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ON M-SPECTRAL SETS AND RATIONALLY INVARIANT SUBSPACES

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1. Introduction. Let H be a separable, infinite dimensional, complex Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. If A is a subalgebra of $L(H)$ a nontrivial A-invariant subspace is a subspace M of H such that $(0)\neq M\neq H$ and such that $AM\subset M$ for each A in A. If A is the algebra of all polynomials in o fixed operator A an A-invariant subspace is exactly an invariant subspace and if A is the commutant of A an A-invariant subspace is exactly an hyperinvariant subspace for A.

One of the most recent tools used to establish the existence of invariant subspaces is the technique introduced by S.Brown in proving that every subnormal operator has a nontrivial invariant subspace [7]. This technique is particularly appealing for the class of operators A for which there exist a bounded open set G in C such that G^- is an M-spectral set for A (i.e. $\|r(A)\| \leqslant M \sup_{\lambda\in G^-} |r(\lambda)|$ for every rational function r with poles off G^-) and such that $G\cap\sigma(A)$ is dominating in G (i.e. $\|h\|_\infty = \sup_{\lambda\in\sigma(A)\cap G} |h(\lambda)|$ for any function in $H^\infty(G)$, the Banach algebra of functions bounded and analytic on G equipped with the supremum norm). Nevertheless, so far, positive results have been obtained only with considerable restrictions on G and/or A; for example A is a contraction and G is the open unit disk in [8],

$\sigma(A)=G^-$ in [13], this latter case but with M=1 being previously solvend in [1]; see also [2], [3], [4], [9]. Another disturbing feature of most of these results is that subspaces invariant only under A are produced while one would expect to obtain $R_{G^-}(A)$-invariant subspaces; that is subspaces invariant under the algebra $R_{G^-}(A)=\{r(A):r$ rational function with poles off $G^-\}$.

By studying representations of $H^\infty(G)$ into $L(H)$ the authors of [9] offered an approach which, when it works, automatically produces an $R_{G^-}(A)$-invariant subspace. In this paper we are able to apply successfully this approach to a fairly broad class of sets G, namely those bounded open sets G such that $R(G^-)$ is pointwise boundedly dense in $H^\infty(G)$ and such that $R(\partial G)=C(\partial G)$ (Theorem 4.1). (For a compact set X in C we denote as usual by C(X) the algebra of complex continuous functions equipped with the sup norm and by R(X) the closure in C(X) of the algebra of rational functions with poles off X). In particular we show (Theorem 8.1) that this class of sets includes any bounded open set G such that G^- has a finite number of holes.

This result generalizes Theorem 4.2 of [9] which covers the case when the boundary of G consists of a finite number of Jordan loops. Our second application refers to the case when $\sigma(A)$ is an M-spectral set for A. We improve Stampfli's result [13] by showing (Theorem 8.2) that for any finite set of holes in $\sigma(A)$ there is a nontrivial subspace of H invariant under any rational function of A with poles in the union of these holes.

The paper is organized as follows. The first four sections are dedicated to the proof of the basic result (Theorem 4.1). To apply Theorem 3.1 of [9] to our situation we need first to extend the representation of $R(G^-)$ into $L(H)$ (obtained from the fact that G^- is M-spectral for A) into a representation of $H^\infty(G)$ and next to show that this extension satisfies the proper hypotheses. These two problems reduce essentially to establishing the w_*-S.O.T. sequential continuity of the corresponding representations. (We say that a representation is w_*-S.O.T. sequential continuous if it maps a sequence of functions that converges to 0 in the weak $*$ topology into a sequence of operators that converges to 0 in the strong operator topology). Of course these continuity difficulties appear more or less explicitly in any application of the S.Brown technique. So far they have been solved by transfering the problem to the unit disk where a result of Nagy-Foiaş (Theorem II 5.4 of [14]) enables one to assume that either A^n or A^{*n} tends to 0 in the strong operator topology. This going back to the unit disk is precisely the basic limitation of previous applications of the S.Brown technique (especially with regard to the type of invariant subspace produced). Our main innovation is to deal directly with the continuity difficulties via a result (Corollary 3.2) which, roughly speaking, generalizes the above theorem of Nagy-Foiaş and enables us to exhibit a nontrivial hyperinvariant subspace for A whenever the w_*-S.O.T. sequential continuity property is not satisfied. The proof of Corollary 3.5. is broken into two steps. In § 2 we show how the w_*-S.O.T. sequential discontinuity implies the existence of nontrivial intertwinings between A and M_z on $R(\partial G)$ on one hand and between A^* and $M_{\bar{z}}$ on $R(\partial G^*)$ on the other hand. In §3 we show how these intertwinings lead to a nontrivial hyperinvariant subspace for A. Here the hypothesis $R(\partial G)=C(\partial G)$ plays a crucial role via the characterization of closed ideals in $C(X)$. The first author was initially lead to Corollary 3.2 via local spectral theory techniques; we briefly sketch this approach of the end of §3.

Section 4 completes the proof of Theorem 4.1. The applications of this theorem that we give rely heavily on the results of [11] on pointwise bounded approximation and Dirichlet algebras. The basic definitions and results that we need are presented in §5 in a form suitable to our purposes; as a tool for generalizing Stampfli's result and in connection with Dirichlet algebras, we develop in §6 a natural (and, we believe, interesting in its own right) partition of the set of holes of a connected compact set in the plane. Before concluding the proof of the applications in §8 we need a few additional results on $H^\infty(G)$ which we present in §7.

2. w_*-S.O.T. sequential discontinuity of representations and intertwinings

Throughout this section G is an arbitrary bounded open set of C, A is an operator in $L(H)$ having G^- as an M-spectral set, A is a subalgebra of $H^\infty(G)$ containing $R(G^-)$, and Φ is a bounded linear map from A into $L(H)$ such that $\Phi(rf)=\Phi(r)\ \Phi(f)=\Phi(f)\Phi(r)$ and $\Phi(r)=r(A)$ for any r rational with poles off G^- and for any f in A. Furthermore we assume that for any μ in G and any f in A the function f_μ defined by

$$f_\mu(\lambda)=\frac{f(\lambda)-f(\mu)}{\lambda-\mu} \quad \lambda\neq\mu \ , \quad f_\mu(\mu)=f'(\mu)$$

also belongs to A. [Though A will be either $R(G^-)$ or $H^\infty(G)$ which clearly satisfy the above conditions we need this general setting to avoid using nearly identical arguments in two different places; note also that Φ is not required to be an homomorphism; in fact it will be a consequence of the results of this section that, under the additional hypotheses of Theorem 4.1, Φ can be assumed to be multiplicative].

Finally, naturally associated with Φ is another representation Φ^* defined as follows. For $X\subset C$ we set $X^*=\{\bar{z}: z\in X\}$ and if f is a function on X we define $f^\sim$ on X^* by $f^\sim(z)=\overline{f(\bar{z})}$; then Φ^* is defined on $A^*=\{f:f^\sim \in A\}$ by $\Phi^*(f)=\Phi(f^\sim)^*$. The following is the key result of this section.

THEOREM 2.1. Suppose that A has noeigenvalues and that Φ^* is not w_*-SOT sequentially continuous. Then there is a nonzero operator T from $R(\partial G)$ into H such that $TM_z=AT$. (M_z denotes the operator of multiplication by z).

Before proving this theorem we establish two lemmas.

LEMMA 2.2. If Φ^* not w_*-SOT sequentially continuous then there exists a sequence of functions f_n in A converging pointwise boundedly to 0 and a sequence of unit vectors x_n in H such that $\Phi(f_n)x_n$ converges weakly to a nonzero vector y.

Proof. If Φ^* is not w_*-SOT sequentially continuous then there exists a sequence f_n in A converging pointwise boundedly to 0 and a vector x in H such that $(\Phi(f_n))^*x$ does not tend to 0. (We use here the obvious fact that the map $f \rightarrow f^\sim$ is an (isometric) weak $*$ homeomorphism of A onto A^*; also recall that for sequences in $H^\infty(G)$ weak $*$ convergence to 0 is equivalent to pointwise bounded convergence to 0.)

By dropping to a subsequence we may assume that $\lim ||(\Phi(f_n))^*x||=a>0$. Let $x_n=(\Phi(f_n))^*x_n/\|(\Phi(f_n))^*x_n\|$; the sequence $\Phi(f_n)x_n$ is bounded and, again by dropping to a subsequence, we may assume that it is weakly convergent to some vector y. The equalities

$(y,x)=\lim\,(\Phi(f_n)x_n,x)=\lim\,(x_n,\Phi(f_n)^*x)=\lim\,||\Phi(f_n)^*x||$

show that $(y,x)\neq 0$; consequently y is nonzero as desired. □

The operator T of Theorem 2.1. will represent a sort of $R(\partial G)$-functional calculus for A but localized on y. The following approximation lemma will enable us to "remove" the undesirable poles of rational functions in $R(\partial G)$ (that is the poles that are in G).

LEMMA 2.3. Let f_n be a sequence in A converging pointwise boundedly to 0 and let φ be a rational function with poles of ∂G. Then there exists a sequence of polynomials P_n such that:

1) $||P_n||_\infty$ tends to 0 as n tends to ∞, and
2) $\varphi(f_n-P_n)$ belongs to A and converges pointwise boundedly to 0 as n tends to ∞.

Proof. Let $\lambda_1,\dots,\lambda_k$ denote the poles of φ which are in G and let $\alpha_1,\dots,\alpha_k$ be their multiplicity. (Of course if φ has no poles in G we set $P_n=0$). By ([12], Chap.V, § 2) we can find a system of polynomials $L_{j,i}$ such that $L_{j,i}^{(\ell)}(\lambda_m)=\delta_{j,\ell}\delta_{i,m}$ for $1\leq i,\ m\leq k$ and $0\leq\ell,\ j\leq\alpha_i$ (here δ is the Kronecker symbol and $h^{(\ell)}$ denotes the ℓ-th derivative of h). Let P_n be the sequence of polynomials defined by

$$P_n=\sum_{i=1}^{k}\sum_{j=0}^{\alpha_i-1} f_n^{(j)}(\lambda_i)L_{j,i}\quad .$$

For each fixed pair j,i the sequence $f_n^{(j)}(\lambda_i)$ tends to 0 as $n\to\infty$ (Cauchy integral formula and Lebesgue dominated convergence theorem); this proves that $\{P_n\}$ satisfies 1). From the definition of the $L_{j,i}$'s it is clear that each λ_i is a zero of order $\geq\alpha_i$ for f_n-P_n. Thus we can write

$$(f_n-P_n)(\lambda)=\prod_{1\leq i\leq k}(\lambda-\lambda_i)^{\alpha_i}g_n(\lambda).$$

By a repeated application of the property that f_μ is in A whenever f is in A and μ in G we see that g_n belongs to A. From the bound $||f_\mu||_\infty\leq 2||f||_\infty$ (dist$(\mu,\partial G))^{-1}$ we obtain that $\{g_n\}$ is a bounded sequence. If z is none of the λ_i's then $g_n(z)$ clearly converges to 0. The convergence to 0 at the λ_i's now follows from the Cauchy integral formula for $g_n(\lambda_i)$. We have $\varphi(\lambda)=\prod_{1\leq i\leq k}(\lambda-\lambda_i)^{-\alpha_i}\psi(\lambda)$ where ψ is a rational function with poles off G^-; therefore $\varphi(f_n-P_n)(=\psi g_n)$ converges pointwise boundedly to 0. □

<u>Proof of Theorem 2.1.</u> Since Φ^* is not w_*-SOT sequentially continuous, by Lemma 2.2 there exist a sequence f_n in A converging weak $*$ to 0 and a sequence of unit vectors x_n such that $\Phi(f_n)x_n$ converges weakly to a nonzero vector y. Let now $\varphi=R/S$ (R,S polynomials) be a rational function with poles off ∂G and let P_n be a sequence of polynomials given by Lemma 2.3 (with respect to φ and f_n). The sequence $\Phi(\varphi(f_n-P_n))x_n$ is bounded. Let u be the weak limit of any of its weak convergent subsequences. We have

$$S(A)u=\text{weak-lim}\ (\Phi(S)\Phi(\varphi(f_{n_k}-P_{n_k})x_{n_k})$$

$$=\text{weak-lim}\ (\Phi(S\varphi(f_{n_k}-P_{n_k})x_{n_k})$$

$$=\text{weak-lim}\ (\Phi(R(f_{n_k}-P_{n_k})x_{n_k})$$

$$=R(A)\text{weak-lim}\ \Phi(f_{n_k}-P_{n_k})x_{n_k}=R(A)y\ .$$

It follows from the equality $S(A)u=R(A)y$ that all weak convergent subsequences have the same limit (otherwise by difference we would have a nonzero vector w such that $S(A)\omega=0$: A would have an eigenvalue). This result together with the metrizability of the weak topology on bounded subsets of H imply that in fact $\Phi(\varphi(f_n-P_n))x_n$ is weak convergent; it also implies that the limit depends only on φ. Let then $T(\varphi)$ denote the limit. The linearity of T is immediate. To extend T to all of $R(\partial G)$ we need a bound on $T(\varphi)$. We claim first that

$$||\varphi(f_n-P_n)||_\infty \leq ||\varphi||_{\partial G}||f_n-P_n||_\infty$$

(where $||\varphi||_{\partial G}=\sup\limits_{\lambda\in\partial G}|\varphi(\lambda)|$). Indeed let ε be any positive number; there is an open neighborhood Ω of ∂G such that the rational function φ is defined on Ω and such that for λ in Ω $|\varphi(\lambda)|\leq||\varphi||_{\partial G}+\varepsilon$; the maximum modulus principle implies that

$$||\varphi(f_n-P_n)||_\infty=\sup_{\lambda\in\Omega\cap G}|\varphi(\lambda)||(f_n-P_n)(\lambda)| \;;$$

therefore we have

$$||\varphi(f_n-P_n)||_\infty\leq(||\varphi||_{\partial G}+\varepsilon)||f_n-P_n||_\infty$$

and the desired result since ε is arbitrary. As a consequence we have

$$||T(\varphi)||\leq||\Phi||\,||\varphi||_{\partial K}\limsup||f_n-P_n||_\infty=C||\varphi||_{\partial K}$$

with $C=||\Phi||\limsup||f_n||$. Thus T can be extended as a bounded linear map from $R(\partial G)$ into H. To prove that T satisfies $TM_z=AT$ let again $\varphi=R/S$ with R,S polynomials. By previous considerations we have $S(A)T(zR/S)=AR(A)y=AS(A)T(R/S)=S(A)AT(R/S)$. Since $S(A)$ is one-to-one this gives $TM_z(R/S)=AT(R/S)$. The equality extends to any φ in $R(\partial G)$ by continuity. □

3. Intertwinings with M_z and hyperinvariant subspaces.

The basic result of this section is the following theorem.

T H E O R E M 3.1. Let X be a compact subset of the complex plane such that $R(X)=C(X)$ and let A be an operator in $L(H)$ for which there exist nonzero operators T and V from, respectively, $R(X)$ and $R(X^*)$ into H such that

(1) $TM_z=AT$ and

(2) $V\tilde{M}_z=A^*V$

where M_z and $\tilde{M}_z$ denote the operator of multiplication by z on $R(X)$ and on $R(X^*)$, respectively. Then A has a nontrivial hyperinvariant subspace.

Before proving this theorem we observe that combined with Theorem 2.1 (applied to Φ and Φ^*) it leads immediately to the following result which as mentionned in the introduction can be seen as a generalization of Theorem II 5.4 of [14] .

C O R O L L A R Y 3.2. Let A be an operator in $L(H)$ and let G be a bounded open set in C such that G^- is an M-spectral set for A and such that $R(\partial G)=C(\partial G)$. Let A be a subalgebra of $H^\infty(G)$ containing $R(G^-)$ and such that for any f in A and μ in G f_μ is in A. Finally let Φ be a bounded linear map from A into $L(H)$ such that $\Phi(rf)=r(A)\Phi(f)=\Phi(f)r(A)$ for any f in A and any rational function r with poles off G^-. Then, if neither Φ nor

ϕ^* are w_*-SOT sequentially continuous, A has a nontrivial hyperinvariant subspace.

The proof of Theorem 3.1 will be broken into a few lemmas. First there is no loss of generality in assuming that neither A or A^* have eigenvalues. We denote by S the set of operators T satisfying (1) and by S^* the set of operators V satisfying (2). We summarize some elementary properties of S in the following lemma whose proof we omit. It is also clear that any statement about S has a dual version about S^*. Though we do not state these dual versions explicitly we will use them freely.

L E M M A 3.3. The set S is a submanifold of $L(R(X),H)$. Moreover for any φ in R(X), any B in the commutant of A, and any T in S the operators TM_φ and BT belong to S.

L E M M A 3.4. Let T belong to S. Then

a) Ker T is a closed ideal of R(X),

b) There exists a closed s(T) in X such that

$\text{Ker } T=\{f\in C(X) : f|s(T)=0\}$.

c) Let r denote the restriction map from C(X) into C(s(T)) $(r(f)=f|s(T))$; then there is an operator $\tilde{T}:C(s(T)) \to H$ such that $T=\tilde{T}r$ and $\tilde{T}M_z=A\tilde{T}$ where M_z is now multiplication by z on C(s(T)),

d) Let $T\neq 0$ in S and let λ_o belong to s(T); then for each $\varepsilon>0$ there is a nonzero T_1 in S such that $s(T_1)\subset\{z\in X: |\lambda-\lambda_o|<\varepsilon\}$.

Proof. a) From $AT=TM_z$ we get $q(A)T=TM_q$ for any polynomial q. Applied to $(p/q)\varphi$ (with $\varphi\in C(X)$, p,q polynomials - q does not vanish on X) this equality gives

$$q(A)T((p/q)\varphi)=TM_q((p/q)\varphi)=T(p\varphi)=TM_p\varphi=p(A)T\varphi \quad .$$

Thus if φ belongs to Ker T then so does $(p/q)\varphi$ (recall that $q(A)$ is one-to-one because A has no eigenvalues). Now any f in C(X) is a uniform limit of rational functions p_n/q_n with poles off X and we have (for φ in Ker T) $T(f\varphi)=\lim T((p_n/q_n)\varphi)=0$. Since Ker T is always a closed subspace this concludes the proof of a).

b) The existence of s(T) follows immediately from the well-known characterization of the closed ideals of C(X).

c) Any intertwining T in S induces an intertwining $\tilde{T}$: $C(X)/\text{Ker } T \to H$ between the operator $\tilde{M}_z$ induced by M_z on C(X)/Ker T and the operator A. To complete the proof observe that (via the factorization of r) C(X)/Ker T can be (isometrically) identified with C(s(T)) and that in this identification $\tilde{M}_z$ becomes M_z on C(s(T)).

d) For any T in S and any φ in C(X) we have $s(TM_\varphi)\subset \text{supp } \varphi$ (because for any ψ vanishing on supp φ we have $TM_\varphi(\psi)=T(0)=0$). Take now φ to be 1 in a neighborhood of λ_o but 0 for $|\lambda-\lambda_o|\geq\varepsilon$. Let $T_1=TM_\varphi$; then by the above remark $s(T_1)$ satisfies the desired inclusion; if $T_1=0$ then $T=TM_{(1-\varphi)}$ and $s(T)$ supp$\subset(1-\varphi)$; therefore $T_1\neq 0$ for λ_o in s(T). □

L E M M A 3.5. Let T belong to S and V to S^*. Then:

a) s(T) cannot be a singleton, and

b) if $s(T)\cap s(V)^*=\Phi$ then $Ran(T)\perp Ran(V)$.

Proof. a) If s(T) is a singleton $\{\lambda_o\}$ then C(s(T)) is one-dimensional and it follows from c) of Lemma 3.4 that λ_o is an eigenvalue for A. This contradicts the assumption that A has empty point spectrum.

b) By c) of Lemma 3.4 and its dual version for S^* we may see T and V as operators defined on C(s(T)) and C(s(V)) respectively. We want to "dualize" the intertwining $A^*V=VM_z$; to avoid the difficulty created by the mixture of Banach space and Hilbert space dualities we proceed directly as follows. For x in H we define a continuous linear functional $V'(x)$ on $C(s(V)^*)$ by

(3) $\langle V'(x), f\rangle=(x,V(\tilde{f}))$, $f\in C(s(V)^*)$ . .

It is easy to check that V' is a bounded linear map from H into $M(s(V)^*)$ (the Banach space of complex Borel measures on $s(V)^*$) and that $V'A=M'_zV'$ where M'_z is multiplication by z on $M(s(V)^*)$ (i.e. $M'_z(\mu)=\upsilon$ with $d\upsilon=z\,d\mu$). Finally the definition of V' makes it clear that Ker $V'=(\text{Ran } V)^{\perp}$. Thus we have to prove that $V'T=0$.

Let $W=V'T$; from the intertwinings $TM_z=AT$ and $V'A=M'_zV'$ we obtain $V'TM_z=V'AT=M'_zV'T$ that is $M'_zW=WM_z$. But the spectra, $s(V)^*$ and $s(T)$, of M'_z and M_z are disjoint; therefore by a well-known result $W=0$. □

<u>Proof of Theorem 3.1</u>. Let T and V be nonzero operators in S and S^* respectively. Since neither $\sigma(T)$ nor $\sigma(V)$ can be singletons we can find λ in s(T) and μ in s(V) such that $\lambda\neq\bar{\mu}$. Applying d) of Lemma 3.4 (and its dual version for S^*) with $\varepsilon<|\lambda-\bar{\mu}|/2$ we obtain nonzero operators T_1 and V_1 respectively in S and S^* such that $s(T_1)\cap s(V_1)^*=\emptyset$. For any operator B commuting with A, BT_1 is in S and satisfies $s(BT_1)\cap s(V_1)^*=\emptyset$. By b) of Lemma 3.5 we have $\text{Ran}(BT_1)\perp\text{Ran } V_1$ and the closed linear span of $\bigcup_{BA=AB} B\,\text{Ran } T_1$ is a nontrivial hyperinvariant subspace for A. □

Remark. As mentionned in the introduction Corollary 3.2 can be proved using local spectral techniques (see [5], for basic definitions and results). In that context Theorem 2.1 is replaced by a result saying that (under the same hypotheses) there exists a nonzero vector y whose local spectrum with respect to A (notation $\sigma_A(y)$) is contained in ∂G. Instead of Lemma 3.4. we have a proposition stating that once we have a nonzero vector y such that $\sigma_A(y) \subset \partial G$ we can find (under the assumption $R(X)=C(X)$) another vector $y_1 \neq 0$ with $\sigma_A(y_1) \subset X \cap \Delta(\lambda_o, \varepsilon)$ ($\lambda_o \in \sigma_A(y)$, ε arbitrary) and such that the local resolvent satisfies a certain growth condition. Finally Lemma 3.5 is replaced by the following two results of local spectral theory.

1. if $\sigma_A(y)=\{\lambda_o\}$ and the local resolvent of y, $\rho_{A,y}(\lambda)$ satisfies $||\rho_{A,y}(\lambda)|| \leq M|\lambda-\lambda_o|^{-k}$ (for some k) then λ_o is an eigenvalue of A.

2. if $\sigma_A(x) \cap (\sigma_{A^*}(y))^* = \emptyset$ (for nonzero vectors x and y) then A has a nontrivial hyperinvariant subspace.

4. <u>The main theorem</u>. We are now ready to prove the central result of the paper which is the following.

T H E O R E M 4.1. Let A be an operator in $L(H)$ and let G be a bounded open set in C such that:

(i) G^- is an M-spectral set for A,

(ii) $\sigma(A) \cap G$ is dominating in G,

(iii) $R(G^-)$ is pointwise boundedly dense in $H^\infty(G)$, and

(iv) $R(\partial G)=C(\partial G)$.

Then there exists a nontrivial $R_{G^-}(A)$-invariant subspace.

Proof. First we claim that we can assume that $G=(G^{-})^{o}$. To prove this assertion, let G satisfy (i) to (iv); we have to show that $G_1=(G^{-})^{o}$ also satisfies (i) to (iv). Since $G_1^{-}=G^{-}$ there is no problem for (i). That G_1 satisfies (iii) and (iv) follow easily from the inclusions $G\subset G_1$ and $\partial G_1\subset\partial G$. These inclusions together with the maximum modulus principle also imply that the inclusion map $H^{\infty}(G_1)\longrightarrow H^{\infty}(G)$ is an isometry; thus $\sigma(A)\cap G$ is dominating in G_1 and consequently so is $\sigma(A)\cap G_1$. Let Φ denote the representation from $R(G^{-})$ into $L(H)$ defined by $\Phi(r)=r(A)$ (Φ is a priori defined only for r rational with poles off G^{-} but, since G^{-} is M-spectral for A, it extends by continuity to all of $R(G^{-})$). By Corollary 3.3. there is no loss of generality in assuming that one - say Φ - of the two representations Φ and Φ^{*} is w_{*}-SOT sequential continuous. Let now h be in $H^{\infty}(G)$; there is a sequence r_n in $R(G^{-})$ converging pointwise to h and such that $||r_n||_{\infty}\leq||h||_{\infty}$ (Theorem 6.9 of [11]). The closed balls of $L(H)$ are compact and metrizable in the W.O.T. (weak operator topology); therefore any subsequence of the sequence $\Phi(r_n)$ has a W.O.T. convergent subsequence. Furthermore since Φ is w_{*}-S.O.T. sequentially continuous and since the W.O.T. is weaker than the S.O.T. all these convergent subsequences have the same limit. Therefore $\Phi(r_n)$ itself is W.O.T. convergent and the limit depends only on h (again because of the w_{*}-W.O.T. sequential continuity of Φ). We set $\Phi(h)=\text{W.O.T. }\lim\Phi(r_n)$; clearly $||\Phi(h)||\leq M||h||_{\infty}$; the linearity of Φ (now defined on $H^{\infty}(G)$) is immediate; finally if s is in $R(G^{-})$ and h as above we have $sh=w_{*}\lim sr_n$; thus $\Phi(sh)=\text{WOT }\lim\Phi(sr_n)=$ $=\text{WOT }\lim\Phi(s)\Phi(r_n)=s(A)\Phi(h)$; similarly we get $\Phi(sh)=\Phi(h)s(A)$. We now apply again Corollary 2.4 (with $A=H^{\infty}(G)$) to assume that either Φ or Φ^* -say Φ- is w_*-SOT sequential continuous as a map from $H^\infty(G)$ to $L(H)$. We can now prove that Φ is an homomorphism. Indeed let g, h in $H^\infty(G)$ and let s_n in $R(G^-)$ converging pointwise boundedely to g; then $s_n h$ converges pointwise boundedly to gh and

$$\Phi(gh)=\text{WOT}\lim \Phi(s_n h)=\text{WOT}\lim \Phi(s_n)\Phi(h)=\Phi(g)\Phi(h).$$

The representation Φ now satisfies all the hypotheses of Theorem 3.1. of [9]; consequently there exists a nontrivial R_G-(A)-invariant subspace. □

5. <u>C-sets and D-sets in S^2</u>. In this section we develop some material on Dirichlet algebras necessary for our applications of Theorem 4.1. Most of it is contained in [11] where characterizations of conditions (3) and (4) (in Theorem 4.1) and of Dirichlicity of R(K) are given in terms of analytic capactiy. Since we need to extend some of these results to compact subsets of the Riemann sphere we recall the basic definitions. We deal with the usual model of the Rieman sphere: $S^2=C\cup\{\infty\}$. Though most of what we say applies to any compact subset of S^2 we are interested only in ordinary compact subsets of C or in complements of bounded open subsets of C (in other words we never consider compact subsets K of S^2 such that $\infty\in\partial K$). The usual definitions of C(K), R(K) and $H^\infty(\overset{o}{K})$ extend obviously to the case $\infty\in\overset{o}{K}$ (considering that ∞ is a pole of a rational function f if 0 is a pole of $f(1/z)$ and that f is analytic at ∞ if $z\to f(1/z)$ is analytic at 0). We say that K is a <u>Dirichlet set</u> (or briefly a <u>D-set</u>) if K is a compact set such that R(K) is a Dirichlet algebra (i.e. Re(R(K)) is dense in $C_R(\partial K)$). We say that the

compact set K is a C-set if $R(\partial K)=C(\partial K)$ and $R(K)$ is pointwise boundedly dense in $H^{\infty}(\mathring{K})$. A circular transformation of S^2 is a map $\varphi: S^2 \longrightarrow S^2$ of the form $\varphi(z)=(az+b)/(cz+d)$ with $ad-bc\neq 0$. If φ is a circular transformation such that $\varphi(K)=K_1$ then the map $f \longrightarrow f\circ\varphi$ maps $R(K_1)$, $R(\partial K_1)$, $H^{\infty}(\partial K_1)$ and $C(\partial K_1)$ isometrically onto, respectively, $R(K)$, $R(\partial K)$, $H^{\infty}(K)$ and $C(\partial K)$, and preserves pointwise convergence. Consequently $\varphi(K)$ is a C-set (resp. a D-set) if and only if K is a C-set (resp. a D-set). Using a circular transformation of the type $\varphi(z)=1/z-z_o$ $(z_o \notin K)$ we see that the following results (known to be true in the case of compact subsets of the plane) remain valid in the case when $\infty \in \mathring{K}$.

P R O P O S I T I O N 5.1. ([11], Theorem 5.1). Let K be a compact subset. Then the following are equivalent:

(i) K is a D-set.

(ii) K is a C-set and each component of $\mathring{K}$ is simply connected.

P R O P O S I T I O N 5.2. ([11], Corollary 9.6). The intersection of countably many, decreasing D-sets is a D-set.

We now turn our attention to characterization of C-sets and D-sets which involve analytic capactiy (denoted by γ). Again we merely adjust results of [11] to our needs. (Here and elsewhere $\Delta(z;\delta)$ is the open disk of radius δ centered at z

P R O P O S I T I O N 5.3. ([11], Theorem 8.9). Let K be a compact subset of S^2 $(\infty \in \partial K)$. Then the following are equivalent

(i) K is a C-set,

(ii) There exists $\delta_o>0$ such that $\gamma(\Delta(z;\delta)\setminus K)=\gamma(\Delta(z;\delta)\setminus\mathring{K})$

for each $\zeta \in \partial K$ and $0<\delta<\delta_0$.

(iii) There exists a σ-curvilinear null set E such that for each $\zeta \in (\partial K) \setminus E$ there exists $r \geq 1$ satisfying

$$\liminf_{\delta \to 0} \frac{\gamma(\Delta(\zeta;\delta) \setminus K)}{\gamma(\Delta(\zeta;\delta) \cap \partial K^{o})} > 0 \ .$$

Proof. When $K \subset C$ this is Theorem 8.9 of [11] (the "localized" version of (ii) clearly implies (iii)).

Suppose now that ∞ belongs to $\overset{\circ}{K}$. Let Δ be an open disk large enough to contain the closure of $S^2 \setminus K$. The transformation $\varphi : \varphi(\zeta) = 1/(\zeta - a)$ (where a is a fixed point in $S^2 \setminus K$) maps $K \cap \Delta^-$ into $K_1 \setminus \Delta_1$ where $K_1 = \varphi(K)$ and $\Delta_1 = \varphi(S^2 \setminus \Delta^-)$; Δ_1 is an open disk whose closure is contained in $\overset{\circ}{K}_1$. Observing that conditions (ii) and (iii) are automatically satisfied on $\partial \Delta_1$ we see that K_1 is a C-set if and only if $K_1 \setminus \Delta_1$ is a C-set. Consequently K is a C-set if and only if $K \setminus \Delta$ is a C-set. The desired equivalences for K now follow from their counterparts for $K \cap \Delta^-$ (observe again that the latter is a compact subset of C for which (ii) and (iii) are automatically satisfied on $\partial \Delta$). $\square$

An immediate consequence of this result is that the characterization of D-sets (Theorem 9.3 of [11]) can be extended to compact sets K such that $\infty \in \overset{\circ}{K}$.

The following lemma will lead us to a somewhat more convenient version of this characterization when K is connected.

L E M M A 5.4. Let K be a compact, connected subset of S^2. Then the following are equivalent:

(i) ∂K is connected.

(ii) each component of $\overset{\circ}{K}$ is simply connected.

Proof. Suppose that a component G of $\overset{\circ}{K}$ is not simply connected and let L_1 and L_2 be two nonempty compact disjoint subsets of S^2 such that $S^2\setminus G=L_1\cup L_2$. From the inclusion $S^2\setminus K\subset S^2\setminus G$ we obtain $\partial K=(\partial K\cap L_1)\cup(\partial K\cap L_2)$. Since ∂K contains each ∂L_i (because $\partial L_i\subset \partial G\subset \partial K$) we have a nontrivial splitting of ∂K into two disjoint compact sets. This proves the implication (i) $\Rightarrow$ (ii).

To prove the converse we use the following observation whose proof we omit. The boundary of a simply connected domain is connected. Suppose now that each component of K is simply connected and let $\partial K=L_1\cup L_2$ be a splitting of ∂K into two disjoint compact sets. By the above observation for each component G of $\overset{\circ}{K}$, ∂G is a connected set; hence we have either $\partial G\subset L_1$ or $\partial G\subset L_2$. Let V_i denote the union of the components G of K such that $\partial G\subset L_i$ $(i=1,2)$ and let $K_i=L_i\cup V_i$. Since $K_1\cap K_2=\emptyset$ and $K=K_1\cup K_2$ the proof will be completed once we show that each K_i is closed. Let then λ_n be a sequence in, say, K_1 that converges to λ. We can assume that $\lambda\notin V_1$ (otherwise we are done). We define a sequence μ_n in L_1 as follows: $\mu_n=\lambda_n$ whenever $\lambda_n\in L_1$; if $\lambda_n\notin L_1$ then λ_n belongs to a component G of $\overset{\circ}{K}$ satisfying $\partial G\subset L_1$; in that case we choose μ_n to be the point of ∂G nearest to λ on the segement with endpoints λ_n and λ. In all cases $\mu_n\in L_1$ and $|\lambda-\lambda_n|\le|\lambda-\lambda_n|$; therefore $\lambda=\lim \mu_n$ and λ belongs to L_1, hence to K_1 as desired. □

We now state a convenient characterization of connected D-sets.

T H E O R E M 5.5. Let K be a connected compact subset of S^2 $(\infty\notin\partial K)$. Then the following are equivalent:

(i) K is a D-set.

(ii) ∂K is connected and there exists $\delta_o>0$ such that, for in ∂K and $0<\delta<\delta_o$, $\gamma(\Delta(z;\delta)\setminus K)\ge\delta/4$.

Proof. That (i) implies (ii) follows from the previous lemma together with the equivalence of (i) and (iii) in Theorem 9.3 of [11].

Now suppose that condition (ii) holds; since $\gamma(\Delta(z;\delta)\cap\partial\mathring{K})\leq\gamma(\Delta(z;\delta))=\delta$ we get at each z of ∂K $\gamma(\Delta(z;\delta)\setminus K)/\gamma(\Delta(z;\delta)\cap\partial\mathring{K})\geq 1/4$; therefore K is a C-set by Proposition 5.3; by Lemma 5.4 each component of $\mathring{K}$ is simply connected; thus K is a D-set (Proposition 5.1). □

6. Dirichlet chains for connected compact sets. In this section K denotes a fixed compact subset in the complex plane. (It will be convenient to consider K as embedded in S^2; the unbounded component of the complement of K will be identified as the hole that contains ∞). Let $\mathcal{G}$ denote the set of holes in K. For each hole H the set $K_H=S^2\setminus H$ is a D-set. (This is a well-known result if $\infty\in H$; a suitable circular transformation transfers the result to any hole-recall that here since K is connected each hole is simply connected). Now if H′ is another hole "touching" H (i.e. $\partial H\cap\partial H'\neq\emptyset$) Theorem 5.5 (or Corollary 9.7 of [11]) shows that $S^2\setminus(H\cup H')$ is still a D-set. This process can be repeated and motivates the following definition. A set $\mathcal{C}$ of holes (i.e. a subset of $\mathcal{G}$) is called a Dirichlet chain (or shorthly a D-chain) for K if the (compact) set $K_{\mathcal{C}}=S^2\setminus\bigcup_{H\in\mathcal{C}} H$ is a D-set. We denote by $\mathcal{F}$ the set of D-chains for K ordered by inclusion. Finally we define the boundary of a D-chain (notation $\partial\mathcal{C}$) to be the boundary of the corresponding D-set $K_{\mathcal{C}}$. Since K is connected (it is the union of a connected compact set with some of its holes) and Dirichlet its boundary is connected. Note that $\partial\mathcal{C}\,(=\partial K_{\mathcal{C}})=(\bigcup_{H\in\mathcal{C}}\partial H)^-$. We begin with an elementary but useful result.

L E M M A 6.1. The union of two Dirichlet chains whose boundaries overlap is a Dirichlet chain.

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two D-chains and let $\mathcal{C}=\mathcal{C}_1 \cup \mathcal{C}_2$; it follows easily from the above observation that $\partial\mathcal{C}=\partial\mathcal{C}_1 \cup \partial\mathcal{C}_2$; therefore $\partial\mathcal{C}$ is connected. The analytic capacity condition ((ii) of Theorem 5.5) is satisfied at any point of the boundary with respect to $K_{\mathcal{C}_1}$ or $K_{\mathcal{C}_2}$; it is therefore also satisfied with respect to $K_{\mathcal{C}_1 \cup \mathcal{C}_2}=K_{\mathcal{C}_1} \cap K_{\mathcal{C}_2}$ because γ is a monotone increasing set function. This concludes the proof. □

We can now prove the main result on Dirichlet chains.

T H E O R E M 6.2. Let K be a connected compact set in C and let G denote its set of holes. Then

(i) any hole in K belongs to a unique maximal D-chain (consequently these maximal D-chains determine an (at most countable) partition of G),

(ii) the boundaries of these maximal D-chains are pairwise disjoint, and

(iii) if $\mathcal{C}$ is a maximal D-chain then $K \cap \mathring{K}_{\mathcal{C}}$ is dominating in $\mathring{K}_{\mathcal{C}}$.

Before proving Theorem 6.2 we observe that the existence of maximal D-chains was already implicitly established in [3], offering a convenient substitute for the transfinite induction argument used in [1] and [13] ; we repeat the proof for completeness.

Proof. (ii) as well as the uniqueness part of (i) follow from Lemma 6.1 and from (i). Let then, to finish the proof of (i), H be a hole in K and let L be the set of D-chains containing H. L is nonempty: $\{H\} \in L$.

Let now $\{\mathcal{C}_i\}_{i \in I}$ be a totally ordered set of D-chains in

L and let $\mathcal{C} = \bigcup_{i \in I} \mathcal{C}_i$. Since G is countable we can write $\mathcal{C} = \bigcup_{n \in N} \mathcal{C}_{i_n}$ where the $\mathcal{C}_{i_n}$ are increasing; we have now $K_{\mathcal{C}} = \bigcap_{n \in N} K_{\mathcal{C}_{i_n}}$ and $K_{\mathcal{C}}$ is a D-set by Proposition 5.2; thus $\mathcal{C}$ is in L and L is an inductive set. Zorn's lemma now concludes the proof of (i). To prove (iii) let f belong to $H^\infty(\overset{o}{K}_{\mathcal{C}})$ where $\mathcal{C}$ is a maximal D-chain and let $s = \sup_{\lambda \in K \cap \overset{o}{K}_{\mathcal{C}}} |f(\lambda)|$. For any $H \in G$ we have either $H \in \mathcal{C}$ or else $H^- \subset \overset{o}{K}_{\mathcal{C}}$ (indeed by (ii) if $H \notin \mathcal{C}$, the maximal D-chain containing H has a boundary disjoint from $\partial K_{\mathcal{C}}$). For any $H \notin \mathcal{C}$ we have

$$\sup_{\lambda \in H} |f(\lambda)| = \sup_{\lambda \in \partial H} |f(\lambda)| \leq s$$

(the equality is a consequence of the maximum modulus principle, the inequality follows from the definition of s combined with the inclusion $\partial H \subset K \cap \overset{o}{K}_{\mathcal{C}}$). Therefore

$$\sup_{\lambda \in (C \setminus K) \cap \overset{o}{K}_{\mathcal{C}}} |f(\lambda)| \left(= \sup_{H \notin \mathcal{C}} \left(\sup_{\lambda \in H} |f(\lambda)|\right)\right) \leq s$$

and

$$s = \sup_{\lambda \in \overset{o}{K}_{\mathcal{C}}} |f(\lambda)|$$ as desired. □

7. <u>Splitting $H^\infty(G)$ when $G = G_1 \cap G_2$</u>. Throughout this section G_1 is a bounded open set in C, G_2 is an open set of S^2 such that $\overline{S^2 \setminus G_2} \subset G_1$, and $G = G_1 \cap G_2$. A specialized version of the following decomposition theorem was already given in [9].

THEOREM 7.1. Let G_1, G_2, and G as above and let A_2 denote the subalgebra of $H^\infty(G_2)$ that consists of those functions in $H^\infty(G_2)$ vanishing at ∞. Then there are projections P_i defined on $H^\infty(G)$ such that:

1) The ranges of P_1 and P_2 are respectively $H^\infty(G_1)$ and A_2, and $P_1+P_2=I$,

2) P_1 and P_2 are norm-continuous, and

3) P_1 and P_2 are weak$_*$-continuous.

Proof. 1) We only outline it since it is a standard application of Cauchy integral techniques. Let $0<\varepsilon< \inf\limits_{\lambda_i\in\partial G_i} |\lambda_1-\lambda_2|$.

By Problem 5K of [6] we can choose (for i=1,2) a system Γ_i of closed rectifiable Jordan curves in G_i such that:

(a) If $V_i=\{\lambda: I(\Gamma_i,\lambda)=\delta_{i,1}\}$ then $V_i^-\subset G_i$ and $\{\lambda\in G_i: d(\lambda,\partial G_i)\geq\varepsilon/4\}\subset V_i$.

(b) The geometrical range of Γ_i is the boundary of V_i.

(c) $I(\Gamma_i\lambda)=-\delta_{i,2}$ whenever $\lambda\in C\setminus V_i^-$.

(Here $I(\Gamma,\lambda)$ denotes the winding number of Γ with respect to λ and $\delta_{i,j}$ the usual Kronecker symbol). For f in $H^\infty(G)$ we set (i=1,2) $f_i(z)=\frac{1}{2\Pi i}\int_{\Gamma_i} f(\zeta)/(\zeta-z)d\zeta$ $(z\in V_i)$. The following facts (all easy consequences of the definition and of the Cauchy integral formula) conclude the proof of 1). (We let $P_i(f)=f_i$.)

- f_i is analytic in V_i and can be analytically extend on G_i.
- $f=f_1+f_2$.
- f_1 is bounded on any compact set contained in G_1 in particular on ∂G_2; thus f_2 is also bounded near ∂G_2 and

consequently belongs to $H^\infty(G_2)$ similarly f_1 belongs to $H^\infty(G_1)$.

- $f_2(\infty)=0$.
- for f in $H^\infty(G_1)$ $f_1=f$ and for f in A_2 $f_2=f$.

2) The embedding $H^\infty(G_1)\subset H^\infty(G)$ is an isometry (by the maximum modulus principle); thus the range of P_1 is norm-closed and the projection P_1 is norm-continuous (and so is P_2).

3) Since $H^\infty(G_1)$ is the dual of a separable Banach space it is enough to prove the sequential w_* continuity of P_1 ([8], Theorem 2.3). Let then f_n convering pointwise boundedly to 0 in $H^\infty(G)$. Then $P_1(f_n)$ is norm bounded and the pointwise convergence to 0 on G_1 follows from the definition together with the uniform convergence to 0 of f_n on compact sets of G. □

Next we wish to show that the notion of dominating set behaves well with respect to that decomposition of $H^\infty(G)$.

Though the result holds without restriction on G_2 we give the proof only in the case that G_2 is connected, sufficient for our applications.

T H E O R E M 7.2. Let G_1 , G_2 and G as above and suppose G_2 connected. A subset S of G that is dominating in both G_1 and G_2 is also dominating in G.

First we establish the following lemma.

L E M M A 7.3. Let S and G as in Theorem 7.2. (that is, S is dominating in G_1 and G_2). Let f in $H^\infty(G)$ such that $f|S=0$. Then f=0.

Proof. Let $f=f_1+f_2$ be the decomposition of f given by Theorem 7.1. We will show that f_1 and f_2 are identically 0. Let $\varepsilon = \inf_{\lambda_i\in\partial G_i} |\lambda_1-\lambda_2|$; by the maximum modulus principle the set

$S_i=\{\lambda\in S: d(\lambda,\partial G_i)\le\varepsilon/4\}$ is still dominating in G_i. Clearly we have $S_1\cup S_2\subset S$, $\bar S_1\subset G_2$, $\bar S_2\subset G_1$ and $\bar S_1\cap\bar S_2=\emptyset$. Let $M_{ij}=\sup |f_j(\lambda)|$.

Then we have

$$M_{11}=M_{12}\leq M_{22}=M_{21}\leq M_{11}$$

(the equalities come from the relation $f_1=-f_2$ on $S_1\cup S_2$ and the inequalities from the fact that S_i is dominating in G_i). Now the equality $M_{12}=M_{22}$ implies that f_2 reaches its maximum at some point of ∂G_1 which is in G_2; thus f_2 must be constant and equal to $0=f_2(\infty)$; now $f_1|S=-f_2|S=0$ and since S is dominating in G_1, $f_1=0$. □

<u>Proof of Theorem 7.2</u>. Proceeding as in Lemma 7.3. we may assume that $S=S_1\cup S_2$ with S_i dominating in G_i, $S_1^-\subset G_2$, $S_2^-\subset G_1$ and $S_1^-\cap S_2^-=\emptyset$. Suppose that S is not dominating in G; then there exists a function of norm one in $H^\infty(G)$ such that $\sup_{\lambda\in S}|f(\lambda)|=\alpha<1$. Any subsequence of $\{f^p\}_{p\ N}$ converges uniformly to 0 on S. Since $||f^p||=1$ for all p we can choose one (denote it f_n) which is weak$_*$ convergent to say g. Of course $g|S=0$. By the previous theorem $f_{1,n}$ and $f_{2,n}$ are weak$_*$ convergent to respectively g_1 and g_2. It follows from Lemma 7.1 that $g_1=g_2=0$. Now the weak$_*$-convergence of $f_{2,n}$ to 0 implies its uniform convergence (to 0) on the compact set S_1. By difference ($f_{1,n}=f-f_{2,n}$) we get that $f_{1,n}$ converges uniformly to 0 on S_1 and consequently $f_{1,n}$ converges to 0 in norm (recall that S_1 is dominating in K_1). Similarly $||f_{2,n}||_\infty$ tends to zero. Thus $||f_n||$ tends to zero in contradiction with the fact that $||f_n||=1$ for all n. □

8. <u>Applications</u>. We are now ready to prove the announced applications of Theorem 4.1.

T H E O R E M 8.1. Let A be an operator in $L(H)$ and let G be a bounded open set in C such that:

a) G^- is a connected, M-spectral set for A,

b) $\partial(A) \cap G$ is dominating in G, and

c) G^- has only a finite number of maximal D-chains; then

there exists a nontrivial $R_{G^-}(A)$-invariant subspace.
Before proving this theorem we make two remarks.

First the assumption that G^- is connected is necessary to talk about D-chains of G^- but is in fact nonrestrictive; indeed if G^- is disconnected then an easy argument using b) shows that $\sigma(A)$ itself is disconnected (with the consequence that A has a nontrivial hyperinvariant subspace). The other observation is that condition c) is obviously satisfied in the case when G^- has only a finite number of holes (without any restriction on their boundaries). Thus Theorem 8.1 substantially generalizes Theorem 4.2 of [9].

Proof of Theorem 8.1. Let $K=G^-$; as in the proof of Theorem 4.1 there is no loss of generality in assuming that $\mathring{K}=G$. It is sufficient to show that K is a C-set: once this is done the conclusion follows from Theorem 4.1. The boundary of K is the union of the boundaries of the maximal D-chains. Thus for each $\zeta \in \partial K$ and δ small enough (recall that the boundaries of the maximal D-chains are disjoint and there are only a finite number of them) we have $\gamma(\Delta(\zeta;\delta) \setminus K) \geq \delta/4$. Since the inequality $\gamma(\Delta(\zeta,\delta) \cap \partial\mathring{K}) \leq \delta$ is always satisfied we obtain that K is a C-set via (iii) of Proposition 5.3. □

We now turn our attention to the case of an operator having its spectrum as an M-spectral set and give the following generalization of the main result of [13].

T H E O R E M 8.2. Let A be an operator in $L(H)$ such that $\sigma(A)$ is a connected M-spectral set for A, let $\mathcal{C}_1,\ldots,\mathcal{C}_n$ be (finitely many) maximal D-chains for $\sigma(A)$, and let $K=S^2\setminus \bigcup_{\substack{H\in\mathcal{C}_i\\ 1\leq i\leq n}} H$. Then there is a non trivial $R_K(A)$-invariant subspace.

Proof. Let $K_i=S^2\setminus \bigcup_{H\in\mathcal{C}_i} H$ and $G=\overset{\circ}{K}$; we can assume without loss of generality that $\mathcal{C}_1$ contains the unbounded component of $\sigma(A)$; an induction argument based on Theorem 7.2 and Part (iii) of Theorem 6.2 shows that $\sigma(A)$ is dominating in $G(=\bigcap_{i=1}^{n}\overset{\circ}{K}_i)$. A similar argument to the one used in the proof of Theorem 8.1, shows that K is a C-set. Certainly $R(G^-)$ (which contains $R(K)$) is pointwise boundedly dense in $H^\infty(G)$; $R(\partial G)=C(\partial G)$ follows from $\partial G\subset\partial K$ and $R(\partial K)=C(\partial K)$. Thus, if $\sigma(A)\subset G^-$ we have the desired conclusion by Theorem 4.1.; on the other hand if $\sigma(A)\not\subset G^-$ an argument of [10] (developped there in the case M=1) can easily be adapted to show that A has a nontrivial hyperinvariant subspace. □

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