdifferential of $u \longrightarrow g(\cdot,u)$. Eqs. (3.8), (3.9), (3.10) can be also be interpreted in the following weak sense

$$(3.11)\qquad \int_Q p\,\psi_t\,dxdt + \int_0^T a(p,\psi)dt + \int_\Sigma \mu\,\psi\,d\sigma dt =$$

$$= -\int_Q \ell(y^*-y_d)\psi\,dxdt - \int_\Omega \nu\,\psi(x,T)dx$$

for all $\psi\in C^1(\bar{Q})$ with $\psi(x,0)=0$. Here $\mu\in L^1(\Sigma)$ and $\nu\in L^2(\Omega)$ satisfy

$$(3.12)\qquad \mu(x,t)\in\partial\beta(y^*(x,t))p\,(x,t) \qquad \text{a.e. } (x,t)\in\Sigma$$

$$(3.13)\qquad \nu(x) \in -\partial\varphi(y^*(\cdot,T))(x) \qquad \text{a.e. } x \in \Omega .$$

We shall consider now the situation in which the control problem (3.1) is governed by the following unilateral problem ("the Signorini problem")

$$(3.14)\qquad \begin{aligned} & y_t + Ay = 0 && \text{in } Q \\ & y\left(\frac{\partial y}{\partial \nu} - u - f\right) = 0; \quad y \geqslant 0, \frac{\partial y}{\partial \nu} - u - f \geqslant 0 && \text{on } \Sigma \\ & y(\cdot,0) = y_o && \text{in } \Omega . \end{aligned}$$

It is well known (see e.g. [5]) that problem (3.14) can be written under the form (1.1) where the graph β is given by

$$(3.15)\qquad \beta(r) = \begin{cases} 0 & \text{if } r > 0 \\ ]-\infty, 0] & \text{if } r = 0 \\ \emptyset & \text{if } r < 0. \end{cases}$$

According to Proposition 1 for existence we must assume that $y_o \in L^2(\Omega)$, $f \in L^2(\Sigma)$ and $y_o(x) \geqslant 0$ a.e. $x \in \Omega$.

The optimality theorem in this case is

THEOREM 2. *Let* (y^*, u^*) *be an optimal pair for problem* (3.1) *with state system* (3.14) *and under assumptions* (i), (ii), (iii). *Then there exists a function* $p \in L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ *with* $p_t \in L^2(0,T;(H^1(\Omega))')$ *and* $\frac{\partial p}{\partial \nu} \in M(\Sigma)$ *satisfying along with* y^* *and* u^* *the following system*

$$(3.16)\qquad y_t^* + Ay^* = 0 \qquad \text{in } Q$$

$$(3.17)\qquad y^*\left(\frac{\partial y^*}{\partial \nu} - u^* - f\right) = 0; \quad y^* \geqslant 0, \frac{\partial y^*}{\partial \nu} - u^* - f \geqslant 0 \qquad \text{in } \Sigma$$

$$(3.18)\qquad y^*(\cdot,0) = y_o \qquad \text{in } \Omega$$

$$(3.19)\qquad p_t - Ap = (y^* - y_d)\ell \qquad \text{in } Q$$

(3.20) $\left(\frac{\partial p}{\partial \nu}\right)_a = 0$ a.e. on $\{(\sigma,t)\in\Sigma\ , y^* > 0\}$.

(3.21) $p = 0$ a.e. on $\{y^* = 0\}\cap\{(\sigma,t)\in\Sigma\ ;\ u^* - \frac{\partial y^*}{\partial \nu} - f > 0\}$.

(3.22) $p(\cdot,T) + \partial\varphi(y^*(\cdot,T)) \ni 0$ on $L^2(\Omega)$.

(3.23) $p(\sigma,t)\in\partial\, g(\sigma,u^*(\sigma,t))$ a.e. $(\sigma,t)\in\Sigma$.

Here $\left(\frac{\partial p}{\partial \nu}\right)_a$ denotes the absolutely continuous part of the measure $\frac{\partial p}{\partial \nu}$ and $M(\Sigma)$ is the space of all bounded Radon measures on $\overline{\Sigma}$.

4. THE APPROXIMATING CONTROL PROCESS

Consider the control problem: Minimize

(4.1) $$\frac{1}{2}\int_Q \ell(x,t)|y-y_d|^2 dxdt + \int_\Sigma (g_\varepsilon(\sigma,u) + \frac{1}{2}|u-u^*|^2)d\sigma dt + \varphi_\varepsilon(y(T))$$

over all $y\in C([0,T];L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$ and $u\in L^2(\Sigma)$ subject to state system (2.6). Here g_ε and φ_ε are defined by

(4.2) $$g_\varepsilon(\sigma,u) = \inf\{|u-v|^2/2\varepsilon + g(\sigma,v);\ v\in R\}$$

and

(4.3) $$\varphi_\varepsilon(y) = \inf\{\|y-z\|^2_{L^2(\Omega)}/2\varepsilon + \varphi(z);\ z\in L^2(\Omega)\},\quad y\in L^2(\Omega).$$

Since as remarked earlier the map $u \longrightarrow y$ is weakly continuous from $L^2(\Sigma)$ to $L^2(0,T;H^1(\Omega))$, problem (4.1) has at least one solution $(y_\varepsilon, u_\varepsilon)$. Using the fact that g_ε, φ_ε and β^ε are differentiable, it follows by a standard device that there exists $p_\varepsilon\in L^2(0,T;H^1(\Omega))\cap C([0,T];L^2(\Omega))$ with $\frac{dp_\varepsilon}{dt}\in L^2(0,T;(H^1(\Omega))')$ such that

$$(p_\varepsilon)_t - Ap_\varepsilon = \ell(y_\varepsilon - y_d) \quad \text{in } Q$$

(4.4) $$\frac{\partial p_\varepsilon}{\partial \nu} + \dot\beta^\varepsilon(y_\varepsilon)p_\varepsilon = 0 \quad \text{in } \Sigma.$$

$$p_\varepsilon(\cdot,T) + \partial\varphi_\varepsilon(y_\varepsilon(\cdot,T)) = 0 \quad \text{in } \Omega$$

(4.5) $$p_\varepsilon(\sigma,t) = \partial g_\varepsilon(\sigma,u_\varepsilon(\sigma,t)) + u_\varepsilon(\sigma,t) - u^*(\sigma,t) \quad \text{a.e. } (\sigma,t)\in\Sigma.$$

Here $\dot\beta^\varepsilon(y) = \frac{d}{dy}\beta^\varepsilon(y)$.

LEMMA 1. For $\varepsilon \longrightarrow 0$ we have

(4.6) $\quad u_\varepsilon \longrightarrow u^*$ strongly in $L^2(\Sigma)$

(4.7) $\quad y_\varepsilon \longrightarrow y^*$ strongly in $C([0,T];\, L^2(\Omega))$
and weakly in $L^2(0,T;H^1(\Omega))$.

Proof. We have

(4.8)
$$\frac{1}{2}\int_Q \ell |y_\varepsilon - y_d|^2\, dxdt + \int_\Sigma (g_\varepsilon(\sigma,u_\varepsilon) + \frac{1}{2}|u_\varepsilon - u^*|^2)\, d\sigma\, dt + \varphi_\varepsilon(y_\varepsilon(T)) \leq$$
$$\leq \frac{1}{2}\int_Q \ell |z_\varepsilon - y_d|^2 dydt + \int_\Sigma g_\varepsilon(\sigma,u^*)\, d\sigma\, dt + \varphi_\varepsilon(z_\varepsilon(T))$$

where z_ε is the solution to

$$(z_\varepsilon)_t + Az_\varepsilon = 0 \quad \text{in } Q$$
$$\frac{\partial z_\varepsilon}{\partial \nu} + \beta^\varepsilon(z_\varepsilon) = u^* + f \quad \text{in } \Sigma$$
$$z_\varepsilon(0) = y_0 \quad \text{in } \Omega.$$

By Proposition 1 it follows that

(4.9) $\quad z_\varepsilon \longrightarrow y^*$ strongly in $C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$.

Then by (4.2) we have

(4.10)
$$\limsup_{\varepsilon \to 0} \Big(\frac{1}{2}\int_Q \ell |y_\varepsilon - y_d|^2 dxdt + \int_\Sigma (g_\varepsilon(\sigma,u_\varepsilon) + \frac{1}{2}|u_\varepsilon - u^*|^2\, d\sigma\, dt +$$
$$+ \varphi_\varepsilon(y_\varepsilon(T))\Big) \leq \frac{1}{2}\int_Q \ell |y^* - y_d|^2 + \int_\Sigma g(\sigma,u^*)\, d\sigma\, dt + \varphi(y^*(T)).$$

In particular it follows that $\{u_\varepsilon\}$ remain in a bounded subset of $L^2(\Sigma)$. Thus selecting a subsequence we may assume that

(4.11) $\quad u_\varepsilon \longrightarrow \tilde{u}^*$ weakly in $L^2(\Sigma)$.

Then by estimate (2.15) we may also suppose that

$$(4.12)\qquad \begin{aligned} y_\varepsilon &\longrightarrow \tilde{y}^* \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \\ (y_\varepsilon)_t &\longrightarrow \tilde{y}^*_t \quad \text{weakly in } L^2(0,T;(H^1(\Omega))'). \end{aligned}$$

In particular it follows that $\tilde{y}^*$ is the solution to Eq.(1.1) where $u = \tilde{u}^*$ and

$$y_\varepsilon(\cdot,T) \longrightarrow \tilde{y}^*(\cdot,T) \text{ weakly in } L^2(\Omega).$$

Since (y^*,u^*) is a minimum point for functional (4.1) and by (4.11)

$$\liminf_{\varepsilon\to 0} \int_\Sigma g_\varepsilon(\sigma,u_\varepsilon)\,d\sigma\,dt \geq \int_\Sigma g(\sigma,\tilde{u}^*)\,d\sigma\,dt$$

(this is an easy consequence of the weak lower semicontinuity of the convex integrand) by (4.10) it follows (4.6). Now using estimates (2.8) and (2.9) where $u = u^\varepsilon$ and arguing as in the proof of Proposition 1 we obtain (4.7).

We notice for later use that by (2.15) and (4.6) it follows that

(4.13) $\{\beta^\varepsilon(y_\varepsilon)\}$ is bounded in $L^2(\Sigma)$ and therefore on some subsequence $\beta^\varepsilon(y_\varepsilon) \longrightarrow u^*+f-\dfrac{\partial y^*}{\partial\nu}$ weakly in $L^2(\Sigma)$.

LEMMA 2 <u>There exists a positive constant</u> C <u>independent of</u> <u>such that</u>

$$(4.14)\qquad \|p_\varepsilon(t)\|_{L^2(\Omega)} + \|p_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq C\,, \quad t\in[0,T]$$

$$(4.15)\qquad \|(p_\varepsilon)_t\|_{L^2(0,T;(H^1(\Omega))')} \leq C$$

$$(4.16)\qquad \int_\Sigma |\dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon|\,d\sigma\,dt \leq C.$$

<u>Proof</u>. Multiplying Eq. (4.4) by p_ε and integrating over

$]t,T[\times\Omega = Q_t$ one gets

(4.17) $$\frac{1}{2}\|p_\varepsilon(t)\|^2_{L^2(\Omega)} + \int_t^T a(p_\varepsilon, p_\varepsilon)ds \leq \frac{1}{2}\|p_\varepsilon(T)\|^2_{L^2(\Omega)} + \int_{Q_t} \ell(y_\varepsilon - y_d)p_\varepsilon\, dxdt.$$

Let γ be a C^1 - approximation to sgn. We multiply Eq.(4.4) by $\gamma(p_\varepsilon)$, use Green's formula and let γ tend to sgn. We have

(4.18) $$\int_\Sigma \dot{\beta}^\varepsilon(y_\varepsilon)\, p_\varepsilon\, d\sigma dt \leq \int_Q |y_\varepsilon - y_d|\, \ell\, dxdt + \int_\Omega |p_\varepsilon(T,x)|\, dx.$$

Next by definition of $\partial\varphi_\varepsilon$ one has

$$-(p_\varepsilon(T), y_\varepsilon(T) - y^*(T) - \rho v) \geqslant \varphi_\varepsilon(y_\varepsilon(T)) - \varphi_\varepsilon(y^*(T) + \rho v)$$

for all $v \in L^2(\Omega)$, $\|v\|_{L^2(\Omega)} \leq 1$ and some $\rho > 0$. Since $y_\varepsilon(T) \longrightarrow y^*(T)$ in $L^2(\Omega)$ and φ is locally bounded on $L^2(\Omega)$, it follows that $\{p_\varepsilon(T)\}$ is bounded in $L^2(\Omega)$. Along with estimates (4.17) and (4.18) the latter implies (4.14), (4.15) and (4.16) as claimed.
In particular, it follows by (4.14) and (4.15) that $\{p_\varepsilon\}$ is precompact in $L^2(0,T;H^{1-\delta}(\Omega))$ for any $0 < \delta < 1$. By the "trace" theorem we may therefore conclude that $\{p_\varepsilon\}$ is compact in $L^2(\Sigma)$.
Hence extracting a sequence (again denoted ε) it follows by Lemma 2 that there exists some function $p \in L^2(0,T;H^1(\Omega))$ with $p_t \in L^2(0,T;(H^1(\Omega))')$ such that

(4.19) $$p_\varepsilon \longrightarrow p \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and strongly in } L^2(0,T;H^{1-\delta}(\Omega))$$

(4.20) $$(p_\varepsilon)_t \longrightarrow p_t \quad \text{weakly in } L^2(0,T;(H^1(\Omega))')$$

(4.21) $$p_\varepsilon \longrightarrow p \quad \text{strongly in } L^2(\Omega).$$

In particular it follows by (4.19) and (4.20) that $p \in C([0,T];L^2(\Omega))$ and

$$(4.22) \qquad p_\varepsilon(t) \longrightarrow p(t) \text{ weakly in } L^2(\Sigma) \text{ for every } t \in [0,T].$$

Finally, estimate (4.16) shows that the set $\{\dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon\}$ is bounded in $L^1(\Sigma)$ and therefore it is weak-star compact in the space $M(\Sigma)$. Thus there exists $\mu_p \in M(\Sigma)$ such that on some subsequence $\varepsilon \longrightarrow 0$,

$$(4.23) \qquad \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon \longrightarrow \mu_p \text{ weak star in } M(\Sigma).$$

Letting ε tend to zero in Eq.(4.4) it follows by (4.19) ~ (4.23) that p is the solution to

$$p_t - Ap = y^* - y_d \quad \text{in } Q$$

$$(4.24) \qquad \frac{\partial p}{\partial \nu} + \mu_p = 0 \quad \text{in } \Sigma$$

$$p(.,T) + \partial\varphi(y^*(.,T)) \ni 0 \quad \text{in } \Omega \;,$$

while by (4.5) we have

$$(4.25) \qquad p(\sigma,t) \in \partial g(\sigma,u^*(\sigma,t)) \quad \text{a.e. } (\sigma,t) \in \Sigma.$$

Eq.(4.24) can be interpreted either in the weak sense or (see (2.4))

$$(4.26) \qquad \frac{d}{dt}(p(t),\psi) - a(p(t),\psi) + \mu_p(\psi) = \int_Q 1(y^* - y_d)\psi \, dxdt$$

$$p(0,T) \in -\partial\varphi(y^*(.,T)), \quad \forall \psi \in C^1(\bar{\Omega})$$

where $(.,.)$ is the inner product of $L^2(\Omega)$ and $\mu_p(\psi)$ is the value of μ_p at $\gamma_o\psi$ (the trace of ψ on Σ).

5. PROOF OF THEOREM 1

If β is a locally Lipschitzian function on R, the generalized gradient $\partial\beta$ in the sense of Clarke of β is defined by (see [6], [11])

$$(5.1) \qquad \partial\beta(y) = \text{conv}\left\{ z \in R;\ z = \lim_{y_n \to y} \dot{\beta}(y_n) \right\} \quad y \in R.$$

(By $\dot{\mathcal{f}}$ we shall denote the ordinary derivative of $\mathcal{f}$).

Let $\mathcal{f}^{\varepsilon}$ be the function defined by (2.5), i.e.,

$$(5.2) \qquad \mathcal{f}^{\varepsilon}(y) = \int_{-\infty}^{\infty} \beta_{\varepsilon}(y-\varepsilon\theta)\,\rho(\theta)\,d\theta\,, \qquad y \in R$$

where $\beta_{\varepsilon} = \varepsilon^{-1}(1-(1+\varepsilon\beta)^{-1})$.

We begin the proof of Theorem 1 with the following technicall lemma (see [2])

LEMMA 3. <u>Let</u> E <u>be a locally compact space and let</u> ν <u>a positive measure on</u> E <u>such that</u> $\nu(E) < \infty$. <u>Let</u> $\{y_{\varepsilon}\} \subset L^1(E)$ <u>be a sequence such that for</u> $\varepsilon \longrightarrow 0$.

$$(5.3) \qquad y_{\varepsilon} \longrightarrow y \quad \text{strongly in } L^1(E) \text{ and}$$

$$(5.4) \qquad \dot{\mathcal{f}}^{\varepsilon}(y_{\varepsilon}) \longrightarrow g \quad \text{weakly in } L^1(E).$$

<u>Then</u>

$$(5.5) \qquad g(x) \in \partial \mathcal{f}(y(x)) \quad \nu\text{-a.e. } x \in E.$$

<u>Proof</u>. By $L^1(E)$ we have denoted the space of all real-valued ν-measurable functions $y(x)$ defined ν-a.e. on E such that

$$|y(x)| \text{ is } \nu\text{-integrable over E.}$$

Selecting a subsequence of $\{y_{\varepsilon}\}$ we may assume that

$$(5.6) \qquad y_{\varepsilon}(x) \longrightarrow y(x) \quad \nu\text{-a.e.} \quad x \in E.$$

Next by (5.4) and the Mazur theorem it follows that

$$(5.7) \qquad g = \lim_{m \to \infty} g_m \quad \text{strongly in } L^1(E)$$

where $\{g_m\} \subset L^1(E)$ are of the form

$$(5.8) \qquad g_m = \sum_{i \in I_m} \alpha_m^i \dot{\beta}^{\varepsilon_i}(y_{\varepsilon_i}).$$

Here I_m is a finite subset of natural numbers in the interval $[m, \infty[$ and $\alpha_m^i \geq 0$, $\sum_{i \in I_m} \alpha_m^i = 1$ are real numbers.

According to (5.7) we may also assume without any loss of generality that

$$(5.9) \qquad g_m(x) \longrightarrow g(x) \quad \nu\text{- a.e. } x \in E.$$

We fix $x \in E$ such that (5.6) and (5.9) hold, and consider a sequence $\{z_n\}$ of real numbers such that $\dot{\beta}(z_n)$ exist and $z_n \longrightarrow y(x)$ for $n \longrightarrow \infty$. We set $y_i = y_{\varepsilon_i}(x)$ and notice that by (5.2) we have

$$(5.10) \qquad \dot{\beta}^{\varepsilon_i}(y_i) = \varepsilon_i^{-1} \int_{-\infty}^{\infty} \beta_{\varepsilon_i}(y_i - \varepsilon_i\theta)\, \dot{\rho}(\theta)\, d\theta.$$

On the other hand, we have

$$\beta(z_i) = \beta((1+\varepsilon_i\beta)^{-1}(y_i - \varepsilon_i\theta)) + \dot{\beta}(z_i)(z_i - (1+\varepsilon_i\beta)^{-1}(y_i - \varepsilon_i\theta)) + \omega_i(\theta)(z_i - (1+\varepsilon_i\beta)^{-1}(y_i - \varepsilon_i\theta))$$

where $\omega_i(\theta) \longrightarrow 0$ for $\delta_i = z_i - (1+\varepsilon_i\beta)^{-1}(y_i - \varepsilon_i\theta) \longrightarrow 0$.

Along with (5.10) the latter yields

$$(5.11) \qquad \dot{\beta}^{\varepsilon_i}(y_i) = \dot{\beta}(z_i) - \dot{\beta}(z_i) \int_{-\infty}^{\infty} \beta_{\varepsilon_i}(y_i - \varepsilon_i\theta)\, \dot{\rho}(\theta)\, d\theta - \varepsilon_i^{-1} \int_{-\infty}^{\infty} \omega_i(\theta)(z_i - (1+\varepsilon_i\beta)^{-1}(y_i - \varepsilon_i\theta))\, \dot{\rho}(\theta)\, d\theta.$$

Since β is locally Lipschitzian, it follows by (5.6) that $\beta_{\varepsilon_i}(y_i - \varepsilon_i\theta) \longrightarrow \beta(y(x))$ uniformly on $[-1,1]$.

On the other hand, z_i can be chosen sufficiently close to y_i in a such a way,

$$|y_i - z_i|/\varepsilon_i \longrightarrow 0 \quad \text{for } i \longrightarrow \infty .$$

Then $\delta_i \longrightarrow 0$ for $i \longrightarrow \infty$ and equality (5.11) yields

$$|\dot{\beta}^{\varepsilon_i}(y_i) - \dot{\beta}(z_i)| \longrightarrow 0 \text{ for } i \longrightarrow \infty .$$

Along with (5.1) and (5.8) the latter implies $g(x) \in \partial\beta(y(x))$ as claimed.

Now we continue the proof of Theorem 1 by observing that by condition (3.3) it follows after some calculation involving formula (5.2),

$$(5.12) \qquad |\dot{\beta}^{\varepsilon}(y)y| \leq C(|\beta^{\varepsilon}(y)| + y^2 + 1)$$

for all $y \in R$ and $\varepsilon > 0$, where C is a positive constant independent of ε.

For each $\varepsilon > 0$ and natural number n one defines

$$E_n^{\varepsilon} = \{(x,t) \in \Sigma ; |y_{\varepsilon}(x,t)| \leq n\} .$$

Since β is locally Lipschitzian we have

$$|\dot{\beta}^{\varepsilon}(y_{\varepsilon}(x,t))| \leq C_n \quad \text{for} \quad (x,t) \in E_n^{\varepsilon} .$$

Let Σ_0 be an arbitrary measurable subset of Σ. We have

$$(5.13) \quad \int_{\Sigma_0} |p_{\varepsilon}(x,t)||\dot{\beta}^{\varepsilon}(y_{\varepsilon}(x,t))| dxdt \leq \int_{\Sigma_0 \cap E_n^{\varepsilon}} |p_{\varepsilon}|(x,t)|\dot{\beta}^{\varepsilon}(y_{\varepsilon}(x,t))| dxdt +$$

$$+ \int_{\Sigma_0 \cap CE_n^{\varepsilon}} |p_{\varepsilon}|(x,t)|\dot{\beta}^{\varepsilon}(y_{\varepsilon}(x,t))| dxdt \leq C_n \int_{\Sigma_0} |p_{\varepsilon}(x,t)| dxdt +$$

$$+ Cn^{-1} \int_{E_n^{\varepsilon} \cap \Sigma_0} |\beta^{\varepsilon}(y_{\varepsilon}(x,t))| \, |p_{\varepsilon}(x,t)| dxdt + C \int_{E_n^{\varepsilon} \cap \Sigma_0} |y_{\varepsilon}(x,t)| dxdt + Cn^{-1} .$$

Since by (4.7), (4.13) and (4.14), $\{y_{\varepsilon}\}$, $\{\beta^{\varepsilon}(y_{\varepsilon})\}$ and $\{p_{\varepsilon}\}$ are bounded in $L^2(\Sigma)$, it follows by inequality (5.13) that the family

$\left\{ \int_{\Sigma_0} p_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon)\,dxdt \right\}$ is equicontinuous. Hence by the Dunford-Pettis criterion, the family $\{p_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon)\}$ is weakly compact in $L^1(\Sigma)$. Then by (4.23) it follows that $\mu_p \in L^1(\Sigma)$ and

(5.14) $$\dot{\beta}^\varepsilon(y_\varepsilon)\, p_\varepsilon \longrightarrow \mu_p \quad \text{weakly in } L^1(\Sigma).$$

On the other hand it follows by (4.7) that $\{y_\varepsilon\}$ is bounded in $L^2(0,T;H^{1/2}(\Gamma))$ and therefore $\{y_\varepsilon\}$ is compact in $L^2(0,T;L^2(\Gamma))$. Thus selecting a subsequence if necessary we have

$$y_\varepsilon(x,t) \longrightarrow y^*(x,t) \quad \text{a.e. } (x,t) \in \Sigma$$

and by Egorov's theorem, for each $\eta > 0$ there exists a measurable subset $E_\eta \subset \Sigma$ such that $m(\Sigma \setminus E_\eta) \le \eta$ (m is the measure on Σ induced by the Lebesgue measure $dxdt$), y_ε is bounded on E_η and

(5.15) $$y_\varepsilon(x,t) \longrightarrow y^*(x,t) \text{ uniformly on } E_\eta.$$

Next, since $\{\dot{\beta}^\varepsilon(y_\varepsilon)\}$ are uniformly bounded on E_η we may assume (extracting further subsequence) that

(5.16) $$\dot{\beta}^\varepsilon(y_\varepsilon) \longrightarrow g \quad \text{weakly in } L^1(E_\eta)$$

(actually weak-star in $L^\infty(E_\eta)$). Then by Lemma 3 it follows that

$$g(x,t) \in \partial\beta(y^*(x,t)) \quad \text{a.e. } (x,t) \in E_\eta .$$

Now by (4.14) it follows that $\{p_\varepsilon\}$ is compact in $L^2(\Sigma)$. Then again by the Egorov theorem we may assume that $p_\varepsilon \longrightarrow p$ uniformly on E_η. Along with (5.15) and (5.16) the latter implies that $\mu_p = g\,p$ on E_η. Hence

$$\mu_p(x,t) \in p(x,t)\,\partial\beta(y^*(x,t)) \qquad \text{a.e. } (x,t) \in E_\eta .$$

Since $m(\Sigma \setminus E_\eta)$ can be made arbitrarily small we may conclude that

$$\mu_p(x,t) \in p(x,t)\,\partial\beta(y^*(x,t)) \quad \text{a.e. } (x,t) \in \Sigma.$$

Thus the conclusions of Theorem 1 follow by Eqs.(4.24), (4.25).

<u>Remark</u>. Arguing as in the proof of Theorem 3 in [2] it follows that for a general locally Lipschitzian mapping one has,

$$(5.17)\qquad (\mu_p)_a\,(x,t) \in p(x,t)\,\partial\beta(y^*(x,t)) \quad \text{a.e.}\,(x,t)\in\Sigma$$

where $(\mu_p)_a \in L^1(\Sigma)$ is the absolutely continuous part of the measure μ_p.

6. PROOF OF THEOREM 2

If β is the graph defined by (3.15) then $\beta_\varepsilon(y) = -\varepsilon^{-1}y^-$ and $\beta^\varepsilon(y) = \varepsilon^{-1}\int_{\varepsilon^{-1}y}^{\infty}(y-\varepsilon\theta)\rho(\theta)\,d\theta$ for $\varepsilon>0$, $y\in R$ respectively,

$$\dot\beta^\varepsilon(y) = \varepsilon^{-1}\int_{\varepsilon^{-1}y}^{\infty}\rho(\theta)\,d\theta .$$

Hence

$$(6.1)\qquad |y_\varepsilon\dot\beta^\varepsilon(y_\varepsilon)p - p_\varepsilon\beta^\varepsilon(y_\varepsilon)| = \Big|p_\varepsilon\int_{\varepsilon^{-1}y_\varepsilon}^{\infty}\rho(\theta)\,d\theta\Big| \le |\dot\beta^\varepsilon(y_\varepsilon)p_\varepsilon|\,\varepsilon .$$

On the other hand, arguing as in [1] we find that

$$(6.2)\qquad |p_\varepsilon\beta^\varepsilon(y_\varepsilon)| \le 2\varepsilon|\dot\beta^\varepsilon(y_\varepsilon)p_\varepsilon|\,(\zeta_\varepsilon + \varepsilon^{-1}|y_\varepsilon|\gamma_\varepsilon) \quad \text{a.e. on } \Sigma$$

where

$$\zeta_\varepsilon(\sigma,t) = \begin{cases} 0 & \text{if } |y_\varepsilon(\sigma,t)| > \varepsilon \\ 1 & \text{if } |y_\varepsilon(\sigma,t)| \le \varepsilon \end{cases}$$

and

$$\gamma_\varepsilon(\sigma,t) = \begin{cases} 0 & \text{if } y_\varepsilon(\sigma,t) > -\varepsilon \\ 1 & \text{if } y_\varepsilon(\sigma,t) \le -\varepsilon . \end{cases}$$

Inasmuch as by (3.14) $\{\beta^\varepsilon(y_\varepsilon)\}$ is bounded in $L^2(\Sigma)$ and by Lemma 2 $\{\dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon\}$ is bounded in $L^1(\Sigma)$ we see by (6.2) that on some subsequence $\varepsilon \longrightarrow 0$ we have

(6.3) $$p_\varepsilon \beta^\varepsilon(y_\varepsilon) \longrightarrow 0 \quad \text{a.e. on } \Sigma .$$

On the other hand, we know that $\beta^\varepsilon(y_\varepsilon) \longrightarrow u^*-f-\frac{\partial y^*}{\partial \nu}$ weakly in $L^2(\Sigma)$ and $p_\varepsilon \longrightarrow p$ strongly in $L^2(\Sigma)$. This implies that the sequence $\{p_\varepsilon \beta^\varepsilon(y_\varepsilon)\}$ is weakly convergent in $L^1(\Sigma)$ to $p(u^* -f-\frac{\partial y^*}{\partial \nu})$ and by (6.3) it follows that

(6.4) $$p(u^* -f-\frac{\partial y^*}{\partial \nu}) = 0 \quad \text{a.e. on } \Sigma$$

and therefore

$$p_\varepsilon \beta^\varepsilon(y_\varepsilon) \longrightarrow 0 \text{ strongly in } L^1(\Sigma).$$

Then by (6.1) we see that

(6.5) $$y_\varepsilon \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon \longrightarrow 0 \text{ strongly in } L^1(\Sigma).$$

Next by the Egorov theorem, for each $\eta > 0$, $\exists\, E_\eta$ a measurable subset of Σ such that $m(\Sigma \setminus E_\eta) \le \eta$, $y_\varepsilon \longrightarrow y^*$ uniformly on E_η and y^* is continuous on E_η (m denotes the Lebesgue measure). Along with (6.5) the latter yields

(6.6) $$\lim_{\varepsilon \to 0} y^* \dot{\beta}^\varepsilon(y_\varepsilon)p_\varepsilon = 0 \text{ strongly in } L^1(E_\eta).$$

Denote by $E_{\eta,\delta}$ the following subset of Σ

$$E_{\eta,\delta} = \{(\sigma,t) \in E_\eta \,; |y^*(\sigma,t)| \ge \delta\} .$$

By (6.6) it follows that for each $\delta > 0$, $\mu_p = 0$ on $E_{\eta,\delta}$ and therefore $(\mu_p)_a = 0$ on $E_\eta \cap \{(\sigma,t)\in\Sigma;\ y^*(\sigma,t) \neq 0\}$. Since $m(\Sigma\setminus E_\eta) \longrightarrow 0$ for $\eta \longrightarrow 0$ we may conclude that

$$(\mu_p)_a = 0 \quad \text{on} \quad \{(\sigma,t)\in\Sigma;\ y^*(\sigma,t) > 0\}.$$

Along with Eqs. (4.24), (4.25) and (6.4) the latter complets the proof of Theorem 2.

<u>Remark</u>. If $\{y_\varepsilon\}$ is a compact subset of $C(\bar{Q})$ then it follows by (6.5) that $\mu_p\, y^* = 0$ and therefore (3.20) becomes

$$\frac{\partial p}{\partial \nu} = 0 \quad \text{on} \quad \{y^* > 0\}.$$

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