

INSTITUTUL
DE
MATEMATICA

INSTITUTUL NATIONAL
PENTRU CREATIE
STIINTIFICA SI TEHNICA

ISSN 0250-3638

MIXED PROBLEMS FOR A CLASS OF NONLINEAR
DIFFERENTIAL HYPERBOLIC SYSTEMS

by

Gheorghe MOROSANU

PREPRINT SERIES IN MATHEMATICS

No.24/1980

BUCURESTI

MIXED PROBLEMS FOR A CLASS OF NONLINEAR
DIFFERENTIAL HYPERBOLIC SYSTEMS

by
Gheorghe MOROȘANU*)

May 1980

*) Faculty of Mathematics, Iasi University, Iasi 6600, Romania

mea 16742

MIXED PROBLEMS FOR A CLASS OF NONLINEAR
DIFFERENTIAL HYPERBOLIC SYSTEMS

GHEORGHE MOROȘANU

Faculty of Mathematics, Iași University, Iași 6600, Romania

1. INTRODUCTION

It is the purpose of the present paper to investigate non-linear differential hyperbolic systems of the form

$$\begin{aligned} \alpha(x) \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} + Au &\ni f(t, x), \\ \beta(x) \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + Bv &\ni g(t, x), \end{aligned} \quad (1.1)$$

for $0 < x < 1$ and $t > 0$

with the initial data

$$u(0, x) = u_0(x); \quad v(0, x) = v_0(x), \quad 0 \leq x \leq 1 \quad (1.2)$$

and the boundary-value conditions

$$(u(t, 0), -u(t, 1)) \in L(v(t, 0), v(t, 1)), \quad t \geq 0. \quad (1.3)$$

For the sake of brevity we simply begin with the enumeration of some basic assumptions which will be used in that follows.

(H₁) Both A and B are maximal monotone graphs in $R \times R$, $R =]-\infty, \infty[$. In other words, there exist two lower semicontinuous proper convex functions $\varphi, \psi : R \rightarrow]-\infty, \infty]$ such that

$A = \partial\varphi$, $B = \partial\psi$. Here $\partial\varphi$ and $\partial\psi$ denote the subdifferentials of φ and ψ . Moreover, we assume that

$$(1) \quad D(\psi) = R, \text{ where } D(\psi) = \{r \in R; \psi(r) < +\infty\}$$

(so in particular ψ is continuous on R).

(11) There exists a function $u^* \in H^1(0,1)$ such that $(u^*(0), u^*(1)) \in R(L)$ (the range of L) and $u^*(x) \in \text{Int } D(\varphi)$ (interior of $D(\varphi)$), for all $x \in [0,1]$. Implicitly, we have assumed $\text{Int } D(\varphi) \neq \emptyset$, that is $D(\varphi)$ is not a singleton.

(H₂) The graph of the (multivalued) function $L: R^2 \rightarrow R^2$, $R^2 = R \times R$ is maximal monotone in $R^2 \times R^2$.

It is well-known that many boundary conditions can be expressed in the general form (1.3). For example, if $L = \partial \ell$, where

$\ell: R^2 \rightarrow]-\infty, \infty]$ is defined by

$$\ell(x,y) = 0, \text{ for } x=a, y=b; \quad a, b \in R \text{ and } = \infty, \text{ otherwise} \quad (1.4)$$

then (1.3) becomes:

$$v(t,0)=a, \quad v(t,1)=b \quad (\text{two-point boundary conditions}).$$

Taking L as the subdifferential of the indicator function of the first bisectrix of the plane we obtain periodic boundary conditions.

(H₃) The functions α and β belong to $L^\infty(0,1)$ and besides there exists some constant $C > 0$ such that $\alpha(x) \geq C$, $\beta(x) \geq C$, for a.e. $x \in]0,1[$.

In the previous papers [3, 4, 12, 13] it has been discussed the existence, regularity and asymptotic behaviour of solutions for hyperbolic systems of the form (1.1) where A and B were assumed to be single-valued functions, $A=A(x,u)$, $B=B(x,v)$ from $]0,1[\times R$ into R that satisfy conditions of Caratheodory type. It is pointed out there the physical significance of such systems.

The multivalued case presented in this paper is motivated by an example given in the last section and perhaps there are many other physical meanings of the problem.

The plan of the paper is the following. In Section 2 we shall state and prove an existence and regularity result (Theorem 2.1). The solution obtained satisfies the system in a generalized sense which will be precised in the statement of Theorem 2.1. In Section 3 we shall discuss on the asymptotic behaviour of solutions.

The last section is devoted to an example of interest in electrical network theory.

2. EXISTENCE AND REGULARITY OF SOLUTIONS

Assume familiarity with the notation, concepts, and basic results of the theory of monotone operators and differential

equations developed on general Banach spaces. All the results in this field we shall use in the sequel without particular references can be found in the books [2, 5, 6]. We also refer the reader e.g. to [2, 5] for the usual notation of function spaces and $W^{k,p}$ spaces. However, we recall that $BV(0,1)$ is the space of all real-valued functions of bounded variation defined on $[0,1]$. $M(0,1)$ denotes the space of all Radon measures over $[0,1]$, i.e. the dual space of $C[0,1]$. It is well-known that every function $v \in BV(0,1)$ generates a measure Dv given by

$$Dv(h) = \int_0^1 h(x) dv(x), \quad \forall h \in C[0,1]$$

and, conversely, every measure can be expressed by means of the Stieltjes integral associated to some function of bounded variation.

In keeping with the notations introduced in Section 1 we define the operator $\tilde{A} : C[0,1] \rightarrow M(0,1)$ by

$$\tilde{A}u = \left\{ \mu \in M(0,1); \mu(u-v) \geq \int_0^1 \varphi(u(x))dx - \int_0^1 \varphi(v(x))dx, \text{ for every } v \in C[0,1] \right\}, \quad (2.1)$$

with

$$D(\tilde{A}) = \{ u \in C[0,1]; \tilde{A}u \neq \phi \}. \quad (2.2)$$

The realization of B on $L^2(0,1)$ will be denoted by \bar{B} . In other

words, \bar{B} represents the subdifferential of the function $\Psi: L^2(0,1)$

$\rightarrow]-\infty, +\infty]$ defined by

$$\Psi(v) = \int_0^1 \psi(v(x)) dx, \text{ if } \psi(v) \in L^1(0,1), \text{ and} \\ = +\infty, \text{ otherwise.}$$

Let X denote the product space $L^2(0,1) \times L^2(0,1)$ endowed with

the usual scalar product and Hilbertian norm. We recall that

$L^2(0,1)$ is a space of equivalence classes, $L^2(0,1) = \mathcal{L}^2(0,1)/\approx$,

where $\mathcal{L}^2(0,1)$ is the space of all functions $u: [0,1] \rightarrow \mathbb{R}$, of

square Lebesgue summable, and as usual $u \approx v$ iff $u(x) = v(x)$, a.e.

$x \in]0,1[$ (with respect to the Lebesgue measure).

Define the multivalued operator $\mathcal{A}: X \rightarrow X$ by: $D(\mathcal{A}) =$

$= \{(u,v) \in X; u \in H^1(0,1), \text{ the equivalence class } v \text{ contains at}$

least a function $v_1 \in BV(0,1)$ and there exists $\mu \in \tilde{\mathcal{A}}u$ such that

$(u(0), -u(1)) \in L(v_1(0), v_1(1))$ and $\mu - Dv_1 \in L^2(0,1)\}$. (2.3)

For each (u,v) fixed in $D(\mathcal{A})$, let $K_{uv} = \{Dv_1\}$, i.e. the set of

all measures Dv_1 given in (2.3). Then, let

$$\mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (-K_{uv} + \tilde{\mathcal{A}}u) \cap L^2(0,1) \\ -u' + \bar{B}v \end{pmatrix}, \text{ for all } (u,v) \in D(\mathcal{A}). \quad (2.4)$$

The space $L^2(0,1)$ is identified with its own dual. The derivative

u' is understood in the sense of distributions. If $h \in L^2(0,1)$

and h contains a function $h_1 \in C[0,1]$ then h is identified with h_1 so that in particular $h(0)$ and $h(1)$ make sense, i.e. $h(0)=h_1(0)$ and $h(1)=h_1(1)$. Then, one has

$$C_0^\infty(0,1) \subset C[0,1] \subset L^2(0,1) \subset M(0,1) \subset \mathcal{D}'(0,1),$$

algebraically and topologically. The fact that $\mu - Dv_1 \in L^2(0,1)$ means that this measure can be extended to a functional belonging to the dual space of $L^2(0,1)$. We also note that the restriction of Dv_1 to $C_0^\infty(0,1)$ coincides to the distribution v' . A natural question is how many elements there are in K_{uv} (see (2.3), (2.4)).

So let $v_1, v_2 \in BV(0,1)$ belong to the equivalence class v . It is immediate that the function $v_1 - v_2$ is identically equal to zero on the set of its continuity points. Then, according to [14, p. 111] it follows that v_1 and v_2 generate the same measure, i.e. $Dv_1 = Dv_2$, if and only if $v_1(0) = v_2(0)$ and $v_1(1) = v_2(1)$. Therefore the set K_{uv} may generally have more than one element. However, if $L^{-1}(u(0), -u(1))$ is a singleton then K_{uv} is a singleton too.

The following lemma is essential in our treatment.

LEMMA 2.1. Assume that hypotheses (H_1) and (H_2) hold. Then, the operator \mathcal{A} defined by (2.3) and (2.4) is maximal monotone.

Proof. First of all, we note that $D(\mathcal{A})$ is nonempty. Indeed, let $v^* \in H^1(0,1)$ such that $(v^*(0), v^*(1)) \in D(L)$ and $(u^*(0), -u^*(1)) \in L(v^*(0), v^*(1))$, where u^* is the function appearing in hypothesis (H_1) (ii). By an elementary argument involving (H_1) it follows that the set $\bigcup_{0 \leq x \leq 1} A u^*(x)$ is bounded. So it is apparent that $(u^*, v^*) \in D(\mathcal{A})$.

The operator \mathcal{A} is monotone. The verification of this fact is only a simple exercise involving the hypotheses and the following formula for integration by parts

$$Dv(h) = v(1)h(1) - v(0)h(0) - \int_0^1 v(x) h'(x) dx, \quad (2.5)$$

for $v \in BV(0,1)$ and $h \in W^{1,1}(0,1)$.

In order to prove the maximality of \mathcal{A} we shall show that, for each pair $(p,q) \in X$, there exists $(u,v) \in D(\mathcal{A})$ such that

$$\begin{aligned} u - K_{uv} + \tilde{A}u &\ni p, \\ v - u' + \bar{B}v &\ni q. \end{aligned} \quad (2.6)$$

It is easy to show (see [13]) that the operator $T : X \rightarrow X$ defined by

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v' \\ -u' \end{pmatrix}, \text{ for every } (u,v) \in D(T), \quad (2.7)$$

$$D(T) = \left\{ (u,v) \in H^1(0,1) \times H^1(0,1); \begin{aligned} &(u(0), -u(1)) \\ &\in L(v(0), v(1)) \end{aligned} \right\} \quad (2.8)$$

is maximal monotone on X . Consequently, for each $\lambda > 0$, the following approximating boundary-value problem

$$u_\lambda - v'_\lambda + A_\lambda u_\lambda = p, \quad \text{a.e. } x \in]0, 1[, \quad (2.9)$$

$$v_\lambda - u'_\lambda + B_\lambda v_\lambda = q, \quad \text{a.e. } x \in]0, 1[,$$

$$(u_\lambda(0) - u_\lambda(1)) \in L(v_\lambda(0), v_\lambda(1)) \quad (2.10)$$

has a unique solution $(u_\lambda, v_\lambda) \in H^1(0, 1) \times H^1(0, 1)$. Here A_λ and B_λ are the Yosida approximations of A and B respectively (see e.g. [2, p. 58]). According to a standard device we do not reproduce here it follows by (2.9) and (2.10) that

$$\{u_\lambda\} \text{ and } \{v_\lambda\} \text{ are bounded in } L^2(0, 1). \quad (2.11)$$

The system (2.9) can be put into the equivalent form

$$\begin{aligned} u'_\lambda &= \partial \psi_\lambda(v_\lambda) + v_\lambda - q, \quad \text{a.e. } x \in]0, 1[, \\ u_\lambda &= \partial \varphi_\lambda^*(v'_\lambda + p - u_\lambda), \quad \text{a.e. } x \in]0, 1[, \end{aligned} \quad (2.12)$$

where φ_λ and ψ_λ mean the regularized functions associated to φ and ψ , respectively, while φ_λ^* is the conjugate function of φ_λ (see e.g. [2, pp. 52, 57]). In order to obtain an estimate for v'_λ it is useful to define as in [4] the following function

$$P_\lambda : R \times R \rightarrow]-\infty, +\infty],$$

$$P_\lambda(r, s) = \psi_\lambda(r) + \frac{1}{2} |r - q|^2 + \varphi_\lambda^*(s + p - u_\lambda). \quad (2.13)$$

Of course, P_λ also depends of x by means of the functions p, q

and u_λ but for the sake of simplicity the variable x is omitted.

It is apparent from (2.12) and (2.13) that

$$(u'_\lambda, u_\lambda) \in \partial P_\lambda(v_\lambda, v'_\lambda) \quad (2.14)$$

We then have

$$\begin{aligned} P_\lambda(v_\lambda, v'_\lambda) &\leq P_\lambda(v^*, z^* - p + u_\lambda) + \\ &+ u'_\lambda(v_\lambda - v^*) + u_\lambda(v'_\lambda - z^* + p - u_\lambda) \leq \\ &\psi(v^*) + \frac{1}{2} |v^* - q|^2 + \varphi_\lambda^*(z^*) + \\ &[(u_\lambda - u^*)(v_\lambda - v^*)]' + (u^*)'(v_\lambda - v^*) + \\ &+ u_\lambda(p - z^* + (v^*)') + u^*(v_\lambda - v^*)', \end{aligned} \quad (2.15)$$

for every $z^* \in R$.

It is convenient to choose $z^* = z_\lambda^*(x) = A_\lambda u^*(x)$. Then by (H_1)

(ii) and the wellknown property $|A_\lambda r| \leq |Ar|$, $\forall r \in D(A)$ we infer that $\{z_\lambda^*\}$ is bounded in $L^\infty(0,1)$. Moreover, (see [5, p. 91])

$$\begin{aligned} \varphi_\lambda^*(z_\lambda^*) &= z_\lambda^* u^* - \varphi_\lambda(u^*) \\ &\leq z_\lambda^* u^* - \varphi((I + \lambda A)^{-1} u^*). \end{aligned} \quad (2.16)$$

But $-\varphi$ is sublinear, therefore we have

$$\varphi_\lambda^*(z_\lambda^*) \leq \text{const.}, \quad \text{a.e. } x \in]0,1[. \quad (2.17)$$

On the other hand, by definition of conjugate function, we have

$$\begin{aligned} P_\lambda(v_\lambda, v'_\lambda) &\geq \psi((I + \lambda B)^{-1} v_\lambda) + u^*(v'_\lambda + p - u_\lambda) \\ &+ \int |v'_\lambda + p - u_\lambda| = \varphi(u^* + \int w_\lambda), \end{aligned} \quad (2.18)$$

Let us note that $\varphi(u^* + \rho w_\lambda) \leq \text{const.}$, for a.e. $x \in]0,1[$ and $\rho > 0$

small enough (see (H_1) (ii)). Thus by combining the inequalities

(2.15), (2.17), (2.18) and integrating from 0 to 1 it follows

$$\{v'_\lambda\} \text{ is bounded in } L^1(0,1). \quad (2.19)$$

By a similar device we also derive

$$\{u'_\lambda\} \text{ is bounded in } L^1(0,1). \quad (2.20)$$

Since

$$u_\lambda(x) = \int_0^1 (x u'_\lambda + u_\lambda) dx - \int_x^1 u'_\lambda(\xi) d\xi$$

from (2.11), (2.19) and (2.20) we derive

$$\{u_\lambda\}, \{v_\lambda\} \text{ are bounded in } C[0,1]. \quad (2.21)$$

Since B is everywhere defined on R it is bounded on bounded sets. Thus using (2.21) and the second equation of system (2.9) we may infer that

$$\{u'_\lambda\} \text{ is bounded in } L^2(0,1). \quad (2.22)$$

We are now able to apply Arzelà-Ascoli Criterion (see (2.21) and (2.22)) and to conclude that, on a subsequence,

$$u_\lambda \longrightarrow u, \text{ as } \lambda \longrightarrow 0, \text{ in } C[0,1]. \quad (2.23)$$

Concerning the sequence $\{v'_\lambda\}$ we have only proved its boundedness in $L^1(0,1)$ and this seems to be the best possible estimate we can

obtain. However, by Helly's principle it follows that, there exists $v_1 \in BV(0,1)$ such that, on a subsequence, we have

$$v_\lambda(x) \longrightarrow v_1(x), \text{ as } \lambda \rightarrow 0, \text{ for every } x \in [0,1]. \quad (2.24)$$

Taking into account (2.21) and (2.24) one gets by Lebesgue dominated convergence theorem

$$v_\lambda \longrightarrow v \text{ strongly in } L^p(0,1), \quad 1 \leq p < \infty. \quad (2.25)$$

Of course, in (2.25) v represents the equivalence class of v_1

with respect to " \approx " while, according to the usual convention,

the class of v_λ was identified with v_λ . Furthermore, we have, on

some subsequences

$$v'_\lambda \longrightarrow Dv_I \text{ vaguely in } M(0,1), \quad (2.26)$$

$$A_\lambda u_\lambda(\cdot) \longrightarrow \mu \text{ vaguely in } M(0,1). \quad (2.27)$$

We assert that $\mu \in \tilde{A}u$. Let $\Phi: L^2(0,1) \longrightarrow]-\infty, +\infty]$ be the

function defined by $\Phi(u) = \int_0^1 \varphi(u(x)) dx$, if $\varphi(u) \in L^1(0,1)$,

and $= +\infty$, otherwise.

We intend to pass to the limit, as $\lambda \rightarrow 0$, in the inequality

$$\begin{aligned} \int_0^1 A_\lambda u_\lambda(x) (u_\lambda(x) - h(x)) dx &\geq \Phi_\lambda(u_\lambda) - \Phi_\lambda(h) \\ &\geq \frac{1}{2\lambda} \|u_\lambda - (I + \lambda A)^{-1} u_\lambda\|_{L^2(0,1)}^2 + \Phi((I + \lambda A)^{-1} u_\lambda(\cdot)) \\ &\quad - \Phi(h), \text{ for every } h \in D(\Phi). \end{aligned} \quad (2.28)$$

From (2.23), (2.27), and (2.29) by a straightforward reasoning one deduces that

$$(I + \lambda A)^{-1} u_{\lambda}(\cdot) \rightarrow u, \text{ as } \lambda \rightarrow 0, \text{ strongly in } L^2(0,1).$$

Now by passing to the limit in (2.28) we conclude that $\mu \in \tilde{A}u$, as claimed. From (2.23), (2.24), and (2.10) one gets

$$(u(0), -u(1)) \in L(v_1(0), v_1(1)). \quad (2.29)$$

On the other hand, from (2.22), (2.23), (2.25), (2.26), and (2.27) one obtains

$$\begin{aligned} u - Dv_1 + \mu &= p \\ v - u' + \bar{B} v &\geq q. \end{aligned}$$

Summarising we have demonstrated that $(u, v) \in D(\mathcal{A})$ and satisfies (2.6). Thereby Lemma 2.1 is completely proved.

Remark 2.1. Let μ belong to $M(0,1)$ and let $q \in BV(0,1)$ so that

$\mu = Dq$. Then we have the decomposition

$$\mu = \mu_a + \mu_s \quad (2.30)$$

where $\mu_a = \dot{q}$, $\mu_s = Dq_s$; $q_s(x) = q(x) - \int_0^x \dot{q}(\xi) d\xi$, that is q_s is

the singular part of q . Here \dot{q} is the ordinary derivative of q

which exists almost everywhere on $]0,1[$, and is in $\mathcal{L}^1(0,1)$.

According to [11, 15], if $\mu \in \tilde{A}u$ then we have

$$\mu_a(x) \in Au(x), \text{ a.e. } x \in]0,1[, \quad (2.31)$$

and

$$\mu_s(u-h) \geq 0, \text{ for every } h \in C[0,1], \text{ such that} \\ h(x) \in \overline{D(\varphi)}, \quad \forall x \in [0,1]. \quad (2.32)$$

The operator \tilde{A} defined by (2.1) and (2.2) is also considered in [1, 16], in a different context.

THEOREM 2.1. Assume that hypotheses (H_1) , (H_2) , and (H_3) hold.
Let f, g be given in $W^{1,1}(0,T; L^2(0,1))$ and let (u_0, v_0) belong to $D(\tilde{A})$ defined by (2.3). Then, there exists a unique pair of functions $(u, v) \in W^{1,\infty}(0,T; L^2(0,1))$ such that

$$u, v \in L^\infty([0, T[\times]0, 1[); \frac{\partial u}{\partial x} \in L^\infty(0, T; L^2(0, 1)); \quad (2.33)$$

for each $t \in [0, T[$, $v(t, \cdot) \in BV(0, 1)$ and we have

$$\alpha(x) \frac{\partial^+ u}{\partial t}(t, x) - \dot{v}_x(t, x) + Au(t, x) \ni f(t, x), \text{ a.e. } x \in]0, 1[, \quad (2.34)$$

$$\beta(x) \frac{\partial^+ v}{\partial t}(t, x) - \frac{\partial u}{\partial x}(t, x) + Bv(t, x) \ni g(t, x), \text{ a.e. } x \in]0, 1[, \quad (2.35)$$

$$D_x v_s(t, \cdot)(u(t, \cdot) - h) \geq 0, \quad \forall h \in C[0, 1]; \quad h(x) \in \overline{D(\varphi)}, \quad 0 \leq x \leq 1; \quad (2.36)$$

in addition u, v satisfy conditions (1.2) and (1.3).

Here $\frac{\partial^+ u}{\partial t}, \frac{\partial^+ v}{\partial t}$ denote the right partial derivatives of u, v :

$[0, T[\rightarrow L^2(0, 1)$; for each $t \in [0, T[$, $\frac{\partial u}{\partial x}(t, \cdot)$ is the distri-

butional derivative of $u(t, \cdot)$ while $\dot{v}_x(t, \cdot)$ denote the ordinary

derivative of $v(t, \cdot)$, which does not coincide to the distri-

butional derivative of $v(t, \cdot)$. The notation \dot{v}_x is chosen to

point out this distinction. Of course, $\frac{\partial u}{\partial x}(t, \cdot)$ coincides to

$\dot{u}_x(t, \cdot)$. Finally $v_s(t, \cdot)$ is the singular part of $v(t, \cdot)$, i.e.

$$v_s(t, x) = v(t, x) - \int_0^x \dot{v}_{\xi}(t, \xi) d\xi,$$

and, $D_x v_s(t, \cdot)$ is the measure generated by $v_s(t, \cdot)$.

Remark 2.1. In fact, in Theorem 2.1, for each t , the equivalence class $v(t, \cdot)$ contains a function $v^1(t, \cdot) \in BV(0, 1)$ such that $u(t, \cdot)$, $v^1(t, \cdot)$ verify (1.3) and (2.36). Clearly $v^1(t, \cdot)$ is determined on $[0, 1]$ apart from a countable set of $x \in [0, 1]$. If $L^{-1}(u(0, t), -u(1, t))$ is a singleton then $v^1(t, \cdot)$ is determined up to the class of functions of bounded variation belonging to $v(t, \cdot)$ and generating the same measure.

Proof of Theorem 2.1. Firstly, we assume that $\alpha(x) = \beta(x) = 1$, a.e. $x \in]0, 1[$. Let us reformulate (1.1), (1.2), (1.3) as the following abstract Cauchy problem on the space $X = L^2(0, 1) \times L^2(0, 1)$,

$$\frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + A \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \ni \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad (2.37)$$

$$\begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \quad (2.38)$$

According to Lemma 2.1 the general theory of ordinary differential equations associated to monotone operators can be applied to obtain partially the assertion made in the statement

of Theorem 1. Thus there exist $u, v \in W^{1,\infty}(0,T; L^2(0,1))$ which satisfy (2.37) and (2.38). Therefore, for each $t \in [0, T[$, $v(t, \cdot)$ contains a function $v^1(t, \cdot) \in BV(0,1)$ such that $u(t, \cdot), v^1(t, \cdot)$ satisfy (1.3), and there exists $\mu(t) \in \tilde{A}u(t, \cdot)$ such that

$$\begin{aligned} \frac{\partial^+ u}{\partial t}(t, \cdot) - D_x v^1(t, \cdot) + \mu(t) &= f(t, \cdot), \text{ in } L^2(0,1), \\ \frac{\partial^+ v}{\partial t}(t, \cdot) - \frac{\partial u}{\partial x}(t, \cdot) + \bar{B}v(t, \cdot) &\ni g(t, \cdot), \text{ in } L^2(0,1). \end{aligned} \quad (2.39)$$

Since $\mu(t) - D_x v^1(t, \cdot) \in L^2(0,1)$ it follows that

$$D_x v^1(t, \cdot) = \mu_s(t), \quad t \in [0, T[. \quad (2.40)$$

Therefore by Remark (2.1), (2.34), (2.35) and (2.36) are fulfilled.

In order to prove (2.33) we intend to use Theorem 3.16 in [6, p. 102]. Let us denote by \mathcal{A}^λ the operator defined on X by

$$\mathcal{A}^\lambda \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -v' + A_\lambda u \\ -u' + B_\lambda v \end{pmatrix} \quad (2.41)$$

with $D(\mathcal{A}^\lambda) = D(T)$ (see (2.8)). A revision of the proof of Lemma

1 shows us that

$$(I + \mathcal{A}^\lambda) \begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow (I + \mathcal{A})^{-1} \begin{pmatrix} p \\ q \end{pmatrix}, \text{ as } \lambda \rightarrow 0, \text{ strongly in } X.$$

Therefore by the above cited theorem the solution (u_λ, v_λ) of

the approximating Cauchy problem

$$\frac{d^+}{dt} \begin{pmatrix} u_\lambda(t) \\ v_\lambda(t) \end{pmatrix} + \mathcal{A}^\lambda \begin{pmatrix} u_\lambda(t) \\ v_\lambda(t) \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad 0 \leq t < T, \quad (2.42)$$

$$\begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} (0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad (2.43)$$

converges in $C([0, T]; X)$ to the solution (u, v) of (2.37), (2.38).

By a standard argument it follows that

$$\left\{ \frac{d}{dt} \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix} \right\} \text{ is bounded in } L^\infty(0, T; X) . \quad (2.44)$$

Thus, by the procedure used in the proof of Lemma 1 that we do not repeat here it may be obtained the following estimates

$$\left\{ \frac{\partial u_\lambda}{\partial x} \right\} \text{ is bounded in } L^\infty(0, T; L^2(0, 1)) , \quad (2.45)$$

$$\{u_\lambda\} \text{ and } \{v_\lambda\} \text{ are bounded in } L^\infty([0, T[\times]0, 1[) . \quad (2.46)$$

From (2.45), (2.47) one obtains (2.33).

The general case when α, β are not identically equal to 1 reduces to the above one. Indeed, it suffices to divide both equations of (1.1) by α and β , respectively, and to observe that the operator

$$\tilde{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{1}{\alpha} K_{uv} + \frac{1}{\alpha} \tilde{A}u \\ -\frac{1}{\beta} u' + \frac{1}{\beta} \tilde{B}v \end{pmatrix} , \text{ with } D(\tilde{A}) = D(A)$$

is maximal monotone on the weighted space $L^2(0, 1; \alpha(x)dx) \times L^2(0, 1; \beta(x)dx)$. The Theorem 2.1 is now completely proved.

Remark 2.2. The roles of A and B can be reversed. Specifically, assumptions (i) and (ii) of (H_1) become

$$(1)' \quad D(\varphi) = R .$$

(11)' There is a function $v_* \in H^1(0,1)$ such that $(v_*(0), v_*(1)) \in D(L)$ and $v_*(x) \in \text{Int } D(\psi)$, $\forall x \in [0,1]$.

Of course, this change produces the reversibility of properties of u and v , where (u,v) represents the solution of (2.37), (2.38).

Remark 2.3. Theorem 1 remains true if B is replaced by a single-valued function $B_1(x,r):]0,1[\times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies Caratheodory conditions and is increasing with respect to the second variable. In fact, with the exception of some minor modifications, the same procedure is applicable to this case.

Remark 2.4. The assumption $D(\varphi) \neq \mathbb{R}$ we made in Theorem 2.1 generates one of the main difficulties of the problem. Of course, assuming $D(\varphi) = \mathbb{R}$ the reader can easily observe that in this simpler case the solution (u,v) is of classical type.

Remark 2.5. Our treatment can also be applied to the systems of the form (1.1) with $2n$ equations, specifically, when A and B are subdifferentials from \mathbb{R}^n into itself. Since this extension does not generate new aspects or difficulties we have confined ourselves to the simplest case of 2 equations.

3. ASYMPTOTIC BEHAVIOUR

We begin this section by assuming that $D(\varphi) = D(\psi) = R$. Then it is not difficult to show by using the procedure in the proof of Lemma 1 that the resolvent of \mathcal{A} is a completely continuous operator (see also [12]) and even more $(I + \mathcal{A})^{-1}$ maps bounded subsets of X into bounded subsets in $H^1(0,1) \times H^1(0,1)$. This fact together with the assumption $0 \in R(\mathcal{A})$ assures (see [9]) the precompactness for the orbits of the semigroup $\{S(t); t \geq 0\}$ generated by $-\mathcal{A}$ and defined on $\overline{D(\mathcal{A})}$ which is actually equal to X . Then, sufficient conditions assuring the strong convergence of solutions in the topology of X , as t approaches ∞ , can be formulated in a manner similar to that of [12]. Moreover, if (u_0, v_0) is taken in $D(\mathcal{A})$ then the associated orbit $\bigcup_{t \geq 0} S(t)(u_0, v_0)$ is bounded in $H^1(0,1) \times H^1(0,1)$ so in particular it is precompact in $C[0,1] \times C[0,1]$.

The last assertion is a consequence of the following simple equality

$$S(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = (I + \mathcal{A})^{-1} \left(S(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \frac{d}{dt} S(t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right), \quad t \geq 0.$$

Under the weaker hypotheses (H_1) we failed in attempt to prove that

$(I + \mathcal{A})^{-1}$ is completely continuous so the technique of Dafermos

and Slemrod [9] cannot be applied to this case. However, in the

sequel we shall state an asymptotic result using the concept of "demipositivity" introduced by R.E.Bruck, Jr. [7] .

DEFINITION 3.1. Let M be a monotone operator on a real Hilbert space H . Then M is demipositive if there exists $y_0 \in M^{-1}0$ which satisfies:

(A) the conditions $x_n \rightarrow x$ weakly, $v_n \in M x_n$, $\{v_n\}$ bounded, and $\lim_{n \rightarrow \infty} \langle v_n, x_n - y_0 \rangle_H = 0$ imply $0 \in Ax$.

Here $\langle \cdot, \cdot \rangle_H$ denotes the scalar product of H .

THEOREM 3.1. Assume that (H_1) , (H_2) , (H_3) hold; $(u_0, v_0) \in \text{Cl } D(A)$; $f, g \in L^1(0, \infty; L^2(0, 1))$; $(0, 0) \in L(0, 0)$, and $A^{-1}0 = \{0\}$, $B^{-1}0 = \{0\}$. Then,

$$u(t, \cdot) \rightarrow 0, \text{ as } t \rightarrow \infty, \text{ strongly in } L^2(0, 1), \quad (3.1)$$

$$v(t, \cdot) \rightarrow 0, \text{ as } t \rightarrow \infty, \text{ weakly in } L^2(0, 1), \quad (3.2)$$

where (u, v) is the weak solution on $[0, \infty[$ of (1.1), (1.2), (1.3) corresponding to $(u_0, v_0; f, g)$.

If in addition $(\dot{u}_0, v_0) \in D(A)$ and $f, g \in W^{1,1}(0, \infty; L^2(0, 1))$

then

$$u(t, \cdot) \rightarrow 0, \text{ as } t \rightarrow \infty, \text{ weakly in } H^1(0, 1),$$

and so in particular strongly in $C[0, 1]$. --- (3.1)'

The symbol " Cl " means the closure in $X = L^2(0, 1) \times L^2(0, 1)$.

A weak solution of (1.1), (1.2), (1.3) on $[0, T]$ is the uniform limit in X of any sequence of solutions obtained by Theorem 2.1 corresponding to initial data in $D(\mathcal{A})$ and to right-hand sides in $W^{1,1}(0, T; L^2(0, 1))$ (see [2, p.134]).

Proof. To prove (3.1) and (3.2) we may assume without any loss of generality that $\alpha(x) = \beta(x) = 1$, a.e. $x \in]0, 1[$ and $f = g = 0$ (see e.g. [12]). Clearly we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathcal{A}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We assert that \mathcal{A} is demipositive. To show this let (u_n, v_n) and (z_n, w_n) be some sequences such that

$$\begin{pmatrix} z_n \\ w_n \end{pmatrix} \in \mathcal{A} \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \text{and} \quad \left\{ \begin{pmatrix} z_n \\ w_n \end{pmatrix} \right\} \text{ is bounded in } X, \quad (3.3)$$

$$u_n \longrightarrow \bar{u}, \quad v_n \longrightarrow \bar{v} \quad \text{weakly in } L^2(0, 1), \quad (3.4)$$

and

$$\lim_{n \rightarrow \infty} \left\langle \begin{pmatrix} z_n \\ w_n \end{pmatrix}, \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\rangle_X = 0. \quad (3.5)$$

By means of (2.5) this becomes

$$\lim_{n \rightarrow \infty} \{ [u_n(0)v_n(0) - u_n(1)v_n(1)] + \mu_n(u_n) + \int_0^1 \delta_n(x)v_n(x)dx \} = 0, \quad (3.6)$$

where $\mu_n \in \tilde{\mathcal{A}}u_n$ and $\delta_n \in \bar{\mathcal{B}}v_n$.

Let us assume that $\varphi(0) = \psi(0) = 0$. Then (3.6) leads us to

$$\lim_{n \rightarrow \infty} \int_0^1 \varphi(u_n(x))dx = \lim_{n \rightarrow \infty} \int_0^1 \psi(v_n(x))dx = 0.$$

This together with (3.4) yields

$$\varphi(\bar{u}(x)) = \psi(\bar{v}(x)) = 0, \text{ a.e. } x \in]0, 1[, \text{ so that}$$

$$\bar{u}(x) = \bar{v}(x) = 0 \text{ a.e. } x \in]0, 1[. \quad (3.7)$$

From (3.5), (3.7) and Lemma 1.3 in [2, p. 42] it follows that

$$(\bar{u}, \bar{v}) \in D(\mathcal{A}).$$

So \mathcal{A} is indeed demipositive and in particular $\mathcal{A}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

The result of Bruck [7] can be applied to obtain the weak convergence of solutions. In order to prove (3.1)' assume $(u_0, v_0) \in D(\mathcal{A})$ and $f, g \in W^{1,1}(0, \infty; L^2(0, 1))$. Let us again consider the problem (2.42), (2.43) this time on $[0, \infty[$. By a standard device one obtains

$$\left\{ \frac{d}{dt} \begin{pmatrix} u_\lambda(t) \\ v_\lambda(t) \end{pmatrix} \right\} \text{ is bounded in } L^\infty(0, \infty; L^2(0, 1)).$$

By the same reasoning as in the proof of Lemma 3.1 one deduces that

$$\frac{\partial u}{\partial x} \in L^\infty(0, \infty; L^2(0, 1)).$$

Therefore $\{u(t, \cdot), t \geq 0\}$ is bounded in $H^1(0, 1)$ and (3.1)' is now proved because we know $u(t, \cdot) \rightarrow 0$, as $t \rightarrow \infty$, weakly in $L^2(0, 1)$.

Then (3.1) follows by a simple completion argument.

Theorem 3.1 is now completely proved.

4. AN EXAMPLE

A mathematical model of transmission line phenomena (see Cooke and Krumme [8]) is the problem

$$L \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} + Ru + e(t, x) = 0, \quad (4.1)$$

$$C \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + Gv = 0$$

for $0 < x < 1, t > 0,$

$$u(x, 0) = -i_0(x); \quad v(x, 0) = v_0(x), \quad 0 \leq x \leq 1, \quad (4.2)$$

$$v(t, 0) = R_0 u(t, 0); \quad -u(t, 1) \in f_0(v(t, 1)), \quad t \geq 0, \quad (4.3)$$

where $i = -u$ represents the current flowing in the line and v

is the voltage across the line; the constants R and R_0, L, G, C

are resistances, inductance, conductance, and capacitance; e is

the voltage per unit length impressed along the line in series

with it. By physical reasons $L > 0, C > 0, R \geq 0, R_0 > 0,$ and $G \geq 0.$

The multivalued function f (whose graph is assumed here to be

maximal monotone in $R \times R$) represents a nonlinear resistance.

Let us require that $u(t, x) = -i(t, x)$ deviates as little as possible

from a prescribed interval $K = [a, b] \subset R.$ To achieve this we shall

interpret the term e as a feedback distributed control.

So let us choose e to be the multivalued function $e(u) = \partial I_K(u).$

In this case (4.1) becomes

$$\begin{aligned} L \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} + Ru + \partial I_K(u) &\ni 0, \\ C \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} + Gv &= 0. \end{aligned} \quad (4.1)'$$

Recall that $I_K(u) = 0$ if $u \in K$, and $= +\infty$, otherwise. Theorem 2.1 can be directly applied to the problem (4.1)', (4.2), (4.3). The control is achieved because (see (2.34)) for each $t \geq 0$, $u(t, x) \in D(\partial I_K) = [a, b]$, a.e. $x \in]0, 1[$.

The above control problem was suggested us by [10, p.21] where the "thermostatic control" for the heat equation is discussed.

ACKNOWLEDGEMENT

The author wishes to thank Professor V. Barbu for many helpful discussions and comments concerning this paper.

REFERENCES

1. S. AIZICOVICI, On an abstract Volterra equation, in "Volterra Equations", Lecture Notes in Math. 737, Springer-Verlag, Berlin/Heidelberg/New York, 1979.
2. V. BARBU, "Nonlinear Semigroups and Differential Equations in Banach Spaces", Noordhoff, Leyden, 1976.
3. V. BARBU, Nonlinear boundary-value problems for a class of hyperbolic systems, Rev. Roumaine Math. Pures Appl. 22 (1977), 155-168.
4. V. BARBU and G. MOROȘANU, Existence for a nonlinear hyperbolic system, Nonlinear Anal., to appear.
5. V. BARBU and T. PRECUPANU, "Convexity and Optimization in Banach Spaces", Sijthoff & Noordhoff, Bucharest, 1978.
6. H. BREZIS, "Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert", North-Holland, Amsterdam, 1973.
7. R. E. BRUCK, JR., Asymptotic convergence of nonlinear contraction semigroups in Hilbert space, J. Funct. Anal. 18 (1975), 15-26.
8. K. L. COOKE and D. W. KRUMME, Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations, J. Math.

Anal. Appl. 24 (1968), 372-387.

9. C. M. DAFERMOS and M. SLEMROD, Asymptotic behavior of non-linear contraction semigroups, J. Funct. Anal. 13 (1973), 97-106.
10. G. DUVAUT and J. L. LIONS, "Inequalities in Mechanics and Physics", Springer-Verlag, Berlin/Heidelberg/New York, 1976.
11. A. D. IOFFE and V. L. LEVIN, Subdifferentials of convex functions, Trudy Moscov. Obšč. 26 (1972), 3-71 (Russian).
12. G. MOROȘANU, Stability of solutions of nonlinear boundary-value problems for hyperbolic systems, Nonlinear Anal., to appear.
13. G. MOROȘANU, Existence for nonlinear differential systems of hyperbolic type, An. Stiint. Univ. "Al. I. Cuza" Iași, Sect. I-a Mat. (N.S.), to appear.
14. F. RIESZ and B. SZ. NAGY, *Leçons d'analyse fonctionnelle*, Gauthier-Villars, Paris, 1965.
15. R. T. ROCKAFELLAR, Integrals which are convex functionals, II, Pacific J. Math. 39 (1971), 439-469.
16. M. SCHATZMAN, Thesis, P. and M. Curie University, January, 1979.

